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Gevrey-smoothness of invariant tori for analytic nearly integrable Hamiltonian systems under Rüssmann's non-degeneracy condition

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Abstract

In this paper we prove Gevrey smoothness of the persisting invariant tori for small perturbations of an analytic integrable Hamiltonian system with Rüssmann's non-degeneracy condition by an improved KAM iteration method with parameters.

Keywords: Gevrey smoothness; Hamiltonian system; KAM iteration; Non-degeneracy condition

1. Introduction

Consider the following Hamiltonian dynamical system:

$$\begin{cases} \dot{q} = H_p(q, p) = h_p(p) + f_p(q, p), \\ \dot{p} = -H_q(q, p) = -f_q(q, p) \end{cases} \tag{1.1}$$

where $H(q, p) = h(p) + f(q, p)$ is the Hamiltonian function, $(q, p) \in T^n \times D$, with T^n being the usual n-dimensional torus and D a bounded connected open domain of R^n. Suppose $h(p)$ and $f(q, p)$ are real analytic on $\bar{D}$ and $\bar{D} \times T^n$.

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If $f = 0$, then the system (1.1) is integrable and has invariant tori $T^n \times \{p_0\}$ for all $p_0 \in D$, on which there exists a linear flow, $p(t) = p_0$, $q(t) = \omega(p_0)t + q_0$ for any $q_0 \in T^n$, with the frequency $\omega(p_0) = h_p(p_0)$. The classical KAM theorem asserts that if the frequency $\omega(p)$ is not degenerate, that is,

$$\det(\partial\omega/\partial p) = \det(h_{pp}) \neq 0, \tag{1.2}$$

then most of the invariant tori can persist when f is sufficiently small [3–10]. Later the result was extended to the case of Rüssmann's non-degeneracy [1,2,15,19], see (1.3). These invariant tori form a parameterized family. How do the invariant tori depend on the parameter? In the analytic case, if the usual non-degeneracy condition (1.2) holds, Pöschel proved that the persisting invariant tori are C^∞-smooth in the frequency parameter [11]. More recently, Popov improved this result and proved that these KAM tori are Gevrey-smooth in their frequencies [12,13]. For some related result, also see [14]. But in the case of Rüssmann's non-degeneracy condition, no result is known about Gevrey-smoothness. In this case, the frequency cannot be regarded as independent parameter and so the previous methods in [9,11,12] are not valid. In this paper, by an improved KAM iteration with parameters, we prove that the Gevrey smoothness of persisting invariant tori for analytic nearly integrable Hamiltonian system is also true in the case of Rüssmann's non-degeneracy.

Let $\Pi \subset R^n$ be a closed bounded set. Denote by $G^\mu(\Pi)$ ($\mu \geqslant 1$) the space of all Gevrey functions in a domain Π of index μ. This means $f \in G^\mu(\Pi)$ iff $f \in C^\infty(\Pi)$ and there exists a constant M such that

$$\sup_{\xi\in\Pi} \left|\partial_\xi^\beta f(\xi)\right| \leqslant M^{|\beta|+1} \beta!^\mu, \quad \forall\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in Z_+^n,$$

where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. Note that the derivatives are understood in Whitney's sense [21].

Remark. Obviously, analytic functions are Gevrey-functions; but Gevrey-function need not be analytic. For $\mu = 1$, the Gevrey function class $G^\mu(\Pi)$ coincides with the class of analytic functions, but for $\mu > 1$, the Gevrey function class is larger.

Theorem 1.1. *Suppose that $h(p)$ and $f(q, p)$ are real analytic on $\bar{D}$ and $T^n \times \bar{D}$, respectively, $\omega(p) = h_p(p) = (\omega_1(p), \omega_2(p), \ldots, \omega_n(p))$ satisfies Rüssamnn's non-degeneracy condition*: $\forall(a_1, a_2, \ldots, a_n) \in R^n \setminus \{0\}$,

$$a_1\omega_1(p) + a_2\omega_2(p) + \cdots + a_n\omega_n(p) \not\equiv 0 \quad \text{on } \bar{D}. \tag{1.3}$$

Then there exist a sufficiently large positive m depending on the function h and the domain D, and a sufficiently small positive constant $\delta > 0$, such that for $\tau > nm - 1$, $\mu > \tau + 2$ and a sufficiently small $\alpha > 0$, if $\|f\| = \sup_{T^n\times D} |f(q, p)| \leqslant \delta\alpha^2$, then, there is a nonempty Cantor set $\Pi(\alpha) \subset D$ with meas$(D \setminus \Pi(\alpha)) \leqslant c\alpha^{\frac{1}{m}}$, and for $\xi \in \Pi(\alpha)$ the Hamiltonian system (1.1) *has an invariant torus T_ξ with the frequency $\omega_*(\xi)$ satisfying the Diophantine condition*:

$$\left|\langle\omega_*(\xi), k\rangle\right| \geqslant \frac{\alpha}{|k|^\tau}, \quad \forall k = (k_1, k_2, \ldots, k_n) \in Z^n \setminus \{0\},$$

where $|k| = |k_1| + |k_2| + \cdots + |k_n|$. *Moreover, the family of invariant tori* $\{T_\xi, \xi \in \Pi(\alpha)\}$ *is* G^μ*-smooth in* ξ. *That means that for each* $\xi \in \Pi(\alpha)$, *the invariant torus* T_ξ *is an embedding torus*: $\Phi(\cdot,\xi): T^n \to D \times T^n$, *and* $\Phi \in G^{1,\mu}(T^n \times \Pi(\alpha))$, *that is,* $\Phi \in C^\infty(T^n \times \Pi(\alpha))$ *and* $\Phi(\theta,\xi)$ *is analytic in* θ *on* T^n *and* G^μ*-smooth in* ξ *on* $\Pi(\alpha)$. *Furthermore, the frequencies* $\omega^* \in G^\mu(\Pi(\alpha))$. *Here* c *is a positive constant depending only on* τ, μ, n *and* ω.

