Reducibility of Slow Quasi−Periodic Linear Systems*

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Abstract

In this note, we prove that the reducibility of analytic quasi-periodic linear systems close to the identity is irrelevant to the size of the base frequencies. More precisely, we consider the quasi-periodic linear systems

$$\dot{X} = (A + B(\theta))X, \quad \dot{\theta} = \lambda^{-1}\omega$$

in $\mathbb{C}^m$ where the matrix A is constant and ω is a fixed Diophantine vector, $\lambda \in \mathbb{R}\backslash\{0\}$. We prove that the system is reducible for typical A if $B(\theta)$ is analytic and sufficiently small (depending on A, ω but not on λ).

1 Introduction and Main Result

Consider quasi-periodic (or q-p for short) linear differential systems close to constant,

$$\dot{X} = (A + B(\theta))X, \quad \dot{\theta} = \omega, \tag{1.1}$$

where A is a $m \times m$ constant matrix, $B(\theta)$ is a small analytic $m \times m$ matrix defined on $\mathbb{T}^n$, the frequencies $\omega = (\omega_1, \cdots, \omega_n)$ are rational independent.

A typical example of q-p linear systems comes from the (continuous-time) q-p Schrödinger operators, which are defined on $L^2(\mathbb{R})$ as

$$(\mathcal{L}y)(t) = -y''(t) + q(\theta + \omega t)y(t),$$

where $q : \mathbb{T}^n \to \mathbb{R}$ is called the potential and $\theta \in \mathbb{T}^n$ is called the phase. It is well-known that the spectrum of $\mathcal{L}$ does not depend on the phase when ω is rational independent, but it is closely related to the dynamics of the Schrödinger equations

$$(\mathcal{L}y)(t) = -y''(t) + q(\theta + \omega t)y(t) = Ey(t), \tag{1.2}$$

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or equivalently the dynamics of the linear systems

$$\dot{X} = V_{E,q}(\theta)X, \quad \dot{\theta} = \omega, \tag{1.3}$$

where

$$V_{E,q}(\theta) = \begin{pmatrix} 0 & 1 \\ q(\theta) - E & 0 \end{pmatrix} \in sl(2, \mathbb{R}).$$

System (1.1) is said to be *reducible*, if there exists a $gl(m, \mathbb{C})$-valued function P defined on $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ such that the change of variables $x \mapsto P(\theta)x$ transforms system (1.1) into a constant system, i.e., a linear system with constant coefficient (or we say that P conjugates system (1.1) to a constant linear system and P is called the conjugation map). If P is C^r (or analytic), we say that system (1.1) is C^r(or analytically) reducible. It is equivalent to say that $P^{-1}((A + B(\theta))P - \partial_\omega P)$ is a constant matrix.

Due to the importance in the theory of dynamical systems and the spectrum theory of the corresponding operators, the reducibility problem of q-p linear systems has received much attention. Floquet theory shows that the periodic linear systems (i.e., n=1) are always reducible, but it is not the case for quasi-periodic linear systems (see [11]).

The reducibility of q-p linear systems (1.1) was initiated by Dinaburg and Sinai [4], who proved that the linear systems (1.3) are reducible for “most” $E > E^*(q, \alpha, \tau)$ which is sufficiently large, if ω is fixed and satisfies the Diophantine condition:

$$|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^n,$$

α, τ are positive constants. The result was generalized by Rüssmann [15] for ω satisfing the Bruno condition. The reducibility of q-p linear systems with coefficients in $gl(n, \mathbb{R})$ was considered by Jorba and Simó [10].

Eliasson [5] proved a full measure reducibility result for q-p linear Schrödinger equations. More precisely, Eliasson proved that (1.3) is reducible for almost all $E > E^*(q, \omega)$ in Lebesgue measure sense, where ω is a fixed Diophantine vector. All the above mentioned results hold for more general systems (1.1) with B sufficiently small. We emphasize that all the above results are perturbative, i.e., E^* (or the smallness of B in (1.1)) depends on the frequency ω through its Diophantine constant α. In case that the frequency is of the form $\frac{1}{\lambda}\omega$, $\alpha \to 0$ as $\lambda \to \infty$. To get the reducibility result, the size of the perturbation has to go to zero when $\lambda \to \infty$.

