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Invariant tori for nearly integrable Hamiltonian systems with degeneracy*

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1. Introduction and main results

In this paper we study the following Hamiltonian systems

$$\frac{dp}{dt} = -H_q(p,q) = -\varepsilon f_q(p,q)\ , \quad \frac{dq}{dt} = H_p(p,q) = h_p(p) + \varepsilon f_p(p,q)\ , \qquad (1.1)$$

with Hamiltonian $H(p,q) = h(p) + \varepsilon f(p,q)$, $(p,q) \in \Omega \times T^n$, $n \geq 2$, where $p = (p_1, p_2, \ldots, p_n)$ are action variables varying over some bounded connected domain Ω in R^n, and $q = (q_1, q_2, \ldots, q_n)$ are conjugate angular variables whose domain T^n is the usual n-torus obtained by identifying the points whose components differ by integer multiples of 2π. Suppose $f(p,q)$ has period 2π in every component of q and $H(p,q)$ is analytic in $\bar{\Omega} \times T^n$, where $\bar{\Omega}$ is the closure of Ω.

For $\varepsilon = 0$ the unperturbed Hamiltonian $H(p,q) = h(p)$ is independent of q, and the equations of motion ar reduced to $\dot{p} = 0$, $\dot{q} = \omega$ with $\omega = h_p(p)$. They have a n-parameter family of invariant tori $\{p_0\} \times T^n$ for $\forall p_0 \in \Omega$ with constant frequencies $\omega(p_0) = (\omega_1(p_0), \omega_2(p_0), \ldots, \omega_n(p_0)) = h_p(p_0)$.

If $h(p)$ satisfies the usual nondegeneracy condition,

$$\det[h_{pp}(p)] \neq 0 \quad \text{for} \quad \forall\, p \in \Omega\ ,$$

or equivalently, $\mathrm{rank}(h_{pp}) = n$ for $\forall\, p \in \Omega$, the well known KAM theorem points out (see [E], [P1]): when ε is sufficiently small, the perturbed Hamiltonian systems (1.1) persist the majority of invariant tori with their frequencies ω satisfying the strong nonresonant conditions or small divisor conditions:

$$|\langle k, \omega\rangle| \geq \frac{\Delta}{|k|^\tau} \quad \text{for all} \quad 0 \neq k \in Z^n\ , \qquad (1.2)$$

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where $|k| = |k_1| + |k_2| + \ldots + |k_n|$, Δ is a small positive constant and $\tau > n - 1$. This result was first announced by Kolmogorov in 1954 [K] and the proof was given by V. Arnold in [A].

Recently, there has been a fair amount of work on the perturbation of degenerate Hamiltonian systems, i.e., $\det[h_{pp}(p)] \equiv 0$. Bruno [B] proved that the majority of invariant tori of unperturbed systems are preserved if $\text{rank}(h_p, h_{pp}) = n$. Chongqing Cheng and Yisui Sun [CS] obtained the existence of invariant tori under the following assumptions:

(1) $\text{rank}(\frac{\partial\omega}{\partial p}) = r$ for all $p \in \Omega$,

(2) there exists a twist curve on the range of any neighbourhood of p_0 for $\forall p_0 \in \Omega$, where "twist curve" means that on it every curvature component is not zero.

H. Rüssmann in [R] announced the following results: systems (1.1) possesses many invariant tori if on Ω

$$a_1 h_{p_1} + a_2 h_{p_2} + \ldots + a_n h_{p_n} \not\equiv 0 \tag{1.3}$$

for any $(a_1, a_2, \ldots, a_n) \in R^n$. This says that h_p does not fall into a hyperplane through the origin. The condition (1.3) is the sharpest one, we have not seen its proof yet. In this paper, we will give a nondegeneracy condition by means of the derivatives of $h(p)$, which is equivalent to the condition (1.3) in analytic case, and under this nondegeneracy condition we obtain the Rüssmann's results for analytic case. Furthermore, the arguments of this paper are available to the nonanalytic case since our nondegeneracy conditions only involve the finite order derivatives of $h(p)$.

Main results

Theorem A. *Suppose that $H = h(p) + \varepsilon f(p, q)$ is analytic in $\bar{\Omega} \times T^n$. If for some $p \in \bar{\Omega}$*

$$\text{rank}\left\{\omega, \frac{\partial^\alpha \omega}{\partial p^\alpha} | \forall \alpha, |\alpha| \le n - 1\right\} = n \ , \tag{1.4}$$

where $\omega(p) = h_p(p)$ and $\frac{\partial^\alpha \omega}{\partial p^\alpha} = (\frac{\partial^\alpha \omega_1}{\partial p^\alpha}, \frac{\partial^\alpha \omega_2}{\partial p^\alpha}, \ldots, \frac{\partial^\alpha \omega_n}{\partial p^\alpha})$, then for $\forall \Delta > 0$ sufficiently small, there exists $\varepsilon_0 = \varepsilon_0(\Delta) > 0$ such that if $|\varepsilon| < \varepsilon_0$, there exists a nonempty Cantor subset $\Omega_\varepsilon \subset \Omega$ such that (1.1) admits a family of invariant tori $\{I_p | p \in \Omega_\varepsilon\}$, whose frequencies $\omega_*(p)$ satisfy $|\omega_*(p) - \omega(p)| \le c\varepsilon$ with c being a constant independent of ε. Moreover, $\text{mes}\,(\Omega - \Omega_\varepsilon) = o(\Delta)$, *where $o(\Delta)$ is infinitively small as $\Delta \to 0$.*

Theorem A can be proved by KAM iterations. The KAM iterations are based on the measure estimates for small divisor condition. Hence, we need the following Theorem B.