Remark. The non-degeneracy condition (1.3) is the sharpest one for KAM theorem, which is first given by Rüssmann in [15]. It means geometrically that the frequency vector ω does not lie on a hyperplane through the origin of R^n. Actually, it follows from [16,17,19] that the Rüssmann's non-degeneracy condition (1.3) is equivalent to that there exists a sufficiently positive integer m depending on h and D such that

$$\operatorname{Rank}\{\omega(p), \partial_p^\beta \omega(p) \mid |\beta| \leqslant m\} = n \quad \text{for all } p \in \bar{D}. \tag{1.4}$$

In Theorem 1.1, the m is the smallest one such that Eq. (1.4) holds. Especially, for the case $m = 1$, the condition (1.3) is equivalent to the Kolmogorov's non-degeneracy condition and our results correspond to those in [12]. Also from [19] we have that the Rüssmann's non-degeneracy condition (1.3) is also equivalent to that there exists a point $p_0 \in \bar{\Omega}$ such that

$$\operatorname{Rank}\{\omega(p_0), \partial_p^\beta \omega(p_0) \mid |\beta| \leqslant n-1\} = n.$$

We will use KAM iteration to prove this theorem; and the outline is the same as in [9]. At first we linearize the Hamiltonian system (1.1) at the invariant tori of the integrable system and then we will consider a parameterized Hamiltonian system instead of the Hamiltonian system (1.1).

For any $\xi \in D$, let $p = \xi + I$ and $q = \theta$. Under the symplectic map,

$$\begin{aligned} H(q,p) &= h(\xi) + \langle h_p(\xi), I\rangle + f_h(I;\xi) + f(\theta, \xi + I) \\ &= e + \langle \omega(\xi), I\rangle + P, \end{aligned}$$

where $e = h(\xi)$, $\omega(\xi) = h_p(\xi)$, $P = P(\theta, I;\xi) = f_h(I,\xi) + f(\theta,\xi+I)$, and $\xi \in D$ is regarded as parameter. Here e is an energy constant and has no influence on the Hamiltonian system, so we usually omit it; ω is called frequency vector; and P is a small perturbation term. The corresponding Hamiltonian system becomes

$$\begin{cases} \dot{\theta} = H_I = \omega(\xi) + P_I(\theta, I;\xi), \\ \dot{I} = -H_\theta = -P_\theta(\theta, I;\xi). \end{cases} \tag{1.5}$$

Thus, persistence of invariant tori for the nearly integrable system (1.1) is reduced to that of invariant tori for the family of Hamiltonian system (1.5) depending on the parameter $\xi \in D$.

Let

$$D(s,r) = \{(\theta, I) \in \mathbf{C}^n \times \mathbf{C}^n \mid |\operatorname{Im}\theta|_\infty \leqslant s,\ |I|_1 \leqslant r\},$$

where $|\operatorname{Im}\theta|_\infty = \max_{1\leqslant i\leqslant n} |\operatorname{Im}\theta_i|$, $|I|_1 = \sum_{1\leqslant i\leqslant n} |I_i|$. Denote

$$\Pi = \{\xi \in D \mid \operatorname{dist}(\xi, \partial D) \geqslant d\}$$

and

$$\Pi_d = \{\xi \in \mathbf{C}^n \mid \operatorname{dist}(\xi, \Pi) \leqslant d\}$$

with $2r \leqslant d = \alpha^{\frac{1}{m}}$, where m is the smallest one satisfying Eq. (1.4). Thus, we have $\operatorname{meas}(D \setminus \Pi) = O(\alpha^{\frac{1}{m}})$ as $\alpha \to 0$. We usually take $r^2 = \epsilon$ with ϵ being the small perturbation scale, thus we can put the higher order of I into the perturbation term. So if ϵ is sufficiently small, $2r \leqslant d$ always holds in the sequel. This technique is usually used to put high order nonlinear terms into perturbation terms and we refer to [9] for details.

Now the Hamiltonian function $H(\theta, I; \xi)$ is analytic in $(\theta, I; \xi)$ on $D(s, r) \times \Pi_d$. We expand $f(\theta, I; \xi)$ as Fourier series with respect to θ and we have

$$f(\theta, I; \xi) = \sum_{k \in Z^n} f_k(I; \xi)\, e^{\mathrm{i}\langle k, \theta\rangle}.$$

Define

$$\|f\|_{D(s,r)\times\Pi_d} = \sum_{k,l} \|f_k\|_{r;d}\, e^{s|k|},$$

where $\|f_k\|_{r;d} = \sup_{|I|_1 \leqslant r, \xi \in \Pi_d} |f_k(I; \xi)|$.

Remark. The norm $\|\cdot\|_{D(s,r)\times\Pi_d}$ was introduced in [10]. In this paper, by using this norm, we simplify the estimate of the Gevrey-norm in KAM steps.

We write $f(z; \xi) \in G^{1,\mu}(\tilde{D} \times \Pi)$ iff $f \in C^\infty(\tilde{D} \times \Pi)$ and $f(z; \xi)$ is analytic in z on $\tilde{D}$ and G^μ-smooth in ξ on Π.

