Examples of Discontinuity of Lyapunov Exponent in Smooth Quasi-Periodic Cocycles*

Yiqian Wang and Jiangong You†
Department of Mathematics, Nanjing University, Nanjing 210093, China
Email: yqwangnju@yahoo.com; jyou@nju.edu.cn

Abstract

We study the regularity of the Lyapunov exponent for quasi-periodic cocycles (T_ω, A) where T_ω is an irrational rotation $x \to x + 2\pi\omega$ on $\mathbb{S}^1$ and $A \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$, $0 \leq l \leq \infty$. For any fixed $l = 0, 1, 2, \cdots, \infty$ and any fixed ω of bounded-type, we construct $D_l \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$ such that the Lyapunov exponent is not continuous at D_l in $\mathcal{C}^l$-topology. We also construct such examples in a smaller Schrödinger class.

1 Introduction and Results

Let X be a $\mathcal{C}^r$ compact manifold. If $T : X \to X$ is an ergodic system with normalized invariant measure μ and $A : X \to SL(2,\mathbb{R})$, we call (T, A) a cocycle. When A is L^∞ (C^l, analytic, respectively), we call (T, A) a L^∞ (C^l, analytic, respectively) cocycle.

For any $n \in \mathbb{N}$ and $x \in X$, we denote

$$A^n(x) = A(T^{n-1}x) \cdots A(Tx)A(x)$$

and

$$A^{-n}(x) = A^{-1}(T^{-n}x) \cdots A^{-1}(T^{-1}x).$$

For fixed (X, T, μ), the (maximum) Lyapunov exponent of (T, A) is defined as

$$L(A) = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n(x)\| d\mu \in [0, \infty).$$

We are interested in the continuity of the Lyapunov exponent $L(A)$ in $\mathcal{C}^l(X, SL(2,\mathbb{R}))$. It is known that $L(A)$ is upper semi-continuous, thus it is continuous at generic A. Especially, it

*This work is supported by NNSF of China (Grant 11031003) and a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

†The corresponding author.

is continuous at A with $L(A) = 0$ and at uniformly hyperbolic cocycles. The most interesting issue is the continuity of $L(A)$ at the points of non-uniformly hyperbolic cocycles, which is bound to depend on the class of cocycles under consideration including its topology. Knill [27] showed that $L : L^\infty(X, SL(2,\mathbb{R})) \to [0,\infty)$ is not continuous if (X,T) is aperiodic (i.e. the set of periodic points is of zero measure). Then Furman proved that if (X,T) is uniquely ergodic, then $L : \mathcal{C}^0(X, SL(2,\mathbb{R})) \to [0,\infty)$ is never continuous at points of non-uniformly hyperbolicity. Motivated by Mañé [28, 29], Bochi [7, 8] further proved that with $T : X \to X$ being a fixed ergodic system, any non-uniformly hyperbolic $SL(2,R)$-cocycle can be approximated by cocycles with zero Lyapunov exponent in the C^0 topology. These results suggest that the discontinuity of L is very common among cocycles with low regularity.

We also mention some other related results on the continuity of the Lyapunov exponent. Furstenberg - Kifer [18] and Hennion [20] proved continuity of the largest Lyapunov exponent of i.i.d random matrices under a condition of almost irreducibility. More recently, C. Bocker-Neto and M. Viana [6] proved that the Lyapunov exponents of locally constant $GL(2,\mathbb{C})$-cocycles over Bernoulli shifts depend continuously on the cocycle and on the invariant probability.

If the base system is a rotation on torus, i.e., $X = \mathbb{T}^n$, $T = T_\omega : x \to x + 2\pi\omega$ with rational independent ω, we call (T_ω, A) a quasi-periodic cocycle. $X = \mathbb{S}^1$ is the most special case. For simplicity, we denote the cocycle (T_ω, A) by (ω, A).

If furthermore $A(x) = S_{v,E}(x)$ is of the form

$$S_{v,E}(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix},$$

we call $(\omega, S_{v,E}(x))$ a quasi-periodic Schrödinger cocycle. This type of cocycles have attracted much attention largely due to their rich background in physics.

Now we recall some positive results for quasi-periodic cocycles (ω, A). In [19] Goldstein and Schlag developed a powerful tool, the Avalanche Principle, and proved that if ω is a Diophantine irrational number and $v(x)$ is analytic, then the Lyapunov exponent $L(E)$ is Hölder continuous provided $L(E) > 0$. Similar results were proved in [13] by Bourgain, Goldstein and Schlag when the underlying dynamics is a shift or skew-shift of a higher dimensional torus. Then Bourgain and Jitomirskaya [12] improved the result of [19] by showing that if ω is an irrational number and the potential $v(x)$ is analytic, then the Lyapunov exponent is jointly continuous on E and ω. This result is crucial to solving the Ten Martini problem in [2]. Similar results were obtained by Bourgain for shifts of higher dimensional tori in [11]. Later, Jitomirskaya, Koslover and Schulteis [22] proved that the Lyapunov exponent is continuous on a class of analytic one-frequency quasiperiodic $M(2,\mathbb{C})$-cocycles with singularities. With this result, they proved continuity of Lyapunov exponent associated with general quasi-periodic Jacobi matrices or orthogonal polynomials on the unit circle in various parameters. Recently, Jitomirskaya and Marx [23] proved the continuity of Lyapunov exponent for all non-trivial singular analytic quasiperiodic cocycles with one-frequency, thus removing the constraints in [22]. Moreover, applications are extended to analytic Jacobi operators with more parameters, which is crucial to determining the Lyapunov exponent of extended Harper's model by Jitomirskaya and Marx [24]. For further results, one is referred to [4, 5, 9, 10, 14, 15, 16, 17, 21, 30].

In conclusion, the Lyapunov exponent of quasi-periodic cocycles is discontinuous in $\mathcal{C}^0$ topology, and continuous in $\mathcal{C}^\omega$ topology.

In [22] the authors proposed to consider the situation between $\mathcal{C}^0$ and $\mathcal{C}^\omega$. Klein [26] studied continuity of Lyapunov exponent on E in the Gevrey case. More precisely, he proved that the Lyapunov exponent of quasi-periodic Schrödinger cocycles in the Gevrey class is continuous at the potentials $v(x)$ satisfying some transversality condition. Recently, Avila and Krikorian [1] restricted their attention to a class of quasi-periodic $SL(2,\mathbb{R})$ cocycles, called ϵ-monotonic cocycles (cocycles satisfying a twist condition). They proved that the Lyapunov exponent is continuous, even smooth in smooth category of ϵ-monotonic quasi-periodic $SL(2,\mathbb{R})$ cocycles.

An interesting question is if the Lyapunov exponent of (ω, A) is always continuous in $\mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$, $l = 1, 2, \cdots, \infty$, as in $\mathcal{C}^\omega(\mathbb{S}^1, SL(2,\mathbb{R}))$ or in ϵ-monotonic quasiperiodic $SL(2,\mathbb{R})$-cocycles.

In this paper, we construct a cocycle $D_l \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$ such that the Lyapunov exponent is not continuous at D_l in $\mathcal{C}^l$-topology for any $l = 1, 2, \cdots, \infty$.

Theorem 1 *Suppose that ω is a fixed irrational number of bounded-type. For any $0 \le l \le \infty$, there exist cocycles $D_l \in \mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$ such that the Lyapunov exponent is discontinuous at D_l in $\mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$.*

Remark 1.1 *Let $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The cocycles we constructed are of the form $\Lambda \cdot R_{\frac{\pi}{2}-\phi(x)}$, where $\phi(x)$ is either a 2π-periodic function corresponding to a cocycle homotopic to the identity (see Figure 1), or a sum of the identity and a 2π-periodic function corresponding to a cocycle non-homotopic to the identity (see Figure 2).*

Remark 1.2 *Theorem 1 shows that the continuity of Lyapunov exponent in $\mathcal{C}^l$-topology ($l = 1, 2, \cdots, \infty$) and C^ω is different. Combining with Avila and Krikorian's result [1], it also shows the continuity of Lyapunov exponent in $\mathcal{C}^l$-topology ($l = 1, 2, \cdots, \infty$) and C^0 is different. It is plausible that the Lyapunov exponent is continuous at an open and dense set in $\mathcal{C}^l$-topology ($l = 1, 2, \cdots, \infty$). Surprisingly, there are no examples of continuity of Lyapunov exponent at non-uniformly hyperbolic cocycles which are homotopic to the identity.*

Remark 1.3 *We say ω is an irrational number of bounded type if there exists $M \ge \frac{\sqrt{5}+1}{2}$, such that for its fractional expansion $\frac{p_n}{q_n}$, $n = 1, 2, \cdots$, it holds that $q_{n+1} < M q_n$, $\forall n$. Technically we need to assume that ω is of bounded type. This is not typical as the set of such numbers is of measure zero. We believe that counterexamples can be constructed for ω in a full measure, even for all real numbers.*

Remark 1.4 *Recently, Jitomirskaya and Marx [25] obtained similar results in complex category $M(2,\mathbb{C})$ by the tools of harmonic analysis.*

From the $SL(2,\mathbb{R})$ examples homotopic to the identity constructed in Theorem 1, it is easy to construct examples in the Schrödinger class by conjugation.[1]

Theorem 2 *Suppose that ω is a fixed irrational number of bounded-type. For any $0 \le l \le \infty$, there exists a C^l periodic function $v(x) = v(x + 2\pi)$ such that the Lyapunov exponent is discontinuous at $S_{v,0}$ in the Schrödinger class, i.e., there exist C^l periodic functions $v_n(x) = v_n(x + 2\pi)$ such that $v_n(x) \to v(x)$ is C^l topology but $L(S_{v_n,0}) \nrightarrow L(S_{v,0})$.*

Outline of the proof of Theorem 1. D_l will be constructed as the limit of a sequence of cocycles $\{A_n(x), n = N, N+1, \cdots\}$ in $\mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$. $\{A_n(x), n = N, N+1, \cdots\}$ possess some kind of finite hyperbolic property, i.e., $\|A_n^{r_n^+}(x)\| \sim \lambda^{r_n^+}$ for most $x \in \mathbb{S}^1$ and $\lambda \gg 1$ with $r_n^+ \to \infty$ as $n \to \infty$, which gives a lower bound estimate $(1-\epsilon)\log\lambda$ of the Lyapunov exponent of the limit cocycle $D_l(x)$ if $\lambda \gg 1$. Then by modifying $\{A_n(x)\}_{n=N}^{\infty}$, we construct another sequence of cocycles $\{\tilde{A}_n(x)\}_{n=N}^{\infty}$ such that $\tilde{A}_n(x) \to D_l(x)$ in $\mathcal{C}^l$-topology as $n \to \infty$. Moreover, for each n, the Lyapunov exponent of $\tilde{A}_n(x)$ is less than $(1-\delta)\log\lambda$ with $1 > \delta \gg \epsilon > 0$ independent of λ, which implies the discontinuity of the Lyapunov exponent at $D_l(x)$.

A key technique in the construction of $A_n(x)$ comes from Young [31], which was derived from Benedicks-Carleson [3]. However, there is a difference between our method and the one in [31]. To construct $A_n(x)$ and $\tilde{A}_n(x)$, we have to start from some cocycle possessing “degenerate” critical points, while the critical points of cocycles in [31] are non-degenerate.

The proof of Theorem 2. For any $0 \le l \le \infty$, assume that $D_{l+\tau}(x) = \Lambda \cdot R_{\frac{\pi}{2}-\phi(x)}$ are cocycles homotopic to the identity constructed in Theorem 1, and $\tau = \tau(\omega)$ is a fixed integer which will be defined later. In the example, $\phi(x)$ can be assumed to satisfy $\max_x |\phi(x)| < \frac{\pi}{10}$. Let $\alpha = (0,1)^T$. Then $D_{l+\tau}(x)\cdot\alpha$ and α are linearly independent for every x, thus the matrix $B_1(x) = (-D_{l+\tau}(x-\omega)\cdot\alpha, \alpha) \in C^{l+\tau}(\mathbb{S}^1, GL(2,\mathbb{R}))$ is non-singular. A direct computation shows that there exist $a(x), c(x) \in C^{l+\tau}(\mathbb{S}^1, \mathbb{R})$ such that

$$B_1(x+\omega)^{-1} D_{l+\tau}(x) B_1(x) = S(x) = \begin{pmatrix} a(x) & -1 \\ c(x) & 0 \end{pmatrix}$$

Here $c(x) > 0$ since the determinant of B_1 does not change sign, and we write $c(x) = e^{f(x)}$. Let $B_2(x) = \begin{pmatrix} e^{d(x)} & 0 \\ 0 & e^{d(x+\omega)} \end{pmatrix}$, where

$$d(x+2\omega) - d(x) = f(x) - [f(x)]. \tag{1.1}$$

Then $B_2(x+\omega)^{-1} S(x) B_2(x)$ has the form $\begin{pmatrix} -v(x) & -1 \\ e^{[f(x)]} & 0 \end{pmatrix}$ where $v(x)$ is uniquely determined by $D_{l+\tau}$. Since 2ω is Diophantine, (1.1) has a solution $d(x) \in C^l(\mathbb{S}^1, \mathbb{R})$ if τ is large enough. It follows that $v(x) \in C^l(\mathbb{S}^1, \mathbb{R})$.

[1]The authors are grateful to A. Avila, Z. Zhang and the referee for pointing out this. The proof given below was proposed by A. Avila and the referee. One can also use Z.Zhang’s trick in [32] to give another proof.

Let $B(x) = B_1(x)B_2(x)$, then $detB(x) = e^{[f]}detB(x+\omega)$ by $D_{l+\tau}(x)B(x) = B(x+\omega)S(x)$. It follows that $e^{[f]} = 1$ since $x \mapsto x + n\omega$ is ergodic in $\mathbb{S}^1$, and consequently $detB(x) = e$ is constant. Let $\tilde{B}(x) = \frac{1}{\sqrt{e}}B(x) \in C^l(\mathbb{S}^1, SL(2,\mathbb{R}))$, we have

$$\tilde{B}(x+\omega)^{-1}D_{l+\tau}(x)\tilde{B}(x) = \begin{pmatrix} -v(x) & -1 \\ 1 & 0 \end{pmatrix} = S_{v,0}.$$

$L(D_l) = L(S_{v,0})$ since Lyapunov exponent is conjugation invariant.

By Theorem 1, there is a sequence of $\tilde{A}_n$ such that $\tilde{A}_n \to D_{l+\tau}$ in $C^{l+\tau}$ topology and $|L(\tilde{A}_n) - L(D_{l+\tau})| > \delta$ for a positive δ when n is large. By the similar argument as above, there exist $\tilde{B}_n(x) \in C^l(\mathbb{S}^1, SL(2,\mathbb{R}))$, $v_n(x) \in C^l(\mathbb{S}^1, \mathbb{R})$, such that $\tilde{B}_n$ conjugates $\tilde{A}_n$ to a Schrödinger cocycle $S_{v_n,0}$ and thus $L(\tilde{A}_n) = L(S_{v_n,0})$. Since $\|\tilde{A}_n - D_{l+\tau}\|_{C^{l+\tau}} \to 0$, we have $\|\tilde{B}_n - \tilde{B}\|_{C^l} \to 0$ and then $\|v_n - v\|_{C^l} \to 0$. On the other side, $|L(S_{v,0}) - L(S_{v_n,0})| > \delta > 0$ when n is large enough. The proof of Theorem 2 is thus finished.

Throughout the paper ω is a fixed irrational number of bounded type (described by the parameter M), l is a fixed positive integer, $\delta = \frac{1}{4}M^{-20} > 0$, $\epsilon = M^{-100} > 0$. N, μ and λ with $\lambda \geq \mu \geq \lambda^{1-\epsilon} \gg N \gg 1$ and $\mu^\epsilon > 2$ denote three large numbers determined later.

2 Some properties of the concatenation of hyperbolic matrices

In this section, we will study the norm of the product of hyperbolic matrices by analyzing the curves of the most contracted directions of them. The analysis in this section is developed from [31]. In the following, all matrices belong to $SL(2,\mathbb{R})$.

A matrix $A \in SL(2,\mathbb{R})$ with $\|A\| > 1$ is called hyperbolic. We denote the unit vectors on the most contracted and expanded direction of A by $s(A)$ and $u(A)$ respectively. That is,

$$|A \cdot s(A)| = \min_{|v|=1} |A \cdot v| = \|A\|^{-1}, \qquad |A \cdot s'(A)| = \max_{|v|=1} |A \cdot v| = \|A\|.$$

It is known that $s \perp u$ and $As \perp Au$. Moreover, for two matrices A and B with $\|A\|, \|B\| > 1$, it is easy to see that $\|BA\| = \|B\| \cdot \|A\|$ if and only if $A(s(A))$ is parallel to $s(B)$. The most contracted direction plays a key role in the growth of the norm of product of hyperbolic matrix sequences.

For a sequence of matrices $\{\cdots, A_{-1}, A_0, A_1, \cdots\}$, we denote

$$A^n = A_{n-1} \cdots A_1 A_0$$

and

$$A^{-n} = A_{-n}^{-1} \cdots A_{-1}^{-1}.$$

Definition 2.1 *For any* $1 \ll \mu \le \lambda$, *we say that the block of matrices* $\{A_0, A_1, ..., A_{n-1}\}$ *is* μ*-hyperbolic if*

$$\text{(i)} \quad \|A_i\| \le \lambda \quad \forall i,$$
$$\text{(ii)} \quad \|A^i\| \ge \mu^{i(1-\epsilon)} \quad \forall i$$

and (i)-(ii) *hold if* $A_0, ..., A_{n-1}$ *is replaced by* $\{A_{n-1}^{-1}, ..., A_0^{-1}\}$.

The next proposition is due to Young [31], which tells us when the concatenation of two hyperbolic blocks is still a hyperbolic block.

Lemma 2.1 *Suppose* C *satisfies* $\|C\| \ge \mu^m$ *with* $\mu \gg 1$. *Assume* $\{A_0, A_1, \cdots, A_{n-1}\}$ *is a* $\mu-$*hyperbolic sequence and* $\angle(s(C^{-1}), s(A^n)) = 2\theta \ll 1$. *Then* $\|A^n \cdot C\| \ge \mu^{(m+n)(1-\epsilon)} \cdot \theta$.

Denote $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and R_θ the rotation by the angle θ, i.e., $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Let $\phi(x)$ be the lift of a $\mathcal{C}^l$ function defined on $\mathbb{S}^1$. Throughout this paper, the matrix A is of the special form $\Lambda \cdot R_{\frac{\pi}{2}-\phi(x)}$.

Let RP^1 be the real projective line and denote the natural projection $\mathbb{R}^2 \to \mathrm{RP}^1$ by $v \to \bar{v}$. For any matrix $A \in SL(2, \mathbb{R})$, define the map $\bar{A} : \mathrm{RP}^1 \to \mathrm{RP}^1$ by $\bar{A} \cdot \bar{v} = \overline{A \cdot v}$. Then we define the projective actions corresponding to $A(x)$ by

$$\Phi_A : \mathbb{S}^1 \times \mathrm{RP}^1 \to \mathbb{S}^1 \times \mathrm{RP}^1, \quad \Phi_A(x, \theta) = (Tx, \bar{A}(x)\theta).$$

Then for $A(x) = \Lambda \cdot R_{\frac{\pi}{2}-\phi(x)}$, we have

$$\Phi_A = \Phi_\Lambda \circ \Phi_{R_{\frac{\pi}{2}-\phi(x)}} : \mathbb{S}^1 \times \mathrm{RP}^1 \to \mathbb{S}^1 \times \mathrm{RP}^1,$$

where $\Phi_\Lambda(x, \theta) = (x, \bar{\Lambda}\theta)$ and $\Phi_{R_{\frac{\pi}{2}-\phi(x)}}(x, \theta) = (Tx, \frac{\pi}{2} - \phi(x) + \theta)$.