Theorem B. *Suppose the mapping $g : x \in \bar{\Omega} \to (g_1(x), g_2(x), \ldots g_n(x))$ is analytic on $\bar{\Omega}$, where Ω is a bounded connected domain in R^n and $\bar{\Omega}$ is its closure.*

Let

$$\Omega_\Delta = \left\{ x \,|\, |k_1 g_1(x) + k_2 g_2(x) + \ldots + k_n g_n(x)| \geq \frac{\Delta}{|k|^\tau}\ ,\ \textit{for all}\ 0 \neq k \in Z^n \right\}$$

with $\tau > n(n-1) - 1$. *If for all* $x \in \bar{\Omega}$

$$\operatorname{rank}\left\{ g, \frac{\partial^\alpha g}{\partial x^\alpha}\ |\forall\, \alpha,\ |\alpha| \leq n-1 \right\} = n\ , \tag{1.5}$$

then for $\forall\, \Delta > 0$ *sufficiently small,* $|\Omega - \Omega_\Delta| \leq c(\mathrm{diam}\Omega)^{n-1}\Delta^{\frac{1}{n-1}}$, *where* c *is independent of* Δ. *So if* Δ *is small enough,* Ω_Δ *is a nonempty Cantor subset of* Ω.

In the proof of Theorem A the KAM procedure is standard, which is described in details in many papers, such as [M], [E], [P1], [P2], [P3] and [CS]. Since the nondegeneracy condition (1.4) is concerned with the high order derivatives and the small divisor condition only holds on a Cantor subset, it is necessary to estimate some Whitney norms (see [W]) in KAM steps, which makes the proof complicated. But this is not essential for the proof, so we omit the details and refer to [E], [P1], [P2], [P3] and [CS]. By the way, thanks to the referees of this paper, they told us that M.B. Sevryuk in [S] gave a quite simpler proof of the Rüssmann's results by using well known results of some papers and the preprint of this paper in ETH-Zürich (July 1994). Also he gave an example to show that the Rüssmann's nondegeneracy condition is also necessary for the results. In this paper we mainly prove Theorem B in the next section. In Sect. 3 we prove the nondegeneracy condition (1.4) is equivalent to the Rüssmann's nondegeneracy condition (1.3) in the analytic case.

Remark 1.1. If $\operatorname{rank}(h_p) = 1$, then the conditions given in [CS] are equivalent to $\det\left(\frac{\partial^j \omega_j}{\partial p_1^j}\right) \neq 0$ for $p \in \Omega$, where $\omega(p) = (\omega_1(p_1), \omega_2(p_1), \ldots, \omega_n(p_1))$, $j = 1, 2, \ldots, n$. It is easy to see that the range of ω cannot lie in any hyperplane in R^n.

Remark 1.2. If $\operatorname{rank}\left(\frac{\partial \omega(p)}{\partial p}\right) = n - 1$, the condition (1.4) is sharper than Bruno's condition. For example, $h(p_1, p_2, \ldots, p_n) = \sqrt{p_1^2 + \ldots + p_{n-1}^2} + p_n$, and then

$$\omega(p) = \left(\frac{p_1}{\sqrt{p_1^2 + \ldots + p_{n-1}^2}}\ , \ldots, \frac{p_{n-1}}{\sqrt{p_1^2 + \ldots + p_{n-1}^2}}, 1 \right).$$

It is easy to verify that $\operatorname{rank}(\omega(p), \frac{\partial \omega}{\partial p}) < n$, but $\operatorname{rank}\{\omega, \frac{\partial^\alpha \omega}{\partial x^\alpha} |\forall\, \alpha,\ |\alpha| \leq 2\} = n$.

Remark 1.3. Since ω is analytic on $\bar{\Omega}$, the equation (1.4) holds for some $p \in \Omega$ implies that (1.4) holds for an open subset $\Omega_* \subset \Omega$ satisfying $\mathrm{mes}(\Omega - \Omega_*) = \underline{0}$. So, in the proof of Theorem A we may suppose (1.4) holds for all $p \in \bar{\Omega}$. Thus, in Theorem A for the subset $\Omega_\varepsilon \subset \Omega$ we can have the measure estimate $\mathrm{mes}(\Omega - \Omega_\varepsilon) \leq c\Delta^{\frac{1}{n-1}}$, where c is a constant independent Δ and ε.

Remark 1.4. Theorem B can be extended to the nonanalytic case that g is a C^s $(s \geq 1)$ continuously differentiable mapping. That is, if for all $x \in \overline{\Omega}$

$$\text{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} \mid \forall\, \alpha,\ |\alpha| \leq s\right\} = n\ ,$$

then Theorem B also holds with $\Delta^{\frac{1}{s}}$ instead of $\Delta^{\frac{1}{n-1}}$. So Theorem A can be extended to C^r smooth situation, where r is a sufficiently large positive integer. But in the nonanalytic case the condition (1.4) is not equivalent to the Rüssmann's condition (1.3). In fact, the Rüssmann's condition is not sufficient in the nonanalytic case since Theorem A does not hold if for all $p \in \Omega$, ω falls into two intersected hyperplanes through the origin.

Remark 1.5. If $\det(\frac{\partial \omega}{\partial p}) = 0$ for all $p \in \Omega$, the range of any small perturbation of ω on Ω may not intersect with the range of ω on Ω. So the frequencies ω_* may not be from the frequencies of invariant tori of unperturbed systems, which is different from the nondegeneracy case $\det(\frac{\partial \omega}{\partial p}) \neq 0$, $\forall\, p \in \Omega$. However, the frequencies ω_* are ε-close to some frequencies of invariant tori of unperturbed systems.

