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Boundedness of solutions for polynomial potentials with C^2 time dependent coefficients

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Abstract. In this paper we prove the existence of invariant curves and thus stability for all time for a class of Hamiltonian systems with time dependent potentials:

$$\frac{d^2x}{dt^2} + V_x(x,t) = 0, x \in R^1$$

where $V(x,t) = \frac{1}{2n+2}x^{2n+2} + \Sigma_{j=0}^{2n} \frac{p_j(t)}{j+1}x^{j+1}, p_j(t+1) = p_j(t), p_j \in C^2, 2n \geq j \geq n+1; p_j \in C^1, n \geq j \geq 0, n \geq 1$.

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1. Introduction and results

The problem of the boundedness of solutions for a time dependent nonlinear differential equation,

$$\frac{dx^2}{dt^2} + x^{2n+1} + \sum_{j=0}^{2n} p_j(t)x^j = 0, p_j(t+1) = p_j(t),\ n \geq 1, \tag{1.1}$$

is first proposed by J. Littlewood. The first result is due to Morris[M]; he proved that for piecewise continuous $p(t)$, all solutions of $\frac{dx^2}{dt^2} + x^3 = p(t)$ are bounded due to the presence of invariant tori. The result was extended to the case of time-periodic polynomial potentials with constant leading coefficient:

$$V(x,t) = \frac{1}{2n+2}x^{2n+2} + \sum_{j=0}^{2n} \frac{p_j(t)}{j+1}x^{j+1},$$

and sufficiently smooth $p_j(t) \in C^{l(n,j)}(S^1)$ by Dieckerhoff and Zehnder[DZ]. The idea of the proof is to show that the corresponding equation $\frac{dx^2}{dt^2} + V_x(x,t) = 0$

possesses invariant tori in the (x, x', t)-space with large energy, and thus proved the boundedness of solutions for all time. In [DZ], they also asked a question that whether the high smoothness requirement on $p_j(t)$ is necessary. In [LL], it was proved that $p_{2n+1} \in C^{6+\epsilon}, p_j \in C^{5+\epsilon}, 0 \le j \le 2n, \epsilon > 0$ are sufficient to guarantee the existence of invariant tori and boundedness. This result gives a uniform bound for the smoothness of all $p_l(t)$. Due to the method, it seems that the smoothness requirement can not be weakened in [LL].

In this paper, by modifying the method of Dieckerhoff and Zehnder[DZ], we prove the same result for equation (1.1) with

$$p_j(t) \in C^2, 2n \ge j \ge n+1, p_j(t) \in C^1, n \ge j \ge 0. \tag{1.2}$$

Remark. In [M], the result holds true if $p(t)$ is piecewise continuous; Recently, M. Levi and J. You[LY] give a counterexample which shows that if coefficients of higher order terms are piecewise continuous the equation may have unbounded solutions; in fact they proved that the equation $\frac{dx^2}{dt^2} + x^{2n+1} + a(t)x^l = 0 (l > n+1)$ with a piecewise smooth $a(t)$ possesses an unbounded solution.

The following is the main results of this paper.

Theorem 1. *Suppose that $p_l(t+1) = p_l(t), 1 \le j \le 2n, n \ge 1$, satisfy the condition (1.2). Then all solutions of (1.1) are bounded, i.e. each solution exists for all $t \in R^1$ and*

$$\sup_{t \in R^1} (|x(t)| + |x'(t)|) < \infty.$$

Moreover, there is a (large) $\omega* > 0$ such that for every irrational number $\omega > \omega*$ satisfying $\left|\omega - \frac{p}{q}\right| \ge c|q|^{-2-\beta}$, for all integers p and $q \ne 0$ with two constant $\beta > 0$ and $c > 0$, there is a quasiperiodic solution of (1.1) having frequencies $(\omega, 1)$; i.e. there is a smooth function $F(\theta_1, \theta_2)$ periodic of period 1, such that $x(t) = F(\theta_1 + \omega t, \theta_2 + t)$ are solutions of the equation.

It is mentioned here that X. Yuan [Yu] got the similar results independently by different method.

