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Quasiperiodic solutions of Duffing's equations [1]

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1. Introduction

It is well known that the long-time behaviour of a time-dependent nonlinear differential equation

$$\frac{d^2x}{dt^2} + f(t,x) = 0, \tag{1.1}$$

where f being periodic or quasi-periodic in t, can be very intricate. For example, there are equations having unbounded solutions but with infinitely many zeroes and with nearly unbounded solutions having randomly prescribed numbers of zeroes and also periodic solutions.

In this paper, we consider the existence of quasi-periodic solutions and the boundedness of all solutions of the following equation:

$$\frac{d^2x}{dt^2} + x^{2n+1} + \sum_{k=0}^{l} x^k p_k(t) = 0, \quad l \le 2n, \tag{1.2}$$

where $p_0, \ldots, p_l$ are quasi-periodic functions with frequencies $\omega_1, \ldots, \omega_m$, i.e., there are l functions $F_k : \mathbf{T}^m \to \mathbf{R}^1$ such that $p_k(t) = F_k(\omega_1 t, \ldots, \omega_m t)$, $k = 0, \ldots, l$, $\mathbf{T}^m = \mathbf{R}^m/\mathbf{Z}^m$.

When $m = 1$, i.e., $p_0, \ldots, p_l$ are periodic functions with the same period, a few mathematicians studied Eq. (1.2). The first result of the existence of quasi-periodic

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solutions for Eq. (1.2) was due to Morris [7]. He proved that there are infinitely many quasi-periodic solutions and the boundedness of all solutions of Eq. (1.2) when $l=0$, $m=1$ and $p_0(t)$ is a continuous function. In 1987, Dieckerhoff and Zehnder in [2] proved the existence of quasi-periodic solutions if $p_0,\dots,p_l$ are sufficiently smooth. When $m=1$, $l\le 1$, Liu in [4, 5] also obtained the same conclusion when p_0 and p_1 are only continuous periodic functions. Recently, Laederich and Levi in [3] proved the existence of quasi-periodic solutions when $p_0,\dots,p_l\in C^{5+\varepsilon}(\mathbf{S}^1)$.

The method of the above-mentioned papers is as follows: By means of transformation theory, the equation is, outside of a large disc D in the $(x,\dot{x})$-plane, transformed into a Hamiltionian equation having the property. The Poincaré mapping of the equation is close to a so-called twist map in $\mathbf{R}^2\setminus D$. By means of Moser's twist theorem one finds invariant curves diffeomorphic to circles and surrounding the origin in the $(x,\dot{x})$-plane. It turns out that the solutions starting at $t=0$ on the invariant curves are quasi-periodic. Moreover, every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x,\dot{x},t)\in\mathbf{R}^2\times\mathbf{R}$, which confines the solutions in the interior and which leads to a bound of these solutions.

In constrast of the case $p_0,\dots,p_l$ are periodic, when $p_0,\dots,p_l$ are quasi-periodic functions, one cannot use the method of the above-mentioned papers to this case because the equation (1.2) is not a time-periodic equation.

On the other hand, Berger and Chen [1] studied the following equation:

$$\ddot{x}-x-x^3=p(t),$$

where $p(t)$ is quasi-periodic in time t. Using the method of the calculus of variation, they obtained a quasi-periodic solution with the same frequencies as $p(t)$.

In this paper, we assume the functions $p_0,\dots,p_l(l\le 2n)$ in Eq. (1.2) are C^∞ and quasi-periodic in time t with basic frequencies $\omega_1,\dots,\omega_m$ which satisfy the Diophantine conditions:

$$|k_1\omega_1+\cdots+k_m\omega_m|\ge\frac{c}{|k|^m},\tag{1.3}$$

where c is a positive number. Under the above assumptions, we will prove the following theorem.