2. Proof of Theorem B

The small divisor condition (1.2) is met when we solve the homological equation in KAM step. By measure estimate it easily follows that for any open domain Ω in R^n, most points satisfy the small divisor condition (1.2) (see [P1]). But it is not true for submanifold. For example, there is not any point in the hyperplane $x_1 = 0$ satisfying (1.2). [PY] obtained some results for one dimensional submanifold. In this section we consider the general submanifold and prove Theorem B. Denote the Lebesgue measure of set Ω by $|\Omega|$. We first prove some lemmas.

Lemma 2.1. *Suppose that $g(x)$ is a m-th differentiable function on the closure $\overline{I}$ of I, where $I \subset R^1$ is an interval. Let $I_h = \{x \mid\ |g(x)| < h, x \in I\}$, $h > 0$. If on I, $|g^{(m)}(x)| \geq d > 0$, where d is a constant, then $|I_h| \leq ch^{\frac{1}{m}}$, where $c = 2(2 + 3 + \ldots + m + d^{-1})$.*

Proof.. Let $I_h^{m-1} = \{x \mid\ |g^{(m-1)}(x)| < h, x \in I\}$. Since

$$|(g^{(m-1)}(x))'| = |g^{(m)}| \geq d > 0\ , \quad x \in I\ ,$$

I_h^{m-1} has at most one connected component and it follows that $|I_h^{m-1}| \leq \frac{2h}{d}$.

Let $I_h^{m-2} = \{x \mid\ |g^{(m-2)}| < h^2\}$. $I - I_h^{m-1} = \{x \mid\ |g^{(m-1)}| \geq h\}$ has at most two connected components $I_{(1)}^{m-1}$ and $I_{(2)}^{m-1}$, and

$$|(g^{(m-2)})'| = |g^{(m-1)}| \geq h\ , \quad x \in I_{(1)}^{m-1} \cup I_{(2)}^{m-1}\ .$$

In the same way, since $I_h^{m-2} \cap I_{(1)}^{m-1}$, $I_h^{m-2} \cap I_{(2)}^{m-1}$ have at most one connected component in $I_{(1)}^{m-1}$ and $I_{(2)}^{m-1}$ respectively, we have

$$|I_h^{m-2} \cap I_{(1)}^{m-1}| \leq 2h \ , \quad |I_h^{m-2} \cap I_{(2)}^{m-1}| \leq 2h \ .$$

Thus,

$$\begin{aligned} |I_h^{m-2}| &\leq |I_h^{m-2} \cap (I - I_h^{m-1})| + |I_h^{m-2} \cap I_h^{m-1}| \\ &\leq |I_h^{m-2} \cap I_{(1)}^{m-1}| + |I_h^{m-2} \cap I_{(2)}^{m-1}| + |I_h^{m-2} \cap I_h^{m-1}| \\ &\leq 4h + 2d^{-1}h = 2(2 + d^{-1})h \ . \end{aligned}$$

Let

$$I_h^1 = \{x| \ \ |g'(x)| < h^{m-1} \ , \quad x \in I\} \ .$$

After $m - 1$ steps inductively we have that

$$|I_h^1| \leq 2(2 + 3 + \ldots + m - 1 + d^{-1})h \ .$$

Since $|(g'(x))^{(m-1)}| \geq d > 0$, $I - I_h^1$ has at most m connected components. Denote these components by $I_{(1)}^1, I_{(2)}^1, \ldots, I_{(m)}^1$, and $I_h^0 = \{x| \ \ |g(x)| < h^m\}$. Then $|I_h^0 \cap I_{(1)}^1| \leq 2h, \ldots |I_h^0 \cap I_{(m)}^1| \leq 2h$. Thus

$$\begin{aligned} |I_h^0| &\leq |I_h^0 \cap (I - I_h^1)| + |I_h^0 \cap I_h^1| \\ &\leq [2m + 2(2 + 3 + \ldots + m - 1 + d^{-1})]h \\ &\leq 2(2 + 3 + \ldots + m + d^{-1})h \leq ch \ . \end{aligned}$$

Noticing that $I_h = I^0_{h^{\frac{1}{m}}}$, it follows $|I_h| \leq ch^{\frac{1}{m}}$.

Below we define a dictionary order of multiple index set. Let

$$Q = \{\alpha \mid \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in Z^n \ , \quad |\alpha| = l \ , \quad \alpha_i \geq 0\} \ ,$$

where l is a positive integer. For $\alpha, \beta \in Q$, we say "$\alpha \prec \beta$" if and only if there is $j \leq n$ such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ as $i < j$. Thus, for the dictionary order "$\prec$" we have

$$\begin{cases} (1) \text{ for any } \alpha, \beta \in Q, \quad \alpha \prec \beta \quad \text{or} \quad \beta \prec \alpha \quad \text{or} \quad \alpha = \beta \ , \\ (2) \text{ if } \alpha \prec \beta \quad \text{and} \quad \beta \prec \gamma, \quad \text{then } \alpha \prec \gamma \ . \end{cases}$$

Lemma 2.2. *Let*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \ , \quad \lambda_i = M^{n-i} \ , \quad i = 1, 2, \ldots, n \ , \quad \text{with } M = ln \ .$$

Then $\alpha \prec \beta$ if and only if $\langle \alpha, \lambda \rangle < \langle \beta, \lambda \rangle$, where $\langle , \rangle$ indicates the usual scalar product.

