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Existence of quasiperiodic solutions and Littlewood's boundedness problem of Duffing equations with subquadratic potentials

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1. Introduction and results

The dynamics of the equation

$$\frac{d^2x}{dt^2} + G'(x) = p(t), \tag{1.1}$$

where $p(t)$ is periodic has been extensively investigated due to its relevance in applications. Much work has been carried out concerning the existence of periodic solutions (see [3, 8] also for further references). In this paper we pay attention to more complicated solutions and the Lagrangian stability problem proposed by Littlewood [5]. As remarked in [5], it has been conjectured that all solutions of Eq. (1.1) are bounded if either

$$\lim_{|x|\to\infty} \frac{G(x)}{x^2} = +\infty \quad \text{(superquadratic case)} \tag{1.2}$$

or

$$\lim_{|x|\to\infty} G'(x)\,\mathrm{sign}\, x = +\infty \quad \text{and} \quad \lim_{|x|\to\infty} \frac{G(x)}{x^2} = 0 \quad \text{(subquadratic case).} \tag{1.3}$$

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There are several papers concerning this problem for the superquadratic case (see [3, 6, 7, 9, 13]. The first contribution to this problem is by Morris. In [9], he has shown that Littlewood's boundedness conjecture is true for a special superlinear Duffing equation

$$\frac{d^2x}{dt^2} + 2x^3 = p(t).$$

Diekerhoff and Zehnder [3], Liu [6], You [13] and Levi [7] further generalized Morris' results, and reached the same conclusions for a more general class of superquadratic Duffing equations.

For the sublinear case, except the paper of Norris [11] for analytic potentials, we are not aware of any boundedness results. In this paper, we will prove the boundedness of solutions for a special but typical case of sublinear Duffing equations with C^1 potential $(1/(\alpha+1))\,|x|^{\alpha+1}$,

$$\frac{d^2x}{dt^2} + x|x|^{\alpha-1} = p(t). \tag{1.4}$$

where $0<\alpha<1$, $p(t)\in C^{(\infty)}(S^1)$ is a continuous 1-periodic function.

We give the results for

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x|x|^{\alpha-1} + p(t), \tag{1.5}$$

which is equivalent to Eq. (1.4).

Theorem. *Suppose that* $p(t)$ *is a* C^∞ *periodic function and* $0<\alpha<1$. *System* (1.5) *possesses infinitely many invariant tori with positive Lebesgue measure in the* (t,x,y)-*space* $S^1\times R^2$, *and each solution of Eq.* (1.5) *on the invariant tori is quasiperiodic. Moreover, all solutions of Eq.* (1.5) *are bounded*; *i.e.*,

$$\sup_{t\in R^1}\left(|x(t)| + \left|\frac{dx(t)}{dt}\right|\right) < +\infty.$$

It follows that for all solution $x(t)$ *of Eq.* (1.4), $\sup_{t\in R^1}(|x(t)|+|dx(t)/dt|)$ *are bounded.*

The idea for proving the boundedness of solutions of system (1.5) is as follows. By means of transformation theory, outside of a large disc $D=\{(x,y)\in R^2: x^2+y^2\le r^2\}$ in (x,y)-plane, (1.5) is transformed into a perturbation of an integrable Hamiltonian system. The Poincaré map of the transformed system is close to a so-called twist map in $R^2\backslash D$. Then Moser's twist theorem guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the (x,y)-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x,y,t)\in R^2\times R$, which confines the solutions in the interior and which leads to a bound of these solutions if the uniqueness of initial value problem holds.

Since there is no theorem available on the uniqueness of the initial value problem for system (1.5), we will not try to prove the uniqueness for our system (1.5) directly. Note that, from the proof below, Eq. (1.5) is equivalent to a smooth system in $R^2\backslash D$ which has uniqueness. Hence system (1.5) has uniqueness.

