POINT SPECTRUM FOR THE QUASI-PERIODIC LONG RANGE OPERATORS

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Abstract. We generalize Gordon type argument to quasi-periodic operators with finite range interaction and prove that these operators have no point spectrum when the rational approximation rate of the base frequency is comparably large. We also show that this kind of argument may not be true in the infinite range interaction cases, where we find some point spectrum.

1. Introduction

We consider the quasi-periodic operators $L_{f,\phi,\alpha}$ acting on $l^2(\mathbb{Z})$:

$$(L_{f,\phi,\alpha}u)_n = \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k u_{n-k} + 2\cos 2\pi(\phi + n\alpha)u_n, \tag{1}$$

where $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, $f(\theta) \in C_h^\omega(\mathbb{T},\mathbb{R})$ with zero average and f_k are fourier coefficients of $f(\theta)$ such that

$$f(\phi) = \sum_{1\le|k|\le D} f_k e^{2\pi \mathrm{i}\phi}, \quad f_{-k} = \bar{f}_k,\ 1 \le D \le \infty. \tag{2}$$

If $D < \infty$ and $f_D \neq 0$, then $f(\theta)$ is a real trigonometric polynomial and we shall call $L_{f,\phi,\alpha}$ operators with finite range interactions. If $D = \infty$, we call $L_{f,\phi,\alpha}$ operators with infinite range interactions or long range operators. In this paper, we discuss the absence and presence of point spectrum of the operators $L_{f,\phi,\alpha}$ with respect to parameters (α, D). To measure how Liouvillean α is, we denote

$$\beta(\alpha) := \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}, \tag{3}$$

where $\frac{p_n}{q_n}$ is the $n-$th convergent of α.

The operator $L_{f,\phi,\alpha}$ is of special interest and importance. Firstly, it is a natural generalization of the almost Mathieu operator (AMO) by letting $D = 1$ and $f_{-1} = f_1 = \lambda^{-1}$ (up to a multiplying constant).

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Secondly, it is the Aubry dual (Fourier type transformation) of classical Schrödinger operator on $l^2(\mathbb{Z})$:

$$(H_{f,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + f(\theta + n\alpha)u_n, \tag{4}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $f(\theta) \in C_h^\omega(\mathbb{T}, \mathbb{R})$. Readers can consult Haro and Puig's recent paper [10] for more relationship between these two operators.

We first recall some results about the point spectrum of the almost Mathieu operator. In studying the problem of absence of point spectrum of the almost Mathieu operator $H_{2\lambda\cos 2\pi,\theta,\alpha}$, Gordon type argument is one of the most efficient ways. The original Gordon's lemma is due to Gordon [9] and was first applied to AMO by Avron and Simon [5]. It says that if $\beta(\alpha) = \infty$, then $H_{2\lambda\cos 2\pi,\theta,\alpha}$ has no point spectrum for any θ, λ. Combined with some finer estimate and the explicit formula of the Lyapunov exponent of AMO, Gordon type argument can actually shows that if $\lambda < e^{\beta/2}$, then $H_{2\lambda\cos 2\pi,\theta,\alpha}$ has no point spectrum for any θ. It was conjectured by Jitomirskaya in [12] that the optimal condition for $H_{2\lambda\cos 2\pi,\theta,\alpha}$ to have purely singular continuous spectrum should be $1 < \lambda < e^\beta$, which still remains open.

On the other hand, it was proved by Avila and Jitomirskaya in [2] that if $\lambda > e^{16\beta/9}$, then $H_{2\lambda\cos 2\pi,\theta,\alpha}$ has Anderson localization (pure point spectrum with exponentially decaying eigen-functions) for a.e. θ. And the optimal condition for AMO to exhibit Anderson localization was conjectured to be $\lambda > e^\beta$ in [2], which also remains open.