Suppose that $A^n(x)$ is hyperbolic for any $x \in I \subset \mathbb{S}^1$. Let $s, u : I \to \mathrm{RP}^1$ be the function

$$s(x) = \overline{s(A^n(x))}, \quad u(x) = \overline{u(A^n(x))}.$$

We also define $s', u' : T^n(I) \to \mathrm{RP}^1$ by

$$s'(x) = \overline{s(A^{-n}(x))}, \quad u'(x) = \overline{u(A^{-n}(x))}.$$

It is not difficult to see that

$$(T^n x, s'(T^n x)) = \Phi_A^n(x, u(x)), \quad (T^n x, u'(T^n x)) = \Phi_A^n(x, s(x)), \quad x \in I. \tag{2.1}$$

Since $\phi(x)$ is $\mathcal{C}^l$, we have that the map $h : (x, \theta) \to \frac{\partial}{\partial\theta}|A^n(x)\hat{\theta}|$ is $\mathcal{C}^l$. Obviously, from the definition of $s(x)$ and $u(x)$, we have $h(x, s(x)) = h(x, u(x)) = 0$. Moreover, since $A^n(x)$ is hyperbolic, we can easily see that if $h(x, \theta) = 0$, then $\frac{\partial h}{\partial\theta}(x, \theta) \neq 0$, where $\hat{\theta}$ denotes the unit vector corresponding to $\theta \in \mathrm{RP}^1$. Thus by Implicit Function Theorem s, u are determined by $h(x,\theta)=0$ with l-order derivatives. Similarly, we can prove that s', u' are of l-order differentiability.

The following lemma gives the estimates on the derivatives of curves of the most contracted direction of hyperbolic matrices.

Lemma 2.2 *Let I be an interval in $\mathbb{S}^1$. Assume that $(A(x),\cdots,A(T^{n-1}x))$ is μ-hyperbolic for each $x\in I$ with $n,\mu\geq\lambda^{1-\epsilon}\gg 1$. Then it holds that*

$$(1)\, |s-\phi(x)|_{\mathbb{C}^1} < 2\mu^{-(1-\epsilon)}, \quad \forall x\in I;$$

$$(2)\, |s'|_{\mathbb{C}^1} < 2\mu^{-(1-\epsilon)} \quad \forall x \in T^n I.$$

The proof can be found in [31] given by Young.

3 The construction of $A_n(x)$

We first construct the counter-examples in finite smooth case. Throughout this paper, $l\in\mathbb{N}$ is arbitrary but fixed, and $N\gg 1$ with $q_N^{-2}<\delta$ and

$$10l\sum_{n=N}^{\infty}\frac{\log q_{n+1}}{q_n}\leq\epsilon. \tag{3.1}$$

For $c_1\in[0,\pi), c_2=c_1+\pi$ and $n\geq N$, we define $\mathcal{C}_0=\{c_1,c_2\}$, $I_{n,1}=[c_1-\frac{1}{q_n^2},\ c_1+\frac{1}{q_n^2}]$, $I_{n,2}=[c_2-\frac{1}{q_n^2},\ c_2+\frac{1}{q_n^2}]$ and $I_n=I_{n,1}\bigcup I_{n,2}$. For $x\in I_n$, we denote the smallest positive integer j with $T^jx\in I_n$ (respectively $T^{-j}x\in I_n$) by $r_n^+(x)$ (respectively $r_n^-(x)$), and define $r_n^{\pm}=\min_{x\in I_n} r_n^{\pm}(x)$. Obviously, $r_n^{\pm}\geq q_n$. Moreover, for $C\geq 1$, we denote by $\frac{I_{n,i}}{C}$ the set $[c_i-\frac{1}{Cq_n^2}, c_i+\frac{1}{Cq_n^2}], i=1,2$ and by $\frac{I_n}{C}$ the set $\frac{I_{n,1}}{C}\cup\frac{I_{n,2}}{C}$.

For any $n>N$, we inductively define $\{\lambda_n\}$ by $\log\lambda_n=\log\lambda_{n-1}-\frac{10l\log q_n}{q_{n-1}}$ where $\lambda_N=\lambda$. It is easy to see that λ_n decrease to some λ_∞ with $\lambda_\infty>\lambda^{1-\epsilon}$ if $\lambda\gg N\gg 1$.

In this section, we will inductively construct a convergent sequence of cocycles $\{A_n(x), n=N,N+1,\cdots\}$ in $\mathcal{C}^l(\mathbb{S}^1,SL(2,\mathbb{R}))$ with some desirable properties. More precisely, we will prove

Proposition 3.1 *There exist $A_n=\Lambda R_{\frac{\pi}{2}-\phi_n(x)}$ with $\phi_n(x)$ the lift of a $\mathcal{C}^l$ function on $\mathbb{S}^1$ $(n=N,N+1,\cdots)$ such that the following properties hold:*

1. $$|\phi_n(x)-\phi_{n-1}(x)|_{C^l}\leq\lambda_n^{-q_{n-1}^{\frac{1}{10}}},\quad \text{if}\quad n>N. \tag{3.2}$$
2. *For each $x\in I_n$, $A_n(x), A_n(Tx),\cdots,A_n(T^{r_n^+(x)-1}x)$ is λ_n-hyperbolic.*

3. Let $s_n(x) = \overline{s(A_n^{r_n^+}(x))}$, $s_n'(x) = \overline{s(A_n^{-r_n^-}(x))}$. *Then we have*

$$(1)_{\rm n} \quad s_n(x) - s_n'(x) = \phi_0(x) \quad \text{ on } \frac{I_n}{10};$$

$$(2)_{\rm n} \quad |s_n(x) - s_n'(x)| \geq \frac{1}{2}|\phi_0(x)| \geq \frac{1}{(20q_n^2)^{l+1}}, \quad x \in I_n \backslash \frac{I_n}{10},$$

where $\phi_0(x)$ *is defined in (3.3) and (3.4).*

Proof. *The construction of* $A_N(x)$: Let $c_1, c_2 \in \mathbb{S}^1$ with $c_1 \in [0,\pi)$, $c_2 = c_1 + \pi$ and δ_0 a small positive number. We define ϕ_0 on $\{x||x-c_1| \leq \delta_0\} \bigcup \{x||x-c_2| \leq \delta_0\}$ as follows.

$$\phi_0(x) = \begin{cases} \phi_{01}(x), & |x-c_1| < \delta_0; \\ -\phi_{02}(x) \text{ (or } \phi_{02}(x)), & |x-c_2| < \delta_0, \end{cases} \tag{3.3}$$

and where

$$\phi_{0i}(x) = \text{sgn}(x-c_i)|x-c_i|^{l+1}, \quad i = 1,2. \tag{3.4}$$

Then we define $\phi(x)$ be a lift of a $\mathcal{C}^l$ function on $\mathbb{S}^1$ satisfying the following.

(a)

$$\phi(x) = \begin{cases} \phi_{01}(x), & |x-c_1| \leq \delta_0; \\ -\phi_{02}(x) \text{ (or } \pi + \phi_{02}(x), \text{ respectively)}, & |x-c_2| \leq \delta_0. \end{cases}$$

(b) $\forall |x-c_i| > \delta_0,\ i = 1,2,\ |\phi(x) - k\pi| > \delta_0^{l+1}$ for any $k \in Z$.

Remark 3.1 *One can either choose* $\phi(x)$ *to be a* 2π *periodic function (see Fig. 1), which corresponds to a cocycle homotopic to the identity, or to be the identity plus a* 2π*-periodic function (see Fig. 2), which corresponds to a cocycle non-homotopic to the identity.*

Let $A = \Lambda \cdot R_{\frac{\pi}{2}-\phi(x)}$ which belongs to $\mathcal{C}^l(\mathbb{S}^1, SL(2,\mathbb{R}))$. From [31], there exists a (large) $\lambda^* > 0$ depending on ϕ, l and ϵ such that

$$\{A(x), \cdots, A(T^{r_N^+(x)-1}x)\} \ \textit{is } \lambda-\textit{hyperbolic}, \quad \forall x \in I_N \tag{3.5}$$

if $\lambda > \lambda^*$.

Let $\bar{s}_N(x) = \overline{s(A^{r_N^+}(x))}$ and $\bar{s}_N'(x) = \overline{s(A^{-r_N^-}(x))}$. Define $e_N(x)$ to be the following 2π-periodic function:

$$e_N(x) = \begin{cases} \phi_0(x) - (\bar{s}_N - \bar{s}_N')(x) & x \in \frac{I_N}{10} \\ h_N^{\pm}(x), & x \in I_N \backslash \frac{I_N}{10} \\ 0, & x \in \mathbb{S}^1 \backslash I_N \end{cases}$$

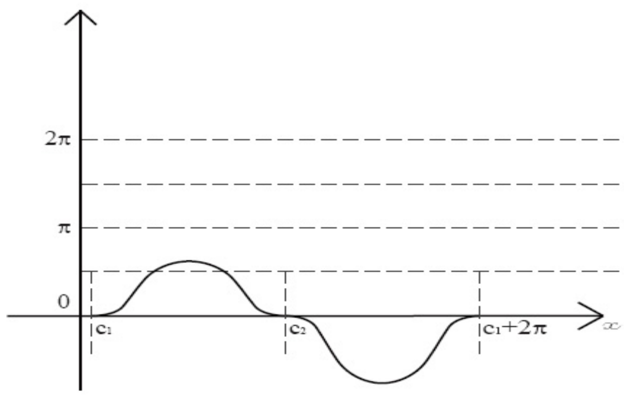

Fig1: homotopic to the Identity

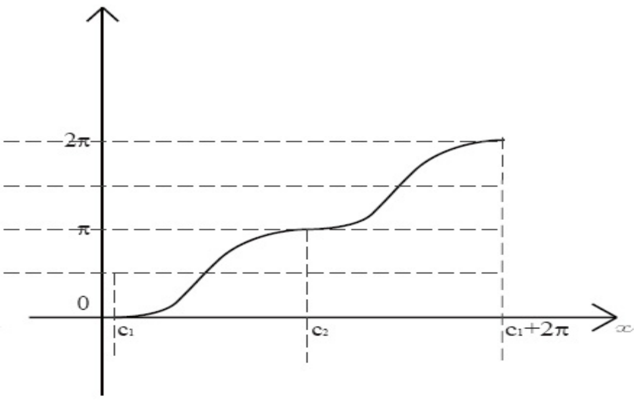

Fig2: non-homotopic to the Identity

where $h_N^{\pm}(x)$, restricted in each interval of $I_N \backslash \frac{I_N}{10}$, is a $\mathbb{C}^l-$function satisfying

$$\frac{d^j h_N^{\pm}}{dx^j}(c_i \pm \frac{1}{10q_N^2}) = \frac{d^j \phi_0}{dx^j}(c_i \pm \frac{1}{10q_N^2}) - \frac{d^j(\bar{s}_N - \bar{s}'_N)}{dx^j}(c_i \pm \frac{1}{10q_N^2})$$

$$\frac{d^j h_N^{\pm}}{dx^j}(c_i \pm \frac{1}{q_N^2}) = 0, \quad i = 1, 2, \quad 0 \leq j \leq l \tag{3.6}$$

$$|h_N^{\pm}(x)| \leq 4\|\phi\|_{\mathbb{C}^1} \cdot \lambda^{-(1-\epsilon)}.$$

From Lemma 2.2, we have $|\phi_0(x) - (\bar{s}_N - \bar{s}'_N)(x)| \leq 4\|\phi\|_{\mathbb{C}^1} \cdot \lambda^{-(1-\epsilon)}$, which implies the existence of $h_N^{\pm}$.

Now we define $A_N = \Lambda \cdot R_{\frac{\pi}{2} - \phi_N(x)}$ where $\phi_N(x) = \phi(x) + e_N(x)$ is a modification of $\phi(x)$. Property 2 listed in Proposition 3.1 for A_N is a consequence of the following lemma.

Lemma 3.1 *For $x \in I_N$, it holds that*

$$A_N^{r_N^+(x)}(x) = A^{r_N^+(x)}(x) \cdot R_{-e_N(x)}$$

and

$$A_N^{-r_N^-(x)}(x) = R_{e_N(T^{-r_N^-(x)}x)} \cdot A^{-r_N^-(x)}(x).$$

Proof. Obviously $T^i x \in \mathbb{S}^1 \backslash I_N$ for $x \in I_N$ and $1 \leq i \leq r_N^+(x) - 1$. Since $A_N(x) = A(x)$ for $x \in \mathbb{S}^1 \backslash I_N$, we have that

$$A_N^{r_N^+(x)}(x) = A^{r_N^+(x)}(x) \cdot (A^{-1}(x) A_N(x)), \quad x \in I_N.$$

From the definition, we have $A_N(x) = A(x) \cdot R_{\phi(x)-\phi_N(x)}$, which implies $A^{-1}(x)A_N(x) = R_{\phi(x)-\phi_N(x)}$. Thus we obtain the first equation. Similarly, we can prove the second equation. □

Proof of Property 2 listed in Proposition 3.1 for A_N From (3.5), for each $x \in I_N$, $A(x)$, $A(Tx), \cdots, A(T^{r_N^+(x)-1}x)$ is λ_N-hyperbolic. It is known that a rotation does not change the norm of a vector. Thus from Lemma 3.1, we know that for each $x \in I_N$, $A_N(x)$, $A_N(Tx), \cdots$, $A_N(T^{r_N^+-1}x)$ is λ_N-hyperbolic, which shows that $s_N(x)$ and $s'_N(x)$ are well-defined.

Subsequently, we have the following conclusion:

Lemma 3.2 *It holds that*

$$e_N(x) = (s_N(x) - s'_N(x)) - (\bar{s}_N(x) - \bar{s}'_N(x)), \quad x \in I_N.$$

Proof. Since a rotation does not change the norm of a vector, for a hyperbolic matrix A and a rotation matrix R_θ, we have

$$s(A \cdot R_{-\theta}) = s(A) + \theta, \quad s(R_\theta \cdot A) = s(A). \tag{3.7}$$

From Lemma 3.1, we have

$$s_N(x) = \bar{s}_N(x) + e_N(x), \quad s'_N(x) = \bar{s}'_N(x).$$

Thus

$$\phi_N(x) - \phi(x) = (s_N(x) - s'_N(x)) - (\bar{s}_N(x) - \bar{s}'_N(x)), \quad x \in I_N,$$

which concludes the proof. □

Property 3 listed in Proposition 3.1 for A_N is a consequence of the next lemma.

Lemma 3.3 *$s_N(x) - s'_N(x)$ coincides with $\phi_0(x)$ on $\frac{I_N}{10}$. Moreover, on $I_N \backslash \frac{I_N}{10}$,*

$$|s_N(x) - s'_N(x) - \phi_0(x)| \leq \frac{1}{(20q_N^2)^{l+1}},$$

if $\lambda > q_N^{8(l+1)} \cdot |\phi\|_{\mathbb{C}^1}$ and $q_N > 20$.

Proof. From the definition of $e_N(x)$, we have $e_N(x) = \phi_0(x) - (\bar{s}_N - \bar{s}'_N)(x)$ on $\frac{I_N}{10}$. Thus by Lemma 3.2, we have for each $x \in \frac{I_N}{10}$, $(s_N - s'_N)(x) = (\bar{s}_N - \bar{s}'_N)(x) + e_N(x) = (\bar{s}_N - \bar{s}'_N)(x) + \phi_0(x) - (\bar{s}_N - \bar{s}'_N)(x) = \phi_0(x)$. More generally, for each $x \in I_N$, we have $(s_N - s'_N)(x) = (\bar{s}_N - \bar{s}'_N)(x) + e_N(x) = \phi_0(x) + (\bar{s}_N - \bar{s}'_N - \phi_0)(x) + e_N(x)$. Hence the last part of this lemma can be obtained from the construction of e_N if $\lambda > q_N^{8(l+1)} \cdot |\phi\|_{\mathbb{C}^1}$ and $q_N > 20$.. □

The construction of A_N is thus finished except the verification of Property 1, which will be done for all n together later. Assuming that $A_N, \cdots, A_{n-1}$ satisfying the properties listed in Proposition 3.1 have been constructed, we then construct A_n.

The construction of A_n: The construction is similar to that of A_N. By inductive assumptions, the sequence $\{A_{n-1}(x), \cdots, A_{n-1}(T^{r^+_{n-1}(x)-1}x)\}$ is $\lambda_{n-1}-$hyperbolic. Moreover, the functions $s_{n-1}(x)$ and $s'_{n-1}(x)$ satisfy:

$$(1)_{\mathrm{n-1}} \quad s_{n-1}(x) - s'_{n-1}(x) = \phi_0(x) \quad \text{on } \tfrac{I_{n-1}}{10};$$

$$(2)_{\mathrm{n-1}} \quad |s_{n-1}(x) - s'_{n-1}(x)| \geq \tfrac{1}{2}|\phi_0(x)| \geq \tfrac{1}{(20q_{n-1}^2)^{l+1}}, \quad x \in I_{n-1} \backslash \tfrac{I_{n-1}}{10}.$$

To construct $A_n(x)$ with desired properties, we need the following lemmas.