Proof.. By the definition of $\alpha \prec \beta$, there is $j \in \{1, 2, \ldots, n\}$ such that $\alpha_j < \beta_j$ and $\alpha_i = \beta_i$ as $i < j$. Noticing that $\alpha_i, \beta_i \leq l$, $\alpha_j + 1 \leq \beta_j$ and $M = ln$, it easily follows that $\langle \alpha, \lambda \rangle < \langle \beta, \lambda \rangle$. By the above first property of the order, this lemma holds.

By the order relation "$\prec$" we rewrite Q as

$$Q = \{\alpha(i) | \, |\alpha(i)| = l, \alpha_j(i) \geq 0 \ , \ i = 1, 2, \ldots, N, \ \alpha(1) \prec \alpha(2) \prec \ldots \prec \alpha(N)\} \ ,$$

where α_j are the components of α and N is the cardinality of Q. Let $n_i = \langle \alpha(i), \lambda \rangle$, by Lemma 2.2 we have $n_1 < n_2 < \ldots n_N$.

Lemma 2.3. *There exist $t_1, t_2, \ldots, t_N$ such that*

$$\det \begin{pmatrix} t_1^{n_1} & t_1^{n_2} & \ldots & t_1^{n_N} \\ t_2^{n_1} & t_2^{n_2} & \ldots & t_2^{n_N} \\ \vdots & \vdots & & \vdots \\ t_N^{n_1} & t_N^{n_2} & \ldots & t_N^{n_N} \end{pmatrix} \neq 0$$

Proof.. This lemma holds obviously, we omit the details.

Lemma 2.4. *Suppose $f(x)$ is a sufficiently smooth function. There exist vectors $\nu(1), \nu(2), \ldots, \nu(N)$ such that, for $\forall \beta \in Q$ there exist constants $c_1, c_2, \ldots, c_N$ such that*

$$\frac{\partial^\beta f}{\partial x^\beta} = \sum_{i=1}^{N} c_I D^l_{\nu(i)} f(x) \ ,$$

where $D^l_\nu f(x)$ indicates l-th direction derivatives of $f(x)$ along ν, $\nu(i)$ and c_i ($i = 1, 2, \ldots, N$) are independent of x and the function $f(x)$.

Proof..

$$D^l_\nu f(x) = \sum_{|\alpha|=l} \frac{\partial^\alpha f}{\partial x^\alpha} \nu^\alpha \ ,$$

where $\nu^\alpha = \nu_1^{\alpha_1} \nu_2^{\alpha_2} \ldots \nu_n^{\alpha_n}$. Let $\nu(t) = (t^{\lambda_1}, t^{\lambda_2}, \ldots, t^{\lambda_n})$. We have

$$D^l_{\nu(t)} f(x) = \sum_{|\alpha|=l} \frac{\partial \alpha f}{\partial x^\alpha} t^{\langle \alpha, \lambda \rangle} = \sum_{i=1}^{N} t^{n_i} \frac{\partial^{\alpha(i)} f}{\partial x^{\alpha(i)}} \ ,$$

where $n_i = \langle \alpha(i), \lambda \rangle$. Let $t = t_j$ and $\nu(j) = \nu(t_j)$, it follows that

$$D^l_{\nu(j)} f(x) = \sum_{i=1}^{N} t_j^{n_i} \frac{\partial^{\alpha(i)} f}{\partial x^{\alpha(i)}}, \quad j = 1, 2, \ldots, N \ . \tag{2.1}$$

Notice that $\det(t_j^{n_i})_{1 \leq i,j \leq N} \neq 0$. By solving the above linear equation systems (2.1), it follows that

$$\left\{ \frac{\partial^{\alpha(i)} f}{\partial x^{\alpha(i)}} \middle| \ i = 1, 2, \ldots, N \right\}$$

can be linearly expressed by

$$\{D^l_{\nu(j)} f(x) | j = 1, 2, \ldots, N\} \ .$$

Since $t_j^{n_i}$ are independent of x and the function $f(x)$, the coefficients of expression are also independent of x and $f(x)$. Thus we prove this lemma.

Proof of Theorem B. Let

$$g\left(\frac{k}{|k|},x\right)=\frac{k_1}{|k|}g_1(x)+\frac{k_2}{|k|}g_2(x)+\ldots+\frac{k_n}{|k|}g_n(x)$$

and

$$\Omega^k=\left\{x|\ \ |g\left(\frac{k}{|k|},x\right)|\le h(|k|)\right\},$$

where $h(|k|)$ will be decided later. Below we estimate the measure of Ω^k.

Let

$$g(\xi,x)=\xi_1g_1(x)+\xi_2g_2(x)+\ldots+\xi_ng_n(x)\ ,\quad (\xi,x)\in D\times\bar{\Omega}\ ,$$

$$A=\mathrm{matrix}(D^{l_1}_{\nu(1)}g,\ldots D^{l_n}_{\nu(n)}g)\ .$$

We have $|\det A|\ge d>0$ on Ω_x, where d is a constant. By the compactness of $\bar{\Omega}$, there are finite such neighbourhoods to cover $\bar{\Omega}$. Without lossing generality, suppose $|\det A|\ge d>0$ on $\bar{\Omega}$. There exists a constant $\bar{d}>0$ such that $|A\xi|\ge\bar{d}>0$ for $\forall(\xi,x)\in D\times\bar{\Omega}$, where $\bar{d}$ only depends on the smallest eigenvalue of A on Ω, and the norm $|A\xi|$ is defined in the same way as $|\xi|$. For any $(\bar{\xi},\bar{x})\in D\times\bar{\Omega}$, there exist $l\le n-1$ and $\nu\in R^n$ such that

$$|D^l_\nu g(\bar{\xi},\bar{x})|\ge\frac{\bar{d}}{n}\ ,$$

where $D^l_\nu g(\xi,x)$ are l-th direction derivatives of $g(\xi,x)$ with respect to x along the vectors ν. So there exists a neighbourhood of $(\bar{\xi},\bar{x})$, $D_{\bar{\xi}}\times\Omega_{\bar{x}}$, such that

$$|D^l_\nu g(x,\xi)|\ge\frac{\bar{d}}{2n}\ ,\quad\forall(\xi,x)\in D_{\bar{\xi}}\times\Omega_{\bar{x}}\ .$$