Theorem. *For Eq.* (1.2), *under the above assumptions of* $p_0,\dots,p_l,l\le 2n$, *we have*
(1) *all solutions are bounded for all time*: $\mathrm{Sup}_{\mathbf{R}^1}(|x(t)|+|\dot{x}(t)|)<\infty$,
(2) *most solutions with large amplitude are quasi-periodic, that is, most initial conditions* (*in the sense of Lebesgue measure*) *with large* $|x(0)|+|\dot{x}(0)|$ *give rise to quasi-periodic solutions*: $x(t)=f(\omega_1t,\omega_2t,\dots,\omega_mt,(\omega+k)t)$ *where* f *is a function on an* $(m+1)$*-torus,* k *is a sufficiently large integer and* $\omega\in(0,1)$ *satisfies the small divisor conditions*:

$$|k_0\omega+k_1\omega_1+\cdots+k_m\omega_m|\ge\frac{c}{(|k_0|+|k_1|+\cdots+|k_m|)^{m+1}},$$

for all $(k_0,\cdots,k_m)\in\mathbf{Z}^{m+1}\setminus\{0\}$.

2. Action and angle variables

Dropping the time-dependent terms, Eq. (1.2) becomes

$$\ddot{x} + x^{2n+1} = 0.$$

Introducing $\dot{x} = y$ we have the system

$$\dot{x} = y, \qquad \dot{y} = -x^{2n+1}, \tag{2.1}$$

which is time-independent Hamiltonian system on $\mathbf{R}^2$:

$$\begin{aligned} \dot{x} &= \frac{\partial}{\partial y} h(x,y), \\ \dot{y} &= -\frac{\partial}{\partial x} h(x,y), \\ h(x,y) &= \frac{1}{2} y^2 + \frac{1}{2n+2} x^{2n+2}. \end{aligned} \tag{2.2}$$

Clearly, h is positive on $\mathbf{R}^2$ except at the only equilibrium point $(x,y) = (0,0)$ of (2.1) where $h = 0$. All the solutions of Eq. (2.2) are periodic with period tending to zero as $h = E$ tends to infinity.

Suppose $(C(t), S(t))$ is the solution of (2.1) satisfying the initial condition $(C(0), S(0)) = (1,0)$. Let $T_* > 0$ be its minimal period.

From Eq. (2.1), these analytic functions satisfy

(1) $S(t+T_*) = S(t)$, $C(t+T_*) = C(t)$ and $C(0) = 1$, $S(0) = 0$;
(2) $\dot{C}(t) = S(t)$, $\dot{S}(t) = -C^{2n+1}(t)$;
(3) $(n+1)S^2(t) + C^{2n+2}(t) = 1$;
(4) $C(-t) = C(t)$, $S(-t) = -S(t)$.

From (2.2), one can easily see that there is a $\delta_0 > 0$ such that $S(t)$ and $C(t)$ are analytic in the strip $\{|\mathrm{Im}\, t| < \delta_0\}$.

The action and angle variables are now defined by the map $\Psi : \mathbf{R}^+ \times \mathbf{S}^1 \to \mathbf{R}^2 \setminus D$, where $(x,y) = \Psi(\rho, \theta)$ with $\rho > 0$ and $\theta(\mathrm{mod} 1)$ is given by the formula

$$\Psi : \; x = a^\alpha \rho^\alpha C(\theta T_*), \qquad y = a^\beta \rho^\beta S(\theta T_*), \tag{2.3}$$

where

$$\alpha = \frac{1}{n+2}, \qquad \beta = 1 - \alpha, \qquad a = \frac{1}{\alpha T_*}.$$

We claim that Ψ is a symplectic diffeomorphism from $\mathbf{R}^+ \times \mathbf{S}^1$ onto $\mathbf{R}^2 \setminus \{0\}$. Indeed, using the Jacobian determinat Δ of Ψ one finds by (3) $|\Delta| = 1$, so that Ψ is measure-preserving. Moreover, since $(C(t)$, and $S(t))$ are solutions of Eq. (2.1) having T_* as their minimal period, one concludes that Ψ is one-to-one and onto. This proves the claim.