Lemma 3.4 *Let $x_0, \ldots, x_m$ be a $T-$orbit with $x_0, x_m \in I_n$ and $x_i \notin I_n$ for $0 < i < m$. Then $\{A_{n-1}(x_0), \ldots, A_{n-1}(x_{m-1})\}$ is λ_n-hyperbolic.*

Proof. The proof is similar to that in [31]. For the sake of the readers, we will give the sketch of the proof. Assume that $0 = j_0 < j_1 \cdots < j_k = m$ are the return times of x_0 to I_{n-1}. Since

$$\angle(s(A_{n-1}^{-j_i}(x_{j_i})), \ \ s(A_{n-1}^{j_{i+1}-j_i}(x_{j_i}))) > \frac{1}{2}|s'_{n-1}(x_{j_i}) - s_{n-1}(x_{j_i})| > \frac{1}{8q_n^{2(l+1)}}, \tag{3.8}$$

from the induction assumption and Lemma 2.1, we obtain that

$$\|A_{n-1}^{j_i}(x_0)\| \geq \lambda_n^{j_i(1-\epsilon)}, \quad i = 1, \ldots, k.$$

□

Let $\bar{s}_n(x) = \overline{s(A_{n-1}^{r_n^+}(x))}$ and $\bar{s}'_n(x) = \overline{s(A_{n-1}^{-r_n^-}(x))}$, $x \in I_n$. Define $e_n(x) \in \mathcal{C}^l$ be the following 2π-periodic function:

$$e_n(x) = \begin{cases} (s_{n-1} - s'_{n-1})(x) - (\bar{s}_n - \bar{s}'_n)(x) & x \in \frac{I_n}{10} \\ h_n^{\pm}(x), & x \in I_n \backslash \frac{I_n}{10} \\ 0, & x \in \mathbb{S}^1 \backslash I_n \end{cases}$$

where $h_n^{\pm}(x)$ is a polynomial of degree $2l+1$ restricted in each interval of $I_n \backslash \frac{I_n}{10}$ satisfying

$$\tfrac{d^j h_n^{\pm}}{dx^j}(c_i \pm \tfrac{1}{10q_n^2}) = \tfrac{d^j(s_{n-1}-s'_{n-1})}{dx^j}(c_i \pm \tfrac{1}{10q_n^2}) - \tfrac{d^j(\bar{s}_n - \bar{s}'_n)}{dx^j}(c_i \pm \tfrac{1}{10q_n^2})$$

$$\tfrac{d^j h_n^{\pm}}{dx^j}(c_i \pm \tfrac{1}{q_n^2}) = 0, \quad i = 1, 2, \quad 0 \leq j \leq l.$$

Define $\phi_n(x) = \phi_{n-1}(x) + e_n(x)$. Let $A_n(x) = \Lambda \cdot R_{\frac{\pi}{2}-\phi_n(x)}$. The property 2 in Proposition 3.1 for A_n can be derived from the following lemma:

Lemma 3.5 *For $x \in I_n$, it holds that*

$$A_n^{r_n^+(x)}(x) = A_{n-1}^{r_n^+(x)}(x) \cdot R_{-e_n(x)}$$

and

$$A_n^{-r_n^-(x)}(x) = R_{e_n(T^{-r_n^-(x)}x)} \cdot A_{n-1}^{-r_n^-(x)}(x).$$

Similar to the proof of Lemma 3.2, we have the following result:

Lemma 3.6 *It holds that*

$$e_n(x) = (s_n(x) - s_n'(x)) - (\bar{s}_n(x) - \bar{s}_n'(x)), \quad x \in I_n.$$

The property 3 in Proposition 3.1 for A_n can be obtained by the following lemma:

Lemma 3.7 *Let $\lambda_n > \max\{8(l+1),\ q_N^{8(l+1)} \cdot |\phi\|_{\mathbb{C}^1}\}$ and $q_N \geq (10+M)^{10}$. Then it holds that $s_n(x) - s_n'(x)$ coincides with $\phi_0(x)$ on $\frac{I_n}{10}$. Furthermore, on $I_n \backslash \frac{I_n}{10}$,*

$$|(s_n(x) - s_n'(x)) - (s_{n-1}(x) - s_{n-1}'(x))| \leq \frac{1}{(20q_n^2)^{l+1}}.$$

Proof. From the definition of $e_n(x)$, we have $e_n(x) = (s_{n-1} - s_{n-1}')(x) - (\bar{s}_n - \bar{s}_n')(x)$ on $\frac{I_n}{10}$, which together with Lemma 3.6 implies that for each $x \in \frac{I_n}{10}$, $(s_n - s_n')(x) = (\bar{s}_n - \bar{s}_n')(x) + e_n(x) = (s_{n-1} - s_{n-1}')(x)$. Since $(s_{n-1} - s_{n-1}')(x) = \phi_0(x)$ on $\frac{I_{n-1}}{10}$ by induction assumption $(1)_{n-1}$, we obtain the first part of the lemma.

For each $x \in I_n \backslash \frac{I_n}{10}$, we have $(s_n - s_n')(x) = (\bar{s}_n - \bar{s}_n')(x) + e_n(x) = (s_{n-1} - s_{n-1}')(x) + (\bar{s}_n - s_{n-1} + s_{n-1}' - \bar{s}_n')(x) + e_n(x)$. Recall that $\lambda_n > \lambda_\infty > \lambda^{1-\epsilon} \gg 1$ and $q_n \leq M q_{n-1}$, we have $\lambda_n^{q_{n-1}} \gg q_n^{2l}$. Hence the last part of this lemma can be obtained from the induction assumption $(2)_{n-1}$ for $(s_{n-1} - s_{n-1}')(x)$ on I_{n-1} and Lemmas 3.8 and 3.9 if $\lambda_n > \max\{8(l+1),\ q_N^{8(l+1)} \cdot |\phi\|_{\mathbb{C}^1}\}$ and $q_n \geq q_N \geq (10+M)^{10}$. □

The property 1 in Proposition 3.1 for all $A_n, n = N, N+1, \cdots$ is obtained by the definition of $\phi_{n-1}(x)$, $\phi_n(x)$ and the following lemmas.

Lemma 3.8 *Let $\lambda, N \gg 1$. Then for $x \in I_n$, $s_{n-1}, s_{n-1}', \bar{s}_n, \bar{s}_n'$ are $\mathcal{C}^l$ curves. Moreover, for any $k \leq \min\{l, r_{n-1}^{+\frac{1}{10}}\}$, it holds that*

$$|\bar{s}_n - s_{n-1}|_{\mathcal{C}^k}, |\bar{s}_n' - s_{n-1}'|_{\mathcal{C}^k} \leq \|\phi_{n-1}\|_k \cdot \lambda^{-\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}}. \tag{3.9}$$

Proof. The lemma is proved in the Appendix. □

Remark 3.2 *In the appendix, we will prove that Lemma 3.8 not only holds for $A = \Lambda R_{\frac{\pi}{2}-\phi(x)}$ defined in this section, but also holds for the one defined in Section 5. So it is applicable when we construct the C^∞ counter-example in Section 5.*

When we construct the finite smooth counter-examples, l is fixed. One can take λ sufficiently large (depending on l) such that

$$|\bar{s}_n - s_{n-1}|_{\mathcal{C}^l}, |\bar{s}'_n - s'_{n-1}|_{\mathcal{C}^l} \leq \lambda^{-(r^+_{n-1})^{\frac{1}{5}}} < \lambda^{-q^{\frac{1}{5}}_{n-1}}, \tag{3.10}$$

holds for all $n > N$.

Lemma 3.9 *For any $x \in \mathbb{S}^1$, it holds that $|e_n(x)|_{\mathcal{C}^l} \leq \lambda_n^{-q^{\frac{1}{10}}_{n-1}}$.*

Proof. From Lemma 3.8, we have that for fixed l and $\lambda, n \gg 1$, $|e_n(x)|_{\mathcal{C}^l} \leq \lambda_n^{-q^{\frac{1}{5}}_{n-1}}$ for $x \in \frac{I_n}{10}$. Consequently from Cramer's rule, $|h^\pm_n(x)|_{\mathcal{C}^l} = O(\lambda_n^{-q^{\frac{1}{5}}_{n-1}})$, which implies $|e_n(x)|_{\mathcal{C}^l} \leq \lambda_n^{-q^{\frac{1}{10}}_{n-1}}$ for $x \in \mathbb{S}^1$. □

By property 1 in Proposition 3.1, $A_n(x)$ converge to a cocycle $D_l(x)$ in $\mathcal{C}^l$-topology. Next we estimate the lower bound of the Lyapunov exponent of $D_l(x)$.

Theorem 3 *The Lyapunov exponent $L(D_l)$ of $D_l(x)$ has a lower bound $(1 - 4\epsilon) \log \lambda$.*

Proof. From the subadditivity of the finite Lyapunov exponent, the finite Lyapunov exponent of a cocycle converges (to the Lyapunov exponent). Thus there exists a large $N_0 \geq N$ such that

$$\left| \frac{1}{N_0} \int_{\mathbb{S}^1} \log \|D_l^{N_0}(x)\| dx - L(D_l) \right| \leq \epsilon.$$

Since $A_n(x)$ converges to $D_l(x)$, there exists a large $N_1 > N_0$ such that for any $n > N_1$, it holds that

$$\left| \frac{1}{N_0} \int_{\mathbb{S}^1} \log \|D_l^{N_0}(x)\| dx - \frac{1}{N_0} \int_{\mathbb{S}^1} \log \|A_n^{N_0}(x)\| dx \right| \leq \epsilon.$$

Thus it is sufficient to prove $\frac{1}{N_0} \int_{\mathbb{S}^1} \log \|A_n^{N_0}(x)\| dx \geq (1 - 3\epsilon) \log \lambda$ for sufficiently large n.

We say that $x \in \mathbb{S}^1$ is nonresonant for $A_n(x)$ if

$$\begin{cases} \operatorname{dist}(T^i x, \mathcal{C}_0) > \frac{1}{q_N^2} & \text{for } \ 0 \leq i < q_N, \\ \\ \operatorname{dist}(T^i x, \mathcal{C}_0) > \frac{1}{q_k^2} & \text{for } q_{k-1} \leq i < q_k, \quad N < k \leq n. \end{cases} \tag{3.11}$$

The set of points with the nonresonant property (3.11) has Lebesgue measure at least $2\pi(1 - \sum_{N \leq k < n} \frac{1}{q_k})$, which is larger than $2\pi(1 - \frac{\epsilon}{2\pi})$ for $N \gg 1$.

Proposition 3.2 *For each $x \in \mathbb{S}^1$ with the nonresonant property (3.11), $(A_n(x), \cdots, A_n(T^{q_n-1}x))$ is $\lambda^{1-\epsilon}$-hyperbolic.*

Proof. Let the trajectory in question be $x, Tx, \cdots$. Let j_0 be the first time it is in I_N, and let n_0 be s.t. $T^{j_0}x \in I_{n_0} \backslash I_{n_0+1}$. In general, let j_i and n_i be defined so that $T^{j_i}x \in I_{n_i} \backslash I_{n_i+1}$, and $T^{j_{i+1}}x$ be the next return of $T^{j_i}x$ to I_{n_i}. Obviously, $j_{i+1} - j_i \geq q_{n_i}$. Moreover, from Proposition 3.1, it holds that $A_n(T^{j_i}x), \cdots, A_n(T^{j_{i+1}-1}x)$ is λ_∞−hyperbolic.

Since $T^{j_i}x \notin I_{n_i+1}$, from $(2)_{\rm n}$ in Proposition 3.1, we have $\angle(s_n(T^{j_i}x), s'_n(T^{j_i}x)) > (\frac{1}{20}|I_{n_i+1}|)^{l+1}$. Similar to (3.8), it holds that $\angle(s(A_n^{-j_i}(T^{j_i}x)),\ s(A_n^{j_{i+1}-j_i}(T^{j_i}x))) > \frac{1}{2}\angle(s_n(T^{j_i}x), s'_n(T^{j_i}x))$. Hence from Lemma 2.1, it follows that

$$\begin{aligned}\|A_n^{j_{i+1}}(x)\| &\geq \|A_n^{j_i}(x)\| \cdot \|A_n^{j_{i+1}-j_i}(T^{j_i}x)\| \cdot \angle(s(A_n^{-j_i}(T^{j_i}x)),\ s(A_n^{j_{i+1}-j_i}(T^{j_i}x))) \\ &\geq \|A_n^{j_i}(x)\| \cdot \lambda_\infty^{j_{i+1}-j_i} \cdot (\tfrac{1}{40}|I_{n_i+1}|)^{l+1}.\end{aligned}$$

Inductively, we have

$$\|A_n^{j_s}(x)\| \geq \|A_n^{j_0}(x)\| \cdot \lambda_\infty^{j_s-j_0} \cdot \prod_{i=1}^{s-1}(\frac{1}{40}|I_{n_i+1}|)^{l+1}.$$

Similar to the proof of (3.5), we have $\|A_n^{j_0}(x)\| \geq \lambda_\infty^{j_0}$. Consequently,

$$\|A_n^{j_s}(x)\| \geq \lambda_\infty^{j_s} \cdot \prod_{i=1}^{s-1}(\frac{1}{40}|I_{n_i+1}|)^{l+1}. \tag{3.12}$$

Suppose $q_{m-1} \leq j_s < q_m, N+1 \leq m \leq n$. The nonresonant property prohibits x_{j_i} from entering I_m for $i < s$. For $k < m$, the number of j_i's such that $n_i = k$ is less than j_s/q_k, since the smallest first return time r_k for $x \in I_k$ satisfies $r_k \geq q_k$ (see the beginning of this section). Moreover at each one of these returns, the distance from c_1 and c_2 is $\geq (\frac{1}{40}|I_{k+1}|)^{l+1} = (40q_{k+1}^2)^{-(l+1)}$.

Then

$$-\frac{1}{j_s}\log\prod_{i=1}^{s-1}(\frac{1}{40}|I_{n_i+1}|)^{l+1} \leq \frac{1}{j_s}\sum_{k=N}^{n-1}\frac{j_s}{q_k}\log(40q_{k+1}^2)^{(l+1)} \leq 4(l+1)\sum_{k\geq N}\frac{\log q_{k+1}}{q_k} < \frac{\epsilon}{2}\ln\lambda_\infty$$

if $\lambda \gg N \gg 1$. Equivalently, we have

$$\prod_{i=1}^{s-1}(\frac{1}{40}|I_{n_i+1}|)^{l+1} \geq \lambda_\infty^{-\frac{\epsilon}{2}j_s}.$$

Thus from (3.12), it holds that

$$\|A_n^{j_s}(x)\| \geq \lambda_\infty^{(1-\frac{\epsilon}{2})j_s}.$$

Now we consider the case $j_s < i < j_{s+1}$. Since $A_n(T^{j_i}x), \cdots, A_n(T^{j_{s+1}}x)$ is $\lambda_\infty-$hyperbolic, from the definition 2.1, it holds that $A_n(T^{j_i}x), \cdots, A_n(T^i x)$ is also $\lambda_\infty-$hyperbolic(without loss of generality, we assume $i - j_s \geq 10$). Then similar to the argument above, we have

$$\|A_n^i(x)\| \geq \lambda_\infty^{(1-\frac{\epsilon}{2})i} \geq \lambda^{(1-2\epsilon)i}, \quad 10 \leq i \leq q_n.$$

This concludes our proof. □

From Proposition 3.2, we have that $\frac{1}{j}\log\|A_n^j(x)\| > (1-2\epsilon)\log\lambda$ for each nonresonant point and $1 \leq j \leq q_n$ with $n \geq N$. Since the measure of the nonresonant point set is not less than $2\pi(1-\frac{\epsilon}{2\pi})$, choose $n > N_0$, $j = N_0$ and this concludes the proof of Theorem 3. □

4 The construction of $\tilde{A}_n(x)$

Recall that ω is bounded type, i.e., $q_{n+1} < Mq_n$ for some $M \geq \frac{\sqrt{5}+1}{2}$. In this section, we will prove the following:

Theorem 4 *There exists a sequence of cocycles $\tilde{A}_n(x)$ such that $\tilde{A}_n(x) \to D_l(x)$ in $\mathcal{C}^l$-topology. Moreover, the Lyapunov exponent of $\tilde{A}_n(x)$ is less than $(1-\delta)\log\lambda$ for any large $n \gg \lambda$.*

To prove Theorem 4, we need the following proposition:

Proposition 4.1 *There exist $\tilde{A}_n$ with the following properties:*

1. *$\tilde{A}_n$ is of the form $\Lambda R_{\frac{\pi}{2}-\tilde{\phi}_n(x)}$ with*

$$|\tilde{\phi}_n(x) - \phi_n(x)|_{C^l} = O(q_n^{-2}). \tag{4.1}$$

2. *For each $x \in I_n$, $\tilde{A}_n(x), \tilde{A}_n(Tx), \cdots, \tilde{A}_n(T^{r_n^+(x)-1}x)$ is λ_n-hyperbolic.*
3. *Let $\tilde{s}_n(x) = s(\tilde{A}_n^{r_n^+}(x))$, $\tilde{s}'_n(x) = s(\tilde{A}_n^{-r_n^-}(x))$. Then we have*

$$\tilde{s}_n(x) = \tilde{s}'_n(x) \quad \text{on } \frac{I_n}{10}.$$

Proof. Let $\tilde{e}_n(x) \in \mathcal{C}^l$ be a 2π-periodic function such that

$$\tilde{e}_n(x) = \begin{cases} (s_n - s'_n)(x) & x \in \frac{I_n}{10} \\ \tilde{h}_n^{\pm}(x), & x \in I_n \backslash \frac{I_n}{10} \\ 0, & \mathbb{S}^1 \backslash I_n, \end{cases}$$

where $\tilde{h}_n^{\pm}(x)$ is a polynomials of degree $2l+1$ restricted on each interval of $I_n \backslash \frac{I_n}{10}$ and satisfies for $i = 1, 2$ and $0 \le j \le l$

$$\frac{d^j \tilde{h}_n^{\pm}}{dx^j}(c_i \pm \frac{1}{10q_n^2}) = \frac{d^j(s_n - s_n')}{dx^j}(c_i \pm \frac{1}{10q_n^2}) \qquad c$$
$$\frac{d^j \tilde{h}_n^{\pm}}{dx^j}(c_i \pm \frac{1}{q_n^2}) = 0.$$

From $(1)_n$ in Proposition 3.1, it holds for $0 \le j \le l$ that $|(s_n - s_n')(x)|_{\mathcal{C}^j} = O(q_n^{-2(l+1-j)})$. Hence from Cramer's rule we have that $|\tilde{h}_n^{\pm}(x)|_{\mathcal{C}^l} = O(q_n^{-2})$. Consequently, $|\tilde{e}_n(x)|_{\mathcal{C}^l} = O(q_n^{-2})$.

Define $\tilde{\phi}_n(x) = \phi_n(x) + \tilde{e}_n(x)$ and $\tilde{A}_n(x) = \Lambda \cdot R_{\frac{\pi}{2} - \tilde{\phi}_n(x)}$. Thus conclusion 1 is proved, which together with the fact that $A_n(x) \to D_l(x)$ in $\mathcal{C}^l$-topology implies that $\tilde{A}_n(x) \to D_l(x)$ in $\mathcal{C}^l$-topology.

Since for each $x \in I_n$, $A_n(x), A_n(Tx), \cdots, A_n(T^{r_n^+(x)-1}x)$ is λ_n-hyperbolic and $\tilde{\phi}_n(x) = \phi_n(x)$ on $\S^1 \backslash I_n$, we see that $\tilde{A}_n(x), \tilde{A}_n(Tx), \cdots, \tilde{A}_n(T^{r_n^+(x)-1}x)$. Thus $\tilde{s}_n(x) = s(\tilde{A}_n^{r_n^+}(x))$ and $\tilde{s}_n'(x) = s(\tilde{A}_n^{-r_n^-}(x))$ are well-defined. Moreover, similar to Lemma 3.2, it holds that $\tilde{s}_n(x) - \tilde{s}_n'(x) = s_n(x) - s_n'(x) - \tilde{e}_n(x)$.

Thus from the definition of $\tilde{e}_n(x)$, it holds that

$$\tilde{s}_n(x) = \tilde{s}_n'(x), \quad x \in \frac{I_n}{10}. \tag{4.2}$$

This ends the proof of the proposition. □

The following observations are useful later for the estimate of the upper bound of the Lyapunov exponent for $\tilde{A}_n(x)$.