Thus we obtain a family of covers of $D\times\bar{\Omega}:\{D_{\bar{\xi}}\times\Omega_{\bar{x}}\quad|\quad\forall(\bar{\xi},\bar{x})\in D\times\bar{\Omega}\}$. Since $D\times\bar{\Omega}$ is a compact set, there exist finite covers $D_1\times\Omega_1,D_2\times\Omega_2,\ldots,D_M\times\Omega_M$ and the corresponding M integers and M vectors $l_1,l_2,\ldots,l_M,\nu_2,\nu_2,\ldots,\nu_M$ such that $\cup_{j=1}^M D_j\times\Omega_j\supset D\times\bar{\Omega}$ and

$$|D^{l_j}_{\nu_j}g(x,\xi)|\ge c_1>0\ ,\quad\forall(\xi,x)\in D_j\times\Omega_j\ .$$

Here Ω_j $(j=1,2,\ldots,M)$ are chosen to be convex sets and c_1 depends on Ω, n and the smallest eigenvalue of A.

Now we fix $k\ne0$ and estimate the measure of Ω^k. Let $h(|k|)=\frac{\Delta}{|k|^\tau}$ with $0<\Delta<\min\{c_1,1\}$ and $\tau>n(n-1)-1$, thus $h(|k|)\le\min\{c_1,1\}$. Then if $l_j=0$, we have $\Omega^k\cap\Omega^j\ne\emptyset$, so we only consider $l_j\ge1$. Let $\frac{k}{|k|}\in D_j$. Then for $x\in\Omega_j$ we have

$$\left|D^l_\nu g\left(\frac{k}{|k|},x\right)\right|\ge c_1>0\ ,\tag{2.2}$$

where we drop the subscripts of l_j and ν_j for simplicity. To estimate $|\Omega^k\cap\Omega_j|$, we first estimate 1-dimension measure of $\Omega^k\cap\Omega_j$ along ν.

Let

$$g(t) = g\left(\frac{k}{|k|}, x_0 + \nu t\right), \quad x_0 \in \partial \Omega_j \quad \Omega_\nu = \{t| \quad x_0 + t\nu \in \Omega_j\}$$

and

$$V_\nu = \left\{t| \quad |g\left(\frac{k}{|k|}, x_0 + \nu t\right)| < h(|k|)\right\},$$

where $\partial\Omega_j$ is the boundary of Ω_j. Since Ω_j is a connected convex neighbourhood, Ω_ν is also connected.

Since $g^{(l)}(t) = D_\nu^l g(\frac{k}{|k|}, x_0 + \nu t)$, by (2.2) we have $|g^{(l)}(t)| \geq c_1$ for $t \in \Omega_\nu$. By Lemma 2.1

$$\begin{aligned} |V_\nu| &\leq 2\left(2 + 3 + \ldots + l + \frac{1}{c_1}\right)[h(|k|)]^{\frac{1}{l}} \\ &\leq 2\left(2 + 3 + \ldots + n - 1 + \frac{1}{c_1}\right)[h(|k|)]^{\frac{1}{n-1}} \\ &\leq c_2[h(|k|)]^{\frac{1}{n-1}}, \quad \text{with} \quad c_2 = 2\left(2 + 3 + \ldots + n - 1 + \frac{1}{c_1}\right). \end{aligned}$$

Thus $|\Omega^k \cap \Omega_j| \leq (\text{diam}\Omega)^{n-1}|V_\nu| \leq (\text{diam}\Omega)^{n-1}c_2[h(|k|)]^{\frac{1}{n-1}}$. Therefore,

$$\begin{aligned} |\Omega^k| &\leq \sum_{j:\frac{k}{|k|}\in D_j} |\Omega^k \cap \Omega_j| \leq Mc_2(\text{diam}\Omega)^{n-1}[h(|k|)]^{\frac{1}{n-1}} \\ &\leq M(\text{diam}\Omega)^{n-1}c_2\frac{\Delta^{\frac{1}{n-1}}}{|k|^{\frac{\tau+1}{n-1}}}. \end{aligned}$$

Since $\Omega_\Delta = \cap_{k\neq 0}(\Omega - \Omega^k)$, we have

$$\begin{aligned} |\Omega - \Omega_\Delta| \leq \sum_{k\neq 0}|\Omega^k| &\leq M(\text{diam}\Omega)^{n-1}c_2\Delta^{\frac{1}{n-1}}\sum_{k\neq 0}\frac{1}{|k|^{\frac{\tau+1}{n-1}}} \\ &\leq c(\text{diam}\Omega)^{n-1}\Delta^{\frac{1}{n-1}}, \end{aligned}$$

where $c = Mc_2\sum_{k\neq 0}\frac{1}{|k|^{\frac{\tau+1}{n-1}}} < +\infty$ for $\tau > (n-1)n - 1$. If $\Delta < (\frac{|\Omega|}{c})^{n-1}$, then $|\Omega - \Omega_\Delta| < |\Omega|$, thus Ω_Δ has positive measure. Moreover, since $\{\frac{k}{|k|}|\forall k \in Z^n\}$ is dense in the set D, Ω_Δ is a Cantor set.