Let $\tau > nm - 1$, $\tau + 2 < \mu < 2\tau + 3$, $\sigma = (\frac{2}{3})^{\frac{l}{l+\tau+1}}$ with $l = \mu - \tau - 2$. Let $\rho_0 = (1 - \sigma)s/10$, $r_0 = r$ and I_n be the n-unit matrix. Denote $W_0 = \operatorname{diag}(\frac{1}{\rho_0} I_n, \frac{1}{r_0} I_n)$. Below, for simplicity we will use the same notation c to indicate different constants, which usually depend on τ, μ, n and ω. With these notations and definitions we have the following result.

Theorem 1.2. *Let τ, μ and W_0 be defined as above. Let $H(\theta, I; \xi) = \langle \omega(\xi), I\rangle + P(\theta, I; \xi)$. Suppose $\omega(\xi)$ and $P(\theta, I; \xi)$ are analytic on Π_d and $D(s, r) \times \Pi_d$, respectively. Let $T = \max_{\xi \in \Pi_d} |\partial\omega/\partial\xi|$. Suppose $\omega(\xi)$ satisfies* (1.3). *Then, there exists $\gamma > 0$, which is independent of ϵ, α, r, s and usually depends on τ, μ, n, d, ω, such that for any* $0 < \alpha < 1$, *if*

$$\|P\|_{D(s,r)\times\Pi_d} = \epsilon \leqslant \gamma\alpha r s^{\tau+1},$$

there is a nonempty Cantor set $\Pi_* \subset \Pi$, and a family of symplectic mappings

$$\Phi_*(\cdot, \cdot; \xi) : D(s/2, r/2) \to D(s, r), \quad \forall \xi \in \Pi_*,$$

satisfying $\Phi_* \in G^{1,\mu}(D(s/2, r/2) \times \Pi_*)$ and, for all $\beta \in Z_+^n$,

$$\left\| W_0 \partial_\xi^\beta (\Phi_* - id) \right\|_{D(s/2,r/2)\times\Pi_*} \leqslant c\gamma^{\frac{1}{n+1}} M^{|\beta|} \beta!^\mu, \tag{1.6}$$

where $\beta! = \beta_1!\beta!\cdots\beta_n!$ and, M and c are constants depending on n, τ, T and μ. Under the symplectic mappings, the Hamiltonian function H has the following form: $H_*(\theta, I;\xi) = H \circ \Phi_*(\theta, I:\xi) = N_*(I;\xi) + P_*(\theta, I:\xi)$, where $N_*(I;\xi) = \langle\omega_*(\xi), I\rangle$, and $P_*(\theta, I;\xi) = O(I^2)$ as $I \to 0$. Hence, the Hamiltonian system (1.5) has a family of invariant tori $\{T_\xi = \Phi_*(T^n, 0;\xi) \mid \xi \in \Pi_*\}$, which is G^μ-smooth in ξ on Π_*, and whose frequencies $\omega_*(\xi)$ satisfy, for all $\xi \in \Pi_*$,

$$\left|\partial_\xi^\beta\left[\omega_*(\xi) - \omega(\xi)\right]\right| \leqslant c\gamma^{\frac{1}{n+1}}\alpha s^{\tau+1}M^{|\beta|}\beta!^\mu, \quad \forall\beta \in Z_+^n, \tag{1.7}$$

and

$$\left|\left\langle\omega_*(\xi), k\right\rangle\right| \geqslant \frac{\alpha}{|k|^\tau}, \quad \forall k \in Z^n \setminus \{0\}. \tag{1.8}$$

Moreover, we have $\operatorname{meas}(\Pi \setminus \Pi_*) \leqslant c\alpha^{1/m}$. *Here the above constants c depend only on τ, μ, n and ω.*

Remark. The KAM Theorem 1.1 can easily follow from the KAM Theorem 1.2 with parameters, and this technique was first introduced by Pöschel in [9].

2. Proof of theorems

In the same way as in [9], we can obtain Theorem 1.1 from Theorem 1.2. So we need only to prove Theorem 1.2. Our method is KAM iteration and the idea is similar to [9–12,17,18,20].

2.1. KAM-step

The procedure of KAM-step is standard; we summarize the result for one KAM step in the following lemma.

Iteration Lemma 2.1. *Let $H(\theta, I;\xi) = N(I;\xi) + P(\theta, I;\xi)$ with $N(I;\xi) = \langle\omega(\xi), I\rangle$. Let $0 < E < 1$ and $0 < \rho < s/5$. Let $K > 0$ satisfy $e^{-K\rho} = E$. Suppose*

$$\left|\left\langle\omega(\xi), k\right\rangle\right| \geqslant \frac{2\alpha}{|k|^\tau}, \quad \forall\xi \in \Pi,\ \forall k \in Z^n,\ 0 < |k| \leqslant K, \tag{2.1}$$

where $0 < \alpha \leqslant \alpha_0$ is fixed. Let $\max_{\xi\in\Pi_d}|\partial\omega/\partial\xi| \leqslant T$, $d = \frac{\alpha}{2TK^{\tau+1}}$. Suppose that

$$\|P\|_{s,r;d} \leqslant \epsilon = \alpha r\rho^{\tau+1}E,$$

where $\|P\|_{s,r;d} = \|P\|_{D(s,r)\times\Pi_d}$. Then, for any $\xi \in \Pi_d$, there exists a symplectic mapping $\Phi(\cdot,\cdot;\xi) : D(s_+, r_+) \to D(s, r)$, such that

$$H_+(\theta, I;\xi) = H \circ \Phi(\theta, I;\xi) = N_+(I;\xi) + P_+(\theta, I;\xi),$$

where $N_+(I;\xi) = \langle\omega_+(\xi), I\rangle$ and P_+ satisfies

$$\|P_+\|_{s_+,r_+;d} \leqslant \epsilon_+ = \alpha_+ r_+\rho_+^{\tau+1}E_+$$

with

$$s_+ = s - 5\rho, \quad \eta = \sqrt{E}, \quad \rho_+ = \sigma\rho, \quad r_+ = \eta r, \quad E_+ = cE^{\frac{2}{3}}, \quad \alpha/2 \leqslant \alpha_+ \leqslant \alpha,$$

where $\sigma = (\frac{2}{3})^{\frac{l}{l+\tau+1}}$ *with* $l = \mu - \tau - 2$. *Moreover,*

$$\left|\omega_+(\xi) - \omega(\xi)\right| \leqslant \frac{\epsilon}{r}, \quad \forall \xi \in \Pi_d. \tag{2.2}$$