An example by Bourgain [3] proves that the Eliasson's perturbative reducibility result is optimal, i.e., the size of the perturbation does depend on the frequencies some how. In this paper, we will prove that the reducibility does not depend on the size of the base frequencies. More precisely, we prove a reducibility result for (1.1) no matter how small the ω is.

In case that $n = 2$, stronger reducibility result called non-perturbative reducibility is available. The non-perturbative reducibility means that the smallness of the perturbation does not depend on the Diophantine constant α. Hou and You [9] proved, besides other results, that non-perturbative reducibility for (1.3). The non-perturbative reducibility and global reducibility of q-p linear mappings were given by Avila and Krikorian [2] , Avila and Jitormirskaya [1]. For more results, see [6], [8], [13], [14].

In this paper, we consider the following family of quasi-periodic skew-product systems with $n \geq 2$

$$\dot{X} = (A + B(\theta))X, \quad \dot{\theta} = \frac{\omega}{\lambda}, \tag{1.4}$$

where $\theta \in \mathbb{T}^n, X \in \mathbb{C}^m, \lambda \in \mathbb{R}\backslash\{0\}$, A is a constant $m \times m$ matrix, $B(\theta)$ is analytic and sufficiently small which does not depend on λ, $\omega = (\omega_1, \cdots, \omega_n)$ is fixed and satisfies the Diophantine condition

$$|\langle k, \omega\rangle| \geq \frac{\alpha}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^n, \tag{1.5}$$

α, τ are positive constants. Scaling the time, systems (1.4) are equivalent to the following systems

$$\dot{X} = \lambda(A + B(\theta))X, \quad \dot{\theta} = \omega. \tag{1.6}$$

For the sake of simplicity, in the following, we denote by $|A|$ the determinant of a $m \times m$ matrix $A = (a_{ij})$, by $\|A\|$ its operator norm which is equivalent to $m \cdot \max|a_{ij}|$. Denote by $\|(v_1, \cdots, v_m)\| = \max_{1\leq j\leq m} |v_j|$ the norm for vectors. For $k \in \mathbb{Z}^n$, denote its module by $|k| = |k_1| + \cdots + |k_n|$. $[a]$ denotes the integer part of a number a. If f is a function, $|f|$ denotes its absolute value. Through this note, we use c to designate positive constant which may take different values when its actual value does not matter.

Suppose that $B(\theta)$ is an analytic gl-valued function defined on

$$W_h(\mathbb{T}^n) = \{\theta \in \mathbb{C}^n | dist(\theta, \mathbb{T}^n) < h\}.$$

Let

$$\|B_{ij}(\theta)\|_h = \sum_{k\in\mathbb{Z}^n} |B_{kij}|e^{|k|h},$$

where

$$B_{ij}(\theta) = \sum_{k\in\mathbb{Z}^n} B_{kij}e^{i\langle k,\theta\rangle}.$$

We will give a reducibility result for typical A. Since the eigenvalues of typical matrices are mutually different and the hyperbolic case is trivial, we consider the case $A = \sqrt{-1}\text{diag}(\mu_1, \mu_2, \cdots, \mu_m)$ without loss of generality. For simplicity, we let $\mu_1, \cdots, \mu_m \in [1, 2]$.

Now we are in the position to state the main result.

Theorem 1 *Suppose that the frequency ω is fixed and satisfies the Diophantine condition (1.5), $\lambda \in \mathbb{R}\backslash\{0\}$ is fixed. Then there exists ε depending on α, h but not on λ such that if $\|B\|_h < \varepsilon$, the system (1.4) is reducible for $(\mu_1, \cdots, \mu_m) \in \mathcal{O}_\lambda \subset [1, 2]^m$, where the measure of $\mathcal{O}_\lambda$ is larger than $1 - c\varepsilon^{\frac{1}{3}}$ for all λ.*

Another variant of Theorem 1 is the following:

Theorem 2 *Suppose that the frequency ω is fixed and satisfies the Diophantine condition (1.5), $A = \sqrt{-1}\mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_m)$ is fixed with μ_i, μ_j different, $\lambda \in \mathbb{R}\backslash\{0\}$. Then there exists ε depending on α, h, A such that if $\|B\|_h < \varepsilon$, the system (1.4) is reducible for all $\lambda \in \mathbb{R}\backslash\{0\}$, but a small set of measure $O(\varepsilon^{\frac{1}{3}})$.*

Remark 1.1 *We thank H. Eliasson for pointing out that Theorem 2 may of more interest in some case.*

Remark 1.2 *Applying Theorem 1 to (1.3), we get a reducibility result for the Schrödinger systems with arbitrarily small frequencies. In the previous results, the smallness of the perturbation does depend on the size of the frequencies, more precisely on λ. Thus the size of the perturbation is not uniform for λ.*

Remark 1.3 *In the case that $|\lambda| \le 1$, the reducibility of (1.4) is covered by the previous result. In this paper, we merely consider the case $|\lambda| \ge 1$. When λ is large, the perturbation $\lambda B(\theta)$ in (1.6) can be large since $B(\theta)$ is independent of λ, which is out of scope of the previous results. Actually, the main innovation of this note is the reducibility for arbitrarily large λ.*

Remark 1.4 *In Theorem 1, the set $\mathcal{O}_\lambda$ does depend on λ, but the lower bound of the measure of $\mathcal{O}_\lambda$ does not depend on λ. In fact, $m(\mathcal{O}_\lambda) = 1 - \min\{O(\varepsilon^{\frac{1}{3}}), O(|\lambda|^{\frac{1}{2}} e^{-a|\lambda|}\varepsilon^{\frac{1}{2}})\}$ where a is a positive number depending on ε. It is easy to see that $m(\mathcal{O}_\lambda)$ tends to 1 as $\lambda \to \infty$.*

2 Outline of the Proof

To prove the reducibility, it is equivalent to find a change of variables $X \to P(\theta)X$, such that the transformed system

$$\dot{X} = (L_\omega P \cdot P^{-1} + \lambda P(A(\theta) + B(\theta))P^{-1})X, \quad \dot{\theta} = \omega$$

is a linear system with constant coefficients, where $L_\omega = \langle \omega, \frac{\partial}{\partial \theta} \rangle$ is the derivative along ω.

We will try to find a transformation close to the identity to finish the job, i.e., we assume that

$$P(\theta) = I + F(\theta),$$

where $F(\theta) \in C_h^\omega(\mathbb{T}^n, g)$ is small. In this case, P^{-1} can be expanded as

$$P^{-1} = I - F + F^2 + O(\|F\|^3).$$

It follows that

$$\begin{aligned}
& L_\omega P\cdot P^{-1}+\lambda P(A(\theta)+B(\theta))P^{-1}\\
= \ & \lambda A(\theta)+L_\omega F+\lambda[F,A(\theta)]+\lambda B(\theta)\\
- \ & L_\omega F\cdot F+\lambda[F,B(\theta)]-\lambda FA(\theta)F+\lambda A(\theta)F^2+\lambda O(\|F\|^3). \qquad (2.1)
\end{aligned}$$

We will find F by Newtonian iteration scheme. Firstly, we solve the linearized equation

$$L_\omega F+\lambda[F,A(\theta)]+\lambda B(\theta)=0, \qquad (2.2)$$

where $[F,A]=FA-AF$. Since A is diagonal, (2.2) is decomposed as

$$L_\omega F_{ij}+\lambda(\mu_j-\mu_i)F_{ij}+\lambda B_{ij}(\theta)=0. \qquad (2.3)$$