2. Canonical transformations

In this section, we consider the time dependent Hamiltonian

$$H = \lambda^a + h_1(\lambda, \tau) + h_{21}(\lambda, \theta, \tau) + h_{22}(\lambda, \theta, \tau). \tag{2.1}$$

We will prove that for a certain class of perturbations $h_{21}(\lambda,\theta,\tau), h_{22}(\lambda,\theta,\tau)$, there is a canonical transformation which transforms (1.2) into an integrable Hamiltonian with a small perturbation for large λ, if we are allowed to lose two times of derivatives in time variable τ.

First we introduce a space of functions $F^k(r)$.

$$F^k(r) = \{f(\lambda,\theta,\tau) | f \in C^\infty(R^+ \times T^1, \cdot), f(\cdot,\tau) \in C^k(\cdot, S^1),\\ sup\lambda^{j-r}|(D_\lambda)^j(D_\theta)^l f(\lambda,\theta,\tau)| < \infty\}$$

We summarize some properties readily verified from the definition:

Lemma 1.
(1) if $r_1 < r_2$, then $F^k(r_1) \subset F^k(r_2)$
(2) if $f \in F^k(r)$, then $(D_\lambda)^j f \in F^k(r-j)$
(3) if $f_1 \in F^k(r_1)$, $f_2 \in F^k(r_2)$, then $f_1 \cdot f_2 \in F^k(r_1+r_2)$
(4) if $f \in F^k(r)$, and satisfies $|f(\lambda,\cdot)| \geq c\cdot\lambda^r$ for $\lambda \geq \lambda_0$, then $\frac{1}{f} \in F^k(-r)$.
(5) if $f \in F^k(r), g \in F^k(s)$, then $f(g) \in F^k(rs)$.

Proof. See [DZ].

For $f \in F^k(r)$, we denote the mean value of f over the $\theta-$ variable by $[f]$: $[f](\lambda,\tau) = \int_0^1 f(\lambda,\theta,\tau)d\theta$ and $A_{\lambda_0} := \{(\lambda,\theta,\tau)|\lambda \geq \lambda_0, (\theta,\tau) \in T^2\}$.

Rearranging the Proof of Proposition 1 in [DZ], we have the following lemma:

Lemma 2. *Let $H = \lambda^a + h_1(\lambda,\tau) + h_{21}(\lambda,\theta,\tau) + h_{22}(\lambda,\theta,\tau)$ with $h_1 \in F^k(c), h_{21} \in F^{k-1}(b_1), h_{22} \in F^k(b_2)$. Assume that $k \geq 2, a > 1, c, b_1, b_2 < a, b_1 \leq 1-(a-b_2)$, then $\exists$ a canonical transformation*

$$\psi_1 : \begin{array}{l} \lambda = \mu + u(\mu,\phi,\tau) \\ \theta = \phi + v(\mu,\phi,\tau) \end{array}$$

with $u \in F^k(1-(a-b_2))$ and $v \in F^k(-(a-b_2))$ such that $A_{\mu^1_+} \subset \psi_1(A_{\mu^1_0}) \subset A_{\mu^1_-}$ for some large $\mu^1_- < \mu^1_0 < \mu^1_+$, which transforms (2.2) into a Hamiltonian vector field $\psi_1^*(X_H) = X_{\bar{H}}$ with the Hamiltonian:

$$\bar{H} = \mu^a + \bar{h}_1(\mu,\tau) + \bar{h}_{21}(\mu,\phi,\tau) + \bar{h}_{22}(\mu,\phi,\tau). \tag{2.2}$$

where $\bar{h}_1 = h_1 + [h_{22}] \in F^k(c_1')$ with $c_1' = \max(c_1, b_2), \hat{h}_{21} \in F^{k-1}(b_1')$ with $b_1' = 1 + b_2 - a, \hat{h}_{22} \in F^k(b_2')$ with $b_2' = b_2 - (a - b_2)$.