In the new coordinates (ρ, θ), the Hamiltonian function

$$h(x,y) = \frac{1}{2} y^2 + \frac{1}{2n+2} x^{2n+2}$$

becomes

$$h \circ \Psi(\rho, \theta) = d_0 \cdot \rho^{2\beta} =: h_0(\rho), \tag{2.4}$$

where $d_0 = a^{2\beta}/(2n+2)$, which is independent of the angle variable θ, so that the system (2.2) becomes very simple in the (ρ, θ)-plane:

$$\dot{\theta} = \frac{\partial h_0}{\partial \rho} = 2\beta d_0 \rho^{2\beta - !}, \qquad \dot{\rho} = -\frac{\partial h_0}{\partial \rho} = 0. \tag{2.5}$$

The full equation (1.2) has the Hamiltonian function

$$h(x, y, t) = \frac{1}{2} y^2 + \frac{1}{2n+2} x^{2n+2} + \sum_{k=0}^{l} \frac{1}{k+1} x^{k+1} p_k(t), \tag{2.6}$$

and, under the symplectic diffeomorphism Ψ, it is transformed into the form

$$\tilde{h}(\rho, \theta, t) = h_0(\rho) + h_1(\rho, \theta, t), \tag{2.7}$$

where

$$h_0(\rho) = d_0 \cdot \rho^{2\beta},$$
$$h_1(\rho, \theta, t) = \sum_{k=0}^{l} \frac{1}{k+1} a^{(k+1)\alpha} \rho^{(k+1)\alpha} C^{k+1}(\theta T_*) p_k(t).$$

The Hamiltonian system (1.2) now becomes more complicated as follows:

$$\begin{aligned} \dot{\theta} &= \frac{\partial \tilde{h}}{\partial \rho} = \frac{\partial h_0}{\partial \rho} + \frac{\partial h_1}{\partial \rho}, \\ \dot{\rho} &= -\frac{\partial \tilde{h}}{\partial \theta} = -\frac{\partial h_1}{\partial \rho}. \end{aligned} \tag{2.8}$$

We shall transform this system, for sufficiently ρ, into a simpler system, in which the θ-depending terms are very small. To this end, an iterate sequence of finitely many canonical transformations of $\mathbf{R}^+ \times \mathbf{S}^1$, which are dependent of time t, will be carried put in the next section.

3. Canonical transformations and propositions

First, for a given constant r, we introduce a space of functions, denoted by $\mathscr{F}_\omega(r)$, in the following:

Definition. For a given constant r, $f(\lambda, \theta, t) \in \mathscr{F}_\omega(r)$ with $\omega = (\omega_1, \dots, \omega_m) \in \mathbf{R}^m$ if f is quasi-periodic in t with frequencies $\omega_1, \dots, \omega_m$ and there exists a function F which is analytic in the domain $\Omega_{\delta_0} = \{(\lambda, \theta, \phi) | |\mathrm{Im}\, \lambda| < \delta_0, \quad \mathrm{Re}\, \lambda > 0, \quad |\mathrm{Im}\, \theta| < \delta_0,$

$|\mathrm{Im}\,\phi| < \delta_0\}$ such that $f(\lambda,\theta,t) = F(\lambda,\theta,\omega_1 t,\ldots,\omega_m t)$ and, for all nonnegative integers $j,l,l_1,\ldots,l_m$, we have

$$\sup_{\lambda\geq\lambda_{j,l,l_1,\ldots,l_m}} (\lambda^{j-r}|(D_\lambda^j)(D_\theta^l)(D_{\phi_1}^{l_1})\cdots(D_{\phi_m}^{l_m})F(\lambda,\theta,\phi)|) < \infty,$$

where $\theta\in\mathbf{S}^1$, $\phi = (\phi_1,\ldots,\phi_m)\in\mathbf{T}^m$ and $\lambda_{j,l,l_1,\ldots,l_m}$ are some positive constants.