One sees that the above equation does not admit a small solution when $i=j$. Our strategy is to find a F such that the off-diagonal terms of the transformed system are smaller, while the diagonal terms of the matrix $B(\theta)$ are kept unsolved and moved to A. The price we have to pay is, from the second step, $A=\sqrt{-1}\mathrm{diag}(\mu_1+a_1(\theta),\cdots,\mu_m+a_m(\theta))$ will depend on θ. Then we have to solve θ-dependent homological equations to keep the Newtonian iteration working. This is the key point in our proof. After finitely many iteration steps (depending on λ), we get a linear system of the form

$$\dot X=\lambda(\widetilde A(\theta)+\widetilde B(\theta))X, \quad \dot\theta=\omega,$$

where $\widetilde A(\theta)$ is diagonal, $\|\widetilde A(\theta)-A\|_h\le 3\varepsilon$, $\|\widetilde B(\theta)\|_{\frac{h}{2}}\le c\varepsilon e^{-c|\lambda|h}$. To this stage, we can remove the θ-dependent terms in $\widetilde A(\theta)$ and get a system

$$\dot X=(\lambda A^*+B^*(\theta))X, \quad \dot\theta=\omega,$$

with a constant matrix A^* and a sufficiently small $\|B^*(\theta)\|_{\frac{h}{2}}<c\varepsilon$. Then we use a standard result to get the reducibility.

3 Key Lemmas

We need the following Lemmas:

Lemma 3.1 *Suppose that a_0 is a non-zero constant, $[a(\theta)]=0, \|a(\theta)\|_h<\varepsilon_0\le \frac{a_0^2}{2(|\omega|+|a_0|)}$ and $\|b(\theta)\|_h<\varepsilon$. Then the equation*

$$L_\omega F+\lambda(a_0+a(\theta))F+\lambda b(\theta)=0 \qquad (3.1)$$

has an approximating solution F^* such that

$$\|F^*\|_h<\frac{2(|\omega|+|a_0|)}{a_0^2}\varepsilon \qquad (3.2)$$

and

$$L_\omega F^* + \lambda(a_0 + a(\theta))F^* + \lambda b(\theta) = \lambda \bar{b}(\theta) \tag{3.3}$$

with

$$\|\bar{b}(\theta)\|_{h-\sigma} < c\varepsilon e^{-\frac{|a_0|\sigma}{|a_0|+|\omega|}|\lambda|}.$$

Proof: Rewrite the linear operator $L_\omega + \lambda(a_0 + a(\theta))$ as an infinite dimensional matrix L, and rewrite F, a, b as infinite dimensional vectors defined by their Fourier coefficients. Let

$$L_N = R_{[-N,N]} L R_{[-N,N]}, \quad b_N = R_{[-N,N]} b,$$

where $R_{[-N,N]}$ is the coordinate restriction to $[-N, N]$, $N \in \mathbb{N}^+$. Let $N = [\frac{|a_0|}{|\omega|+|a_0|}|\lambda|]$, then the diagonal of L_N is dominated, and thus $L_N^{-1} < \frac{2(|\omega|+|a_0|)}{a_0^2|\lambda|}$. It follows that $L_N F + \lambda b_N = 0$ has a solution $F_N = \lambda L_N^{-1} b_N$. The analytic function $F^*(\theta)$ corresponding to $(\lambda L_N^{-1} b_N, 0)$ is the desired approximating solution. It is obvious that

$$\|F^*\|_h < \frac{2(|\omega| + |a_0|)}{a_0^2}\|b\|_h$$

and

$$\|\bar{b}(\theta)\|_{h-\sigma} = \|R_{\mathbb{Z}\setminus[-N,N]}(a(\theta)F^* + b(\theta))\|_{h-\sigma} < c\varepsilon e^{-\frac{|a_0|\sigma}{|a_0|+|\omega|}|\lambda|}.$$

Lemma 3.2 (Refined Kuksin's Lemma) *Assume furthermore*

$$|i\langle k, \omega\rangle + \lambda a_0| \geq \frac{\gamma}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^n \tag{3.4}$$

and

$$\|a(\theta)\|_h < \varepsilon_0 \leq c\frac{|a_0|\alpha\sigma^{\tau+n}}{|\omega| + |a_0|}. \tag{3.5}$$