Proof. Following the footsteps in [DZ], we define the canonical transformation implicitly by

$$\psi_1 : \begin{array}{l} \lambda = \mu + \dfrac{\partial}{\partial\theta} S(\mu,\theta,\tau), \\ \phi = \theta + \dfrac{\partial}{\partial\mu} S(\mu,\theta,\tau) \end{array}$$

where $S(\mu,\theta,\tau)$ is a generating function which will be determined later. Denote $h_0(\lambda,\tau)=\lambda^a+h_1(\lambda,\tau), \nu=\frac{\partial}{\partial\theta}S$, then

$$\bar{H}(\mu,\theta,\tau)=h_0(\mu+\nu,\tau)+h_{21}(\mu+\nu,\theta,\tau)+h_{22}(\mu+\nu,\theta,\tau)+\frac{\partial}{\partial\tau}S.$$

By Taylor's formula we can write

$$\bar{H}(\mu,\theta,\tau)=h_0(\mu,\tau)+\frac{\partial h_0}{\partial\mu}(\mu,\tau)\nu+h_{22}(\mu,\theta,\tau)+R \tag{2.10}$$

where

$$\begin{aligned}R=&\frac{\partial}{\partial\tau}S+h_{21}(\mu+\nu,\theta,\tau)+\int_0^1(1-s)h_0''(\mu+s\nu,\tau)\nu^2\,ds\\&+\int_0^1 h_{22}'(\mu+s\nu,\theta,\tau)\nu\,ds.\end{aligned}$$

Let $\nu=\frac{h_{22}-[h_{22}]}{D_\mu h_0}, S=\int_0^\theta \nu\,d\theta$, we have

$$\bar{H}(\mu,\theta,\tau)=h_0(\mu,\tau)+[h_{22}]+R. \tag{2.11}$$

Denote $R_1=\frac{\partial}{\partial\tau}S, R_2=h_{21}(\mu+\nu,\theta,\tau), R_3=\int_0^1(1-\tau)h_0''(\mu+s\nu,\tau)\nu^2\,ds+\int_0^1 h_{22}'(\mu+s\nu,\theta,\tau)\nu\,ds.$

In view of [DZ], we know $R_3\in F^k(b_2-(a-b_2)); S\in F^k(1+b_2-a)$ and thus $R_1\in F^{k-1}(1+b_2-a)$. Obviously, $R_2\in F^{k-1}(b_1)\subset F^{k-1}(b_1')$. Then, $R_1+R_2\in F^{k-1}(b_1')$. Set $\bar{h}_1=h_1+[h_{22}], \bar{h}_{21}=R_1+R_2, \bar{h}_{22}=R_3$. The proof is finished.

From the proof of Lemma 2, we know that if we want to make $h_{22}(\mu,\theta,\tau)$ smaller, the desired canonical transformation depends only on $h_{22}(\mu,\theta,\tau)$. So we have the following stronger result.

Proposition 1. *Under the assumptions of Lemma 2, It concludes that* $\exists$ *a canonical transformation* ψ_2

$$\psi_2:\begin{array}{l}\lambda=\mu+u(\mu,\phi,\tau)\\ \theta=\phi+v(\mu,\phi,\tau)\end{array}$$

satisfying $A_{\mu_+^2}\subset\psi_2(A_{\mu_0^2})\subset A_{\mu_-^2}$ *for large* $\mu_-^2<\mu_0^2<\mu_+^2$, *which transforms Hamiltonian (1.2) into the following form:*

$$\hat{H}(\mu,\phi,\tau)=\mu^a+\hat{h}_1(\mu,\tau)+\hat{h}_{21}(\mu,\phi,\tau)+\hat{h}_{22}(\mu,\phi,\tau), \tag{2.3}$$

with $\hat{h}_1\in F^k(c), \hat{h}_{21}\in F^{k-1}(1+b_2-a)), \hat{h}_{22}\in F^k(-\delta), \delta>0.$

Proof. Note that the main difference between the transformed Hamiltonian $\bar{H}$ in Lemma 2 and the required Hamiltonian $\hat{H}$ is $b_2' = b_2 - (a - b_2)$ may not be small enough. Fortunately, the term $\bar{h}_{22} \in F^k(b_2')$, and $\bar{H}$ also satisfies the assumptions of Lemma 2. So we can use Lemma 2 again without loss of more derivatives. j applications of Lemma 2 lead to a perturbation term $\hat{h}_{22} \in F^2(b_2^j), b_2^j = b_2^j - (a - b_2^{j-1}), b_2^j < b_2^{j-1}$. Taking j sufficiently large, we have $-\delta \le b_2^j < 0$. The composition of the transformations is our ψ_2.