Lemma 4.1 *Suppose A and B are two hyperbolic matrices such that $\|A\| = \lambda_1^m$ and $\|B\| = \lambda_2^n$ with $m, n > 0$ and $\lambda_1, \lambda_2 \gg 1$. If $A(s(A)) \parallel u(B)$, then $\|BA\| \le 2\max\{\lambda_1^m \cdot \lambda_2^{-n},\ \lambda_2^n \cdot \lambda_1^{-m}\}$.*

Proof. For any hyperbolic matrix A, it holds that $s(A) \perp u(A)$ and $A(s(A)) \perp A(u(A))$. From $A(s(A)) \parallel u(B)$, we have $A(u(A)) \parallel s(B)$. For any vector $v \in \mathbb{R}^2$, let $v = v_1 \oplus v_2$ with respect to $s(A) \oplus u(A)$. Then

$$Av = Av_1 \oplus Av_2 = (|Av_1| \cdot u(B)) \oplus (|Av_2| \cdot s(B)).$$

Consequently

$$BAv = (|Av_1| \cdot B(u(B))) \oplus (|Av_2| \cdot B(s(B))).$$

Thus we have

$$|BAv| \le \lambda_2^n \cdot \lambda_1^{-m}|v_1| + \lambda_1^m \cdot \lambda_2^{-n}|v_2| \le 2\max\{\lambda_1^m \cdot \lambda_2^{-n},\ \lambda_2^n \cdot \lambda_1^{-m}\}|v|.$$

This concludes the proof of this lemma. □

Lemma 4.2 *For any interval $I \in \mathbb{S}^1$ with $0 < |I| < \pi/4$, let $r = \min_{x\in I}\min\{i > 0 | T^i x \ (mod\ 2\pi) \in I\}$ and $\hat{r} = \max_{x\in \frac{I}{10}}\min\{i > 0|T^i x\ (mod\ 2\pi) \in \frac{I}{10}\}$. Then $\delta \le \frac{r}{\hat{r}}$.*

Proof. Without loss of generality, we can assume $I = [0, a]$. Let $m = \min\{k|T^{q_k} I \bigcap I \ne \emptyset\}$. Since $T^{q_n}0, T^{q_{n+1}}0$ are on the different side of 0 and $\lim_{n\to\infty}|T^{q_n}0| = 0$, it follows that there is a $k_0 > 0$ such that $|T^{q_{m+k_0+1}}0| < |T^{q_{m+k_0}}0| \le \frac{1}{10}|I|$. It follows that $I \subset T^{q_m} I \cup T^{q_{m+k_0}}\frac{I}{10} \cup T^{q_{m+k_0+1}}\frac{I}{10}$ (mod 1). Thus

$$\frac{r}{\hat{r}} \ge \frac{q_m}{q_{m+k_0+1}} \ge M^{-(k_0+1)}.$$

Since $M > \frac{\sqrt{5}+1}{2}$, one can see that $k_0 < 9$. It follows that $\delta \le \frac{r}{\hat{r}}$. □

Lemma 4.3 *For any interval $I \in \mathbb{S}^1$ with $0 < |I| < \pi/4$, let $r_1 = \max_{x\in I}\min\{i > 0|T^i x\ (mod\ 2\pi) \in I\}$ and $r_2 = \min_{x\in I}\min\{i > 0|T^i x\ (mod\ 2\pi) \in I + \pi\}$. Then there is positive integer $k_1 < 18$ such that $r_1 \le k_1 r_2$.*

Proof. Without loss of generality, we assume $I = [0, a]$. From the proof of Lemma 4.2, we have that there is a positive integer $k_0 < 9$ such that $T^{k_0 r_2}\pi \in [\pi, \pi + \frac{a}{2}]$, which implies that $T^{2k_0 r_2}\pi(mod\ 2\pi)\ \in [0, a]$. The proof is completed by setting $k_1 = 2k_0$. □

From Lemmas 4.2 and 4.3, we can easily obtain the following:

Corollary 4.1 *Let $\min r_n(x) = \min_{x\in I_n}\min\{i > 0|T^i x\ (mod\ 2\pi) \in I_n\}$ and $\max r_n(x) = \max_{x\in\frac{1}{10}I_n}\min\{i > 0|T^i x\ (mod\ 2\pi) \in \frac{1}{10}I_n\}$. Then $M^{-k_1-1} \le \frac{\min r_n(x)}{\max r_n(x)} \le 1$.*

Proof of Theorem 4 Let $\cdots < n_{j-1} < n_j < n_{j+1} < \cdots$ be the returning times of $x \in I_n/10$ to $I_n/10$. Moreover, we denote n_{j+} be the first returning time of $x \in I_n$ to I_n after n_j. Similarly, we denote by n_{j-} the last returning time of $x \in I_n$ to I_n before n_j. Obviously, it holds that $n_{j-1} \le n_{j-} < n_j$ and $n_j < n_{j+} \le n_{j+1}$.

Since $T^{n_j}x \in [c_1 - \frac{1}{2q_n^2}, c_1 + \frac{1}{2q_n^2}]$, (4.2), Lemma 4.1 and Corollary 4.1 are applicable. Set $d_3 = \frac{1}{2}M^{-k_1-1}$. From the definition of $\tilde{A}_n$, we have $\|\tilde{A}_n(x)\| \le \lambda$ for each x. Consequently,

$$\begin{aligned}
&\|\tilde{A}_n(T^{n_{j+}}x)\cdots\tilde{A}_n(T^{n_j}x)\cdots\tilde{A}_n(T^{n_{j-}}x)\|\\
&\le 2\max\{\|\tilde{A}_n^{n_{j+}-n_j}(T^{n_j}x)\|\cdot\|\tilde{A}_n^{n_j-n_{j-}}(T^{n_{j-}}x)\|^{-1}, \|\tilde{A}_n^{n_{j+}-n_j}(T^{n_j}x)\|^{-1}\cdot\|\tilde{A}_n^{n_j-n_{j-}}(T^{n_{j-}}x)\|\}\\
&\le 2\max\{\|\tilde{A}_n^{n_{j+}-n_j}(T^{n_j}x)\|, \|\tilde{A}_n^{n_j-n_{j-}}(T^{n_{j-}}x)\|\}\\
&\le 2\lambda^{\max\{n_{j+}-n_j, n_j-n_{j-}\}} \le \lambda^{(1-d_3)(n_{j+}-n_{j-})},
\end{aligned} \tag{4.3}$$

which implies

$$\|\tilde{A}_n(T^{n_{j+1}}x)\cdots\tilde{A}_n(T^{n_{j-1}}x)\| \le \lambda^{n_{j+1}-n_{j-1}-d_3(n_{j+}-n_{j-})} \le \lambda^{(n_{j+1}-n_j)(1-d_3^2)}.$$

Thus we have, for any k,

$$\|\tilde{A}_n(T^{n_k}x)\cdots\tilde{A}_n(x)\| < \lambda^{\sum_{j=0}^{k}(n_{j+1}-n_j)(1-d_3^2)} = \lambda^{n_k(1-d_3^2)}.$$

In other words, we have shown that the Lyapunov exponent of $\tilde{A}_n(x)$ will be less than $(1-d_3^2)\log\lambda = (1-\frac{1}{4}M^{-2(k_1+1)})\log\lambda$. The proof is finished since k_1 can be less than 18. □

Proof of Theorem 1 for finite order differentiability: The proof for the case $l=0$ can be found in [7, 8, 10, 17, 27, 30]. For $l>0$, from the definition of $A_n(x)$ and $\tilde{A}_n(x)$, we have that in any neighborhood of $D_l(x)$, there exists a cocycle $\tilde{A}_n(x)$ with the Lyapunov exponent less than $(1-\delta)\log\lambda$. From Theorem 3, we know that $L(D_l(x))$ is larger than $(1-4\epsilon)\log\lambda$. The discontinuity is obvious since $\delta > 4\epsilon$. □

5 The proof for the $\mathcal{C}^\infty$ case: a sketch

In this section, we will prove Theorem 1 for the $\mathcal{C}^\infty$ case. The basic idea is same as the finite smooth case. We will pay our attention to the difference between the two cases.

In the following, we will first follow the steps in Section 3 to construct a sequence of C^∞ cocycles which are $\mathcal{C}^1$-convergent. Then we will prove that it actually converges in $\mathcal{C}^\infty$ topology.

Recall $\epsilon = M^{-100} \ll \delta = \frac{1}{4}M^{-20}$ defined in the introduction. Assume $\lambda \gg e^{q_N^{a+1}} \gg 1$ with $0<a<\frac{1}{10}$. For $n\ge N$, define λ_{n+1} such that $\lambda_{n+1}^{q_{n+1}} = \lambda_n^{q_{n+1}}\cdot e^{-(10q_{n+1}^2)^a}$ with $\lambda_N = \lambda^{1-\epsilon}$. From the definition of λ_n, we have $\lambda_n^{q_n} \ge \lambda_{n-1}^{q_n}\cdot e^{-q_n^{2a}} \ge \lambda_{n-2}^{q_n}\cdot e^{-q_n\cdot q_{n-1}^{2a-1}} \ge \cdots \ge \lambda^{q_n}\cdot\lambda_N^{-c_3\cdot q_n^{2a}} \ge \lambda_N^{(1-2\epsilon)q_n}$ if $\lambda\gg 1$, where $c_3>0$ is a constant. It implies that λ_n decrease to $\lambda_\infty \ge \lambda^{1-2\epsilon}$.

Construction of $A_N(x)$ Let $c_1, c_2 \in \mathbb{S}^1$ with $c_1\in[0,\pi)$, $c_2 = c_1+\pi$ and δ_0 a small positive number. We define ϕ_0 on $\{x||x-c_1|\le\delta_0 \text{ or } |x-c_2|\le\delta_0\}$ as follows.

$$\phi_0(x) = \begin{cases} \phi_{01}(x), & |x-c_1|<\delta_0; \\ -\phi_{02}(x) \text{ (or } \phi_{02}(x)), & |x-c_2|<\delta_0, \end{cases} \tag{5.1}$$

where

$$\phi_{0i}(x) = \operatorname{sgn}(x-c_i)e^{-\frac{1}{|x-c_i|^a}}, \quad i=1,2. \tag{5.2}$$

Let $A(x) = \Lambda\cdot R_{\frac{\pi}{2}-\phi(x)}$, where ϕ is the lift of a $\mathcal{C}^\infty$ periodic function on $\mathbb{S}^1$ satisfying

(a)

$$\phi(x) = \begin{cases} \phi_{01}(x), & |x-c_1|\le\delta_0; \\ -\phi_{02}(x) \text{ (or } \pi+\phi_{02}(x), \text{ respectively)}, & |x-c_2|\le\delta_0. \end{cases}$$

(b) $\forall |x-c_i|>\delta_0,\ i=1,2,\ |\phi(x)-k\pi|>e^{-\frac{1}{\delta_0^a}}$ for any $k\in\mathbb{Z}$.

Using the same argument as that in finite smooth case, we have that

$$A(x),\cdots,A(T^{r_N^+(x)-1}x) \text{ is } \lambda- \text{ hyperbolic sequence.} \tag{5.3}$$

By Lemma 2.2,

$$|\bar{s}_N(x)-\bar{s}'_N(x)-\phi_0(x)|\le \|\phi\|\cdot\lambda^{-1} \tag{5.4}$$

for $x\in I_N$.

Let $e_N(x)\in\mathcal{C}^\infty$ be a 2π-periodic function such that $e_N(x)=\phi_0(x)-(\bar{s}_N(x)-\bar{s}'_N(x))$ for $x\in I_N$.

Lemma 5.1 *For any $n\ge N$, there exists $f_n\in\mathcal{C}^\infty$ be a 2π-periodic function such that*

$$f_n(x):\begin{cases} =1, & x\in\frac{I_n}{10},\\ \in[0,1], & x\in I_n\backslash\frac{I_n}{10}\\ =0, & x\in\mathbb{S}^1\backslash I_n\end{cases} \tag{5.5}$$

and

$$\left|\frac{d^r f_n(x)}{dx^r}\right|\le q_n^{3r},\quad 0\le r\le [q_n^{\frac{1}{10}}]. \tag{5.6}$$

The proof will be given in the Appendix.

Let $\hat{e}_N(x)=e_N(x)\cdot f_N(x)$ and $\phi_N(x)=\phi(x)+\hat{e}_N(x)$ for $x\in\mathbb{S}^1$. Define $A_N(x)=\Lambda\cdot R_{\frac{\pi}{2}-\phi_N(x)}$. Obviously, $A_N(x)=A(x)\cdot R_{-\hat{e}_N(x)}$. Then from (5.3), we obtain that, for any $x\in I_N$, $A_N(x),\cdots,A_N(T^{r_N^+(x)-1}x)$ is λ-hyperbolic sequence and $(s_N-s'_N)(x)=(\bar{s}_N-\bar{s}'_N)(x)+\hat{e}_N(x)$, which implies $s_N(x)-s'_N(x)=\phi_0(x)$ on $\frac{I_N}{10}$. (5.4) implies that $|\hat{e}_N(x)|_{\mathcal{C}^1}\le\|\phi\|_l\cdot\lambda^{-(r_N^+)^{\frac{1}{4}}}$ in I_N. Thus we have $|s_N(x)-s'_N(x)|\ge\frac{1}{2}\cdot e^{-(10\cdot q_N^2)^a}$ on $I_N\backslash\frac{I_N}{10}$ if $\lambda>e^{(10\cdot q_N^2)^a}\cdot\|\phi\|$.

Inductively, we assume that $A_N(x),\cdots,A_{n-1}(x)$ have been constructed such that for $N\le i\le n-1$,

$(a)_i$ $|\phi_i(x)-\phi_{i-1}(x)|_{\mathcal{C}^1}\le\lambda_i^{-q_{i-1}^{\frac{1}{10}}}$ for $x\in I_i$, $i>N$;
$(b)_i$ $A_i(x),\cdots,A_i(T^{r_i^+(x)-1}x)$ is λ_i-hyperbolic for $x\in I_i$;
$(c)_i$ $s_i(x)-s'_i(x)=\phi_0(x)$ for $x\in\frac{I_i}{10}$ and $|s_i(x)-s'_i(x)|\ge\frac{1}{2}\cdot e^{-(10\cdot q_i^2)^a}$ for $x\in I_i\backslash\frac{I_i}{10}$.

We now Construct of $A_n(x)$. From $(b)_{n-1}$, we have

$$\|A_{n-1}^{r_{n-1}^+(x)}(x)\|\cdot e^{-(10q_{n-1}^2)^a}\ge\lambda_{n-1}^{q_{n-1}(1-\epsilon)}\cdot e^{-(10q_{n-1}^2)^a}\ge\lambda_n^{(1-\epsilon)q_n},\quad x\in I_{n-1}.$$

Combining this with $(c)_{n-1}$, we obtain that

$$A_{n-1}(x), \cdots, A_{n-1}(T^{r_n^+(x)-1}x) \text{ is } \lambda_n\text{−hyperbolic}, \quad x \in I_n. \tag{5.7}$$

Same as the finite smooth case (see (3.10)), from Lemma 3.8 we have

$$|(s_{n-1}(x) - s'_{n-1}(x)) - (\bar{s}_n(x) - \bar{s}'_n(x))|_{\mathcal{C}^1} \leq \lambda_{n-1}^{-q_{n-1}^{\frac{1}{5}}}, \quad x \in I_n. \tag{5.8}$$

Define a 2π-periodic function $e_n(x) \in \mathcal{C}^\infty$ such that

$$e_n(x) = (s_{n-1}(x) - s'_{n-1}(x)) - (\bar{s}_n(x) - \bar{s}'_n(x)) \quad x \in I_n.$$

Define $\hat{e}_n(x) = e_n(x) \cdot f_n(x)$ where f_n is defined in Lemma 5.1, $\phi_n(x) = \phi_{n-1}(x) + \hat{e}_n(x)$ and $A_n(x) = \Lambda \cdot R_{\frac{\pi}{2}-\phi_n(x)}$. Obviously, $A_n(x) = A_{n-1}(x) \cdot R_{-\hat{e}_n(x)}$. Then from (5.7), we obtain that, for any $x \in I_n$, $A_n(x), \cdots, A_n(T^{r_n^+(x)-1}x)$ is λ_n-hyperbolic sequence and $(s_n - s'_n)(x) = (\bar{s}_n - \bar{s}'_n)(x) + \hat{e}_n(x)$, which implies $s_n(x) - s'_n(x) = \phi_0(x)$ on $\frac{I_n}{10}$. (5.6) and (5.8) imply $|\hat{e}_n(x)|_{\mathcal{C}^1} \leq q_n^3 \cdot \lambda_{n-1}^{-q_{n-1}^{\frac{1}{4}}}$, $x \in I_n$. Thus we have $|s_n(x) - s'_n(x)| \geq \frac{1}{2} \cdot e^{-(10 \cdot q_n^2)^a}$ on $I_n \backslash \frac{I_n}{10}$.

In conclusion, we have

$(a)_n$ $|\phi_n(x) - \phi_{n-1}(x)|_{\mathcal{C}^1} \leq \lambda_n^{-q_{n-1}^{\frac{1}{10}}}$ for $x \in I_n$;
$(b)_n$ $A_n(x), \cdots, A_n(T^{r_n^+(x)-1}x)$ is λ_n-hyperbolic for $x \in I_n$;
$(c)_n$ $|s_n(x) - s'_n(x)| = \phi_0(x)$ for $x \in \frac{I_n}{10}$ and $|s_n(x) - s'_n(x)| \geq \frac{1}{2} \cdot e^{-(10 \cdot q_n^2)^a}$ for $x \in I_n \backslash \frac{I_n}{10}$.

All the construction above is same as the finite smooth case. From $(a)_n$, one sees that A_n converges to a cocycle $D_\infty(x)$. From $(b)_n$, one sees that the Lyapunov exponent of $D_\infty(x)$ has a lower bound $\log \lambda_\infty > (1 - 4\epsilon) \log \lambda$. The additional work we should do is to prove that A_n converge to a cocycle D_∞ in any $C^k, k = 1, 2, \cdots$ topology.

By Lemma 3.8 and Lemma A.10, we have

Lemma 5.2 *Let $\lambda \gg N \gg 1$. For $n > N$ and $0 \leq k \leq [(r_{n-1}^+)^{\frac{1}{10}}]$, it holds that*

$$\left| \frac{d^k(\bar{s}_n - s_{n-1})}{dx^k} \right| + \left| \frac{d^k(\bar{s}'_n - s'_{n-1})}{dx^k} \right| \leq 2\lambda^{-\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}} \cdot \|\phi_0\|_k.$$

Corollary 5.1 *$A_N(x), A_{N+1}(x), \cdots,$ is convergent to $D_\infty(x)$ in $\mathcal{C}^\infty$-topology.*

Proof. It is equivalent to prove that $\phi_n(x),\ n = N, N+1, \cdots$ converge in any $\mathcal{C}^k$ topology. For any fixed $k \in \mathbb{N}$, we take $n_1(k)$ so that $k \leq [(r_{n-1}^+)^{\frac{1}{10}}]$ if $n \geq n_1(k)$. From the definition of $\phi_n(x)$, we have $\phi_n(x) - \phi_{n-1}(x) = \hat{e}_n(x)$ where

$$\hat{e}_n(x) = (\bar{s}_n(x) - s_{n-1}(x) + \bar{s}'_n(x) - s'_{n-1}(x)) f_n(x) = e_n(x) f_n(x)$$

With the help of Lemma 5.2, we have

$$\left|\frac{d^r e_n(x)}{dx^r}\right| \le 2\lambda^{-\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}} \cdot \|\phi_0\|_r,\ 0 \le r \le k.$$

This together with (5.6) implies that

$$\begin{aligned} &\left|\tfrac{d^r \hat{e}_n(x)}{dx^r}\right| \le \textstyle\sum_{|L_1|+|L_2|=r} |D^{L_1} e_n(x)| \cdot |D^{L_2} f_n(x)| \\ &\le 2(r+1)! \cdot \|\phi_0\|_r \cdot q_n^{3r} \cdot \lambda^{-\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}} \le 2(k+1)! \cdot \|\phi_0\|_k \cdot (M \cdot r_{n-1}^+)^{3k} \cdot \lambda^{-\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}}. \end{aligned} \tag{5.9}$$

Take $n_2(k)$ so that $2(k+1)! \cdot \|\phi_0\|_k \cdot (M \cdot r_{n-1}^+)^{3k} \le \lambda^{\frac{1}{6}(r_{n-1}^+)^{\frac{2}{3}}}$ if $n \ge n_2(k)$. Then for any $n \ge \max\{n_1(k), n_2(k)\}$, it holds that

$$|\phi_n(x) - \phi_{n-1}(x)|_{C^k} = |\hat{e}_n(x)|_{\mathbb{C}^k} \le \lambda^{-\frac{1}{6}(r_{n-1}^+)^{\frac{2}{3}}},$$

Hence $\{A_n(x)\}_{n=N}^{\infty}$ converges in $\mathbb{C}^k$-topology for any $k \in \mathbb{N}$. This concludes the proof. □

Construction of $\tilde{A}_n(x)$ Next we will construct the sequence $\tilde{A}_n(x),\ n = N, N+1, \cdots$, which is also $\mathcal{C}^\infty$-convergent to D_∞, but the Lyapunov exponent of each $\tilde{A}_n(x)$ possesses an upper bound less than $(1-\delta)\log\lambda$.