3. Equivalence with Rüssmann's nondegeneracy condition

In this section we prove the nondegeneracy condition (1.4) and the condition (1.3) given by Rüssmann are equivalent in analytic case. Let G be the image of the analytic mapping $g(x) = (g_1(x), \ldots, g_n(x))$ on $\bar{\Omega}$. We will prove that, if the condition (1.4) is not satisfied, then G must lie on a hyperplane through the origin.

Theorem 3.1. *Suppose that for any positive integer* $l \leq n-1$ *the set of vector function* $\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \, \forall\, \alpha, \ |\alpha| \leq l\}$ *has constant rank on* Ω*, i.e., for any* $l \leq n-1$ *its rank is a constant on* Ω*. If on* Ω

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \, \forall\, \alpha, \ |\alpha| \leq n-1\right\} \equiv s \ ,$$

then

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \, \forall\, \alpha, \ |\alpha| \leq s-1\right\} \equiv s$$

on Ω *and for* $|\beta| \geq s$*,* $\frac{\partial^\beta g}{\partial x^\beta}$ *can be linearly expressed by*

$$\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \, \forall\, \alpha, \ |\alpha| \leq s-1\right\} .$$

Proof.. Suppose $g \neq 0$ on Ω. If $\operatorname{rank}\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall\, \alpha, |\alpha| \leq 1\} = \operatorname{rank}\{g\} \equiv 1$, then for $\forall\, \alpha, |\alpha| = 1$, there exists $k_\alpha(x)$ such that $\frac{\partial^\alpha g}{\partial x^\alpha} = k_\alpha(x) g(x)$. Since $g \neq 0$, then $k_\alpha \in C^\infty$. By derivating the above equation, it follows that for $\forall\, \alpha, |\alpha| \geq 2$, $\frac{\partial^\alpha g}{\partial x^\alpha} = k_\alpha(x) g(x)$, where k_α is a differentiable function. Thus we-have $\operatorname{rank}\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall\, \alpha, |\alpha| \leq n-1\} = 1$. This is impossible unless $s = 1$. If $s > 1$, then $\operatorname{rank}\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall\, \alpha, |\alpha| \leq 1\} \equiv r > 2$. If

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall\, \alpha, |\alpha| \leq 2\right\} \equiv r \ ,$$

for any $x \in \Omega$, there exists a neighbourhood of x, $B_x \subset \Omega$, and vectors $\alpha_1, \alpha_2, \ldots, \alpha_r$, $|\alpha_i| \leq 1$ $(i = 1, 2, \ldots, r)$ such that on B_x

$$\operatorname{rank}\left\{\frac{\partial^{\alpha_1} g}{\partial x^{\alpha_1}}, \frac{\partial^{\alpha_2} g}{\partial x^{\alpha_2}}, \ldots, \frac{\partial^{\alpha_r} g}{\partial x^{\alpha_r}}\right\} \equiv r \ .$$

So for $\forall\, \beta, |\beta| = 2$, there exist functions $k_1(x)$, $k_2(x)$, ..., $k_r(x)$ such that

$$\frac{\partial^\beta g}{\partial x^\beta} = k_1(x) \frac{\partial^{\alpha_1} g}{\partial x^{\alpha_1}} + k_2(x) \frac{\partial^{\alpha_2} g}{\partial x^{\alpha_2}} + \ldots + k_r(x) \frac{\partial^{\alpha_r} g}{\partial x^{\alpha_r}} \ .$$

Since $k_1, k_2, \ldots, k_r$ can be obtained by solving the above nonsignular linear equation systems, it is easy to see that $k_i \in C^\infty(B_x)$. In the same way as the above we have $\operatorname{rank}\{g, \frac{\partial^\alpha g}{\partial x^\alpha} |\forall\, \alpha, |\alpha| \leq n-1\} = r$. So if $r < s$, this contradicts and $\operatorname{rank}\{g, \frac{\partial^\alpha g}{\partial x^\alpha} |\forall\, \alpha, |\alpha| \leq 2\} \geq r + 1 \geq 3$.

After at most $s - 1$ steps, we have

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} |\forall\, \alpha, |\alpha| \leq s-1\right\} \equiv s \text{ on } \Omega \ .$$

From the above it is easy to see that for all β, $|\beta| \geq s$, $\frac{\partial^\beta g}{\partial x^\beta}$ can be expressed by the set of vector function

$$\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \, \forall\, \alpha, \ |\alpha| \leq s-1\right\} \ .$$

Theorem 3.2. *Suppose that for any* $l \le n-1$ *the set of vector function* $\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha, |\alpha| \le l\}$ *has constant* rank *on* Ω. *If on* Ω

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha, |\alpha| \le n-1\right\} \equiv n-1 ,$$

then G *must lie on a* $(n-1)$*-hyperplane through the origin.*

Proof.. By Theorem 3.1,

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha, |\alpha| \le n-2\right\} \equiv n-1 \tag{3.1}$$

on Ω. For any $x \in \Omega$ there exists a nonzero vector function $k(x)$ such that

$$\left\langle k, \frac{\partial^\alpha g}{\partial x^\alpha} \right\rangle \equiv 0 \text{ on } \Omega \quad \text{for } \forall \alpha, |\alpha| \le n-1 . \tag{3.2}$$