Assume that Hamiltonian (2.1) satisfies the assumptions in Lemma 2. Applying Proposition 1 one time leads a transformed Hamiltonian (2.3). Set $\tilde{h}_{21} = 0$, and $\tilde{h}_{22} = \hat{h}_{21} + \hat{h}_{22} \in F^{k-1}(1 + b_2 - a)$. Applying Proposition 1 again to $\tilde{H} = \mu^a + \tilde{h}_1(\mu, \tau) + \tilde{h}_{21}(\mu, \phi, \tau) + \tilde{h}_{22}(\mu, \phi, \tau)$, we have the following

Proposition 2. *Under the assumption of Lemma 2, there exists a canonical transformation* ψ_3

$$\psi_3 : \begin{array}{l} \lambda = \mu + u(\mu, \phi, \tau) \\ \theta = \phi + v(\mu, \phi, \tau) \end{array}$$

satisfying $A_{\mu_+^3} \subset \psi_3(A_{\mu_0^3}) \subset A_{\mu_-^3}$ *for large* $\mu_-^3 < \mu_0^3 < \mu_+^3$, *which transforms Hamiltonian (1.2) into the following form:*

$$\hat{H}(\mu, \phi, \tau) = \mu^a + \hat{h}_1(\mu, \tau) + \hat{h}_2(\mu, \phi, \tau), \tag{2.4}$$

with $\hat{h}_1 \in F^k(c), \hat{h}_2 \in F^{k-2}(b'')$ *with* $b'' = b' - (a - 1) \le b_2 - 2(a - 1)$.

In the next section, we will see although the perturbation in Proposition 2 may be still not small for large μ, it is sufficient for our problem.

3. The proof of Theorem 1

Equation (1.1) is equivalent to the following systems

$$\frac{dx}{dt} = \frac{\partial h}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial h}{\partial x}, \tag{3.1}$$

where $h(t, x, y) = \frac{1}{2}y^2 + V(t, x)$. In order to rewrite $h(t, x, y)$ in the form of (2.1), we introduce the action angle variables.

Note that the equation $\frac{d^2x}{dt^2} + x^{2n+1} = 0$ is equivalent to the vector field

$$\frac{dx}{dt} = \frac{\partial h}{\partial y}; \quad \frac{dy}{dt} = -\frac{\partial h}{\partial x}, \tag{3.2}$$

with $h(x, y) = \frac{1}{2}y^2 + \frac{1}{2(n+1)}x^{2n+2}$. Clearly, $h(x, y) \ge 0$, and $h = 0$ iff $(x, y) = (0, 0)$. Obviously, all solutions of (3.2) are periodic.

If $(c(\tau), s(\tau))$ is a solution of (3.2) with $(c(0), s(0)) = (1, 0), \tau_0$ is its minimal period , it's easy to prove the following lemma:

Lemma 3.

(1)$c(\tau), s(\tau) \in C^\infty(R), s(\tau+\tau_0) = s(\tau), c(\tau+\tau_0) = c(\tau), c(0) = 1, s(0) = 0$
(2)$c'(\tau) = s(\tau), s'(\tau) = -[c(\tau)]^{2n+1}$
(3)$(n+1)s(\tau)^2 + [c(\tau)]^{(2n+2)} = 1$
(4)$c(-\tau) = c(\tau), s(-\tau) = -s(\tau)$