Let $\tilde{e}_n(x) = -(s_n(x) - s_n'(x)) \cdot f_n(x)$ be a $\mathcal{C}^\infty$ class $2\pi-$periodic function such that it is $-(s_n(x) - s_n'(x))$ on $\frac{I_n}{10}$ and vanishes outside I_n. From $(c)_n$, we have that $\tilde{e}_n(x) = \phi_0(x) \cdot f_n(x)$. Then we define $\tilde{\phi}_n(x) = \phi_n(x) + \tilde{e}_n(x)$.

Lemma 5.3 *For* $0 \le k \le [q_n^a]$ *and* $x \in I_n$*, it holds that*

$$|\phi_0^{(k)}(x)| \le e^{-\frac{q_n^{2a}}{4}}.$$

The proof can be found in the Appendix.

Take $n_3(k)$ so that $k \le [q_n^a]$ if $n \ge n_3(k)$. Combining (5.6) with Lemma 5.3, we have $|\tilde{e}_n(x)|_{\mathbb{C}^k} \le e^{-\frac{q_n^{2a}}{8}}$ if $n \ge n_3(k)$. It follows that $\tilde{A}_n(x) = \Lambda \cdot R_{\frac{\pi}{2} - \tilde{\phi}_n(x)}$ is convergent to $D_\infty(x)$ in $\mathcal{C}^\infty$-topology.

In the same way as in Section 4, we can obtain that $(1-\delta)\log\lambda$ is the upper bound of the Lyapunov exponent for $\tilde{A}_n(x)$, while the lower bound of the Lyapunov exponent of $D_\infty(x)$ is $(1-4\epsilon)\log\lambda$, which produces the discontinuity since $4\epsilon < \delta$. The proof of Theorem 1 in $\mathcal{C}^\infty$ case is thus finished.

A Appendix

In the Appendix, we will give the proofs of Lemmas 3.8 and 5.2.

A.1 Some lemmas.

Before proving Lemmas 3.8 and 5.2, we firstly give some lemmas as preparations.

Lemma A.1 *Suppose that* $\{A_0, A_1, ..., A_{n-1}\}$ *is* μ*-hyperbolic. Let* $s_i = s(A^i)$, $i = 1, 2, \cdots, n$. *Then for* $\mu \gg 1$, *we have*

$$\text{(a) } \angle(s_i, s_n) \leq \mu^{-2i(1-\epsilon)+3\epsilon}, \qquad \text{(b) } |A^i s_n| \leq \mu^{-i(1-3\epsilon)+3\epsilon}.$$

Proof. Let $u_i = u(A^i)$. To prove (a), we write $s_i = v_1 \oplus v_2$ respecting $s_{i+1} \oplus u_{i+1}$. Then we have

$$|\sin \angle(s_i, s_{i+1})| \cdot |A^{i+1} \cdot u_{i+1}| = |A^{i+1} \cdot v_2| \leq |A^{i+1} \cdot s_i|$$

$$\leq \mu^{1+\epsilon} \cdot |A^i \cdot s_i| \leq \mu^{1+\epsilon} \cdot \mu^{-i(1-\epsilon)}.$$

On the other hand, $|A^{i+1} \cdot u_{i+1}| \geq \mu^{(i+1)(1-\epsilon)}$. Thus we obtain $|\angle(s_i, s_{i+1})| \ll 1$ and $|\angle(s_i, s_{i+1})| \approx |\sin \angle(s_i, s_{i+1})| \leq \mu^{-2i(1-\epsilon)+2\epsilon}$, which implies

$$\angle(s_i, s_n) \leq \sum_{j=i}^{n-1} \angle(s_i, s_{i+1}) \leq \mu^{-2i(1-\epsilon)+3\epsilon}.$$

To prove (b), we write $s_n = v_3 \oplus v_4$ respecting $s_i \oplus u_i$. Then we have

$$|A^i v_3| \leq \mu^{-i(1-\epsilon)}$$

and

$$|A^i v_4| = |\sin \angle(s_i, s_n)| \cdot |A^i u_i|$$

$$\leq \mu^{-2i(1-\epsilon)+3\epsilon} \cdot \mu^{i(1+\epsilon)} = \mu^{-i(1-3\epsilon)+3\epsilon}.$$

□

Let $A \in SL(2, \mathbb{R})$, $\theta \in \mathrm{RP}^1$ and $\bar{A}\theta = \psi$. It holds that

$$|(D\bar{A})_\theta| = \frac{1}{|A\hat{\theta}|^2}. \tag{A.1}$$

Then it follows that

$$\hat{\theta} = A^{-1} \cdot A\hat{\theta} = |A\hat{\theta}| \cdot A^{-1}\hat{\psi} = |A\hat{\theta}| \cdot |A^{-1}\hat{\psi}| \cdot \hat{\theta},$$

which implies that $|A\hat{\theta}| \cdot |A^{-1}\hat{\psi}| = 1$, where $\hat{\theta}$ and $\hat{\psi}$ are the unit vectors corresponding to θ and ψ.

For $x \in I_n$, let $x_0 = x$, $x_{i+1} = Tx_i$, $x'_0 = T^{r_n^+ - 1}x$, $x'_{i+1} = T^{-1}x'_i$. Define

$$\theta_0 = s_n(x_0) = \overline{s_n(A_n^{r_n^+}(x))}, \quad \theta_{j+1} = \overline{A_n(x_j)}\theta_j$$

and

$$\theta'_0 = s'_n(x'_0) = \overline{s_n(A_n^{-r_n^+}(x'_0))}, \qquad \theta'_{j+1} = \overline{A_n^{-1}(x'_j)}\theta'_j,$$

$j = 0, 1, \cdots, r_n^+ - 1$.

Let

$$f(\lambda, \theta) := \lambda^2 \cdot g^{-1}(\lambda, \theta) := |D\bar{\Lambda}(\theta)| = \frac{1}{|\Lambda\hat{\theta}|^2} = \frac{\lambda^2}{\sin^2\theta + \lambda^4\cos^2\theta}. \tag{A.2}$$

Thus from (A.1), we have, for $i > j$,

$$\begin{aligned} \textstyle\prod_{t=j+1}^{i} f_t := \prod_{t=j+1}^{i} f(\lambda, \frac{\pi}{2} + \theta_{t+1}) = \left|(D\overline{A_n^{-(i-j)}}(x_i))_{\theta_i}\right| \\ = \frac{1}{|(D\overline{A_n^{(i-j)}}(x_j))_{\theta_j}|} = |A_n^{(i-j)}(x_j) \cdot \hat{\theta}_j|^2. \end{aligned} \tag{A.3}$$

From (b) of Lemma A.1, we have

$$\prod_{t=0}^{j-1} f_t \le \lambda^{-2j(1-3\epsilon)}. \tag{A.4}$$

Similarly we have

$$\textstyle\prod_{t=j+1}^{i} f'_t := \prod_{t=j+1}^{i} f(\lambda, \theta'_t + \frac{\pi}{2} - \phi_n(x'_{t-1})) = |A_n^{(i-j)}(x'_j) \cdot \hat{\theta}'_j|^2, \qquad \prod_{t=0}^{j-1} f'_t \le \lambda^{-2j(1-3\epsilon)}. \tag{A.5}$$

Now we give estimates for $\prod_{t=j+1}^{i} f(\lambda, \frac{\pi}{2} + \theta_{t+1})$ and $\prod_{t=j+1}^{i} f(\lambda, \theta'_t + \frac{\pi}{2} - \phi_n(x'_{t-1}))$.

Lemma A.2 *Let $\lambda \gg 1$. Then for $a_k = f(\lambda, \frac{\pi}{2} + \theta_k)$ and $a'_k = f(\lambda, \theta'_k + \frac{\pi}{2} - \phi(x'_k))$, $0 \le k \le r_n^+ - 1$, it holds that*

$$|a_{i-1} \cdots a_j|, \quad |a'_{i-1} \cdots a'_j| \le \lambda^{-(i-j)} \cdot g_j(1), \tag{A.6}$$

where $g_x(r) = \max\{\hat{g}_x(r), 1\}$,

$$\hat{g}_x(r) = \begin{cases} (\phi(1/4M^2x^2))^{-2c_7r^2}, & x \ge 0, \\ 1, & x = 0, \end{cases}$$

for $x \ge 0$, $r \in \mathbb{N}$ and $c_7 > 0$ depending only on M.

Proof. We only give estimates for a_k and the estimates for a'_k are similar. From (A.4), we have $|a_{i-1} \cdots a_0| \le \lambda^{-2i(1-3\epsilon)}$, together with $|a_{j-1} \cdots a_0| \ge \lambda^{-2j}$, which implies that

$$|a_{i-1} \cdots a_j| \le \lambda^{-2i(1-3\epsilon)+2j} = \lambda^{-(i-j)} \cdot g_j(1)^2 \cdot \lambda^{-(i-j)+6\epsilon i} \cdot g_j(1)^{-2} \le \lambda^{-(i-j)} \cdot g_j(1)^2.$$

It is trivial that (A.6) holds if $i - j > \frac{6\epsilon}{1-6\epsilon} j$. Thus we only need to consider the case $i - j \le \frac{6\epsilon}{1-6\epsilon} j$.

For any $k \geq 1$, define $n(k)$ be the integer such that $q_{n(k)} \leq k < q_{n(k)+1}$, where we define $q_0 = 1$ for convenience. Then $T^k x (\text{mod } 2\pi) \notin I_{n(k)+1}$ since $r^+_{n(k)+1} \geq q_{n(k)+1}$, which implies $|T^k x - c_s| \geq \frac{1}{q^2_{n(k)+1}}$, $s = 1, 2$. From $q_{n(k)+1} \leq M \cdot q_{n_k}$, it follows that $|T^k x - c_s| \geq \frac{1}{M^2 \cdot q^2_{n(k)}} \geq \frac{1}{M^2 k^2}$. From the assumption, we get that for $k \in [j, i-1] \subset [j, 2j]$,

$$|T^k x - c_s| \geq \frac{1}{4M^2 j^2}. \tag{A.7}$$

Define $S(m) = \{k \in [j, i-1] | T^k x \ (\text{mod } 2\pi) \in I_m, \ m \geq N\}$. Let

$$m^* = \max\left\{ m \,\middle|\, \max\{k | k \in S(m)\} - \min\{k | k \in S(m)\} \geq \frac{9}{10} \cdot |i - j| \right\}$$

if it exists.

If m^* exists, let $k_1 = i_1, i_2, \cdots, i_t = k_2$ are all the points in $[j, i]$ such that $T^{i_s} x \in I_{m^*}$, $1 \leq s \leq t$. Then t is a constant depending only on M. In fact, without loss of generality, let $i - j \geq 60$ since otherwise, $t \leq 60$. Since ω is of bounded type, similar to the proof of Corollary 4.1, we know that there exists a constant $c_6 = c_6(M) > 0$ such that $c_6 \leq \frac{\min t_m(x)}{\max t_{m+1}(x)} \leq 1$, where $\min t_m(x) = \min_{x \in I_m} \min\{i > 0 | T^i x \ (mod\ 2\pi) \in I_m\}$ and $\max t_m(x) = \max_{x \in I_m} \min\{i > 0 | T^i x \ (mod\ 2\pi) \in I_m\}$.

Then if $t \geq [\frac{1}{c_6}] + 1$, one sees that $S(m^* + 1) \neq \emptyset$. Moreover, it holds that $\max t_{m^*+1}(x) > [\frac{1}{30}(i - j)] + 1$. Otherwise, we have that

$$\max\{k \in S(m^*+1)\} \in [i-1-(\frac{1}{30}(i-j)+1), i-1], \quad \min\{k \in S(m^*+1)\} \in [j, j+\frac{1}{30}(i-j)+1],$$

which contracts the definition of m^*. Thus it follows that $\min t_{m^*}(x) \geq [\frac{c_6}{30}(i - j)]$. Then we obtain that $t \leq [\frac{30}{c_6}] + 1$.

From the definition of I_{m^*}, we have

$$|A^{i_{s+1} - i_s}(T^{i_s} x) \cdot \hat{\theta}_{i_s}| \geq \lambda_{m^*}^{i_{s+1} - i_s} \geq \lambda_\infty^{i_{s+1} - i_s}, \quad 0 \leq s \leq t - 1.$$

From the construction of ϕ_n, it holds that $\phi_{m^*}(x) = \phi_n(x)$ on $I_{m^*} \backslash I_n$. Then from $(2)_{\text{n}}$ in Proposition 3.1 and (A.7), we have

$$|s_{m^*}(T^{i_s} x) - s'_{m^*}(T^{i_s} x)| \geq \frac{1}{2} \phi(1/4M^2 j^2),$$

which, by Lemma 2.1, implies

$$\begin{aligned}
&|A^{i_{s+1} - i_s}(T^{i_s} x) \cdot A^{i_s - i_{s-1}}(T^{i_{s-1}} x) \cdot \hat{\theta}_{i_{s-1}}| \\
&\geq \tfrac{1}{4} |A^{i_{s+1} - i_s}(T^{i_s} x) \cdot \hat{\theta}_{i_s}| \cdot |A^{i_s - i_{s-1}}(T^{i_{s-1}} x) \cdot \hat{\theta}_{i_{s-1}}| \cdot |s_{m^*}(T^{i_s} x) - s'_{m^*}(T^{i_s} x)| \\
&\geq \tfrac{1}{4} \lambda_\infty^{i_{s+1} - i_{s-1}} \cdot \phi(1/4M^2 j^2), \qquad 0 \leq s \leq t - 1.
\end{aligned}$$

Consequently, we see that

$$|A^{i-j}(T^jx)\cdot\hat{\theta}_j| \geq |A^{i_t-i_0}(T^{i_0}x)\cdot\hat{\theta}_{i_0}|\cdot\lambda^{-\frac{1}{10}(i-j)} \geq \tfrac{1}{4}\lambda_\infty^{k_2-k_1}\cdot\phi(1/4M^2j^2)\cdot\lambda^{-\frac{1}{10}(i-j)}$$

$$\geq \tfrac{1}{4}\lambda_\infty^{\frac{9}{10}(i-j)}\cdot\phi^{t+1}(1/4M^2j^2)\cdot\lambda^{-\frac{1}{10}(i-j)} \geq \lambda^{\frac{2}{3}(i-j)}\cdot\phi^{c_7}(1/4M^2j^2)$$

if $\lambda \gg 1$, which implies (A.6) with $c_7 = \max\{[\frac{30}{c_6}]+1, 60\}$.

Otherwise, if m^* does not exist. Let $t' \geq 0$ be the number of items in the set $S(N)$. Without loss of generality, we assume $t' \geq 2$ and $i-j \geq 60$. Let $k_3 = j_1 < j_2 < \cdots < j_{t'} = k_4$ be all the points in $[j,i]$ such that $T^{j_s}x \in I_N,\ 1 \leq s \leq t'$. Then similar to the above argument, $\max t_N(x) > [\frac{1}{30}(i-j)]+1$, otherwise m^* will exist. It implies $\min t_N(x) \geq [\frac{c_6}{30}(i-j)]$. Thus $t' \leq [\frac{30}{c_6}]+1$. Then (A.6) can be proved similarly by the above argument. ☐
[2]

We also need the following estimate for $f,\ g$ which are defined in (A.2).

Lemma A.3 *For any $i \geq 1$, it holds that*

$$|\partial_\theta^i f(\lambda,\theta)| \leq 4^i\cdot(i!)^2\cdot\lambda^{2i}|f(\lambda,\theta)|. \tag{A.8}$$

Proof. From the expression of f in (A.2), $\partial_\theta^i f(\lambda,\theta)$ can be written as the sum of the following terms:

$$\lambda^2\cdot k!\cdot g^{-(k+1)}\cdot\partial_\theta^{i_1}g\cdots\partial_\theta^{i_k}g,$$

where $1 \leq k \leq i$, $i_1+\cdots+i_k = i$ with $i_1,\cdots,i_k > 0$ and the number of the terms in the sum is $i!$. When $i_s = 1$, then $|g^{-1}|\cdot|\partial_\theta^{i_s}g| = \frac{|2\sin\theta(\lambda^2-\lambda^{-2})\cdot\lambda^2\cos\theta|}{\sin^2\theta+\lambda^4\cos^2\theta}$. Since $|\frac{\lambda^2\sin\theta\cos\theta}{\sin^2\theta+\lambda^4\cos^2\theta}| \leq 1$, we have $|g^{-1}|\cdot|\partial_\theta g| \leq 4\cdot\lambda^2$.

If $i_s > 1$, then $|g^{-1}|\cdot|\partial_\theta^{i_s}g| \leq \frac{|\lambda^4\sin(2\theta+\frac{\pi}{2}(i_s-1))|}{\sin^2\theta+\lambda^4\cos^2\theta} \leq \lambda^4$ since $\sin^2\theta+\lambda^4\cos^2\theta > 1$ for $\lambda > 1$. In conclusion, we have $|g^{-1}|\cdot|\partial_\theta^{i_s}g| \leq 4\lambda^{2i_s}$. Thus it follows that

$$\lambda^2|g^{-k-1}\cdot\partial_\theta^{i_1}g\cdots\partial_\theta^{i_k}g| \leq \lambda^2\cdot|g|^{-1}\cdot\prod_{s=1}^{k}|g|^{-1}|\partial_\theta^{i_s}g| \leq 4^i\cdot\lambda^2\cdot|g|^{-1}\cdot\lambda^{2i} = 4^i\cdot|f|\cdot\lambda^{2i}.$$

Thus $|\partial_\theta^i f(\lambda,\theta)| \leq 4^i\cdot(i!)^2\cdot\lambda^{2i}\cdot|f|$ since $k \leq i$. This ends the proof of the lemma. ☐

[2] Lemma A.2 describe some kind of "sub-exponential growth property" in the following sense. Let $\phi(x)$ be defined as in Sections 3 or 5, then one can see that there exists $0 < c < 1$ such that

$$|a_{i-1}\cdots a_j| \leq g_j(1) \leq \lambda^{j^c}$$

for any $0 \leq j \leq i \leq r_n^+ - 1$. For example, for $\phi(x)$ defined as in Section 5, we can set $c = 2a$.

A.2 Upper bound estimates for θ_j and θ'_j

In this section, we will give upper bound estimates for derivatives of θ_j and θ'_j which are defined in the last section.