There exists a neighbourhood of x, B_x, and $\frac{\partial^{\alpha_1} g}{\partial x^{\alpha_1}}, \frac{\partial^{\alpha_2} g}{\partial x^{\alpha_2}}, \dots, \frac{\partial^{\alpha_{n-1}} g}{\partial x^{\alpha_{n-1}}}$ such that they are linearly independent on B_x. Thus $k(x)$ can be obtained by solving a nonsingular linear equation system of $n-1$ variables, so $k(x) \in C^\infty(B_x)$. By derivating the above equations (3.2), it follows that

$$\left\langle k_{x_i}, \frac{\partial^\alpha g}{\partial x^\alpha} \right\rangle \equiv 0 \text{ on } B_x, \forall \alpha, |\alpha| \le n-2 ,$$

$i = 1, 2, \dots, n$. By (3.1) it follows that on B_x, $\operatorname{rank}\{k, k_{x_1}, \dots, k_{x_n}\} \equiv 1$ and there exist functions $c_1(x), c_2(x), \dots, c_n(x)$ such that $k_{x_i} = c_i k$. By integrating these equations, we have

$$\begin{aligned} k(x) &= \mathrm{e}^{\int_{x_1^0}^{x_1} c_1(x) dx_1} k(x_1^0, x_2, \dots, x_n) = \bar{c}_1(x) k(x_1^0, x_2, \dots, x_n) , \\ k(x) &= \mathrm{e}^{\int_{x_2^0}^{x_2} c_2(x) dx_2} k(x_1, x_2^0, \dots, x_n) = \bar{c}_2(x) k(x_1, x_2^0, \dots, x_n) , \\ &\dots \qquad \dots \\ k(x) &= \mathrm{e}^{\int_{x_n^0}^{x_n} c_n(x) dx_n} k(x_1, x_2, \dots, x_n^0) = \bar{c}_n(x) k(x_1, x_2, \dots, x_n^0) , \end{aligned}$$

where $x^0 \in B_x$. Combining all the above equations, it follows that

$$\begin{aligned} k(x) &= \bar{c}_1(x) k(x_1^0, x_2, \dots, x_n) = \dots \\ &= \bar{c}_1(x) \bar{c}_2(x_1^0 x_2, \dots, x_n) \dots \bar{c}_n(x_1^0, \dots, x_{n-1}^0, x_n) k(x^0) \\ &= \tilde{c}(x) k(x^0) , \end{aligned}$$

with $\tilde{c}(x) \neq 0$. By (3.2) it follows that $\langle g, k(x^0) \rangle \equiv 0$ on B_x. Since g is analytic on Ω, so $\langle g, k(x^0) \rangle \equiv 0$ on Ω. This implies that G lies on the following hyperplane through the origin:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0 ,$$

where $a_i = k_i(x^0)$, $i = 1, 2, \dots, n$.

Theorem 3.3. *If*

$$\max_{x\in\Omega} \operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} \mid \forall\,\alpha,\ |\alpha| \le n-1\right\} = n-1\ , \tag{3.3}$$

then G lies on a $(n-1)$-hyperplane through the origin.

Proof.. Let

$$r_l = \max_{x\in\Omega} \operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} \mid \forall\,\alpha,\ |\alpha| \le l\right\}$$

and

$$E_l = \left\{x \mid \operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} \mid \forall\,\alpha,\ |\alpha| \le l\right\} = r_l\right\},$$

where l is a positive integer. By Remark 1.3, we have E_l is an open subset of Ω and $\Omega - E_l$ has zero measure. Let $E = \cap_{l\le n-1} E_l$. Then E is an open set and $mes(\Omega - E) = 0$. So there exists a neighbourhood $B \subset E$ such that on B

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} \mid \forall\,\alpha,\ |\alpha| \le 1\right\} \equiv r_l$$

for all $l \le n-1$. From (3.3),

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} \mid \forall\,\alpha,\ |\alpha| \le n-1\right\} \equiv n-1 \text{ on } B\ .$$

By Theorem 3.2, it follows that on B, G lies on a $(n-1)$-hyperplane through the origin. Since g is analytic on Ω, we have G is also on this $(n-1)$-hyperplane.

Furthermore, we can prove the following results.

Theorem 3.4. *If*

$$\max_{x\in\Omega} \operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} \mid \forall\,\alpha,\ |\alpha| \le n-1\right\} = s < n\ , \tag{3.4}$$

then G must lie on a s-hyperplane through the origin on Ω.

Proof.. By (3.4) there exist $\bar g = (\bar g_1, \bar g_2, \dots, \bar g_s)$ such that

$$\max_{x\in\Omega} \operatorname{rank}\left\{\bar g, \frac{\partial^\alpha \bar g}{\partial x^\alpha} \mid \forall\,\alpha, |\alpha| \le n-1\right\} = s < n\ , \tag{3.5}$$

where $\bar g_i$ $(i = 1, 2, \dots, s)$ are some s components of g. Let the rest components be $\bar g_{s+1}, \dots, \bar g_n$. Let $\tilde g = (\bar g_1, \bar g_2, \dots, \bar g_s, \bar g_j)$ $(j \ge s+1)$. We apply Theorem 3.3 to the function $\tilde g$ and obtain $\tilde g$ is on a s-hyperplane of R^{s+1} through the origin, i.e., there exist $c_1^j, c_2^j, \dots, c_s^j, c_{s+1}^j$ such that on Ω

$$c_1^j \bar g_1 + c_2^j \bar g_2 + \ldots + c_s^j \bar g_s + c_{s+1}^j \bar g_{s+1} \equiv 0\ .$$

By (3.5) $\bar g_1, \bar g_2, \dots, \bar g_s$ are linearly independent for some $x \in \Omega$, so $c_{s+1}^j \ne 0$, $j = s+1, \dots, n$. This implies that G is on $n-s$ different $(n-1)$-hyperplanes through the origin. Thus we prove this theorem.