Then equation (3.1) has an solution F with

$$\|F\|_{h-2\sigma} < c\frac{\varepsilon}{\gamma\sigma^{\tau+n}}. \tag{3.6}$$

Proof: Let F^* be the solution of (3.3). In order to get a solution of (3.1), we further solve the following equation:

$$L_\omega \widetilde{F}(\theta) + \lambda(a_0 + a(\theta))\widetilde{F}(\theta) + \lambda\bar{b}(\theta) = 0. \tag{3.7}$$

Let

$$\tilde{a}(\theta) = \sum_{k\neq 0} \frac{a_k}{i\langle k, \omega\rangle} e^{i\langle k,\theta\rangle}.$$

By (1.5), one sees that $\tilde{a}(\theta)$ is well defined in the domain $W_h(\mathbb{T}^n)$.
Set

$$u(\theta) = e^{\lambda\tilde{a}(\theta)}\widetilde{F}(\theta)$$

and

$$G(\theta) = e^{\lambda \tilde{a}(\theta)} \bar{b}(\theta),$$

then (3.7) is changed into

$$L_\omega u + \lambda a_0 u + \lambda G(\theta) = 0.$$

Expanding $u(\theta), G(\theta)$ into the Fourier series, we get

$$(i\langle k, \omega\rangle + \lambda a_0) u_k + \lambda G_k = 0. \tag{3.8}$$

Solving the equation (3.8), we obtain:

$$u_k = \frac{-\lambda G_k}{i\langle k, \omega\rangle + \lambda a_0}.$$

In order to give desired estimate for $\widetilde{F}(\theta)$, we first estimate $\tilde{a}(\theta)$:

$$\begin{aligned} \|\tilde{a}(\theta)\|_{h-\sigma} &= \|\sum_{k\neq 0} \frac{a_k}{i\langle k, \omega\rangle} e^{i\langle k, \theta\rangle}\|_{h-\sigma} \\ &\leq \frac{\|a\|_h}{\alpha} (\sup_{|k|} |k|^\tau e^{-|k|\sigma}) \\ &\leq \frac{c\|a\|_h}{\alpha \sigma^{\tau+n-1}}, \end{aligned}$$

then

$$\begin{aligned} \|G(\theta)\|_{h-\sigma} &= \|e^{\lambda \tilde{a}(\theta)} \bar{b}(\theta)\|_{h-\sigma} \\ &\leq e^{\frac{c|\lambda|\|a\|_h}{\alpha\sigma^{\tau+n-1}}} \|\bar{b}(\theta)\|_{h-\sigma}. \end{aligned}$$

In view of (3.4), we have

$$\begin{aligned} \|u\|_{h-2\sigma} &= \sum_k |u_k| e^{|k|(h-2\sigma)} \\ &\leq \frac{c|\lambda|}{\gamma\sigma^{\tau+n-1}} e^{\frac{c|\lambda|\|a\|_h}{\alpha\sigma^{\tau+n-1}}} \|\bar{b}(\theta)\|_{h-\sigma}. \end{aligned}$$

It follows that

$$\begin{aligned} \|\widetilde{F}(\theta)\|_{h-2\sigma} &= \|e^{-\lambda \tilde{a}(\theta)} u(\theta)\|_{h-2\sigma} \\ &\leq c\frac{|\lambda|}{\gamma\sigma^{\tau+n-1}} e^{-|\lambda|(\frac{|a_0|}{|\omega|+|a_0|}\sigma - \frac{c\|a\|_h}{\alpha\sigma^{\tau+n-1}})}\varepsilon \qquad (3.9) \\ &\leq c\frac{\varepsilon}{\gamma\sigma^{\tau+n}}, \qquad (3.10) \end{aligned}$$

invoking (3.5). Moreover, from (3.2) and (3.10), we have (3.6)

$$\|F\|_{h-2\sigma} = \|F^* + \widetilde{F}\|_{h-2\sigma} \leq c\frac{\varepsilon}{\gamma\sigma^{\tau+n}}.$$

■

Remark 3.1 *The Lemma refines a result by Kuksin [12]. The proof is simpler. Using this Lemma, it is promising to get the quasi–periodic solutions of the derivative nonlinear Schrödinger equation.*