Furthermore, let $\alpha_+ = \alpha - \frac{\epsilon}{2r}K^{\tau+1}$ *and denote*

$$\tilde{\Pi} = \left\{\xi \in \Pi \;\middle|\; \left|\langle\omega_+(\xi), k\rangle\right| < \frac{2\alpha_+}{|k|^\tau},\ \forall K < |k| \leqslant K_+\right\}$$

and $\Pi_+ = \Pi \setminus \tilde{\Pi}$, *then*

$$\left|\langle\omega_+(\xi), k\rangle\right| \geqslant \frac{2\alpha_+}{|k|^\tau}, \quad \forall \xi \in \Pi_+,\ \forall k \in Z^n \text{ with } 0 < |k| \leqslant K_+, \tag{2.3}$$

where $K_+ > 0$ *such that* $e^{-\rho_+ K_+} = E_+$. *Let* $T_+ = T + \frac{3\epsilon}{dr}$ *and* $d_+ = \frac{\alpha_+}{2T_+K_+^{\tau+1}}$. *If* $d_+ \leqslant \frac{2}{3}d$, *then* $\max_{\xi\in\Pi_{d_+}} |\partial\omega_+/\partial\xi| \leqslant T_+$, *where* Π_{d_+} *is the complex* d_+*-neighborhood of* Π_+. *Moreover, we have* $\|P_+\|_{s_+,r_+;d_+} \leqslant \epsilon_+$. *Thus, the above result also holds for* H_+ *in place of* H.

Remark. The above lemma is actually one step in our KAM iteration. Once this lemma holds for the Hamiltonian H, it also holds for the transformed Hamiltonian H_+, and so the KAM step can iterate.

Proof. The proof of this lemma is standard KAM step and we divide it into several parts.

A. Truncation. Let $R = P(\theta, 0; \xi) + \langle P_I(\theta, 0; \xi), I\rangle$. It follows easily that $\|R\|_{s,r;d} \leqslant 2\|P\|_{s,r;d} \leqslant 2\epsilon$. Write $R = \sum_{k\in Z^n} R_k(I;\xi)e^{\mathrm{i}\langle k,\theta\rangle}$ and let

$$R^K = \sum_{|k|\leqslant K} R_k(I;\xi)e^{\mathrm{i}\langle k,\theta\rangle}.$$

By definition, we have

$$\left\|R - R^K\right\|_{s-\rho,r;d} \leqslant 2\epsilon e^{-K\rho}.$$

B. Construction of the symplectic map. The symplectic map is generated by a Hamiltonian flow map at 1-time. We will find a Hamiltonian function F and define the symplectic map by $\Phi = X_F^t|_{t=1}$. It follows

$$H \circ \Phi = N_+ + \{N, F\} + R^K - [R] + P_+,$$

where $[R]$ is the average of R on T^n, $N_+ = N + [R] = \langle I, \omega_+\rangle$, $\{\cdot,\cdot\}$ is the Poisson bracket, and

$$P_+ = \left(R - R^K\right) + \int_0^1 \left\{(1-t)\{N, F\} + R, F\right\} \circ X_F^t\, dt + (P - R) \circ \Phi.$$

We want to find F such that

$$\{N, F\} + R^K - [R] = 0. \tag{2.4}$$

Let $\{F_k\}$ and $\{R_k\}$ be Fourier coefficients of F and R with respect to θ. By the assumption (2.1), we have

$$\big|\langle\omega(\xi), k\rangle\big| \geqslant \frac{\alpha}{|k|^\tau}, \quad \forall \xi \in \Pi_d,\ \forall 0 < |k| \leqslant K.$$

So we have

$$F_k = \frac{1}{\mathrm{i}\langle\omega(\xi), k\rangle} R_k, \quad 0 < |k| \leqslant K,$$

and $F_k = 0$ with $k = 0$ or $|k| > K$.

By Lemma A.1 in Appendix A, we have

$$\big\|F(\theta, I; \xi)\big\|_{r,s-\rho;d} \leqslant \frac{n^\tau \epsilon}{\alpha \rho^\tau}.$$

C. Estimates for the symplectic map. Let

$$W = \operatorname{diag}\big(\rho^{-1} I_n, r^{-1} I_n\big).$$

By Lemma A.1, we have

$$\|W X_F\|_{r,s-2\rho;d} \leqslant \frac{n^\tau \epsilon}{\alpha r \rho^{\tau+1}} = n^\tau E.$$

Thus, if $0 < \eta \leqslant \frac{1}{8}$, and $n^\tau E \leqslant \frac{1}{8}$, then, for all $\xi \in \Pi_d$ we have

$$\Phi = X_F^1 : D(r\eta, s - 3\rho) \to D(2r\eta, s - 2\rho).$$

So

$$\big\|W(\Phi - id)\big\|_{s-5\rho,\eta r;d} \leqslant n^\tau E, \qquad \big\|W(\mathcal{D}\Phi - Id)W^{-1}\big\|_{s-5\rho,\eta r;d} \leqslant n^\tau E,$$

where $\mathcal{D}$ is the differentiation operator with respect to (θ, I).