Next we want to see what happens to both sides of the above questions when we generalize AMO to the operators $L_{f,\phi,\alpha}$. Our first question will be:

Question 1**:** When will the operators $L_{f,\phi,\alpha}$ still possess some point spectrum and when not?

We shall see below that the answers to Question 1 may be different according to choice of the parameters (α, D).

For any $1 \le D \le \infty$, Bourgain and Jitomirskaya [6] proved that if α is Diophantine, and λ is large enough, then $L_{\lambda^{-1}f,\phi,\alpha}$ has Anderson localization for a.e. ϕ. This result was later generalized by Avila-Jitomirskaya [3], who proved that if $\beta(\alpha) < \infty$, $\lambda > \lambda_0(\beta, f)$, then $L_{\lambda^{-1}f,\phi,\alpha}$ has Anderson localization for a.e. ϕ.

With the help of Gordon type argument, we have the following answers to Question 1 in the opposite direction of the above results.

Theorem 1.1. *Suppose that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $D < \infty$, denote $\|f\|_1 = \sum_{1\leq|k|\leq D}|f_k|$. If*

$$\beta(\alpha) > 2D\ln\frac{\|f\|_1}{|f_D|}, \tag{5}$$

then for

$$\lambda < \frac{1}{4}\big(|f_D|e^{\frac{\beta}{2D}} - \|f\|_1\big), \tag{6}$$

$L_{\lambda^{-1}f,\phi,\alpha}$ has no point spectrum for any $\phi \in \mathbb{T}$.

Remark 1.1. *Theorem 1.1 holds automatically when $\beta = \infty$ and $D < \infty$.*

We emphasize that Gordon type argument is not necessarily to be restricted to quasi-periodic Schrödinger operators, readers can consult [7, 8] for more discussions and applications of Gordon type argument on Schrödinger operators (including operators which are not quasi-periodic). Instead of proving Theorem 1.1 directly, we would like to study the following operators:

$$(H^D_{V,\theta,\alpha}u)_n = \sum_{1\leq|k|\leq D} f_k u_{n-k} + V(\theta+n\alpha)u_n,\ n\in\mathbb{Z} \tag{7}$$

and generalize the Gordon type argument to the above operators. In the operators $H^D_{V,\theta,\alpha}$, V is assumed to be Lipschitz continuous on $\mathbb{T}$, $f : \mathbb{T} \to \mathbb{R}$ is a real trigonometric polynomial, with $f_{-k} = \bar{f}_k$, $f_D \neq 0$. Denote $\|V\|_\infty = \sup_{\theta\in\mathbb{T}}|V(\theta)|$, then we have:

Theorem 1.2. *For Lipschitz continuous potential V, if*

$$\beta(\alpha) > 2D\ln\ \frac{2\|V\|_\infty + \|f\|_1}{|f_D|}\ , \tag{8}$$

then $H^D_{V,\theta,\alpha}$ has purely continuous spectrum for any $\theta \in \mathbb{T}$.

Theorem 1.2 is no longer applicable when $D = \infty$, therefore, the remaining question will be:

Question 2**:** If $\beta = \infty$ and $D = \infty$, could $L_{f,\phi,\alpha}$ possess some point spectrum?

We give an answer to this question in the following theorem:

Theorem 1.3. *Let $h > 0$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, then there exists $\varepsilon_0 = \varepsilon_0(h)$ such that we have the following:*

(a): *There is a local dense set of* $f \in C_h^\omega(\mathbb{T},\mathbb{R})$ *in* $\left\{f \in C_h^\omega(\mathbb{T},\mathbb{R}) : \|f\|_h \leq \varepsilon_0\right\}$ *such that* $L_{f,\phi,\alpha}$ *has point spectrum with exponentially decay eigenfunctions for some* $\phi \in \mathbb{T}$.

(b): *Moreover, if* $\beta(\alpha) = \infty$, *then there also exists a global dense set of* $f \in C_h^\omega(\mathbb{T},\mathbb{R})$, *such that* $L_{f,\phi,\alpha}$ *has no point spectrum for any* $\phi \in \mathbb{T}$.