From this definition, one can easily prove the following:

Lemma 3.1. *The following conclusions hold*:
(1) *if* $r_1 < r_2$, *then* $\mathscr{F}_\omega(r_1)\subset\mathscr{F}_\omega(r_2)$;
(2) *if* $f\in\mathscr{F}_\omega(r)$, *then* $(D_\lambda^j)f\in\mathscr{F}_\omega(r-j)$;
(3) *if* $f_1\in\mathscr{F}_\omega(r_1)$ *and* $f_2\in\mathscr{F}_\omega(r_2)$, *then* $f_1\cdot f_2\in\mathscr{F}_\omega(r_1+r_2)$;
(4) *if* $f\in\mathscr{F}_\omega(r)$ *satisfies* $|f(\lambda,\cdot,\cdot)|\geq c\lambda^r$ *for* $\lambda\geq\lambda_0$, *then* $1/f\in\mathscr{F}_\omega(-r)$.

Proof. See [2] for the details. □

For $f\in\mathscr{F}_\omega(r)$, denote the mean value over the θ-variable by $[f]$; i.e.,

$$[f] := \int_0^1 f(\lambda,\theta,\cdot)\,\mathrm{d}\theta.$$

If $\lambda_0 > 0$, then define the domain

$$A_{\lambda_0} = \{(\lambda,\theta,\phi)\,|\,\lambda\geq\lambda_0,\ (\theta,\phi)\in\mathbf{S}^1\times\mathbf{T}^m\}.$$

Note that $A_{\lambda_0}\subset\mathbf{R}^+\times\mathbf{S}^1\times\mathbf{T}^m$.

Now we consider the Hamiltonian function $h(\rho,\theta,t)$ defined by Eq. (2.7),

$$h(\rho,\theta,t) = h_0(\rho) + h_1(\rho,\theta,t),$$

where $h_1(\rho,\theta,t)$ is quasi-periodic in t with frequencies $\omega_1,\ldots,\omega_m$. In order to prove the existence of quasi-periodic solutions for the system (2.8), we need the following main proposition.

Proposition 3.1. *Let*

$$H = \lambda^a + h_1(\lambda,t) + h_2(\lambda,\theta,t),$$

with $h_1\in\mathscr{F}_\omega(c)$ *and* $h_2\in\mathscr{F}_\omega(b)$. *Assume* $a > 1$, $b < a$ *and* $c < a$. *Then, there is a canonical diffeomorphism* Ψ *depending quasi-periodically on* t *of the form*

$$\Psi : \lambda = \mu + U(\mu,\phi,t), \qquad \theta = \phi + V(\mu,\phi,t),$$

with $U\in\mathscr{F}_\omega(1-(a-b))$ *and* $V\in\mathscr{F}_\omega(-(a-b))$ *such that* $A_{\mu_+}\subset\Psi(A_{\mu_0})\subset A_{\mu_-}$ *for some large* $\mu_- < \mu_0 < \mu_+$. *Moreover, the transformed Hamiltonian vectorfield* $\Psi^*(X_H) = X_{\hat H}$ *is of the form*

$$\hat H = \mu^a + \hat h_1(\mu,t) + \hat h_2(\mu,\phi,t),$$

where $\hat{h}_1 \in \mathscr{F}_\omega(c_1)$ *with* $c_1 = \max\{c, b\}$, *is given by*

$$\hat{h}_1(\mu, t) = h_1 + [h_2],$$

and where $\hat{h}_1 \in \mathscr{F}_\omega(b_1)$. *The constant* b_1 *is smaller than* b *and is given by*

$$b_1 = b + \max\{1, b\} - a.$$

Proof. The proof is similar to the proof of the corresponding proposition in [2]. We shall look for the required transformation Ψ given by means of a generating function $S(\mu, \theta, t)$, so that Ψ is defined by

$$\Psi : \lambda = \mu + \frac{\partial S}{\partial \theta}, \qquad \phi = \theta + \frac{\partial S}{\partial \mu}. \tag{3.1}$$