We firstly derive out the recursive expression for θ_j and θ'_j. From (A.2) and the definition of θ_j, θ'_j, we have

$$\begin{aligned}\frac{d\theta_j}{dx} &= f(\lambda, \tfrac{\pi}{2} + \theta_{j+1}) \cdot \tfrac{d\theta_{j+1}}{dx} + \phi'_n(x_j),\\ \frac{d\theta'_j}{dx} &= f(\lambda, \theta'_{j+1} - \phi_n(x'_j) + \tfrac{\pi}{2}) \cdot (\tfrac{d\theta'_{j+1}}{dx} - \phi'_n(x'_j)),\end{aligned}$$

or equivalently,

$$\begin{aligned}\frac{d\tilde{\theta}_j}{dx} &= f(\lambda, \tilde{\theta}_{j+1} + \phi_n(x_{j+1})) \cdot (\tfrac{d\tilde{\theta}_{j+1}}{dx} + \phi'_n(x_{j+1})),\\ \frac{d\theta'_j}{dx} &= f(\lambda, \theta'_{j+1} + \tilde{\phi}_n(x'_{j+1})) \cdot (\tfrac{d\theta'_{j+1}}{dx} + \tilde{\phi}'_n(x'_{j+1})),\end{aligned}$$

where $\tilde{\theta}_j = \frac{\pi}{2} + \theta_j - \phi_n(x_j)$ and $\tilde{\phi}_n(x'_{j+1}) = \frac{\pi}{2} - \phi_n(x'_j)$. For convenience, we will still use notations θ_j and $\phi_n(x'_{j+1})$ to denote $\tilde{\theta}_j$ and $\tilde{\phi}_n(x'_{j+1})$. Then we obtain

$$\begin{aligned}\frac{d\theta_j}{dx} &= f(\lambda, \theta_{j+1} + \phi_{n-1}(x_{j+1})) \cdot (\tfrac{d\theta_{j+1}}{dx} + \phi'_{n-1}(x_{j+1})) := f_{j+1} \cdot (\tfrac{d\theta_{j+1}}{dx} + \phi'_{n-1}(x_{j+1})),\\ \frac{d\theta'_j}{dx} &= f(\lambda, \theta'_{j+1} + \phi_{n-1}(x'_{j+1})) \cdot (\tfrac{d\theta'_{j+1}}{dx} + \phi'_{n-1}(x'_{j+1})) := f'_{j+1} \cdot (\tfrac{d\theta'_{j+1}}{dx} + \phi'_{n-1}(x'_{j+1})),\end{aligned} \tag{A.9}$$

where $0 \le j \le r^+_{n-1} - 2$.

By (A.9), we have

$$\begin{aligned}\frac{d\theta_j}{dx} &= f_{j+1} \cdot (\tfrac{d\theta_{j+1}}{dx} + \phi'_{n-1}(x_{j+1}))\\ &= f_{j+1} \cdot (f_{j+2} \cdot (\tfrac{d\theta_{j+2}}{dx} + \phi'_{n-1}(x_{j+2})) + \phi'_{n-1}(x_{j+1}))\\ &= f_{j+1} \cdot (f_{j+2}(\cdots(\tfrac{d\theta_i}{dx} + \phi'_{n-1}(x_j))\cdots) + \phi'_{n-1}(x_{j+1}))\\ &= \textstyle\sum_{i=j+1}^{r^+_n-1} \prod_{t=j+1}^{i} f_t \cdot \phi'_{n-1}(x_i) + \prod_{i=j+1}^{r^+_n-1} f_i \cdot \frac{d\theta_{r^+_n-1}}{dx}\\ &:= \textstyle\sum_{i=j+1}^{r^+_n-1} F_{j,i}(x, \theta_{j+1}, \cdots, \theta_i) + \prod_{i=j+1}^{r^+_n-1} f_i \cdot \frac{d\theta_{r^+_n-1}}{dx}\\ &:= F_j(x, \theta_{j+1}, \cdots, \theta_{r^+_n-1}) + \textstyle\prod_{i=j+1}^{r^+_n-1} f_i \cdot \frac{d\theta_{r^+_n-1}}{dx}.\end{aligned} \tag{A.10}$$

Similarly, $\frac{d\theta'_j}{dx}$ can be written as the form

$$\frac{d\theta'_j}{dx} = F'_j + \prod_{i=j+1}^{r^+_n-1} f'_i \cdot \frac{d\theta'_{r^+_n-1}}{dx} \tag{A.11}$$

with $F'_j = F'_j(x, \theta'_{j+1}, \cdots, \theta'_{r_n^+-1})$.

From (2.1) and the fact that $S(A) \perp U(A)$ for any hyperbolic matrix A, it holds that

$$\frac{d\theta_0}{dx} = \frac{d\theta'_{r_n^+-1}}{dx}, \quad \frac{d\theta'_0}{dx} = \frac{d\theta_{r_n^+-1}}{dx}. \tag{A.12}$$

From (A.12),(A.10) and (A.11), we have

$$\frac{ds_n}{dx} = F_0(x, \theta_0(x), \cdots, \theta_{r_n^+-1}(x)) + \prod_{i=r_n^+-1}^{0} f_i(\lambda, x, \theta_i) \cdot \frac{ds'_n}{dx}$$

and

$$\frac{ds'_n}{dx} = F'_0(x, \theta'_0(x), \cdots, \theta'_{r_n^+-1}(x)) + \prod_{i=r_n^+-1}^{0} f'_i(\lambda, x, \theta'_i) \cdot \frac{ds_n}{dx}.$$

Thus

$$\left(1 - \prod_{i=r_n^+-1}^{0} f_i \cdot \prod_{i=r_n^+-1}^{0} f'_i\right) \cdot \frac{ds_n}{dx} = F_0 + F'_0 \prod_{i=r_n^+-1}^{0} f_i \tag{A.13}$$

and

$$\left(1 - \prod_{i=r_n^+-1}^{0} f_i \cdot \prod_{i=r_n^+-1}^{0} f'_i\right) \cdot \frac{ds'_n}{dx} = F'_0 + F_0 \prod_{i=r_n^+-1}^{0} f'_i. \tag{A.14}$$

Similarly, from (A.10) and (A.11), we have

$$\begin{aligned} \tfrac{d\theta_j}{dx} &= \textstyle\sum_{j\le k\le r_n^+-1} \left(\prod_{j\le i\le k} f_i\right) b_k(x) + \prod_{j\le k\le r_n^+-1} f_k \cdot \frac{d\theta_0}{dx} \\ &:= \textstyle\sum_{j\le k\le r_n^+-1} F_{j,k}(x, \theta_j, \cdots, \theta_k) + \prod_{j\le k\le r_n^+-1} f_k \cdot \frac{d\theta'_0}{dx} \\ &:= F_j(x, \theta_j, \cdots, \theta_{r_n^+-1}) + \textstyle\prod_{j\le k\le r_n^+-1} f_k \cdot \frac{d\theta'_0}{dx} \end{aligned} \tag{A.15}$$

and

$$\begin{aligned} \tfrac{d\theta'_j}{dx} &= \textstyle\sum_{j\le k\le r_n^+-1} \left(\prod_{j\le i\le k} f'_i\right) b'_k(x) + \prod_{j\le k\le r_n^+-1} f'_k \cdot \frac{d\theta_0}{dx} \\ &:= \textstyle\sum_{j\le k\le r_n^+-1} F'_{j,k}(x, \theta'_j, \cdots, \theta'_k) + \prod_{j\le k\le r_n^+-1} f'_k \cdot \frac{d\theta_0}{dx} \\ &:= F'_j(x, \theta'_j, \cdots, \theta'_{r_n^+-1}) + \textstyle\prod_{j\le k\le r_n^+-1} f'_k \cdot \frac{d\theta_0}{dx} \end{aligned} \tag{A.16}$$

with $b_k = -\phi'(T^k x)$ and $b'_k = \phi'(T^{r_n^+-k-1} x)$.

Now we give estimates for θ_j and θ_j'. For convenience, we use multi-index notation

$$D^K F := \frac{\partial^{k_1}\cdots\partial^{k_m}F}{\partial x_1^{k_1}\cdots\partial x_m^{k_m}},$$

for function $F = F(x_1, \cdots, x_m)$ where $K = (k_1, \cdots, k_m)$, $|K| := k_1 + \cdots + k_m$.

Lemma A.4 *Let $Y = (y_1, \ \cdots, y_t)$ and $L = (0, l_1, \cdots, l_t)$. Assume that $G(\lambda, Y)$ satisfies that for any $(l_1, \cdots, l_t)$,*

$$|D^L G(\lambda, Y)| \leq 4^{|L|} \cdot (|L|!)^2 \cdot \lambda^{2|L|} \cdot \|G\|. \tag{A.17}$$

Define $\Theta = (\theta_1, \cdots, \theta_t)$ and $\Gamma(x) = (\gamma(x+\eta_1), \cdots, \gamma(x+\eta_t))$ with $\eta_i \in \mathbb{R}$. Then for $\hat{G}(\lambda, x, \Theta) = G(\lambda, \Gamma(x) + \Theta)$ and $\hat{L} = (0, l_0, l_1, \cdots, l_t)$, we have

$$|D^{\hat{L}} \hat{G}(\lambda, x, \Theta)| \leq 4^{|\hat{L}|} \cdot (|\hat{L}|!)^2 \cdot P_{t+|\hat{L}|}^{|\hat{L}|} \cdot \|\gamma\|_{l_0} \cdot \lambda^{2|\hat{L}|} \cdot \|G\|. \tag{A.18}$$

Proof. From the condition (A.17), we have that

$$\begin{aligned} |D^{\hat{L}} \hat{G}(\lambda, x, \Theta)| &\leq \left| \tfrac{\partial^{l_0}}{\partial x^{l_0}} \left(\tfrac{\partial^{l_1+\cdot+l_t}}{\partial y^{l_1}\cdots\partial y^{l_t}} G(\lambda, \Gamma(x) + \Theta) \right) \right| \\ &\leq \sum_{m_{1,1}+\cdots+m_{t,k_t}=l_0} \left| \frac{\partial^{k_1+l_1+\cdots+k_t+l_t}}{\partial y_1^{k_1+l_1}\cdots y_t^{k_t+l_t}} G(\lambda, \Gamma(x)+\Theta) \right| \cdot \|\gamma\|_{\mathbb{C}^{m_{1,1}}} \cdots \|\gamma\|_{\mathbb{C}^{m_{t,k_t}}} \\ &\leq 4^{|\hat{L}|} \cdot (|\hat{L}|!)^2 \cdot P_{t+|\hat{L}|}^{|\hat{L}|} \cdot \|\gamma\|_{l_0} \cdot \lambda^{2|\hat{L}|} \cdot \|G\|, \end{aligned} \tag{A.19}$$

where we use the fact that the number of the terms in the sum is not more than $P_{t+|\hat{L}|}^{|\hat{L}|}$ and denote $\|\gamma\|_{\mathbb{C}^{m_{1,1}}} \cdots \|\gamma\|_{\mathbb{C}^{m_{t,k_t}}}$ by $\|\gamma\|_{l_0}$.

□

Remark A.1 *In the following, for a function $h = h(x)$, we sometimes denote by*

$$\|h\|_k = \max_{\{k_1+\cdot+k_m=k\}} \prod_{1\leq i\leq m} \|h\|_{\mathbb{C}^{k_i}}.$$

It is easy to see that

$$\|h\|_{k_1} \cdot \|h\|_{k_2} \leq \|h\|_{k_1+k_2}. \tag{A.20}$$

From Lemmas A.2 and A.4, we have the following estimates:

Lemma A.5 *Let $K = (k_j, \cdots, k_{r_n^+-1})$ with $k_j + \cdots + k_{r_n^+-1} = k$ and*

$$\Theta(x) = (\theta_{j+1}(x), \cdots, \theta_{r_n^+-1}(x)),$$

$$\Theta'(x) = (\theta'_{j+1}(x), \cdots, \theta'_{r_n^+-1}(x)),$$

where $\theta_j(x)$, $\theta'_j(x)$ are defined as above, and $0 \le j \le r_{n-1}^+$. Then we have

$$|D^K(\prod_{i=j}^{r_n^+-1} f_i)(\lambda, x, \Theta)|, \ |D^K(\prod_{i=j}^{r_n^+-1} f_i)(\lambda, x, \Theta')| \le 4^k \cdot (k!)^2 \cdot \|\phi_n\|_{k_j} \cdot P^k_{r_n^+-j+k} \cdot \lambda^{-(r_n^+-j)+2k} \cdot g_j(1)$$

$$|D^K F_{j,i}(\lambda, x, \Theta)|, \ |D^K F'_{j,i}(\lambda, x, \Theta')| \le 4^k \cdot (k!)^2 \cdot \|\phi_n\|_{k_j+1} \cdot P^k_{i-j+k} \cdot \lambda^{-(i-j)+2k} \cdot g_j(1). \tag{A.21}$$

Proof. From (A.8) and Lemma A.4, we have

$$|D^K(\prod_{i=j}^{r_n^+-1} f_i)(\lambda, x, \Theta)| \le 4^k \cdot (k!)^2 \cdot \|\phi_n\|_{k_j+1}\lambda^{2k} \cdot P^k_{i-j+k}|(\prod_{i=j}^{r_n^+-1} f_i)(\lambda, x, \Theta)|.$$

From Lemma A.2, we know that

$$|(\prod_{i=j}^{r_n^+-1} f_i)(\lambda, x, \Theta)| < \lambda^{-(i-j)} \cdot g_j(1).$$

Then

$$|D^K(\prod_{i=j}^{r_n^+-1} f_i)(\lambda, x, \Theta)| \le 4^k \cdot (k!)^2 \cdot \|\phi_n\|_{k_j} \cdot P^k_{r_n^+-j+k} \cdot \lambda^{-(r_n^+-j)+2k} \cdot g_j(1).$$

The other estimates in (A.18) can be proved by the same method.

□

The following element lemma can make the proof of Lemma A.7 simpler:

Lemma A.6 *Let $\lambda \gg 1$. If for any $r \in \mathbb{N}$, $f_1(x)$ and $f_2(x)$ satisfy $|\frac{d^r f_i}{dx^r}| \le |\phi|_r \cdot r^r \cdot \lambda^{8r^2} \cdot g_n(r)$, $\quad i = 1, 2$. Then we have*

$$|\frac{d^r(f_1 \cdot f_2)}{dx^r}| \le |\phi|_r \cdot r^r \cdot \lambda^{8r^2} \cdot g_n(r).$$

Proof. This can be easily proved from the definition of g_n (See Lemma A.2) and the definition of $|\phi|_r$ (See (A.20)). □

The following estimates on the upper bound of derivatives of θ_j, θ'_j are important for the proof of Lemmas 3.8 and 5.2.

Lemma A.7 *Let $\lambda \gg N \gg 1$. Then if $n \geq N$, $x \in I_n$, $0 < j \leq r_n^+ - 1$ and $1 \leq r \leq l$, it holds that*

$$|\theta_0|_{\mathbb{C}^r},\ |\theta_0'|_{\mathbb{C}^r} \leq |\phi_n|_r \cdot r^r \cdot \lambda^{r^4}, \quad |\theta_j|_{\mathbb{C}^r},\ |\theta_j'|_{\mathbb{C}^r} \leq |\phi_n|_r \cdot r^r \cdot \lambda^{r^4} \cdot g_j(r). \tag{A.22}$$

Proof. For $r = 1$, from (A.12)-(A.14), (A.21) with $j, k = 0$ implies that $|\frac{d\theta_0}{dx}|$, $|\frac{d\theta_0'}{dx}| \leq 2|\phi_n|_1$, thus the first part in (A.22) is obtained. From (A.15) and (A.21) we have

$$|\frac{d\theta_j}{dx}| \leq \sum_{j \leq k \leq r_n^+ - 1} \lambda^{-(k-j)} \cdot g_j(1) \cdot (2|\phi_n|_1)$$

$$\leq 4g_j(1) \cdot |\phi_n|_1 \leq |\phi_n|_1 \cdot \lambda \cdot g_j(1)$$

if $\lambda \gg 1$. Similar estimate can be obtained for $\frac{d\theta_j'}{dx}$, Hence the proof for the case $r = 1$ is finished.

Assume (A.22) holds true for the case $0 < i \leq r$. Now we prove the first part of (A.22) for the case $r + 1$. Later we will consider the second part of it.

Let $L = (r_1, l_0, \cdots, l_k)$ and $L_t = (l_{t,1}, \cdots, l_{t,l_t})$, $0 \leq t \leq k$ with $r_1 + |L_0| + \cdots + |L_k| = r$. From (A.21), we have

$$\begin{aligned} |\frac{d^r F_{0,k}}{dx^r}| &\leq \sum |D^L F_{0,k}| \cdot |D^{L_0}\theta_0| \cdots |D^{L_k}\theta_k| \\ &\leq 4^r \cdot (r!)^2 \cdot P_{k+r}^r \cdot \lambda^{-k+2r} \cdot \|\phi_n\|_{r_1+1} \cdot |D^{L_0}\theta_0| \cdots |D^{L_k}\theta_k|, \quad k = 0, 1, \cdots. \end{aligned} \tag{A.23}$$

From inductive assumptions, one sees that

$$|D^{L_t}\theta_t| \leq |\phi_n|_{l_{t,1}} \cdot l_{t,1}^{l_{t,1}} \cdot \lambda^{l_{t,1}^4} \cdot g_t(l_{t,1}) \cdot \cdots \cdot |\phi_n|_{l_{t,l_t}} \cdot l_{t,l_t}^{l_{t,l_t}} \cdot \lambda^{l_{t,l_t}^4} \cdot g_t(l_{t,l_t}).$$

Obviously we have $g_j(r_1) \cdot g_j(r_2) \leq g_j(r_1 + r_2)$. From the fact that $\sum\limits_{\substack{1 \leq t \leq k \\ 1 \leq u \leq l_t}} l_{t,u} = r$, we obtain

$$\prod_{\substack{1 \leq t \leq k \\ 1 \leq u \leq l_t}} g_t(l_{t,u}) \leq g_t\Big(\sum_{\substack{1 \leq t \leq k \\ 1 \leq u \leq l_t}} l_{t,u}\Big) \leq g_t(r) \leq g_k(r).$$

Similarly, we have

$$\prod_{\substack{1 \leq t \leq k \\ 1 \leq u \leq l_t}} l_{t,u}^{l_{t,u}} \leq r^r, \qquad \prod_{\substack{1 \leq t \leq k \\ 1 \leq u \leq l_t}} |\phi_n|_{l_{t,u}} \leq |\phi_n|_{r-r_1}, \qquad \prod_{\substack{1 \leq t \leq k \\ 1 \leq u \leq l_t}} \lambda^{l_{k,l_k}^4} \leq \lambda^{r^4}.$$

Thus (A.23) implies

$$|\frac{d^r F_0}{dx^r}| \leq (r+1)^{r+1} \cdot \lambda^{r^4+2r} \cdot |\phi_n|_{r+1} \cdot \sum_{1 \leq j \leq r_n^+} 4^r \cdot (r!)^2 \cdot P_{j+r}^r \cdot g_j(r) \cdot \lambda^{-j}.$$

Thus provided

$$\sum_{1\le j\le r_n^+} 4^r\cdot (r!)^2\cdot P_{j+r}^r\cdot g_j(r)\cdot \lambda^{-j}\le \lambda^{3r^3}, \tag{A.24}$$

we can obtain

$$|\frac{d^r F_0}{dx^r}|\le \frac{1}{2}(r+1)^{r+1}\cdot \lambda^{(r+1)^4}\cdot |\phi_n|_{r+1}.$$

From the definition of ϕ in Section 3 and $g_x(r)$, we have $g_x(r)\le (2Mx)^{4c_7lr^2}$ with $r\le l$. Thus it holds that

$$4^r\cdot (r!)^2,\quad (x+r)^r,\quad g_x(r)\le \lambda^{\frac{1}{4}x},$$

if $x\ge r^3$ and $\lambda\gg 1$.