Remark 1.2. *Gordon's lemma gives a criterion for absence of point spectrum for all phases, this can be generalized to finite range interaction case, see Theorem 1.1, however, Theorem 1.3* (a) *shows that this is not always true when* $D = \infty$.

2. Preliminaries

2.1. **Fibered rotation number.** An analytic quasi-periodic $SL(2,\mathbb{R})$ cocycle (α, A) is a skew product defined as

$$\begin{aligned}(\alpha, A): \qquad & \mathbb{T}\times\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}^2\\ & (\theta, v) \mapsto (\theta+\alpha, A(\theta)\cdot v),\end{aligned}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $A \in C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$.

Denote by $\mathbb{P}^1$ be the projection space of $\mathbb{R}^2$ and by $\pi : \mathbb{R} \to \mathbb{P}^1$ the projection $\pi(x) = e^{2\pi i x}$. Assume that $A(\cdot)$ is furthermore homotopic to the identity. The map

$$\begin{aligned}F : \mathbb{T}\times\mathbb{P}^1 &\to \mathbb{T}\times\mathbb{P}^1\\ (\theta, v) &\mapsto (\theta+\alpha, \frac{A(\theta)v}{\|A(\theta)v\|})\end{aligned}$$

admits a continuous lift $\widetilde{F} : \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$ of the form $\widetilde{F}(\theta, x) = (\theta+\alpha, x+f(\theta,x))$ such that $f(\theta, x+1) = f(\theta,x)$ and $\pi(x+f(\theta,x)) = A(\theta)\pi(x)/\|A(\theta)\pi(x)\|$. We say that $\widetilde{F}$ is a lift for the cocycle (α, A). Since $\theta \mapsto \theta+\alpha$ is uniquely ergodic on $\mathbb{T}$, then for every $(\theta, x) \in \mathbb{T}\times\mathbb{R}$ the limit

$$rot_f(\alpha, A) := \lim_{n\to\pm\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(\widetilde{F}^k(\theta, x)) \qquad \mathrm{mod}\mathbb{Z},$$

exists, is independent of (θ, x), the chosen lift $\widetilde{F}$, and the convergence is uniform in (θ, x) [13]. We call it the fibered rotation number of (α, A).

Recall that two cocycles $(\alpha, A^i), i = 1, 2$ are conjugated means that there exists $B \in C^\omega(2\mathbb{T}, SL(2,\mathbb{R}))$ such that

$$A^1(\theta) = B(\theta+\alpha)A^2(\theta)B(\theta)^{-1}.$$

Denote it by $(\alpha, A^1) = Ad_B(\alpha, A^2)$. Fibered rotation number is a conjugacy invariant in the following sense:

Proposition 2.1 (Krikorian, [14]). *Let $(\alpha, A^i) \in \mathbb{R}\backslash\mathbb{Q}\times C^\omega(\mathbb{T}, SL(2,\mathbb{R}))$, $i = 1, 2$ be two conjugated quasi-periodic cocycles. If the conjugacy $B \in C^\omega(2\mathbb{T}, SL(2,\mathbb{R})$ and has degree k, then*

$$rot_f(\alpha, A^1) = rot_f(\alpha, A^2) + \frac{1}{2}k\alpha \quad mod\ \mathbb{Z},$$

3. Finite range interaction case

In this section, we generalize Gordon type argument to the quasi-periodic operators with finite range interaction $H^D_{V,\theta,\alpha}$ defined in (7). Consider the eigenvalue equations of the corresponding operators:

$$\sum_{1\le|k|\le D} f_k u_{n-k} + V(\theta + n\alpha)u_n = Eu_n.$$

Since $f_{-D} \neq 0$, this equation can be rewritten as a skew-product system of order $2D$.