Remark 1. *The existence of infinite many periodic solutions is a direct conclusion of Poincáre–Birkhoff twist fixed-point theorem. Since there are many results in this direction, we omit it.*

Remark 2. *It is enough to assume that $p(t)$ is sufficiently smooth, say, $p(t)\in C^{(4)}(S^1)$, if one uses a strengthened Moser's twist theorem by Herman* [4]. *It seems that the smoothness requirement of $p(t)$ for system* (1.5) *is essential if we use the KAM theorem since the potential is not smooth. For smooth potentials, such as $x^2(1+x^2)^{-1/3}$, one could expect to have a similar result for $p(t)\in C^0(S^1)$.*

Remark 3. *The relation between the existence of periodic solutions and the Lagrangian stability has been investigated in* [2]. *For superquadratic potentials, examples were given in* [5, 7, 10] *with both periodic solutions and unbounded solutions. It is an open problem if there is a similar example for subquadratic Duffing equations.*

2. Proof of the theorem

Firstly, we consider the unperturbed equation

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=-x|x|^{\alpha-1},\ \ \alpha>0, \tag{2.1}$$

which is a time-independent Hamiltonian system with Hamiltonian

$$h(x,y)=\frac{1}{2}y^2+\frac{1}{\alpha+1}|x|^{\alpha+1}.$$

Clearly, h is positive on R^2 except at the unique equilibrium point $(x,y)=(0,0)$ where $h=0$. Note that $h(x,y)=E$ is a first integral of the system (2.1), hence, all the solutions of Eq. (2.1) are periodic with period tending to zero as E tends to infinity.

Suppose $(S(t),C(t))$ is the solution of Eq. (2.1) satisfying the initial condition:

$$(S(0),C(0))=(0,1).$$

Let $T_0>0$ be its minimal period. From system (2.1), there is a solution $(S(t),C(t))$ satisfying

(i) $S(t+T_0)=S(t),\ C(t+T_0)=C(t)$ with $S(0)=0,\ C(0)=1,$
(ii) $S(t)\in C^2(R^1),\ C(t)\in C^1(R^1),\ d/dt\,S(t)=C(t),\ d/dt\,C(t)=-S^{\infty}(t),$
(iii) $\frac{1}{2}C^2(t)+1/(\alpha+1)S^{\alpha+1}(t)=\frac{1}{2},$

The action and angle variables are now defined by the mapping $\Psi_1: R^+\times S^1\to R^2/\{0\}$; $(\rho,\theta)\mapsto(x,y)$; where $\rho>0$ and $\theta\,(\mathrm{mod}\,1)$ is given by the formula

$$\Psi_1:\ x=d^b\rho^bS(\theta T_0),\quad y=d^{\alpha/2}\rho^{\alpha/2}C(\theta T_0),$$

where $b=2/(\alpha+3)$, $a=2-2b$, $d=2/(aT_0)$. It is easy to see that $\frac{1}{2}<b<a<1$ if $0<\alpha<1$.

In the new coordinates, system (2.1) becomes

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}=\frac{\partial h}{\partial\theta}(t,\theta,\rho),\quad \frac{\mathrm{d}\theta}{\mathrm{d}t}=ad^a\rho^{a-1}+\frac{\partial h}{\partial\rho}(t,\theta,\rho), \tag{2.2}$$

where $h(t,\theta,\rho)=a^b\rho^bS(\theta T_0)p(t)$ is periodic in θ, t with period 1 (see [3]).

In order to use Moser's twist invariant curve theorem, the perturbation must be sufficiently smooth in the angle variable. Unfortunately, the perturbation $h(\rho,\theta,t)$ in system (2.2) belongs only to C^2 with respect to the angle variable θ. To overcome this difficulty we change the role of θ and t variables as in [1,7,9]. Let Ψ_2: $r=d^a\rho^a+d^b\rho^bS((T_0/2\pi)\theta)p(t)$, $\tau=\theta$, $o=t$. This change of variables leads to the following Hamiltonian system:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r}=d^{-1}r^{1/a-1}+\frac{\partial g(r,\phi,\tau)}{\partial r},\quad \frac{\mathrm{d}r}{\mathrm{d}\tau}=-\frac{\partial g(r,\phi,\tau)}{\partial\phi}. \tag{2.4}$$

where the Hamiltonian $\rho=d^{-1}r^{1/a}+g(r,\phi,\tau)$ is the inverse function of $r=d^a\rho^a+d^b\rho^bS((T_0/2\pi)\theta)p(t)$ with respect to ρ, and t, θ are treated as parameters. It is easy to see that this function is well defined for sufficiently large ρ, and $\rho(r,\phi,\tau)$ is C^2 smooth in $\tau=\theta$. In system (2.4), $\tau=\theta$ serves as new time, and $\phi=t$ serves as the new angle variable.