Abbreviating

$$h_0(\lambda, t) := \lambda^a + h_1(\lambda, t), \qquad v := \frac{\partial}{\partial \theta} S, \tag{3.2}$$

we have the transformed Hamiltonian vectorfield $\Psi^*(X_H) = X_{\hat{H}}$, $\hat{H}$ expressed in the variables (μ, θ) instead of (μ, ϕ):

$$\hat{H}(\mu, \theta, t) = h_0(\mu + v, t) + h_2(\mu + v, \theta, t) + \frac{\partial}{\partial t} S. \tag{3.3}$$

By Taylor's formula we write

$$\hat{H}(\mu, \theta, t) = h_0(\mu, t) + h_0'(\mu, t) v + h_2(\mu, \theta, t) + R, \tag{3.4}$$

where

$$R = \frac{\partial}{\partial \theta} S + \int_0^1 (1 - \tau) h_0''(\mu + \tau v, t) v^2 \, d\tau + \int_0^1 h_2'(\mu + \tau v, \theta, t) v \, d\tau + \frac{\partial}{\partial t} S, \tag{3.5}$$

where the prime stands for the derivative in λ. We now determine v from the equation

$$h'(\mu) v + h_2(\mu, \theta, t) - [h_2](\mu, t) = 0, \tag{3.6}$$

so that

$$v = \frac{1}{h_0'(\mu)} ([h_2] - h_2), \tag{3.7}$$

and therefore,

$$S = \int_0^\theta \frac{1}{h_0'(\mu)} ([h_2] - h_2) \, d\theta. \tag{3.8}$$

Consequently,

$$\hat{H}(\mu, \theta, t) = h_0(\mu, t) + [h_2](\mu, t) + R. \tag{3.9}$$

Lemma 3.2. *Let S be defined by Eq.* (3.8). *Then the formula* (3.1) *defines a symplectic diffeomorphism* Ψ *depending on time t of the form*

$$\Psi : \lambda = \mu + U(\mu,\phi,t), \qquad \theta = \phi + V(\mu,\phi,t), \tag{3.10}$$

with $U \in \mathscr{F}_\omega(1-(a-b))$ *and* $V \in \mathscr{F}_\omega(-(a-b))$.

Proof. The proof of this lemma is very similar to the proof of corresponding lemma in [2]. For the reader's convinence, we write the details of the proof below.

First, we show that

$$S \in \mathscr{F}_\omega(1-(a-b)). \tag{3.11}$$

Indeed, since $a > c$ we have, for μ sufficiently large,

$$\tfrac{1}{2}a\mu^{a-1} \le h_0'(\mu,t) \le 2a\mu^{a-1}. \tag{3.12}$$

By (3) and (4) of Lemma 3.1, we have $S \in \mathscr{F}_\omega(1-(a-b))$.

Next, we solve the second equation of (3.1) for θ and write

$$\phi = \theta + \frac{\partial}{\partial\mu}S = \theta + g(\mu,\theta,t). \tag{3.13}$$

By Eq. (3.11), $g \in \mathscr{F}_\omega(-(a-b))$. For the inverse we set

$$\theta = \phi + V(\mu,\phi,t).$$

In view of Eq. (3.13) we have for v the equation

$$V(\mu,\phi,t) = -g(\mu,\phi+V). \tag{3.14}$$

If μ is large, the $|D_\theta g| \le \frac{1}{2}$ so that V is uniquely determined by the contraction principle. Moreover, by the implicit function theorem V is analytic in A_{μ_0} for some large μ_0. From the proof of Lemma 2 in [2], we can prove $V \in \mathscr{F}_\omega(-(a-b))$.

We insert $\theta = \phi + V$ into the first equation of (3.1) and define U by

$$\lambda = \mu + \frac{\partial S}{\partial\theta}(\mu,\phi+V,t) = \mu + U(\mu,\phi,t). \tag{3.15}$$

Since $\partial S/\partial\theta \in \mathscr{F}_\omega(1-(a-b))$ in view of Eq. (3.9) and since $V \in \mathscr{F}_\omega(-(a-b))$, one concludes that $U \in \mathscr{F}_\omega(1-(a-b))$.