D. Estimates of error terms. Let $\alpha_+ = \alpha - \frac{K^{\tau+1}\epsilon}{2r}$. If $\epsilon \leqslant \frac{\alpha r}{K^{\tau+1}}$, we have

$$\big|\langle\omega_+(\xi), k\rangle\big| \geqslant \frac{2\alpha_+}{|k|^\tau}, \quad \forall \xi \in \Pi,\ \forall 0 \neq |k| \leqslant K.$$

Thus, by the definition of $\tilde{\Pi}$, it follows easily that (2.3) holds. Thus, small divisor condition for the next step holds.

Let $r_+ = \eta r$, $\rho_+ = \sigma\rho$. By Lemmas A.2, A.3 and A.4, it follows that

$$\|P_+\|_{s_+,r_+;d} < c\left[\frac{\epsilon^2}{\alpha r\rho^{\tau+1}} + \left(\eta^2 + e^{-K\rho}\right)\epsilon\right],$$

and

$$\left|\omega_+(\xi) - \omega(\xi)\right| \leqslant \frac{\epsilon}{r}, \quad \forall \xi \in \Pi_d.$$

Suppose $d_+ \leqslant \frac{2}{3}d$. Then, by the Cauchy estimates we have

$$\left|\partial\left(\omega_+(\xi) - \omega(\xi)\right)/\partial\xi\right| \leqslant \frac{3\epsilon}{dr}, \quad \forall \xi \in \Pi_{d_+}.$$

Let $T_+ = T + \frac{3\epsilon}{dr}$. Then $\max_{\xi\in\Pi_{d_+}} |\partial\omega_+/\partial\xi| \leqslant T_+$.

Moreover, if $\alpha \leqslant 2\alpha_+$, it follows that

$$\|P_+\|_{s_+,r_+;d} \leqslant c\epsilon E \leqslant c\alpha_+ r_+ \rho_+^{\tau+1} E^{\frac{3}{2}} = \alpha_+ r_+ \rho_+^{\tau+1} E_+,$$

where $E_+ = cE^{\frac{3}{2}}$ with c a constant depending only on n, τ. Thus, it follows that $\|P_+\|_{s_+,r_+;d} < \epsilon_+$.

Note that here the constants c only depend on n, τ, μ, and ω, and are independent of KAM steps. □

2.2. Iteration

Now we choose some suitable parameters so that the above iteration can go on infinitely.

At the initial step, let $\rho_0 = (1-\sigma)s/10$, $r_0 = r$, $\epsilon_0 = \alpha_0 r_0 \rho_0^{\tau+1} E_0$. Let K_0 satisfy $e^{-K_0\rho_0} = E_0$. $\alpha_0 = \alpha > 0$, $\omega_0 = \omega$, $T_0 = T = \max_{\xi\in\Pi_d} |\partial\omega/\partial\xi|$. Denote

$$\Pi_0 = \left\{\xi \in \Pi \;\middle|\; \left|\langle\omega_0(\xi), k\rangle\right| \geqslant \frac{2\alpha}{|k|^\tau},\ \forall 0 < |k| \leqslant K_0\right\}.$$

Chose $d = \alpha^{\frac{1}{m}}$. Note that this choice for d is only for measure estimate for parameter and has no conflict with the assumption in Theorem 1.2 since we can use a smaller d.

Let $d_0 = \frac{\alpha_0}{2T_0 K_0^{\tau+1}} \leqslant d$ and $\eta_0 = E_0^{\frac{1}{2}}$. Assume the above parameters are all well defined for j. Then, we define $\rho_{j+1} = \sigma\rho_j$, $r_{j+1} = \eta_j r_j$ and $E_{j+1} = cE_j^{\frac{3}{2}}$, $\alpha_{j+1} = \alpha_j - \frac{\epsilon_j}{2r_j} K_j^{\tau+1}$. Define ϵ_{j+1}, η_{j+1}, K_{j+1}, and d_{j+1} in the same way as the previous step.

Since $E_j = cE_{j-1}^{\frac{3}{2}}$, and $x_j = K_j\rho_j = -\ln E_j$, if E_0 is sufficiently small such that $-\ln c/\ln E_j \leqslant (1-\sigma)3/2$, it follows that $3/2 \leqslant \frac{K_{j+1}}{K_j} \leqslant 3/(2\sigma)$. Thus, $d_{j+1} \leqslant \frac{2}{3}d_j$ and so the assumption $d_+ \leqslant \frac{2}{3}d$ in KAM steps hold. Suppose $\max_{\xi\in\Pi_{d_j}} |\partial\omega_j/\partial\xi| \leqslant T_j$. Let $T_{j+1} = T_j + \frac{3\epsilon_j}{d_j r_j}$. Then we have $\max_{\xi\in\Pi_{d_{j+1}}} |\partial\omega_{j+1}/\partial\xi| \leqslant T_{j+1}$.

Again, by the choice of σ, it follows easily that $\rho_{j+1} x_{j+1}^{\frac{l}{\tau+1}} \geqslant \rho_j x_j^{\frac{l}{\tau+1}}$. By induction, it is easy to see that if E_0 is sufficiently small such that $\rho_0 x_0^{\frac{l}{\tau+1}} \geqslant 1$, we have $\rho_j x_j^{\frac{l}{\tau+1}} \geqslant 1$ for all $j \geqslant 1$.