The following lemma gives an estimate of the perturbation g in system (2.4).

Lemma 2.1. *Let $r=\rho^a+\rho^bf(\theta,t)$ with $\frac{1}{2}<b<a<1$ be the Hamiltonian of system (2.2). Denote by $\rho=r^{1/a}+g(r,\theta,t)$ its inverse with respect to ρ for sufficiently large ρ. Then, for $r>r^*$,*

$$|D_r^iD_t^jg(r,\theta,t)|<const\cdot r^{((b+1)/a)-1-i}, \tag{2.5}$$

uniformly on t,θ, where the const. is independent of r.

Proof. In the following we denote by c the absolute constant and different constants are denoted by the same letter c unless their value is of more than transient interest.

Rewrite

$$r=d^{-1}(r^{1/a}+g)^a+(r^{1/a}+g)^bf(\theta,t), \tag{2.6}$$

into the following form:

$$d^{-1}g\int_0^1(r^{1/a}+sg)^{a-1}\,\mathrm{d}s+(r^{1/a}+g)^bf=0. \tag{2.7}$$

If r is large, g is well determined by the contraction principle. Moreover, by the implicit function theorem g is smooth in r for large r.

It is easy to see that $|g(r,\theta,t)|<c\cdot r^{(b+1)/a-1}$. Inductively, we assume that $|D_r^ig(r,\theta,t)|<const\cdot r^{(b+1)/a-1-i}$ holds for $i\le n-1$. Applying D_r^n to Eq. (2.7), the left-hand

side is the algebraic sum of terms

$$(D_r^{i_1} g)\left(D_r^{i_2}\int_0^1 (r^{1/a}+sg)^{a-1}\,\mathrm{d}s\right)$$

with $i_1+i_2=n$; i.e.,

$$(D_r^{i_1} g)\int_0^1 (r^{1/a}+sg)^{a-1-j}\prod_{k=1}^{j} D_r^{jk}(r^{1/a}+sg)\,\mathrm{d}s$$

and

$$(r^{1/a}+g)^{b-v}\prod_{k=1}^{v} D_r^{v_k}(r^{1/a}+g)f(t,\theta)$$

with $1\le j_k\le n,\ 1\le j\le n,\ \sum_{k=1}^{j} j_k=i_2,\ i_1+i_2=n,\ 1<v_k\le n,\ 1\le v\le n,\ \sum_{k=1}^{v} v_k=n.$ We move all the other terms to the right-hand side of the equation except the highest-order terms of the derivatives of g. It is easy to see that the coefficient of the highest-order term $D_r^n g$ is given by

$$\int_0^1 a(r^{1/a}+sg)^{a-1}+a(a-1)g(r^{1/a}+sg)^{a-2}s\,\mathrm{d}s+b(r^{1/a}+g)^{b-1}f>\tfrac{1}{2}ar^{1-1/a},$$

for large r. For the remaining terms, $i_1, j_k, v_k\le n-1$. By inductive assumption, we have

$$\left|D_r^{i_1} g\int_0^1 (r^{1/a}+sg)^{a-1-j}\prod_1^{j} D_r^{j_k}(r^{1/a}+sg)\,\mathrm{d}s\right|\le cr^{b/a-n+1}$$

and

$$\left|(r^{1/a}+g)^{b-v}\prod_1^{v} D_r^{v_k}(r^{1/a}+g)f(t,\theta)\right|\le cr^{b/a-n+1}.$$

Thus $|D_r^n g|\le c(r^{(b+1)/a)-n}$.