To finish the proof of the lemma one can see easily that the map ψ has a right inverse of the same type as ψ defined on A_{μ_-} for some large μ_- and that it is injective on A_{μ_0} for μ_0 large. □

For the transformed Hamiltonian function $\hat{H}$ expressed in the variables μ,ϕ, we have

$$\hat{H}(\mu,\phi,t) = h_0(\mu,t) + [h_2](\mu,t) + R, \tag{3.16}$$

where $R = R_1 + R_2 + R_3$, with

$$R_1 = \frac{\partial S}{\partial t}(\mu, \phi, t),$$
$$R_2 = \int_0^1 (1-\tau) h_0''(\mu + \tau U, t) U^2 \, d\tau, \tag{3.17}$$
$$R_3 = \int_0^1 h_2'(\mu + \tau U, \phi + V) U \, d\tau.$$

Lemma 3.3.
(1) $R_1 \in \mathscr{F}_\omega(1-(a-b))$;
(2) $R_2 \in \mathscr{F}_\omega(b-(a-b))$;
(3) $R_3 \in \mathscr{F}_\omega(b-(a-b))$.
So $R \in \mathscr{F}_\omega(b_1)$ *with* $b_1 = b + \max(1,b) - a$.

Proof. This proof is similar to the proof of the corresponding lemma in [2]. We omit it here. □

In view of Lemmas 3.2 and 3.3 and in view of Eq. (3.3) the proof of Proposition 3.1 is finished setting $\hat{h}_2 = \rho$. □

If H satisfies the assumptions of Proposition 3.1, then for any given number $d > 0$ there is an integer $j = j(a,b,d)$ so that after j successive applications of the proposition we find the corresponding perturbation term $\hat{h}_2$ belongs to $\mathscr{F}_\omega(b_j)$ with $b_j < -d$. Now, we can assume the Hamiltonian function is of the form

$$H = h_1(\mu, t) + h_2(\mu, \theta, t), \tag{3.18}$$

where $h_1 \in \mathscr{F}_\omega(a)$ and $h_2 \in \mathscr{F}_\omega(-d)$ with $d > 0$. Moreover, we assume $h_1'(\mu, t) \geq (1/2)a\mu^{a-1}$ for $\mu \geq \mu_0$ uniformly in t with μ_0 is a positive constant and $|h_2(\mu,\theta,t)| \leq M \leq 1$ on the domain Ω_{δ_0} for some $0 < \delta_0 < 1$.

4. Proof of Theorem

In this section, we will prove the theorem stated in Section 1 of this paper. In the proof, the following result for the existence of quasi-periodic solutions of Hamiltonian system will be used.

Lemma 4.1. *Consider the perturbation* H *of an integrable Hamiltonian* h^0, $H(p,q,t) = h^0(p) + h^1(p,q,t)$, h^0 *is real analytic and nondegrenerate and* h^1 *is of class* C^r *and quasi-periodic in* t *with frequencies* $\omega = (\omega_1, \ldots, \omega_m)$ *satisfy the Diophantine conditions* (1.3), $r > 0$ *sufficiently large*, $(p,q,t) \in \mathbf{R}^n \times \mathbf{T}^n \times \mathbf{R}^1$.

Then, if $|h^1|_{C^r}$ *is sufficiently small, there is a diffeomorphism* Φ *on some Cantor set in* $\mathbf{R}^n \times \mathbf{T}^n \times \mathbf{R}$ *in the sense of Whitney such that under this diffeomorphism, the*

Hamiltonian equations with Hamiltonian H is transformed into

$$\dot{\eta} = 0, \qquad \dot{\theta} = \eta.$$

Φ is quasi-periodic in t with the same frequencies $\omega = (\omega_1, \ldots, \omega_m)$. Moreover, for any $v_1, \ldots, v_n \in \mathbf{R}^n$ satisfying

$$|\ell_1 v_1 + \cdots + \ell_n v_n + k_1\omega_1 + \cdots + k_m\omega_m| \geq \frac{c}{(|k| + |\ell|)^{m+n}}, \quad (\ell, k) \in \mathbf{Z}^{n+m} \setminus \{0\},$$

there are quasi-periodic solutions with frequencies $v_1, \ldots, v_n, \omega_1, \ldots, \omega_m$.