Consider the quasi-periodic system on $W_h(\mathbb{T}^n)$:

$$\dot{X} = \lambda(A + B(\theta))X, \quad \dot{\theta} = \omega, \tag{3.11}$$

where $A = \sqrt{-1}\mathrm{diag}(\mu_1 + a_1(\theta), \cdots, \mu_m + a_m(\theta))$. Assume that

$$\|a_1(\theta)\|_h, \cdots, \|a_m(\theta)\|_h < \varepsilon \ll \rho, \quad \|B\| < \varepsilon \ll \varepsilon_0.$$

Let $\Pi = \{(\mu_1, \mu_2, \cdots, \mu_m) \in [1,2]^m : |\mu_j - \mu_i| > \rho, 1 \leq i, j \leq m\}$ with $\rho = \varepsilon^{\frac{1}{3}}$. The measure of the set Π is larger than $(1 - c\varepsilon^{\frac{1}{3}})^m$.

Lemma 3.3 *There is a F with $\|F\|_h < c\varepsilon$ such that the change of variables $X \to (I + F(\theta))X$ transforms (3.11) into*

$$\dot{X} = \lambda(\widetilde{A}(\theta) + \widetilde{B}(\theta))X, \quad \dot{\theta} = \omega, \tag{3.12}$$

where

$$\widetilde{A}(\theta) = \sqrt{-1}\mathrm{diag}(\mu_1 + \tilde{a}_1(\theta), \cdots, \mu_m + \tilde{a}_m(\theta)),$$
$$\|\tilde{a}_i\|_h < 3\varepsilon, \quad \|\widetilde{B}\|_{\frac{h}{2}} < c\varepsilon e^{-\frac{\rho h|\lambda|}{2(\rho+2|\omega|)}}.$$

Proof: Applying Lemma 3.1, we get an approximating solution $F^*(\theta)$ for

$$L_\omega F_{ij} + \lambda[(\mu_i - \mu_j) + a_i(\theta) - a_j(\theta)]F_{ij} + \lambda B_{ij}(\theta) = 0, \quad i \neq j,$$

with estimate $\|F^*_{ij}\|_h < c\varepsilon$. Let $B_1 = (\bar{B}_{ij})_{i\neq j}$ where $\bar{B}_{ij} = L_\omega F_{ij} + \lambda[(\mu_i - \mu_j) + a_i(\theta) - a_j(\theta)]F_{ij} + \lambda B_{ij}(\theta)$. By Lemma 3.1, $\|B_1\|_{h-\sigma} < c\varepsilon e^{-\frac{\rho\sigma}{\rho+|\omega|}|\lambda|}$. Let

$$B_2 = -\frac{1}{\lambda}L_\omega F^* \cdot F^* + [F^*, B(\theta)] - F^*A(\theta)F^* + A(\theta)(F^*)^2 + O(\|F^*\|^3).$$

It is easy to see that $\|B_2\|_h < c\varepsilon^2$. In view of (2.1), the system (3.11) is transformed into

$$\dot{X} = \lambda(A_1(\theta) + B_1(\theta) + B_2(\theta))X, \quad \dot{\theta} = \omega, \tag{3.13}$$

by $X \to (I + F^*(\theta))X$, where $A_1(\theta) = A(\theta) + \mathrm{diag}(B_{11}(\theta), \cdots, B_{mm}(\theta))$. Applying Lemma 3.1 $[\log_2(-\frac{\rho h|\lambda|}{2(\rho+2|\omega|)\ln\varepsilon} + 1)] + 1$ times, we arrive at (3.12) with the estimate

$$\|\widetilde{B}(\theta)\|_{\frac{h}{2}} < c\varepsilon e^{-\frac{\rho h|\lambda|}{2(\rho+2|\omega|)}}, \quad \|\tilde{a}_i\|_h < 3\varepsilon, \quad \|F\|_h < c\varepsilon.$$

4 Eliminating the θ-dependent Terms in $\widetilde{A}(\theta)$

In this section, we transform (3.12) to

$$\dot{X} = (\lambda A^* + B^*(\theta))X, \quad \dot{\theta} = \omega, \tag{4.1}$$

where A^* is a constant matrix, $B^*(\theta)$ is small.