Differentiating Eq. (2.6) with respect to t, we have

$$a(r^{1/a}+g)^{a-1}g_t'+b(r^{1/a}+g)^{b-1}fg_t'+(r^{1/a}+g)^b f_t'=0.$$

It is equivalent to

$$(a(r^{1/a}+g)^{a-b}+bf)g_t'+gf_t'+r^{1/a}f_t'=0. \tag{2.8}$$

It follows that $|g_t'|\le const\cdot r^{((b+1)/a)-1}$. Inductively, we assume that Eq. (2.5) holds for $i+j\le n-1$. Applying $D_r^iD_t^{j-1}$ to the Eq. (2.8) with $i+j=n$, the left-hand side is the sum of terms

$$(a(r^{1/a}+g)^{a-b}+bf)(D_r^iD_t^jg), \tag{a1}$$

$$D_r^{i_2}D_t^{j_2}(a(r^{1/a}+g)^{a-b}+bf)(D_r^{i_1}D_t^{j_1+1}g) \tag{a2}$$

and

$$(D_r^iD_t^{j-1})(gf_t'+r^{1/a}f_t') \tag{a3}$$

with $i_1+i_2=i,\ j_1+j_2=j-1,\ i+j=n,\ i_2+j_2\geq 1$. Note that Eq. (a2) is the sum of terms

$$b(D_r^{i_1}D_t^{j_1+1}g)(D_r^{i_2}D_t^{j_2}f)$$

and

$$(D_r^{i_1}D_t^{j_1+1})g(r^{1/a}+g)^{a-b-s}\prod_{k=1}^{s}(D_r^{i_2^k}D_t^{j_2^k})(r^{1/a}+g),$$

with $0\leq i_2^k\leq i_2,\ 0\leq j_2^k\leq j_2,\ 1\leq i_2+j_2\leq n-1,\ \sum_{k=1}^s i_2^k=i_2,\ \sum_{k=1}^s j_2^k=j_2,\ 1\leq s\leq n-1$.

All the other terms are moved to the right-hand side of the equation, except the highest-order term of the derivatives of g, $(a(r^{1/a}+g)^{a-b}+bf)D_r^iD_t^jg$. Then we have the estimate (2.5).

In the following, by rearranging the proof in [3], we can strengthen Proposition 2.1 in [3] as follows

Proposition 2.1. *Let $H=\lambda^{a_0}+h_1(\lambda,\tau)+h_{21}(\lambda,\phi,\tau)+h_{22}(\lambda,\phi,\tau)$ satisfy*

$$|D_\lambda^ih_1|<c\lambda^{a_1-i},\quad |D_\lambda^iD_\phi^jh_{21}|<c\lambda^{a_2+1-a_0-i},\quad |D_\lambda^iD_\phi^jh_{22}|<c\lambda^{a_2-i},$$

for large λ, where $a_1,a_2<a_0$. Assume that $h_{21}(\lambda,\phi,\cdot)\in C^{n-1}$, $h_{22}(\lambda,\phi,\cdot)\in C^n$ with respect to τ. Then, for large λ, there is a canonical transformation Ψ depending periodically on τ, which transforms H into the following:

$$\hat{H}=\mu^{a_0}+\hat{h}_1(\mu,\tau)+\hat{h}_{21}(\mu,\theta,\tau)+\hat{h}_{22}(\mu,\theta,\tau)$$

with

$$|D_\lambda^i\hat{h}_1|<c\lambda^{a_1-i},\quad |D_\lambda^iD_\theta^j\hat{h}_{21}|<c\lambda^{a_2+1-a_0-i},\quad |D_\lambda^iD_\theta^j\hat{h}_{22}|<c\lambda^{a_2+1-a_0-i},$$

and $\hat{h}_{21}(\lambda,\phi,\cdot)$, $\hat{h}_{22}(\lambda,\phi,\cdot)\in C^{n-1}$ with respect to τ.

Proof. See the appendix.

Remark. *This strengthened proposition is crucial for this paper. In* [12], *it is used to weaken the smoothness restriction in* [3].