A direct computation shows that

$$\begin{aligned}
&\textstyle\sum_{1\le j\le r_n^+} 4^r\cdot (r!)^2\cdot P_{j+r}^r\cdot g_j(r)\cdot \lambda^{-j}\\
&\le 2\lambda\textstyle\int_1^{+\infty}(x+r)^r\cdot 4^r\cdot (r!)^2\cdot g_x(r)\cdot \lambda^{-x}dx\\
&\le 2\lambda(\textstyle\int_1^{r^3}(x+r)^r\cdot 4^r\cdot (r!)^2\cdot g_x(r)\cdot\lambda^{-x}dx+\int_{r^3}^{\infty}\lambda^{-\frac{1}{4}x}dx)\\
&\le 2\lambda(\textstyle\int_1^{r^3}(40r^4)^r\cdot g_{r^3}(r)dx+4\lambda^{-\frac{1}{4}r^3})\\
&\le 4\lambda\cdot (40r^4)^r\cdot r^3\cdot g_{r^3}(r)\le \lambda^{3r^3}
\end{aligned}$$

if $\lambda\gg 1$. Thus we have proved (A.24). The same estimate holds true for $\prod_{0\le i\le r_n^+-1} f_i$ and $\prod_{0\le i\le r_n^+-1} f_i\cdot \prod_{0\le i\le r_n^+-1} f_i'$. By Lemma A.6, we get same estimates for $\frac{d^{r+1}\theta_0}{dx^{r+1}}$, The estimate for θ_0 is thus finished. The estimate for θ_0' is obtained by the same method.
Next we estimate θ_j, $1\le j\le r_n^+-1$. From (A.21), we obtain

$$\begin{aligned}
&\left|\tfrac{d^{r+1}F_{j,k}}{dx^{r+1}}\right|\le \textstyle\sum |D^L F_{j,k}|\cdot |D^{L_j}\theta_j|\cdots |D^{L_k}\theta_k|\\
&\le \textstyle\sum P_{k-j+r}^r\cdot 4^r\cdot (r!)^2\cdot g_j(1)\cdot \lambda^{2r}\cdot \lambda^{-(k-j)}\cdot (r+1)^{(r+1)}\cdot \lambda^{r^4}\cdot |\phi_n|_{r+1}\cdot g_k(r)\\
&\le (r+1)^{r+1}\cdot \lambda^{r^4+2r}\cdot |\phi_n|_{r+1}\cdot g_j(r+1)[\lambda^{-(k-j)}\cdot P_{k-j+r}^r\cdot 4^r\cdot (r!)^2\cdot g_k(r)\cdot g_j^{-1}(r)].
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\tfrac{d^{r+1}F_j}{dx^{r+1}}\right| \quad &\le (r+1)^{r+1}\cdot \lambda^{(r+1)^4}\cdot |\phi_n|_{r+1}\cdot g_j(r+1)\cdot\\
&\lambda^{-4r^3}\cdot \textstyle\sum_{j\le k\le r_n^+-1}[\lambda^{-(k-j)}\cdot P_{k-j+r}^r\cdot 4^r\cdot (r!)^2\cdot g_k(r)\cdot g_j^{-1}(r)].
\end{aligned}$$

Similar to the estimate for $\left|\frac{d^{r+1}F_0}{dx^{r+1}}\right|$, we can prove that if $\lambda \gg 1$,

$$\sum_{j\le k\le r_n^+-1}[\lambda^{-(k-j)}\cdot P_{k-j+r}^{r}\cdot 4^r\cdot (r!)^2\cdot g_k(r)\cdot g_j^{-1}(r)]\le \lambda^{4r^3}.$$

The same estimates hold true for $\prod_{j\le i\le r_n^+-1} f_i\cdot \frac{d\theta_0}{dx}$. Thus with the help of Lemma A.6, we finish the proof for θ_j and the one for θ_j' is similar. Thus we finish the proof for the case $r+1$. This concludes the lemma. □

Remark A.2 *The above estimate still hold true if ϕ_n in section 3 is replaced by ϕ_n in section 5.*

A.3 The proof of Lemma 3.8

We only give the proof for the first part of Lemma 3.8, the second part can be proved by same method. The following estimates will be used later.

Lemma A.8 *For $0\le j\le r_n^+-1$, it holds that*

$$|\bar\theta_j-\theta_j|\le \lambda^{-2r_{n-1}^+(1-3\epsilon)+2j}. \tag{A.25}$$

Proof. From Lemma A.1 we have

$$|\bar s_n - s_{n-1}|\le \mu^{-2r_{n-1}^+(1-\epsilon)+3\epsilon}\le \lambda^{-2r_{n-1}^+(1-3\epsilon)}. \tag{A.26}$$

Recall that for any linear map $L:\mathbb{R}^2\to\mathbb{R}^2$ and $w\in\mathbb{R}^2$ with $|w|=1$, it holds that $|(D\bar L)_{\bar w}|=\frac{1}{|Lw|^2}$, where $w\in \mathrm{RP}^1$ corresponds to w. From the fact that $\|A(x)\|=\lambda$ for any $x\in\mathbb{S}^1$, we have $|(D\bar A)_{\bar w}|\le\lambda^2$ for any $w\in\mathbb{R}^2$ with $|w|=1$.

From the definition of θ_j, $\bar\theta_j$, we know that $\theta_j=\overline{A_{n-1}(x_{j-1})}\,\theta_{j-1}$ and $\bar\theta_j=\overline{A_{n-1}(x_{j-1})}\,\bar\theta_{j-1}$. Moreover, $\theta_0=s_{n-1}$ and $\bar\theta_0=\bar s_n$. Thus

$$|\bar\theta_j-\theta_j|=|\bar\Lambda\cdot(\phi_{n-1}(x_{j-1})+\bar\theta_{j-1})-\bar\Lambda\cdot(\phi_{n-1}(x_{j-1})+\theta_{j-1})|\le\|D\bar\Lambda\|\cdot|\bar\theta_{j-1}-\theta_{j-1}|\le\lambda^2\cdot|\bar\theta_{j-1}-\theta_{j-1}|$$

From (A.26), we then obtain

$$|\bar\theta_j-\theta_j|\le\lambda^2\cdot|\bar\theta_{j-1}-\theta_{j-1}|\le\cdots\le\lambda^{2j}\cdot|\bar\theta_0-\theta_0|\le\lambda^{-2r_{n-1}^+(1-3\epsilon)+2j}.$$

□

Similar to (A.9), we have

$$\begin{aligned}\frac{d\tilde\theta_j}{dx} &= f(\lambda,\tilde\theta_{j+1}+\phi_{n-1}(x_{j+1}))\cdot(\frac{d\tilde\theta_{j+1}}{dx}+\phi_{n-1}'(x_{j+1})),\\ \frac{d\tilde{\bar\theta}_j}{dx} &= f(\lambda,\tilde{\bar\theta}_{j+1}+\phi_{n-1}(x_{j+1}))\cdot(\frac{d\tilde{\bar\theta}_{j+1}}{dx}+\phi_{n-1}'(x_{j+1})),\end{aligned}$$

where $\tilde{\theta}_j = \frac{\pi}{2} + \theta_j - \phi_{n-1}(x_j)$ and $\tilde{\bar{\theta}}_j = \frac{\pi}{2} + \bar{\theta}_j - \phi_{n-1}(x_j)$ with $\theta_0 = s_{n-1}$ and $\bar{\theta}_0 = \bar{s}_n$. For convenience, we will still use notations θ_j and $\bar{\theta}_j$ to denote $\tilde{\theta}_j$ and $\tilde{\bar{\theta}}_j$. Then we obtain, for $0 \le j \le r^+_{n-1} - 2$ and $1 \le s \le r^+_{n-1} - 1 - j$,

$$\begin{aligned}\tfrac{d\theta_j}{dx} &= \textstyle\sum_{i=j+1}^{j+s} \prod_{t=j+1}^{i} f_t \cdot \phi'_{n-1}(x_i) + \prod_{t=1}^{s} f_{j+t} \cdot \tfrac{d\theta_{j+s}}{dx} \\ &:= \textstyle\sum_{i=j+1}^{j+s} H_{j,i}(x, \theta_{j+1}, \cdots, \theta_{j+s}) + \prod_{t=1}^{s} f_{j+t} \cdot \tfrac{d\theta_{j+s}}{dx} \\ &:= H_j(x, \theta_{j+1}, \cdots, \theta_{j+s}) + \textstyle\prod_{t=1}^{s} f_{j+t} \cdot \tfrac{d\theta_{j+s}}{dx},\end{aligned}$$

where $f_t = f(\lambda, \theta_t + \phi_{n-1}(x_t))$.

Similarly, $\frac{d\bar{\theta}_j}{dx}$ can be written as the form

$$\frac{d\bar{\theta}_j}{dx} = \bar{H}_j + \prod_{t=1}^{s} \bar{f}_{j+t} \cdot \frac{d\bar{\theta}_{j+s}}{dx}$$

with $\bar{H}_j = H_j(x, \bar{\theta}_{j+1}, \cdots, \bar{\theta}_{j+s})$.

To prove Lemma 3.8, it is sufficient to prove

Lemma A.9 *Let $0 \le k \le \min\{l, r^{+\frac{1}{10}}_{n-1}\}$, $0 \le j \le \frac{1}{2} r^+_{n-1}$, $s = [(r^+_{n-1})^{\frac{2}{3}}]$ with $n \gg 1$. Then it holds that*

$$|\frac{d^k\theta_j}{dx^k} - \frac{d^k\bar{\theta}_j}{dx^k}| \le \|\phi_{n-1}\|_k \cdot (r^+_{n-1})^{4k^2} \cdot \lambda^{-s+2k} \cdot k^k \cdot \lambda^{k^4} \cdot g_{s+j}(k). \tag{A.27}$$

Remark. *Note that $\theta_0 = s_{n-1}$ and $\bar{\theta}_0 = \bar{s}_n$, Lemma 3.8 follows from Lemma A.9 by taking $j = 0$, where we use the fact that $g_{s+j}(k) \le (2Mr^+_{n-1})^{4c_7(l+1)^3} \le \lambda^{\frac{1}{3}s}$ if $\lambda, n \gg 1$.*

Proof of Lemma A.9 The proof for the case $k = 0$ can be obtained by lemma A.8.
For the case $k > 0$, from Lemma A.7, one sees that, for $K = (K_{j+1}, \cdots, K_{j+s})$ with $K_i = (k_{i,1}, \cdots, k_{i,l_i})$, $j + 1 \le i \le j + s$,

$$|D^{K_i}\theta_i| \le \textstyle\prod_{t=1}^{l_i} k_{i,t}^{k_{i,t}} \cdot \lambda^{k_{i,t}^4} \cdot \|\phi_{n-1}\|_{k_{i,t}} \cdot g_i(k_{i,t}) \le |K_i|^{|K_i|} \cdot \lambda^{|K_i|^4} \cdot \|\phi_{n-1}\|_{|K_i|} \cdot g_i(|K_i|), \tag{A.28}$$

which, together with (A.21), implies

$$
\begin{aligned}
&|\tfrac{d^{k-1}}{dx^{k-1}}(\textstyle\prod_{i=1}^{s} f_{j+i})| \\
\leq &\sum_{\substack{L=(k_0,l_1,\cdots,l_s)\\ k_0+k_{1,1}+\cdots+k_{s,l_s}=k-1}} |D^L(\prod_{i=1}^{s} f_{j+i})\cdot D^{k_{j+1}}\theta_{j+1}\cdots D^{k_{j+s}}\theta_{j+s}| \\
\leq &\sum_{\substack{L=(k_0,l_1,\cdots,l_s)\\ k_0+k_{1,1}+\cdots+k_{s,l_s}=k-1}} 4^{|L|}\cdot(|L|!)^2\cdot\|\phi_{n-1}\|_{k_0}\cdot P_{s+|L|}^{|L|}\cdot\lambda^{-s+2|L|}g_{j+s}(1)\cdot \\
&\quad (k-k_0)^{k-k_0}\cdot\lambda^{(k-k_0)^4}\cdot\|\phi_{n-1}\|_{k-k_0}\cdot g_{j+s}(k-1-k_0) \\
\leq &P_{s+1+k}^{k}\cdot 4^k\cdot(k!)^2\cdot k^k\cdot\lambda^{k^4}\|\phi_{n-1}\|_k\cdot P_{s+k}^{k}\cdot\lambda^{-s+2k}\cdot g_{j+s}(k) \\
\leq &(s+1+k)^{2k}\cdot 4^k\cdot(k!)^2\cdot k^k\cdot\lambda^{k^4}\cdot\|\phi_{n-1}\|_k\cdot\lambda^{-s+2k}\cdot g_{j+s}(k).
\end{aligned}
$$

Since $k+s\ll r_{n-1}^{+}$, the above estimates imply that, for $n\gg 1$,

$$
|\frac{d\theta_j}{dx}-H_j|_{\mathbb{C}^k},\ |\frac{d\bar{\theta}_j}{dx}-\bar{H}_j|_{\mathbb{C}^k}\leq\frac{1}{2}\cdot r_{n-1}^{+4k^2}\cdot k^k\cdot\lambda^{k^4}\cdot\|\phi_{n-1}\|_k\cdot\lambda^{-s+2k}\cdot g_{j+s}(k). \tag{A.29}
$$

Hence to prove Lemma A.9, it is sufficient to estimate $|\frac{d^k\bar{H}_j}{dx^k}-\frac{d^kH_j}{dx^k}|$.

Assume (A.27) holds for k. We now prove (A.27) holds for $k+1$. Let J_j be the set for all the pairs $(S_j,\ K_t)$ such that $S_j=(s_j,\cdots,s_{j+s}), K_t=(k_{t,1},\cdots,k_{t,s_t})$ with $0\leq j\leq\frac{1}{2}r_{n-1}^{+},\ j+1\leq t\leq j+s,\ k_{t,1},\cdots,k_{t,s_t}\geq 1$ and $s_j+|K_{j+1}|+\cdots+|K_{j+s}|=k$. Then we have

$$
\begin{aligned}
&|\tfrac{d^kH_j}{dx^k}-\tfrac{d^k\bar{H}_j}{dx^k}| \\
\leq &\textstyle\sum_{J_j}|D^{S_j}\bar{H}_j\cdot D^{K_{j+1}}\bar{\theta}_{j+1}\cdots D^{K_{j+s}}\bar{\theta}_{j+s}-D^{S_j}H_j\cdot D^{K_{j+1}}\theta_{j+1}\cdots D^{K_{j+s}}\theta_{j+s}| \\
\leq &\textstyle\sum_{J_j}\Big(|D^{S_j}(\bar{H}_j-H_j)\cdot D^{K_{j+1}}\theta_{j+1}\cdots D^{K_{j+s}}\theta_{j+s}|+ \\
&\quad\textstyle\sum_{1\leq t\leq s}|D^{S_j}H_j\cdot D^{K_{j+1}}\theta_{j+1}\cdots D^{K_{j+t}}(\bar{\theta}_{j+t}-\theta_{j+t})\cdots D^{K_{j+s}}\bar{\theta}_{j+s}|\Big) \\
:= &\textstyle\sum_{J_j}(E_0+\sum_{1\leq t\leq s}E_t).
\end{aligned}
$$

From (A.28), we have

$$
|D^{K_{j+1}}\theta_{j+1}\cdots D^{K_{j+s}}\theta_{j+s}|\leq k^k\cdot\lambda^{k^4}\cdot g_{j+s}(k)\cdot\|\phi_{n-1}\|_{k-s_j}.
$$

Let $e_1 = (1, 0, \cdots, 0), \cdots, e_t = (0, \cdots, 1, \cdots, 0), \cdots, e_s = (0, \cdots, 1)$. Then from (A.18) and (A.25),

$$|D^{S_j}(\bar{H}_{ji} - H_{ji})| \leq \sum_{1\leq t\leq s} \|D^{S_j+e_t} H_{ji}\| \cdot |\bar{\theta}_{j+t} - \theta_{j+t}|$$

$$\leq \sum_{1\leq t\leq s} 4^{|S_j|+1} \cdot ((|S_j|+1)!)^2 \cdot \|\phi_{n-1}\|_{|S_j|+1} \cdot P^{|S_j|+1}_{i-j+|S_j|+1} \cdot \lambda^{-(i-j)+2(|S_j|+1)} \cdot g_j(1) \cdot \lambda^{-2(r^+_{n-1}(1-3\epsilon)-(j+t))}$$

$$\leq \|\phi_{n-1}\|_{|S_j|+1} \cdot (8s(|S_j|+1))^{|S_j|+3} \cdot \lambda^{-r^+_{n-1}+2(k+1)} \cdot g_j(1).$$

In the above, we use the fact thats $j + t < \frac{2r^+_{n-1}}{3}$ and $\epsilon < \frac{1}{10}$, $\lambda \gg 1$.