Let $F_j = \frac{\epsilon_j}{d_j r_j}$. It follows that $F_j = 2T_j x_j^{\tau+1} e^{-x_j}$. Suppose $T_j \leqslant T + 1$. Then we have $\sum_{j\geqslant 0} F_j \leqslant c x_0^{-1}$. Thus, if E_0 is sufficiently small such that $c x_0^{-1} \leqslant \frac{1}{3}$, then $T \leqslant T_{j+1} \leqslant T + 1$.

Let

$$\Pi_{j+1} = \left\{ \xi \in \Pi_j \;\middle|\; \left|\langle \omega_{j+1}(\xi), k\rangle\right| \geqslant \frac{2\alpha_{j+1}}{|k|^\tau},\ \forall K_j < |k| \leqslant K_{j+1} \right\}.$$

Denote $\Pi_{d_j} = \{\xi \in \mathbf{C}^n \mid \operatorname{dist}(\xi, \Pi_j) \leqslant d_j\}$ and $D_j = D(s_j, r_j)$ for simplicity. Note that here and below the notation Π_{d_j} is different from the previous one Π_d.

By the KAM-step, for all $\xi \in \Pi_{d_j}$ we have symplectic mappings

$$\Phi_j(\cdot,\cdot;\xi) : D(r_{j+1}, s_{j+1}) \to D(r_j, s_j)$$

satisfying

$$\left\| W_j(\Phi_j - id) \right\|_{s_{j+1}, r_{j+1}; d_{j+1}} \leqslant n^\tau E_j$$

and

$$\left\| W_j(\mathcal{D}\Phi_j - Id) W_j^{-1} \right\|_{s_{j+1}, r_{j+1}; d_{j+1}} \leqslant n^\tau E_j.$$

Let $\Phi^j = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{j-1}$, and $H_j = H \circ \Phi^j = N_j + P_j$, where $N_j = \langle \omega_j(\xi), I\rangle$. Then we have $|\omega_{j+1} - \omega_j| \leqslant \frac{\epsilon_j}{r_j}$, for all $\xi \in \Pi_{d_j}$. Moreover, $\|P_j\|_{s_j, r_j; d_j} \leqslant \epsilon_j$.

2.3. Convergence of the iteration

Now we prove convergence of the KAM-iteration. In the same way as in [9,10], it follows that, if $c^{\frac{1}{2}} E_0 \leqslant \frac{1}{2}$, then

$$\left\| W_0 \mathcal{D}\Phi^j W_j^{-1} \right\|_{D_j \times \Pi_{d_j}} \leqslant \prod_{i=1}^{j} \left(1 + n^\tau E_j\right) < 2.$$

So, we have

$$\left\| W_0\left(\Phi^j - \Phi^{j-1}\right) \right\|_{D_j \times \Pi_{d_j}} \leqslant c E_j,$$

and

$$\left\| W_0 \mathcal{D}\left(\Phi^j - \Phi^{j-1}\right) \right\|_{D_j \times \Pi_{d_j}} \leqslant c E_j.$$

By the Cauchy's estimates we have

$$\left\| W_0 \partial_\xi^\beta \left(\Phi^j - \Phi^{j-1}\right) \right\|_{D_j \times \Pi_j} \leqslant \frac{c E_j \beta!}{d_j^{|\beta|}},$$

$$\left\| W_0 \partial_\xi^\beta \mathcal{D}\left(\Phi^j - \Phi^{j-1}\right) \right\|_{D_j \times \Pi_j} \leqslant \frac{c E_j \beta!}{d_j^{|\beta|}},$$

and

$$\left|\partial_\xi^\beta(\omega_{j+1}-\omega_j)\right|_{\Pi_j} \leqslant \frac{c\epsilon_j\beta!}{r_j d_j^{|\beta|}}.$$

Let $J_j^\beta = \frac{cE_j\beta!}{d_j^{|\beta|}}$ and $L_j^\beta = \frac{c\epsilon_j\beta!}{r_j d_j^{|\beta|}}$. Now we estimate J_j^β and L_j^β for all $\beta \in Z_+^n$. Again

$$\alpha_{j+1} = \alpha_j - \frac{\epsilon_j}{2r_j}K_j^{\tau+1} = \alpha_j\left(1 - \frac{1}{2}x_j^{\tau+1}e^{-x_j}\right).$$

It follows that if E_0 is sufficiently small, then

$$\prod_{j=0}^{\infty}\left(1 - \frac{1}{2}x_j^{\tau+1}e^{-x_j}\right) = 1 - O\left(x_0^{-1}\right) \geqslant \frac{1}{2}.$$

Thus, $\frac{1}{2}\alpha_0 \leqslant \alpha_j \leqslant \alpha_0$. Obviously, we have $\frac{1}{2}\alpha_j \leqslant \alpha_{j+1} \leqslant \alpha_j$. Thus, the assumption $\alpha_+/2 \leqslant \alpha_+ \leqslant \alpha$ holds. By $\frac{1}{2}\alpha_j \leqslant \alpha_{j+1}$ and the definition of α_{j+1}, it follows that

$$\epsilon_j \leqslant \frac{\alpha_j r_j}{K_j^{\tau+1}}$$

and so the assumption $\epsilon \leqslant \frac{\alpha r}{K^{\tau+1}}$ holds in KAM step.

Let $E_0 = (\frac{10}{1-\sigma})^{\tau+1}\gamma$. By the above discussion, if γ is sufficiently small, under the assumptions of Theorem 1.2, the assumptions of the iteration lemma hold for H at the first step. Then the KAM step can go on infinitively.