Proposition 2.1 can be used two times to our Hamiltonian $\rho = d^{-1} r^{1/a} + g(r,\phi,\tau)$ since $g(\cdot,\tau) \in C^2$ with respect to τ. It follows that

Lemma 2.2. *Let $\rho = r^{1/a} + g(r,\phi,\tau)$ be the function defined in Lemma 2.1. Then there is a canonical transformation Ψ_3 depending periodically on τ, which transforms ρ into*

$$\bar{\rho} = d^{-1} r^{1/a} + g_1(r,\tau) + g_2(r,\phi,\tau) \tag{2.9}$$

with

$$|D_r^i g_1(r,\tau)| < const\, r^{((b+1)/a)-1-i}, \quad |D_\tau^i D_\phi^j g_2(r,\phi,\tau)| < const\, r^{1/2-i} \tag{2.10}$$

and $g_1(r,\tau) \in C^1$, $g_2(r,\phi,\tau) \in C^0$ with respect to τ. For simplicity, here we still denote by (r,ϕ,τ) the coordinates of the transformed Hamiltonian $\bar{\rho}$.

Proof. Using Lemma 2.2 twice leads to the estimates. □

In view of Lemmas 2.1 and 2.2, system (1.5) is transformed into a new Hamiltonian system with the Hamiltonian (2.9) satisfying Eq. (2.10) by $\Psi = \Psi_3 \circ \Psi_2 \circ \Psi_1$. Denote by $\Phi^\tau(\phi,r) = (\bar{\phi}(\tau,\phi,r),\ \bar{r}(\tau,\phi,r))$ the solutions of Eq. (2.9) with $\Phi^0(\phi,r) = (\phi,r)$. Φ^1 is the so-called Poincare mapping of Eq. (2.9).

Lemma 2.3. *Suppose that all hypotheses in the main theorem are satisfied. Then the Poincare mapping Φ^1 of Eq. (2.9) is of the following form*:

$$\theta_1 = \theta + f_0(r) + f_1(r,\phi), \quad r_1 = r + f_2(r,\phi), \tag{2.11}$$

with $f_0(r) = d^{-1} r^{1/a-1} + \int_0^1 (\partial/\partial r) g_1(r,s)\,\mathrm{d}s$. Moreover, we have the following estimates:

$$|D_\phi^k D_r^j f_1(r,\phi)| < r^{b/a-1-j(2-1/a)}, \quad |D_\phi^k D_r^j f_2(r,\phi)| < r^{1/2-j(2-1/a)}, \tag{2.12}$$

for r sufficiently large.

Proof. Let $f_0(r,\tau) = d^{-1} r^{1/a-1}\tau + \int_0^\tau (\partial/\partial r) g_1(r,s)\,\mathrm{d}s$. It follows that Φ^τ has the following form:

$$r(\tau) = r + B(r,\phi,\tau), \qquad \phi(\tau) = \phi + f_0(r,\tau) + A(r,\phi,\tau).$$

The integral equation $\Phi^t(r,\phi) = Id + \int_0^t X_H \circ \Phi^s\,\mathrm{d}s$ is equivalent to the following equations for A and B:

$$A(r,\phi,\tau) = \frac{1}{a}\left(\frac{1}{a} - 1\right) \int_0^\tau \int_0^1 (r + tB)^{1/a-2} B\,\mathrm{d}t\,\mathrm{d}s$$

$$\times \int_0^\tau \int_0^1 \left(\frac{\partial^2}{\partial r^2} g_1\right)(r + tB,\tau) B\,\mathrm{d}t\,\mathrm{d}s$$

$$+\int_0^\tau\left(\frac{\partial}{\partial r}g_2\right)(r+B,\phi+f_0+A)\,\mathrm{d}t\,\mathrm{d}s,$$

$$B(r,\phi,\tau)=-\int_0^\tau\left(\frac{\partial}{\partial\phi}g_2\right)(r+B,\phi+f_0+A)\,\mathrm{d}s. \tag{2.13}$$

Since the existence and uniqueness theorem holds for system (2.9), we know A,B are well determined. Moreover, they depend smoothly on r,ϕ. It is easy to see that $f_1=A(r,\phi,1)$, $f_2=B(r,\phi,1)$. Note that $D_r^iD_\phi^j g_m(\phi+f_0+A,r+B)$ $(m=1,2)$ is the sum of terms

$$(D_1^sD_2^kg_m)\left(\prod_{l=1}^{k}D_r^{i_l}D_\phi^{j_l}(\phi+f_0+A)\right)\left(\prod_{l=k+1}^{k+s}D_r^{i_l}D_\phi^{j_l}(r+B)\right),$$

with $1\le s+k\le i+j$, $\sum_{l=1}^{s+k}l_l=i$, $\sum_{l=1}^{k+s}j_l=j$. The required estimates can be inductively verified from Eqs. (2.13) and (2.10).