This result is similar to the celebrated KAM theorem. The proof can be found in [6].

We introduce the notation $[f]$ for a quasi-periodic function $f(t)$ with frequencies $\omega_1, \ldots, \omega_m$. From the definition of quasi-periodic functions, we have

$$f(t) = \sum_{k \in \mathbf{Z}^n} a_k \mathrm{e}^{\mathrm{i}<\omega,k>t}. \tag{4.1}$$

We denote by $\overline{f}$ the constant term a_0 of f in Eq. (4.1).

Now, we consider the Hamiltonian H defined by Eq. (3.18)

$$H = h_1(\mu, t) + h_2(\mu, \theta, t), \tag{4.2}$$

where $h_1 \in \mathscr{F}_\omega(a)$ and $h_2 \in \mathscr{F}_\omega(-d)$ with $d > 0$ sufficiently large.

In order to apply Lemma 4.1 to Eq. (4.2), we introduce a symplectic transformation T:

$$\lambda = \mu,$$
$$\phi = \theta + \int_0^t \left(\frac{\partial h_1}{\partial \mu}(\mu, \tau) - \overline{\frac{\partial h_1}{\partial \mu}} \right) \mathrm{d}\tau,$$

Under the transformation T, the Hamiltonian H is transformed into the following form

$$\mathscr{H}(\lambda, \phi, t) = \hat{h}_1(\lambda) + \hat{h}_2(\lambda, \phi, t), \tag{4.3}$$

where $\hat{h}_1(\lambda) = \overline{h_1} \in \mathscr{F}_\omega(a)$ and $\hat{h}_2 = h_2(\lambda, \phi + \int_0^t (\overline{\partial h_1}/\partial\mu - \partial h_1/\partial\mu(\lambda, \tau))\, \mathrm{d}\tau, t)$.

We give the estimate of $\hat{h}_2$.

Lemma 4.2. *For $f(\rho, \theta, t) \in \mathscr{F}_\omega(r), \theta = \phi + V(\rho, t)$ with $V(\rho, t) \in \mathscr{F}_\omega(s)$, $s < 1$. Let $F(\rho, \phi, t) = f(\rho, \phi + V(\rho, t), t)$. We have*

$$\sup(\rho^{-r+k+(k+n)s} |D_\rho^k D_\phi^l D_t^n F|) < \infty, \tag{4.4}$$

for all $\rho \geq \rho_0$.

Proof. From the definition of F, it follows that $D_\phi^l D_t^n F$ is a finite sum of terms which is of the following form

$$\partial_\phi^{l+l'} \partial_t^{n'} f(\rho, \phi + V, t) \cdot \partial_t^{n_1} V \cdot \cdots \cdot \partial_t^{n_{l'}} V,$$

where $n' + n_1 + \cdots + n_{l'} = n$ and $l' \le n$. Hence, $D_\rho^k D_\phi^l D_t^n F$ is the sum of the following terms:

$$\partial_\rho^p \partial_\phi^{l+l'+l''} \partial_t^{n'} f \cdot \partial_\rho^{p_1} V \cdots \partial_\rho^{p_{l''}} V \cdot D_\rho^{k'} (\partial_t^{n_1} V \cdots \partial_t^{n_{l'}} V),$$

where $p + p_1 + \cdots + p_{l''} + k' = m, n_1 + \cdots + n_{l'} + n' = n, l'' \le m$ and $l' \le n$.