Since

$$\mu - 1 = \tau + 1 + l, \quad d_j = \alpha_j/\left(2T_j K_j^{\tau+1}\right), \quad \rho_j x_j^{\frac{l}{\tau+1}} \geqslant 1 \quad \text{and} \quad \frac{\alpha}{2} \leqslant \alpha_j \leqslant \alpha,$$

we have

$$\begin{aligned}
J_j^\beta &\leqslant c\left(\frac{2T_j}{\alpha_j}\right)^{|\beta|}\frac{\beta!E_j}{d_j^{|\beta|}} \leqslant c\beta!x_j^{(\tau+1+l)|\beta|}e^{-x_j} \\
&\leqslant c\left(\frac{4(T+1)}{\alpha}\right)^{|\beta|}\beta!\left[x_j^{\beta_1}/e^{\frac{x_j}{(n+1)(\mu-1)}}\cdots x_j^{\beta_n}/e^{\frac{x_j}{(n+1)(\mu-1)}}\right]^{\mu-1}e^{-\frac{x_j}{n+1}} \\
&\leqslant cM^{|\beta|}\beta!^\mu E_j^{\frac{1}{n+1}},
\end{aligned}$$

where $M = 4(T+1)[(n+1)(\mu-1)]^{\mu-1}/\alpha$, and, c only depends on n, α, μ. In the same way as the above, it follows that

$$L_j^\beta \leqslant 2c\alpha M^{|\beta|}\beta!^\mu E_j^{\frac{1}{n+1}}\rho_j^{\tau+1}.$$

Let $D_* = D(0, \frac{1}{2}s)$, $\Pi_* = \bigcap_{j\geqslant 0} \Pi_j$ and $\Phi_* = \lim_{j\to\infty} \Phi^j$. Thus, for any $\beta \in Z_+^n$ we have

$$\left\| W_0 \partial_\xi^\beta (\Phi_* - id) \right\|_{D_*\times\Pi_*} \leqslant cM^{|\beta|} \beta!^\mu E_0^{\frac{1}{n+1}}.$$

In the same way, we have

$$\left\| W_0 \partial_\xi^\beta (D\Phi_* - Id) \right\|_{D_*\times\Pi_*} \leqslant cM^{|\beta|} \beta!^\mu E_0^{\frac{1}{n+1}}.$$

Since Φ^j is affine in I, we have convergence of $\partial_\xi^\beta \Phi^j$ to Φ_* on $D(r/2, s/2)$ and

$$\left\| W_0 \partial_\xi^\beta (\Phi_* - id) \right\|_{D(s/2,r/2)\times\Pi_*} \leqslant cM^{|\beta|} \beta!^\mu E_0^{\frac{1}{n+1}}, \quad \forall \beta \in Z^n. \tag{2.5}$$

Since $E_0 = (\frac{10}{1-\sigma})^{\tau+1}\gamma$, this proves (1.6).

Let $\omega_* = \lim_{j\to\infty} \omega_j$. We have

$$\left|\partial_\xi^\beta (\omega_* - \omega)\right|_{\Pi_*} \leqslant c\alpha M^{|\beta|} \beta!^\mu E_0^{\frac{1}{n+1}} \rho_0^{\tau+1}.$$

Moreover, $|\langle \omega_*(\xi), k\rangle| \geqslant \frac{2\alpha_*}{|k|^\tau}$, for all $0 \neq k \in Z^n$ and $\xi \in \Pi_*$, where $\alpha_* = \lim_{j\to\infty} \alpha_j$ with $\frac{1}{2}\alpha_0 \leqslant \alpha_* \leqslant \alpha_0$. Thus, (1.7) and (1.8) hold.

Let m be the smallest integer such that Eq. (1.4) holds and $|\beta| \leqslant m$. Since $T \leqslant T_j \leqslant T+1$, from $\frac{d_{j+1}}{d_j} = \frac{\alpha_{j+1}T_j}{\alpha_j T_{j+1}}(\frac{K_j}{K_{j+1}})^{\tau+1}$ it follows that $\frac{T}{2(T+1)}(\frac{2\sigma}{3})^\tau \leqslant \frac{d_{j+1}}{d_j} \leqslant (\frac{2}{3})^\tau$.

It follows that $J_{j+1}^\beta / J_j^\beta \leqslant cE_j^{\frac{1}{2}}$, where c depends on β. If $|\beta| \leqslant m$, in the same way as the above, we have

$$\left\| W_0 \partial_\xi^\beta (\Phi_* - id) \right\|_{D(s/2,r/2)\times\Pi_*} \leqslant cE_0. \tag{2.6}$$

Similarly, we have

$$\sum_{j\geqslant 1} L_j^\beta \leqslant cE_0, \quad \forall |\beta| \leqslant m.$$

So,

$$\left|\partial_\xi^\beta (\omega_{j+1} - \omega)\right|_{\Pi_j} \leqslant cE_0, \quad \forall j \geqslant 1,\ \forall |\beta| \leqslant m. \tag{2.7}$$

2.4. Estimates of measure for the parameter sets

Now we estimate the Lebesgue measure of the set Π_*, for which the small divisor condition holds in the KAM iteration. By the KAM step, we have

$$\Pi \setminus \Pi_* = \bigcup_{j\geqslant -1} \tilde{\Pi}_j,$$

where

$$\tilde{\Pi}_j = \left\{ \xi \in \Pi_j \;\middle|\; \left|\langle \omega_{j+1}(\xi), k\rangle\right| \leqslant \frac{2\alpha_{j+1}}{|k|^{\tau}},\ \forall K_j < |k| \leqslant K_{j+1} \right\}$$

and $K_{-1} = 0$. By the equivalent Rüssmann's non-degeneracy condition (1.4) and the estimate (2.7), if E_0 is sufficiently small, then for all $j \geqslant 0$ the frequency $\omega_j(\xi)$ also satisfies (1.4). So by Lemma A.5 (see [19]) we have

$$\begin{aligned} \operatorname{meas}(\tilde{\Pi}_j) &\leqslant c\big[\operatorname{diam}(\Pi)\big]^{n-1} \sum_{K_j < |k| \leqslant K_{j+1}} \big(\alpha_j/|k|^{\tau+1}\big)^{\frac{1}{m}} \\ &\leqslant c\big[\operatorname{diam}(\Pi)\big]^{n-1} \alpha^{\frac{1}{m}} \sum_{K_j < |k| \leqslant K_{j+1}} 1/|k|^{\frac{\tau+1}{m}}. \end{aligned}$$