Let Ψ_4: $\phi=\phi$, $\mu=f_0(r)$, then Φ_4 transforms map (2.11) into the following form:

$$\hat\Phi^1:\ \phi_1=\phi+\mu+\hat f(\mu,\phi),\quad \mu_1=\mu+\hat g(\mu,\phi), \tag{2.14}$$

where $\hat f(\mu,\phi)=f_1(r,\phi)$, $\hat g(\mu,\phi)=\int_0^1 f_0'(r+sf_2)\cdot f_2\,\mathrm{d}s$ with $r=f_0^{-1}(\mu)$.

Lemma 2.4. *For sufficiently large μ,*

$$|D_\phi^kD_\mu^j\hat f(\mu,\phi)|<c\mu^{\varepsilon_0},\quad |D_\phi^kD_\mu^j\hat g(\mu,\phi)|<c\mu^{\varepsilon_0},\quad \varepsilon_0=\frac{-2\alpha}{1-\alpha}<0. \tag{2.15}$$

Proof. A direct calculation leads to the estimates. We omit the details here. □

Lemma 2.5. $\Phi^1(\theta,\mu)$ *has the intersection property for sufficiently large μ; i.e., if an embedded circle C in $S^1\times R^1$ is homotopic to a circle $\mu=const$, then $\Phi^1(C)\cap C\neq 0$.*

Proof. Since Eq. (2.4) is a Hamiltonian system, the Poincare mapping of Eq. (2.4) has the intersection property (see [3]). We know that Φ^1 is conjugated to the Poincare mapping of (2.4) through Ψ for sufficient large μ. It follows that Φ^1 has the intersection property.

Now, we are in a position to prove the existence of invariant circles for Φ^1 by using Moser's twist theorem [10].

Theorem (Moser [10]). *Suppose that $F_1(u,v),F_2(u,v)\in C^{333}$ are periodic in u with the period* 1, *P is a mapping from the annulus $S^1\times[a,b]$ to $S^1\times R^1$, of the form*

$$u_1=u+v+F_1(u,v),\quad v_1=v+F_2(u,v). \tag{2.16}$$

Assume that P has the intersection property. Then there is a real number δ, such that if

$$|F_1|_{C^{333}}+|F_2|_{C^{333}}<\delta, \tag{2.17}$$

then for any irrational number $\lambda \in [a+\beta, b-\beta]$ *satisfying the condition*

$$\left|\lambda - \frac{m}{n}\right| > \gamma \frac{1}{n^{2+\beta}}, \quad \text{for some constants } \beta, \gamma \in]0,1[,$$

and for all integers m *and* n *with* $n>0$, *there is a closed* C^1 *curve* Γ_λ *with the following properties*:

(i) Γ_λ *is invariant under the mapping* P;

(ii) *The mapping* P *restricted on* Γ *is conjugate to the rotation* $\theta \to \theta + \lambda$.

Proof. Since the mapping $\hat{\Phi}^1$ satisfies the assumptions of Moser's twist theorem, it follows that $\hat{\Phi}^1$ possesses a sequence of invariant circles Γ_λ tending to infinity. Thus, $\Psi_4 \circ \Gamma_\lambda \circ \Psi_4^{-1}$ are invariant circles of the Poincare map of system (2.9) with the same property. The solutions of system (2.9) starting from those circles form invariant tori T_λ^2 in the phase space $(\tau, \phi, r) \in T^2 \times R^+$. Thus $\Psi^{-1} \circ T_\lambda^2 \circ \Psi$ are invariant tori of system (1.5) in the phase space $(t, x, y) \in S^1 \times R^2$. Those invariant tori have positive Lebesgue measure, and the union of their interior is R^2. Since system (2.9) is a smooth system with uniqueness and system (1.5) is conjugated to Eq. (2.9) by Ψ for $|x|+|y|$ sufficiently large, the uniqueness of the initial value problem also holds for system (1.5) for sufficiently large $|x|+|y|$. It follows that any solution of system (1.5) must stay within one of its invariant tori, and thus is bounded. The proof is thus completed. □

Appendix A

A.1. Proof of Proposition 2.1

Following the notations and the proof in [3], we first introduce a space of functions $F^k(r)$.