The action and angle variables are defined by the map $\Psi : R^+ \times S^1 \to R^2 \setminus \{0\}$ of the form:

$$\Psi : x = a^\alpha \lambda^\alpha c(\theta\tau_0), y = a^\beta \lambda^\beta s(\theta\tau_0), \tag{3.3}$$

where $\alpha = 1/(n+2), \beta = 1-\alpha, a = \frac{1}{\alpha\tau_0}$. We claim that Ψ is a symplectic diffeomorphism from $R^+ \times S^1$ onto $R^2 \setminus \{0\}$. Indeed, in view of (2), (3) in Lemma 3, one finds that the determinant of Jacobian of Ψ is 1. In the new coordinates, the Hamiltonian function of (3.2) becomes

$$h \circ \Psi(\lambda, \theta) = d \cdot \lambda^{2\beta} = h_0(\lambda), d = \frac{a^{2\beta}}{2n+2}.$$

The transformed system is

$$\frac{d\theta}{dt} = \frac{\partial h_0}{\partial \lambda} = 2\beta d\lambda^{2\beta-1}, \frac{d\lambda}{dt} = -\frac{\partial h_0}{\partial \theta} = 0. \tag{3.4}$$

The Hamiltonian of (3.1)

$$h(t, x, y) = \frac{1}{2}y^2 + \frac{1}{2(n+1)}x^{2(n+1)} + \sum_{l=1}^{2n} \frac{p_l(t)}{l+1} y^{l+1}, \tag{3.5}$$

is transformed by Ψ into the following:

$$H_1(t, \lambda, \theta) = d\lambda^{2\beta} + G_1(t, \lambda, \theta) + G_2(t, \lambda, \theta), \tag{3.6}$$

with

$$\begin{aligned} G_1(\lambda, \theta, t) &= \sum_{j=1}^{n+1} a^{\alpha j} c^j(\theta\tau_0) P_{j-1}(t) \lambda^{\alpha j} \in F^1(\frac{n+1}{n+2}), \\ G_2(\lambda, \theta, t) &= \sum_{j=n+2}^{2n+1} a^{\alpha j} c^j(\theta\tau_0) P_{j-1}(t) \lambda^{\alpha j} \in F^2(\frac{2n+1}{n+2}), \end{aligned} \tag{3.7}$$

Applying Proposition 2 to our Hamiltonian $H_1(t,\lambda,\theta)$, we know that there is a transformation ψ_3, which transforms (3.6) into a new Hamiltonian

$$H_2(t,\lambda,\theta)=\lambda^a+h_1(t,\lambda)+h_2(t,\lambda,\theta), \tag{3.8}$$

with $a=\frac{2n+2}{n+2}$, $h_1\in F^2(\frac{2n+1}{n+2})$, $h_2\in F^0(\frac{1}{n+2})$ for large λ. The corresponding equation is

$$\begin{aligned}\frac{d\theta}{dt}&=a\lambda^{a-1}+\frac{\partial h_1}{\partial\lambda}+\frac{\partial h_2}{\partial\lambda},\\ \frac{d\lambda}{dt}&=-\frac{\partial h_2}{\partial\theta}.\end{aligned} \tag{3.9}$$

Let $\psi_4:\mu=a\lambda^{a-1}$. The coordinates transformation $\Phi=\psi_4\circ\psi_3\circ\Psi:(x,y,t)\to(\theta,\mu,t)$ transforms system (3.1) into the following form:

$$\begin{aligned}\frac{d\theta}{dt}&=\mu+f_1(t,\mu)+f_2(t,\mu,\theta),\\ \frac{d\mu}{dt}&=f_3(t,\mu,\theta),\end{aligned} \tag{3.10}$$

where $f_1\in F^1(\frac{n-1}{n})$, $f_2\in F^0(-\frac{n+1}{n})$, $f_3\in F^0(-\frac{1}{n})$.

Denote $r(\mu,t)=\mu t+\int_0^t f_1(\mu,s)\,ds$, $(\mu(t),\theta(\tau))=\phi^t(\mu,\theta)$ the flow of (3.5) with $\phi^0=id$, and let $r(\mu)=r(\mu,1)$. Let

$$\mu(t)=\mu+B(\mu,\theta,t),$$

$$\theta(t)=\theta+r(\mu,t)+A(\mu,\theta,t).$$

Then the integral equation for the flow of (3.5) is equivalent to the following equations for A and B.