Notice that if the condition (1.4) does not hold, then the condition (3.4) holds. By Theorem 3.4 it follows that the condition (1.4) is equivalent to the Rüssmann's condition (1.3).

Remark 3.1. If

$$\max_{x\in\bar{\Omega}} \operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha,\ |\alpha| \le n-1\right\} = n\ , \tag{3.6}$$

then there exists an integer $\bar{N} > 0$ such that

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha,\ |\alpha| \le \bar{N}\right\} \equiv n \text{ on } \bar{\Omega}\ . \tag{3.7}$$

Proof.. Assume this result does not hold. Then for $\forall\, l = 1, 2, \ldots$, there exists $x_l \in \bar{\Omega}$ such that at x_l

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha,\ |\alpha| \le l\right\} < n\ . \tag{3.8}$$

Since $\bar{\Omega}$ is bounded domain, there exists a convergent subsequence of $\{x_l\}_{l\ge 1}$. Without loss of generality suppose the sequence itself is convergent and $\lim_{l\to\infty} x_l = x_0 \in \bar{\Omega}$. Now we conclude that for $x = x_0$

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha,\ |\alpha| < +\infty\right\} < n\ . \tag{3.9}$$

If (3.9) is not true, then there exists $\bar{l} > 0$ such that for $x = x_0$

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha,\ |\alpha| \le \bar{l}\right\} = n\ .$$

Thus there is a neighbourhood of x_0, B_{x_0}, such that on B_{x_0},

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha,\ |\alpha| \le \bar{l}\right\} \equiv n\ .$$

There exists sufficiently large $l > \bar{l}$ such that $x_l \in B_{x_0}$, so for $x = x_l$,

$$\operatorname{rank}\left\{g, \frac{\partial^\alpha g}{\partial x^\alpha} | \forall \alpha,\ |\alpha| \le l\right\} = n\ .$$

This contradicts with (3.8) and then (3.9) holds.

For $\forall\, \alpha_1, \alpha_2, \ldots, \alpha_n \in Z^n$ let

$$f(x) = \det\left(\frac{\partial^{\alpha_1} g}{\partial x^{\alpha_1}}, \frac{\partial^{\alpha_2} g}{\partial x^{\alpha_2}}, \ldots, \frac{\partial^{\alpha_n} g}{\partial x^{\alpha_n}}\right).$$

By (3.9), $f(x_0) = 0$. Derivating $f(x)$ and using (3.9) again, it follows that for $\forall\, \alpha$, $\frac{\partial^\alpha f}{\partial x^\alpha} = 0$ for $x = x_0$. Since $f(x)$ is analytic on $\bar{\Omega}$, so $f(x) \equiv 0$ on Ω. This contradicts with the condition (3.6), and thus the result is proved.

Remark 3.2. By Remark 1.3, on some local domain we can obtain a more accurate measure estimate for the parameter set, where the invariant tori exist. By Remark 1.3 and Remark 3.1, for the subset Ω_ε of Ω in Theorem A we may have the estimates $|\Omega - \Omega_\varepsilon| \leq c\Delta^{\frac{1}{N}}$, where $\bar{N}$ is a positive integer.

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References

[A] Arnol'd, V.I.: Proof of A.N. Kolmogorov's theorem on the preservation of quasi periodic motions under small perturbations of the Hamiltonian. Russian Math. Surveys **18**(5) (1963), 9–36

[B] Bruno, A.D.: Analytic form of differential equations. Transactions of the Moscow Math. Soc. **25** (1971), 131–288

[CS] Cheng, Ch.-Q., Sun, Y.-S.: Existence of KAM tori in degenerate Hamiltonian systems. J. Diff. Equa. **114** (1994), 288–335

[E] Eliasson, L.H.: Perturbations of stable invariant tori for Hamiltonian systems. Ann. Sc. Norm. Super. Pisa, Cl. Sci. IV **15** (1988), 115–147

[K] Kolmogorov, A.N.: On the persistence of quasi-periodic motions under small perturbations of the Hamiltonian. Doklady Akad. Nauk USSR **98** (1954), 527–530

[M] Moser, J.: On invariant curves of area-preserving mappings of an annulus. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. 2, Nr. 1 (1962), 1–20

[P1] Pöschel, J.: On elliptic lower dimensional tori in Hamiltonian systems. Math. Z. **202** (1989), 559–608

[P2] Pöschel, J.: Integrability of Hamiltonian systems on Cantor sets. Commun. Pure. Appl. Math. **35** (1982), 653–696

[P3] Pöschel, J.: A KAM-theorem for some nonlinear partial differential equations. Preprint ETH Zürich (1994)

[Py] Pyartli, A.S.: Diophantine approximations on submanifolds of Euclidean space. Funct. Anal. Appl. **3** (1969), 303–306

[R] Rüssmann, H.: On twist Hamiltonians. Talk on the Colloque International: Mécanique céleste et systèmes hamiltoniens. Marseille (1990)

[W] Whitney, H.: Analytical extensions of differentiable functions defined in closed sets. Trans. A.M.S. **36** (1934), 63–89

[S] Sevryuk, M.B.: KAM-stable Hamiltonians. J. Dyn. Control Syst. **1** (1995), 351–366