We say $f(\lambda, \phi, \tau) \in F^k(r)$ if $f \in C^\infty(R^+ \times T^1, \cdot)$, $f \in C^k(\cdot, T^1)$, and

$$\sup \lambda^{j-r} |(D_\lambda)^j (D_\phi)^l f(\lambda, \phi, \tau)| < \infty$$

for $\lambda > \lambda_{ijk}$.

For $f \in F^k(r)$, we denote the mean value over the ϕ-variable by $[f]$: $[f](\lambda, \tau) = \int_0^1 f(\lambda, \phi, \tau)\,\mathrm{d}\phi$. Define the canonical transformation ψ implicitly by

$$\begin{aligned} \lambda &= \mu + \frac{\partial}{\partial \phi} S(\mu, \phi, \tau), \\ \theta &= \phi + \frac{\partial}{\partial \mu} S(\mu, \phi, \tau), \end{aligned} \tag{3.1}$$

where $S(\mu, \phi, \tau)$ is a generating function. Denote $h_0(\lambda, \tau) = \lambda_a + h_1(\lambda, \tau)$, $v = (\partial/\partial\phi) S$, then

$$H(\mu, \phi, \tau) = h_0(\mu + v, \tau) + h_{21}(\mu + v, \phi, \tau) + h_{22}(\mu + v, \phi, \tau) + \frac{\partial}{\partial \tau} S.$$

By Taylor's formula we can write

$$H(\mu,\phi,\tau)=h_0(\mu,\tau)+\frac{\partial h_0}{\partial\mu}(\mu,\tau)\mu+h_{22}(\mu,\phi,\tau)+R, \tag{3.2}$$

$$R=\frac{\partial}{\partial t}S+h_{21}(\mu+v,\phi,\tau)+\int_0^1(1-s)h_0''(\mu+sv,\tau)v^2\,\mathrm{d}s$$
$$+\int_0^1 h_{22}'(\mu+sv,\phi,\tau)\mu\,\mathrm{d}s.$$

Let

$$v=\frac{h_{22}-[h_{22}]}{D_\mu h_0(\mu,t)},\qquad S=\int_0^\phi v\,\mathrm{d}\phi,$$

therefore

$$\hat{H}(\mu,\theta,\tau)=h_0(\mu,\tau)+[h_{22}]+R(\mu,\phi,\tau), \tag{3.3}$$

where $\phi=\theta+v(\mu,\theta,\tau)$ is the inverse of Eq. (3.1). Denote $R_0=h_{20}(\mu+v,\theta,\tau)$, $R_1=\partial/\partial\tau$ $S, R_2=h_{21}(\mu+v,\theta,\tau)$, $R_3=\int_0^1(1-\tau)h_0''(\mu+sv,\tau)v^2\,\mathrm{d}s+\int_0^1 h_{22}'(\mu+sv,\theta,\tau)v\,\mathrm{d}s$, with $\phi=\theta+v(\mu,\theta,\tau)$. It was proved in [3], $R_3\in F^k(a_2-(a_0-a_2))$, $S\in F^k(1+a_2-a_0)$. Obviously, $R_2\in F^{n-1}(1+(a_2-a_0))$. Then $H_{21}=R_1+R_2\in F^{n-1}(1+a_2-a_0)$. $H_{22}=R_3\in F^n(2a_2-a_0)$.

j applications of the above argument lead to a perturbation term $\hat{h}_{22}\in F^n(a_2^j), a_2^j=2a_2^{j-1}-a_0, a_2^j<a_2^{j-1}$. Lemma 2.2 is proved by taking j_0 such that $a_2^{j_0}<1+a_2-a_0$. □

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