Consequently, we obtain

$$E_0 \quad \leq \|\phi_{n-1}\|_{k+1} \cdot (k+1)^{k+1} \cdot \lambda^{(k+1)^4} \cdot (r^+_{n-1})^{k+4} \cdot \lambda^{-r^+_{n-1}+2(k+1)} \cdot g_{j+s}(k+1).$$

From (A.22) and the inductive assumption for the case k, we have the following estimate:

$$|D^{K_{j+t}}(\bar{\theta}_{j+t} - \theta_{j+t})| \leq \left|\frac{\partial^{k_{j+t,1}}\bar{\theta}_{j+t}}{\partial x^{k_{j+t,1}}} \cdots \frac{\partial^{k_{j+t,s_{j+t}}}\bar{\theta}_{j+t}}{\partial x^{k_{j+t,s_{j+t}}}} - \frac{\partial^{k_{j+t,1}}\theta_{j+t}}{\partial x^{k_{j+t,1}}} \cdots \frac{\partial^{k_{j+t,s_{j+t}}}\theta_{j+t}}{\partial x^{k_{j+t,s_{j+t}}}}\right|$$

$$\leq \sum_{1\leq m\leq s_{j+t}} \left|\frac{\partial^{k_{j+t,1}}\theta_{j+t}}{\partial x^{k_{j+t,1}}} \cdots \frac{\partial^{k_{j+t,m}}(\bar{\theta}_{j+t}-\theta_{j+t})}{\partial x^{k_{j+t,m}}} \cdots \frac{\partial^{k_{j+t,s_{j+t}}}\bar{\theta}_{j+t}}{\partial x^{k_{j+t,s_{j+t}}}}\right|$$

$$\leq \sum_{1\leq m\leq s_{j+t}} \left(\prod_{r\neq m} (k_{j+t,r})^{k_{j+t,r}} \cdot \lambda^{k^4_{j+t,r}} \cdot g_{j+t}(k_{j+t,r}) \cdot \|\phi_{n-1}\|_{k_{j+t,r}}\right) \cdot$$

$$\|\phi_{n-1}\|_{k_{j+t,m}} \cdot (r^+_{n-1})^{4k^2_{j+t,m}} \cdot \lambda^{-s+2k_{j+t,m}} \cdot (k_{j+t,m})^{k_{j+t,m}} \cdot \lambda^{k^4_{j+t,m}} \cdot g_{j+t}(k_{j+t,m})$$

$$\leq |K_{j+t}|^{|K_{j+t}|} \cdot \lambda^{|K_{j+t}|^4} \|\phi_{n-1}\|_{|K_{j+t}|} \cdot (r^+_{n-1})^{4k^2_{j+t,m}} \cdot \lambda^{-s+2k_{j+t,m}} \cdot g_{j+t}(|K_{j+t}|).$$

It, together with (A.28) and (A.22), implies if $k < \min\{l, r^{+\frac{1}{10}}_{n-1}\}$,

$$E_t \quad \leq s \cdot 4^{|S_j|} \cdot (|S_j|!)^2 \cdot \|\phi_{n-1}\|_{|S_j|} \cdot P^{|S_j|}_{s+|S_j|} \cdot \lambda^{2|S_j|} \cdot g_j(1) \cdot$$

$$\|\phi_{n-1}\|_{k+1-|K_{j+t}|} \cdot (k+1-|K_{j+t}|)^{k+1-|K_{j+t}|} \cdot \lambda^{(k+1-|K_{j+t}|)^4} \cdot g_{j+s}(k+1-|K_{j+t}|) \cdot$$

$$|K_{j+t}|^{|K_{j+t}|} \cdot \lambda^{|K_{j+t}|^4} \|\phi_{n-1}\|_{|K_{j+t}|} \cdot (r^+_{n-1})^{4k^2_{j+t,m}} \cdot \lambda^{-s+2k_{j+t,m}} \cdot g_{j+t}(|K_{j+t}|)$$

$$\leq \|\phi_{n-1}\|_{k+1} \cdot (8(s+k+1))^{4|S_j|} \cdot (r^+_{n-1})^{4k^2_{j+t,m}} \cdot (k+1)^{k+1} \cdot \lambda^{(k+1)^4} \cdot g_{j+s}(k+1) \cdot \lambda^{-s+2k_{j+t,m}+2|S_j|}.$$

Since $k_{i+t,r} \geq 1$ for any t, r, it holds that

$$k_{i+t,m} + s_{i+t} - 1 \leq k_{i+t,m} + \sum_{1\leq r\leq s_{i+t}, r\neq m} k_{i+t,r} = |K_{i+t}|.$$

Moreover, for any $1 \le u \le s$, $s_{i+u} \le |K_{i+u}|$. Consequently, from $s_i + \sum_{1\le u\le s} |K_{i+u}| \le k$, we have

$$
\begin{aligned}
|S_i| + k_{i+t,m} - 1 &= s_i + \sum_{1\le u\le s, u\ne t} s_{i+u} + s_{i+t} + k_{i+t,m} - 1 \\
&\le s_i + \sum_{1\le u\le s, u\ne t} |K_{i+u}| + |K_{i+t}| \\
&\le s_i + \sum_{1\le u\le s} |K_{i+u}| \le k.
\end{aligned}
$$

Thus from the fact that $8(s+k+1) < r^+_{n-1}$ and $|S_j|, k_{i+t,m} \le k$, we have

$$E_t \le \|\phi_{n-1}\|_{k+1} \cdot (r^+_{n-1})^{4(k^2+k)} \cdot (k+1)^{k+1} \cdot \lambda^{(k+1)^4} \cdot g_{j+s}(k+1) \cdot \lambda^{-s+2(k+1)}.$$

Then

$$
\begin{aligned}
&\textstyle\sum_{J_i}(E_0 + \sum_{1\le t\le s} E_t) \\
&\le P^k_{s+k} \cdot [\|\phi_{n-1}\|_{k+1} \cdot (k+1)^{k+1} \cdot \lambda^{(k+1)^4} \cdot (r^+_{n-1})^{k+4} \cdot \lambda^{-r^+_{n-1}+2(k+1)} \cdot g_{j+s}(k+1) \\
&+ s\|\phi_{n-1}\|_{k+1} \cdot (r^+_{n-1})^{4(k^2+k)} \cdot (k+1)^{k+1} \cdot \lambda^{(k+1)^4} \cdot g_{j+s}(k+1) \cdot \lambda^{-s+2(k+1)}] \\
&\le \tfrac{1}{2}\|\phi_{n-1}\|_{k+1} \cdot (r^+_{n-1})^{4(k+1)^2} \cdot \lambda^{-s+2(k+1)} \cdot (k+1)^{k+1} \cdot \lambda^{(k+1)^4} \cdot g_{s+j}(k+1),
\end{aligned}
$$

where we use the fact that $s \cdot P^k_{s+k} \le (s+k)^{k+1} \le r^{+4k}_{n-1}$. It, together with (A.29), implies (A.27) for the case $k+1$. Thus we finish the proof of Lemma A.9.

□

A.4 Proof of Lemma 5.1 and 5.3

Proof of Lemma 5.1 Define

$$\psi(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

Let

$$w_1 = \begin{cases} w_0(x), & x \le 0 \\ w_0(-x), & x > 0, \end{cases}$$

where $w_0(x) = \frac{\psi(x+2)}{\psi(x+2)+\psi(-x-1)}$.

Then we define f_n be a π-periodic function such that

$$f_n(x) = w_1(10q_n^2(x - c_1)), \quad x \in [c_1 - \frac{\pi}{2}, c_1 + \frac{\pi}{2}].$$

We will check f_n satisfy (5.5) and (5.6). Without loss of generality, we assume $x - c_1 \leq 0$. Then

$$f_n(x) = \frac{\psi(10q_n^2(x-c_1)+2)}{\psi(10q_n^2(x-c_1)+2)+\psi(-10q_n^2(x-c_1)-1)}. \tag{A.30}$$

If in addition $|x-c_1| \leq \frac{1}{10q_n^2}$, then $-1 \leq -10q_n^2(x-c_1)-1 \leq 0$. Thus $\psi(-10q_n^2(x-c_1)-1)=0$, which implies $f_n(x)=1$.

For $x \in [\frac{\pi}{2}-c_1, \frac{\pi}{2}+c_1]\backslash I_n$, $|10q_n^2(x-c_1)| \geq 10$. Then for $x-c_1 \leq 0$, it holds that $10q_n^2(x-c_1)+2 \leq -8$, which implies $\psi(10q_n^2(x-c_1)+2)=0$. Hence $f_n(x)=0$.

Combining these with the fact that $0 \leq w_0(x) \leq 1$ for any x, we obtain (5.5).

To deal with (5.6), we first estimate $\psi^{(r)}(x)$ for $r \in \mathbb{N}$. Obviously, $\psi^{(r)}(0)=0$. For $x \neq 0$, by direct computations, we have

$$\begin{aligned} |\psi^{(r)}(x)| &= |\textstyle\sum_{l_1+\cdots+l_s=r} e^{-\frac{1}{x^2}} \cdot (-x^{-2})^{(l_1)} \cdots (-x^{-2})^{(l_s)}| \\ &\leq \textstyle\sum_{l_1+\cdots+l_s=r}(l_1+1)!\cdots(l_s+1)! \cdot e^{-\frac{1}{x^2}} \cdot x^{-(2s+l_1+\cdots+l_s)} \\ &\leq r! \cdot (2r)! \cdot e^{-\frac{1}{x^2}} \cdot x^{-3r} \leq ((2r)!)^2 e^{-\frac{1}{x^2}} \cdot x^{-3r}. \end{aligned} \tag{A.31}$$

In the last inequality, we use the facts that the number of terms in the sum is not more than $r!$ and that $k_1! \cdot k_2! \leq (k_1+k_2)!$.

Next we estimate the maximum of the function $\psi_r(x) = e^{-\frac{1}{x^2}} \cdot x^{-3r}$ for $x > 0$.

Let

$$\psi_r'(x) = (2x^{-3} - 3rx^{-1}) \cdot \psi_r(x) = 0.$$

We obtain the unique extreme point

$$x_r^* = (\frac{2}{3r})^{\frac{1}{2}}. \tag{A.32}$$

Since $\psi_r(x) \to 0$ as x tends to 0 or ∞, x_r^* is the unique maximum point for ψ_r on $x > 0$. It is easy to see that

$$|\psi_r(x_r^*)| = e^{-\frac{3r}{2}} \cdot (\tfrac{2}{3r})^{-\frac{3}{2}r} \leq r^{2r}.$$

Thus we obtain

$$|\psi^{(r)}(x)| \leq ((2r)!)^2 \cdot r^{2r} \leq (2r)^{6r}. \tag{A.33}$$

From the definition, we have $f_n^{(r)}(x) = (10q_n^2)^r \cdot w_0^{(r)}(y)$ with $y = 10q_n^2(x-c_1)$. From the fact that w_1 is even, we only need to consider $y \leq 0$. for $y \leq -2$, $\psi(y+2)=0$ or equivalently $w_0(y)=0$, it is sufficient to consider the situation $-2 \leq y \leq 0$.

If $y \in [-2, -\frac{3}{2}]$, it holds that $-y-1 \in [\frac{1}{2}, 1]$, which implies that $\psi(-y-1) \geq \min_{y\in[\frac{1}{2},1]}$. Otherwise, if $y \in [-\frac{3}{2}, 0]$, we have $y+2 \in [\frac{1}{2}, 2]$, then $\psi(y+2) \geq \min_{y\in[\frac{1}{2},2]} \psi(y)$. In conclusion, we obtain

$$\min_{y\in[-2,0]}(\psi(y+2)+\psi(-y-1)) \geq \min_{y\in[\frac{1}{2},2]} \psi(x) = e^{-4}. \tag{A.34}$$

Thus

$$|w_0^{(r)}(y)| \le \sum_{|R|=r} |\psi_R(y)|,$$

where $R = (r_1, l_1, \cdots, l_s)$ and

$$\psi_R(y) = (\psi_2 + \psi_{-1})^{-(1+s)} \cdot \psi_2^{(r_1)} \cdot (\psi_2 + \psi_{-1})^{(l_1)} \cdots (\psi_2 + \psi_{-1})^{(l_s)}$$

with $\psi_2 = \psi(y+2)$ and $\psi_{-1} = \psi(-y-1)$.

From (A.33) and (A.34), we have

$$\begin{aligned} |\psi_R| &\le e^{4(1+s)} \cdot (2r_1)^{6r_1} \cdot 2^s \cdot \prod_{1\le i\le s} (2l_i)^{6l_i} \\ &\le e^{4(1+s)} \cdot 2^s \cdot (2r)^{6r} \le (8r)^{6r}. \end{aligned}$$

Thus $|w_0^{(r)}(y)| \le (r+1)! \cdot (8r)^{6r}$, which leads that

$$|f_n^{(r)}(x)| \le (10 \cdot q_n)^{2r} \cdot (r+1)! \cdot (8r)^{6r} \le (q_n)^{2r} \cdot (8r)^{8r} \le (q_n)^{3r}$$

if $r \le [q_n^{\frac{1}{10}}]$. □

Proof of Lemma 5.3 Similar to (A.31) in the proof of Lemma 5.1, we obtain that

$$|\phi_0^{(r)}(x)| \le ((2r)!)^2 e^{-\frac{1}{x^a}} \cdot x^{-3r}.$$

From (A.32), $x_r^* = (\frac{2}{3r})^{\frac{1}{2}}$ is the unique extreme point for the function $e^{-\frac{1}{x^2}} \cdot x^{-3r}$. Since for $0 \le r \le [q_n^a]$, it holds that $x_r^* > q_n^{-2}$. Thus on I_n, if $n \gg 1$, we have

$$|\phi_0^{(r)}(x)| \le ((2r)!)^2 e^{-\frac{1}{q_n^{-2a}}} \cdot q_n^{6r} \le e^{-\frac{q_n^{2a}}{4}}.$$

□

A.5 Proof of Lemma 5.2

First we have the following estimate for $\|\phi_n\|_k$.

Lemma A.10 *For any $n, k \in \mathbb{N}$ with $n \ge N$, it holds that*

$$\|\phi_n\|_k \le ((3k)! \cdot k^k \cdot \lambda^{k^4})^{n-N+1} \cdot \prod_{t=N}^{n} q_t^{3k} \cdot \|\phi_0\|_k. \tag{A.35}$$

Proof. Let L_k be the set for all integer vectors $K = (k_1, \cdots, k_m)$ with $m \ge 1$, $k_1, \cdots, k_m \ge 1$ and $|K| = k$.

For $n = N$, from (A.20) and Lemmas 5.1 and A.7 it is easy to see that

$$\begin{aligned}
\|\phi_N\|_k &= \max_{K\in L_k}\prod_{1\le i\le m}\|\phi_N\|_{\mathbb{C}^{k_i}} \le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_0\|_{\mathbb{C}^{k_i}} + \|\phi_N-\phi_0\|_{\mathbb{C}^{k_i}})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_0\|_{\mathbb{C}^{k_i}} + \|(s_N-s_0)\cdot f_N\|_{\mathbb{C}^{k_i}})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_0\|_{\mathbb{C}^{k_i}} + \|\phi_0\cdot f_N\|_{\mathbb{C}^{k_i}} + \|s_N\cdot f_N\|_{\mathbb{C}^{k_i}})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(1+(k_i+1)!\cdot q_N^{3k_i}(1+k_i^{k_i}\cdot\lambda^{k_i^4}))\|\phi_0\|_{k_i}\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(k_i+2)!\cdot q_N^{3k_i}\cdot k_i^{k_i}\cdot\lambda^{k_i^4}\|\phi_0\|_{k_i}\\
&\le (3k)!\cdot k^k\cdot\lambda^{k^4}\cdot q_N^{3k}\cdot\|\phi_0\|_k.
\end{aligned}$$

Thus we prove (A.35) for the case $n = N$.
Assume (A.35 holds true for the cases $N, \cdots, n$, we will prove it holds for the case $n+1$.
From (A.20), the inductive assumption and Lemmas 5.1 and A.7, we have

$$\begin{aligned}
&\|\phi_{n+1}\|_k = \max_{K\in L_k}\prod_{1\le i\le m}\|\phi_{n+1}\|_{\mathbb{C}^{k_i}} \le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_n\|_{\mathbb{C}^{k_i}} + \|\phi_{n+1}-\phi_n\|_{\mathbb{C}^{k_i}})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_n\|_{\mathbb{C}^{k_i}} + \|(s_{n+1}-s_n)\cdot f_{n+1}\|_{\mathbb{C}^{k_i}})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_n\|_{\mathbb{C}^{k_i}} + \|s_n\cdot f_{n+1}\|_{\mathbb{C}^{k_i}} + \|s_{n+1}\cdot f_{n+1}\|_{\mathbb{C}^{k_i}})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_n\|_{k_i} + (k_i+1)!\cdot q_{n+1}^{3k_i}\cdot\|s_n\|_{\mathbb{C}^{k_i}} + (k_i+1)!\cdot q_{n+1}^{3k_i}\cdot\|s_{n+1}\|_{\mathbb{C}^{k_i}})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}(\|\phi_n\|_{k_i} + (k_i+1)!\cdot q_{n+1}^{3k_i}\cdot k_i^{k_i}\cdot\lambda^{k_i^4}\|\phi_{n-1}\|_{k_i} + (k_i+1)!\cdot q_{n+1}^{3k_i}\cdot k_i^{k_i}\cdot\lambda^{k_i^4}\|\phi_n\|_{k_i})\\
&\le \max_{K\in L_k}\prod_{1\le i\le m}3(k_i+1)!\cdot q_{n+1}^{3k_i}\cdot k_i^{k_i}\cdot\lambda^{k_i^4}(\|\phi_n\|_{k_i} + \|\phi_{n-1}\|_{k_i})\\
&\le \tfrac{1}{2}\cdot(3k)!\cdot k^k\cdot\lambda^{k^4}\cdot q_{n+1}^{3k}\cdot(\|\phi_n\|_{k_i} + \|\phi_{n-1}\|_{k_i}) \le ((3k)!\cdot k^k\cdot\lambda^{k^4})^{n-N+2}\cdot\prod_{t=N}^{n+1}q_t^{3k}\cdot\|\phi_0\|_k.
\end{aligned}$$

Thus we complete the proof. □

Now we prove Lemma 5.2. For any fixed $k \ge 1$, we take $n_0(k)$ such that $(r_{n-1}^+)^{\frac{1}{10}} > k$ if $n \ge n_0(k)$.

From the definition of ϕ_0 in section 5, it follows that

$$|g_{s+i}(k)| \le \exp(4M^2(s+i)^{+2a}\cdot 2c_7k^2) \le \lambda^{2(s+i)^{2a}\cdot k^2}$$

if $\lambda, n \gg 1$. Since $2a < 1$, from the definition of s and k, we know that

$$(r_{n-1}^+)^{3k(r_{n-1}^+)^{\frac{1}{10}}}\cdot\lambda^{2k+2s^{2a}\cdot k^2}\cdot k^k\cdot\lambda^{k^4} \le \lambda^{\frac{1}{4}s} \le \lambda^{\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}}.$$

if $\lambda, n \gg 1$.

Taking $i=0$ and $l=\infty$ in Lemma A.9, we have

$$|\tfrac{d^k s_n}{dx^k} - \tfrac{d^k s_{n-1}}{dx^k}| = |\tfrac{d^k \theta_0}{dx^k} - \tfrac{d^k \bar{\theta}_0}{dx^k}|$$

$$\leq \|\phi_{n-1}\|_k \cdot \lambda^{\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}} \cdot \lambda^{-s}$$

$$\leq \|\phi_{n-1}\|_k \cdot \lambda^{-\frac{2}{3}(r_{n-1}^+)^{\frac{2}{3}}}.$$

if $\lambda, N \gg 1$. Then by Lemma A.10, we have

$$|\frac{d^k s_n}{dx^k} - \frac{d^k s_{n-1}}{dx^k}| \leq ((3k)! \cdot k^k \cdot \lambda^{k^4})^{n-N+1} \cdot \prod_{t=N}^{n} q_t^{3k} \cdot \|\phi_0\|_k \cdot \lambda^{-\frac{2}{3}(r_{n-1}^+)^{\frac{2}{3}}}$$

if $\lambda, N \gg 1$.

Since $r_{n-1}^+ \geq q_{n-1} \geq (\sqrt{2})^{n-1}$, it follows that $n \leq \log_{\sqrt{2}} r_{n-1}^+ + 1$. On the other hand, since $q_{n+1} \leq M \cdot q_n$, one sees that $q_n \leq M^n$. Then it follows that

$$\prod_{t=N}^{n} q_t \leq \prod_{t=N}^{n} M^t \leq M^{n^2}.$$

Combining these with $\lambda, n \gg 1$ and $k \leq (r_{n-1}^+)^{\frac{1}{10}}$, one sees that

$$((3k)! \cdot k^k \cdot \lambda^{k^4})^{n-N+1} \cdot \textstyle\prod_{t=N}^{n} q_t^{3k}$$

$$\leq ((3k)! \cdot k^k \cdot \lambda^{k^4})^{\log_{\sqrt{2}}(r_{n-1}^+ + 1)} \cdot M^{6k \log_{\sqrt{2}} r_{n-1}^+ + 1)}$$

$$\leq \lambda^{\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}},$$

which implies that

$$|\frac{d^k s_n}{dx^k} - \frac{d^k s_{n-1}}{dx^k}| \leq \lambda^{-\frac{1}{3}(r_{n-1}^+)^{\frac{2}{3}}} \cdot \|\phi_0\|_k.$$

We conclude Lemma 5.2.

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