From Lemma 3.1, we have

$$\partial_\rho^{p_1} V \cdots \partial_\rho^{p_{l''}} V \in \mathscr{F}_\omega(l''s - (p_1 + \cdots + p_{l''})),$$
$$\partial_t^{n_1} V \cdots \partial_t^{n_{l'}} V \in \mathscr{F}_\omega((n_1 + \cdots + n_{l'})s),$$
$$\partial_\rho^p \partial_\phi^{l+l'+l''} \partial_t^{n'} f \in \mathscr{F}_\omega(r - p).$$

Hence,

$$\begin{aligned}
&\partial_\rho^p \partial_\phi^{l+l'+l''} \partial_t^{n'} f \cdot \partial_\rho^{p_1} V \cdots \partial_\rho^{p_{l''}} V \cdot \partial_t^{n_1} V \cdots \partial_t^{n_{l'}} V \\
&\quad \in \mathscr{F}_\omega(r - p + l''s - (p_1 + \cdots + p_{l''}) + (n_1 + \cdots + n_{l'})s - p') \\
&\quad = \mathscr{F}_\omega(r - m + (l'' + n_1 + \cdots + n_{l'})s) \subset \mathscr{F}_\omega(r - m + (n + m)s),
\end{aligned}$$

which proves the statement of this lemma. □

We are now in a position to prove the theorem.

Proof of Theorem. For any point $(x_0, y_0) \in \mathbf{R}^2$, we can choose a $\lambda_0 \in \mathbf{R}^+$, $\lambda_0 > \frac{1}{2} y_0^2 + [1/2(n+1)] x_0^{2(n+1)}$, sufficiently large such that the conditions of Proposition 3.1 and Lemma 4.2 are met. Hence, the Hamiltonian (2.6) is transformed into Eq. (4.3), i.e.,

$$\mathscr{H}(\lambda, \phi, t) = h_1(\lambda) + h_2(\lambda, \phi, t).$$

where $h_1(\lambda) \in \mathscr{F}_\omega(a), a = 2\beta$ and h_2 has the estimate (4.4) with $r = -d$.

In order to prove the solution $x(t)$ of Eq. (1.2) with initial value (x_0, y_0) is bounded, we introduce a symplectic transformation

$$\lambda = \lambda_0 + p, \qquad \phi = q,$$

where $p \in [1,2] \subset \mathbf{R}^+$. Under this transformation, the Hamiltonian (4.5) is transformed into the following form

$$\mathbf{H}(p, q, t) = \mathscr{H}(\lambda_0 + p, q, t) = h_1(\lambda_0 + p) + h_2(\lambda_0 + p, q, t), \tag{4.6}$$

where $(p, q) \in [1,2] \times \mathbf{S}^1$, $\lambda_0 > 0$ sufficiently large. The transformed Hamiltonian system is:

$$\dot{p} = -\frac{\partial h_2}{\partial q}, \qquad \dot{q} = \frac{\partial h_1}{\partial p} + \frac{\partial h_2}{\partial p}. \tag{4.7}$$

From the above discussions, we have

$$h_1(\lambda_0 + p) = d_0 \cdot (\lambda_0 + p)^{2\beta} + g_1(\lambda_0 + p),$$

where $g_1 \subset 8n\mathscr{F}_\omega((2n+1)\alpha)$. Hence, when λ_0 is sufficiently large, we have

$$\tfrac{1}{2} d_0(\lambda_0 + 2)^{2\beta-2} \le \left| \frac{\partial^2 h_1}{\partial p^2} \right| \le 1, \quad 1 \le p \le 2, \tag{4.8}$$

$h_2(\lambda_0 + p, q, t)$ has the estimate (4.4). Because $p_0, \dots, p_l \in C^\infty$, we can choose d defined in Eq. (3.18) sufficiently large such that $|h_2|_{C^r}$ is very small. This implies the Hamiltonian (4.6) satisfies the conditions of Lemma 4.1. By Lemma 4.1, there is a closed invariant curve $\gamma \in \mathbf{R}^2$ which is near to $p = p_0$, for some $p_0 \in (1, 2)$, such that the solutions with the initial points on γ are quasi-periodic with basic frequencies $\omega, \omega_1, \dots, \omega_m$. These solutions are bounded for all $t \in \mathbf{R}^1$. Hence, from the uniqueness of solutions, the solution which initial at (x_0, y_0) is bounded, i.e., $|\dot{x}(t)| + |x(t)| < \infty$. This completes the proof of the theorem. □

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