Since $\tau > mn - 1$, we deduce

$$\begin{aligned} \operatorname{meas}(\Pi \setminus \Pi_*) &\leqslant c\big[\operatorname{diam}(\Pi)\big]^{n-1} \alpha^{\frac{1}{m}} \sum_{0 \neq k \in Z^n} 1/|k|^{\frac{\tau+1}{m}} \\ &\leqslant c\big[\operatorname{diam}(\Pi)\big]^{n-1} \alpha^{\frac{1}{m}}. \end{aligned}$$

Appendix A

In this section we state several lemmas. Some of the lemmas describe properties of the norm $\|\cdot\|_{s,r}$. The proofs are very similar to [10] and even simpler; so we omit them.

Lemma A.1. *Let $f(\theta, I)$ be analytic on $D(s,r)$. Then $\|f_\theta\|_{s-\rho,r} \leqslant \frac{1}{e\rho}\|f\|_{s,r}$ and $\|f_I\|_{s,r-\sigma} \leqslant \frac{1}{\sigma}\|f\|_{s,r}$ for $0 < \rho < s$ and $0 < \sigma < r$.*

Lemma A.2. *Let $f(\theta, I)$ and $g(\theta, I)$ be analytic on $D(s,r)$. Then*

$$\|fg\|_{s,r} \leqslant \|f\|_{s,r}\|g\|_{s,r}.$$

Lemma A.3. *Let $F(\theta, I)$ and $G(\theta, I)$ be analytic on $D(s,r)$. For $0 < \rho < s$ and $0 < \sigma < r$ we have*

$$\big\|\{F, G\}\big\|_{s-\rho,r-\sigma} \leqslant \frac{2}{\rho\sigma}\|F\|_{s,r}\|G\|_{s,r}.$$

Lemma A.4. *Let $F(\theta, I)$ be analytic on $D(s-\rho, r)$ and affine linear in I. Let $0 < \rho < s/3$. If $\|F\|_{s-\rho,r} \leqslant \rho r/6e$, then $X_F^t : D(s-3\rho, r/2) \to D(s-2\rho, r)$, for $0 \leqslant t \leqslant 1$. Moreover,*

$$\|G \circ \Phi\|_{s-3\rho,r/2} \leqslant 2\|G\|_{s,r}.$$

Proof. The following proof is actually given in [10]. Let $\Phi = X_F^1$. By the Lie series expansion we have

$$G \circ \Phi = \sum_{l \geqslant 0} ad_F^l G,$$

where

$$ad_F^0 G = G, \qquad ad_F^l G = \{ad_F^{l-1} G, F\}, \quad l = 1, 2, \ldots.$$

Let $\rho' = \rho/l$, $\sigma' = r/(2l)$, $D_l = D(s - 2\rho - l\rho', r - l\sigma')$. We conclude

$$\begin{aligned}
\|ad_F^l G\|_{s-3\rho, r/2} &= \|ad_F^l G\|_{D_l} \\
&\leqslant \|\langle F_\theta, ad_F^{l-1} G_I\rangle\|_{D_l} + \|\langle F_I, ad_F^{l-1} G_\theta\rangle\|_{D_l} \\
&\leqslant \frac{1}{\rho}\|F\|_{s-\rho,r}\frac{2l}{r}\|ad_F^{l-1} G\|_{D_{l-1}} + \frac{1}{r}\|F\|_{s-\rho,r}\frac{l}{\rho}\|ad_F^{l-1} G\|_{D_{l-1}} \\
&\leqslant l(3\|F\|_{s-\rho,r}/\rho r)\|ad_F^{l-1} G\|_{D_{l-1}} \\
&\leqslant [l(3\|F\|_{s-\rho,r}/\rho r)]^l \|G\|_{s,r}.
\end{aligned}$$

By Stirling's formula, $l^l/l! \leqslant e^l$ for $l \geqslant 1$. So we obtain

$$\begin{aligned}
\|G \circ \Phi\|_{s-3\rho, r/2} &\leqslant \sum_{l \geqslant 0} \frac{1}{l!}\|ad_f^l G\|_{s-2\rho, r/2} \\
&\leqslant \sum_{l \geqslant 0} (3e\|F\|_{s-\rho,r}/\rho r)^l \|G\|_{s,r} \\
&\leqslant 2\|G\|_{s,r}. \qquad \square
\end{aligned}$$

Lemma A.5. *Let* $0 \neq k \in Z^n$ *and*

$$\Pi_k = \{\xi \in \Pi \mid |\langle \omega(\xi), k\rangle| \leqslant \alpha/|k|^\tau\}.$$

If $\omega(\xi)$ *satisfies the equivalent Rüssmann's non-degeneracy condition* (1.4)*, then*

$$\operatorname{meas}(\Pi_k) \leqslant c[\operatorname{diam}(\Pi)]^{n-1}(\alpha/|k|^{\tau+1})^{\frac{1}{m}}.$$

For the proof of this lemma see [16,19].

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