Lemma 4.1 *If $\varepsilon \leq c\frac{\alpha\rho h^{\tau+n}}{\rho+2|\omega|}$, then the system (3.12) can be transformed to the system (4.1) with*

$$A^* = \sqrt{-1}diag(\mu_1^*, \cdots, \mu_m^*), \quad \mu_j^* = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \mu_j + \tilde{a}_j(\theta)\ d\theta$$

and

$$\|B^*\|_{\frac{h}{2}} \leq c\varepsilon^{\frac{2}{3}}.$$

Proof: Let $\bar{a}_j(\theta) = \sum_{k\neq 0} \frac{(\widetilde{a}_j)_k}{i\langle k,\omega\rangle} e^{i\langle k,\theta\rangle}$ be the solution of $\partial_\omega \bar{a} = \widetilde{a}$, and

$$\bar{A}(\theta) = diag(\bar{a}_1(\theta), \cdots, \bar{a}_m(\theta)),$$

which is well defined by (1.5). The change of variables $X = e^{\lambda\bar{A}(\theta)}Y$ transforms (3.12) into

$$\dot{Y} = (\lambda A^* + B^*(\theta))Y, \quad \dot{\theta} = \omega,$$

where

$$B^*(\theta) = \lambda e^{\lambda\bar{A}(\theta)}\widetilde{B}(\theta).$$

In order to give the estimate of $B^*(\theta)$, we first estimate $\bar{A}(\theta)$

$$\begin{aligned} \|\bar{A}(\theta)\|_{\frac{h}{2}} &= m\cdot\max_{1\leq j\leq m}\|\sum_{k\neq 0}\frac{(\tilde{a}_j)_k}{i\langle k,\omega\rangle}e^{i\langle k,\theta\rangle}\|_{\frac{h}{2}} \\ &\leq \frac{m\cdot 3\varepsilon}{\alpha}(\sup_{|k|}|k|^\tau e^{-\frac{|k|h}{2}}) \\ &\leq \frac{c\varepsilon}{\alpha h^{\tau+n-1}}, \end{aligned}$$

then

$$\begin{aligned} \|B^*(\theta)\|_{\frac{h}{2}} &= \|\lambda e^{\lambda\bar{A}(\theta)}\widetilde{B}(\theta)\|_{\frac{h}{2}} \\ &\leq c|\lambda|e^{-|\lambda|(\frac{\rho h}{2(\rho+2|\omega|)}-\frac{c\varepsilon}{\alpha h^{\tau+n-1}})}\varepsilon. \end{aligned}$$

If $\varepsilon \leq \frac{c\alpha\rho h^{\tau+n}}{\rho+2|\omega|}$. It follows that

$$\|B^*(\theta)\|_{\frac{h}{2}} \leq c\frac{\rho+2|\omega|}{\rho h}\varepsilon \leq c\varepsilon^{\frac{2}{3}}.$$

∎

5 Proof of the Main Result

The change of variables $\Phi = e^{\lambda\bar{A}(\theta)} \circ (I+F)$ transforms the system (1.6) to (4.1), where F and $e^{\lambda\bar{A}(\theta)}$ are defined in Lemma 3.3 and Lemma 4.1.
Let

$$\sigma(A^*) = \{\sqrt{-1}\mu_j^* : j = 1, \cdots, m\}, \quad |\mu_j^* - \mu_j| \leq 3\varepsilon,$$

$$h^* = \frac{h}{2}, \quad \varepsilon^* = \|B^*\|_{h^*} \leq c\varepsilon^{\frac{2}{3}}.$$

Theorem 3 *Suppose that the frequency ω is fixed and satisfies the Diophantine condition (1.5). Then there exists a positive real number ε^* depending on h^*, such that if $\|B^*\|_{h^*} < \varepsilon^*$, the measure of the set of $(\mu_1^*, \cdots, \mu_m^*)$ for which the system (4.1) is non-reducible is less than $c(\varepsilon^*)^{\frac{1}{2}}$.*

Proof: See, i.g., [7]. ■

A direct application of Theorem 3 leads to the main result of this paper.

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