$$\begin{aligned}A(\mu,\theta,t)&=\int_0^t B(\mu,\theta,s)\,ds+\int_0^t\int_0^1\frac{\partial f_1}{\partial\mu}(\mu+s_1B,s)B\,ds_1\,ds\\ &\quad+\int_0^t f_2(\mu+B,\theta+r+A,s)\,ds,\\ B(\mu,\theta,t)&=\int_0^t f_3(\mu+B,\theta+r+A,s)\,ds.\end{aligned} \tag{3.11}$$

From the existence and uniqueness theorem for solutions ([CL]), we know that A,B exist uniquely. Moreover, A,B are smooth in initial values (μ,θ), since the right side equation of (3.5) smoothly depends on (μ,θ)-variables and the smoothness of solutions with respect to initial values is independent of t smoothness.

It is easy to see that the time 1 map ϕ^1 of the flow ϕ of (3.10) is of the form:

$$\phi^1:\theta_1=\theta+r(\mu)+\hat f(\mu,\theta),\mu_1=\mu+\hat g(\mu,\theta).$$

with $\hat{f} = A(\mu,\theta,1), \hat{g} = B(\mu,\theta,1)$. Note that $D_\mu^i D_\theta^j f_m(\phi+r+A, r+B)(m=1,2,3)$ is the sum of terms

$$(D_1^s D_2^k f_m)\cdot(\prod_{l=1}^{k} D_\mu^{i_l} D_\theta^{j_l}(\theta+r+A))\cdot(\prod_{l=k+1}^{k+s} D_\mu^{i_l} D_\theta^{j_l}(\mu+B)), \tag{3.12}$$

with $1 \leq s+k \leq i+j, \sum_{l=1}^{s+k} l_l = i, \sum_{l=1}^{k+s} j_l = j$. The following lemma can be inductively verified from (3.11) and (3.12).

Lemma 4. *For every pair (i, j) of nonnegative integers we have*

$$|D_\mu^i D_\theta^j \hat{f}(\mu,\theta)|, |D_\mu^i D_\theta^j \hat{g}(\mu,\theta)| \leq \mu^{-\frac{1}{n}-j},$$

For sufficiently large μ .

Now we are in the position to prove the existence of invariant circle for ϕ^1 by Moser's twist theorem [Mo].

Theorem A (Moser). *Suppose that $F_1(u,v), F_2(u,v) \in C^{333}$ are periodic in u with the period 1, and P is a mapping from the annulus $S^1 \times [a,b]$ to $S^1 \times R^1$, of the form*

$$u_1 = u + r(v) + F_1(u,v), \qquad v_1 = v + F_2(u,v), \tag{3.13}$$

and P has the intersection property. It concludes that there is a real number δ, such that if

$$|F_1|_{c^{333}} + |F_2|_{c^{333}} < \delta, c_0^{-1} < \frac{dr}{dv} < c_0, \quad c_0 > 0 \tag{3.14}$$

then for any irrational number $\lambda \in [a+\beta, b-\beta]$ satisfying the condition
$|\lambda - \frac{m}{n}| > \gamma\frac{1}{n^{2+\beta}}$, for some constants $\beta, \gamma \in [0,1]$, and for all integers m and n with $n > 0$, there is a closed C^1 curve Γ_λ with the following properties
i)Γ_λ is invariant under the mapping P;
ii) The mapping P restricted on Γ is conjugate to the rotation $\theta \to \theta + \lambda$.

Proof of theorem 1. Since the mapping ϕ^1 meets the assumptions of Moser's twist theorem, it follows that for $\omega \geq \omega^*$, ω^* sufficiently large , and $|\omega - p/q| \geq c|q|^{-2-\beta}$ for two constants $\beta > 0$ and $c > 0$ and for all integers p and $q \neq 0$, there is an embedding $\Gamma : S^1 \to A_{\mu_0}$ of a circle, which is differentially close to the injection map j of the circle $\{\omega\} \times S^1 \to A_{\mu_0}$, and which is invariant under the map ϕ^1. Moreover , on this invariant curve the map ϕ^1 is conjugated to a rotation with rotation number ω:

$$\phi^1 \circ \Gamma(s) = \Gamma(s+\omega) \quad (mod \ \ 1).$$

And the solutions of the equation (3.10) starting at time $t = 0$ on this invariant curve determine a torus in the space $(\mu,\theta,\tau) \in A_{\mu_0} \times S^1$. The solution on this torus is quasiperiodic which corresponds to the quasiperiodic solution of (1.1). This proves the second part of Theorem 1. In order to prove the first part of Theorem 1, just observe that , in the original coordinates, every point $(x, x') \in R^2$ is in the interior of some invariant curve of the time 1 map of the flow surrounding the origin. The solution starting from (x, x') is therefore confined in the interior of the time periodic cylinder corresponding to the invariant curve and hence is bounded. This ends the proof of Theorem 1.

4. Remarks

1. In the previous proof we assume that the leading coefficient is constant. In the case that the leading coefficient is not constant, i.e., $V(t,x) = \sum_{j=0}^{2n+1} \frac{1}{j+1} P_j(t) x^{j+1}$, $p_{2n+1} > 0$, by a coordinates transformation, the leading coefficient $P_{n+1}(t)$ can be remitted to the term of power one up to losing of two times derivatives in t. Thus the same result holds true for $p_{2n+1} \in C^3(S^1)$ is an immediate conclusion of Theorem 1.

In fact, let $x = p_{2n+1}^{\frac{-1}{2n+4}} \cdot y, \tau = \int_0^t p_{2n+1}^{\frac{1}{n+2}}(s)\, ds$, and substitute it in (1.1), we get

$$\frac{d^2y}{d\tau^2} + y^{2n+1} + \sum_{l=0}^{2n} p_l(t) p_{2n+1}^{-\frac{3+l}{2n+4}}(t) y^l + P(p_{2n+1}, p'_{2n+1}(t), p''_{2n+1}(t))y = 0, \quad (4.1)$$

where

$$\begin{aligned} P(\tau) &= P(p_{2n+1}(t), p'_{2n+1}(t), p''_{2n+1}(t)) \\ &= (\frac{1}{2n+4})(\frac{1}{2n+4} + 1) p_{2n+1}^{\frac{-2}{n+2}-2}(t)(p'_{2n+1}(t))^2 \\ &\quad - \frac{1}{2n+4} p_{2n+1}^{\frac{-2}{n+2}-1}(t) p''_{2n+1}(t) \end{aligned}$$

with $t = t(\tau)$. It is easy to see that $P \in C^{n-2}$ if $p_{2n+1} \in C^n$,

In the following we prove that the coefficients in (4.1) are periodic with respect to the new time variable τ.

It is sufficient to prove that $\exists \tau^*, \ni t(\tau + \tau^*) = t(\tau) + 1$. In fact , set $\tau^* = \int_0^1 p_{2n+1}^{\frac{1}{n+2}}(s)\, ds$, so $\tau + \tau^* = \int_0^{t(\tau+\tau^*)} p_{2n+1}^{\frac{1}{n+2}}(s)\, ds$. Meanwhile,

$$\begin{aligned} \tau + \tau^* &= (\int_0^{t(\tau)} + \int_0^1) p_{2n+1}^{\frac{1}{n+2}}(s)\, ds = (\int_0^{t(\tau)} + \int_{t(\tau)}^{t(\tau)+1}) p_{2n+1}^{\frac{1}{n+2}}(s)\, ds \\ &= \int_0^{t(\tau)+1} p_{2n+1}^{\frac{1}{n+2}}(s)\, ds. \end{aligned}$$

That is $\int_0^{t(\tau)+1} p_{2n+1}^{\frac{1}{n+2}}(s)\, ds = \int_0^{t(\tau+\tau*)} p_{2n+1}^{\frac{1}{n+2}}(s)\, ds$, it follows that $t(\tau + \tau*) = t(\tau) + 1$.

2. In fact, The coefficient $p_0(t), p_1(t) \in C^0$ is sufficient to guarantee the boundedness (See [LB],[M]). So according to last remark, boundedness result for $p_{2n+1} \in C^2$ is available.

3. Although our equation has only twice smoothness in t, the invariant curve of the corresponding Poincare mapping is smooth enough.

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