$$X(n) = A(\theta + n\alpha, E, \alpha)X(n-1),$$

where

$$X(n) = \begin{pmatrix} u_{n+D} \\ u_{n+D-1} \\ \cdots \\ u_{n-D+1} \end{pmatrix} \in \mathbb{C}^{2D},$$

and

$$A(\theta, E, \alpha) = \tag{9}$$
$$\begin{pmatrix} -\frac{f_{-D+1}}{f_{-D}} & -\frac{f_{-D+2}}{f_{-D}} & \cdots & \frac{E-V(\theta)}{f_{-D}} & \cdots & -\frac{f_{D-1}}{f_{-D}} & -\frac{f_D}{f_{-D}} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots \\ \cdots & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}$$

The proof of Theorem 1.2 will follow the proof of the original Gordon's lemma. In the following, we denote $A_n = A\big(\theta + (n-1)\alpha\big)$ and

$$M_n(\theta, A, \alpha) = \begin{cases} A_n A_{n-1} \cdots A_1, & n > 0 \\ Id, & n = 0 \\ A^{-1}_{-n-1} \cdots A^{-1}_{-1} A^{-1}_0, & n < 0 \end{cases}$$

Set

$$X(n) = M_n(\theta, E, \frac{p}{q})X(0),\ X(0) \in \mathbb{C}^{2D},$$

first we prove that:

Lemma 3.1. *The following estimate holds:*

$$\max_{0<|i|\leq 2D, i\in\mathbb{Z}}\{\|X(iq)\|\}\geq \frac{1}{2D}\|X(0)\|.$$

Proof. Denote $M=M_q(\theta,E,\frac{p}{q})$ for short. Since M is q periodic, then $M^i=M_{iq}(\theta,E,\frac{p}{q})$ and $M^iX(0)=X\big(iq\big)$. By Cayley-Hamilton Theorem, there exist $c_0,c_1,\cdots,c_{2D}\in\mathbb{C}$ such that

$$c_{2D}M^{2D}+c_{2D-1}M^{2D-1}+\cdots+c_1M+c_0Id=0.$$

Suppose that

$$|c_j|=\max_{0\leq k\leq 2D,k\in\mathbb{N}}|c_k|$$

for some $j\in\{0,\cdots,2D\}$, without loss of generality, we assume that $|c_j|=1$. Notice that $M\in SL(2D,\mathbb{C})$ is invertible, then

$$c_{2D}M^{2D-j}+c_{2D-1}M^{2D-1-j}+\cdots+c_jId+\cdots+c_1M^{1-j}+c_0M^{-j}=0.$$

Consequently, we have

$$\begin{aligned}
&\|X(0)\|\\
\leq\ &\|c_{2D}M^{2D-j}X(0)+\cdots+c_{j+1}MX(0)\\
&+c_{j-1}M^{-1}X(0)+\cdots+c_1M^{1-j}X(0)+c_0M^{-j}(0)\|\\
\leq\ &\|X\big((2D-j)q\big)\|+\cdots+\|X\big(q\big)\|\\
&+\|X\big(-q\big)\|+\cdots+\|X\big((1-j)q\big)\|+\|X\big((-j)q\big)\|.
\end{aligned}\tag{10}$$

There are exact $2D$ terms in (10), thus at least one of them must be larger than $\frac{1}{2D}\|X(0)\|$. Notice that j takes values in $0,\cdots,2D$, then we conclude that there exists $i\in\{\pm1,\cdots,\pm2D\}$ such that $\|X\big(iq\big)\|\geq\frac{1}{2D}\|X(0)\|$. □

Now we approximate a quasi-periodic operator by periodic operators, suppose that $\alpha_k=\frac{p_k}{q_k}$ is the best rational approximation of α, for each $k\in\mathbb{N}$, set

$$A_n^{(k)}=A\big(\theta+(n-1)\frac{p_k}{q_k}\big),\ M_n^{(k)}=M_n(\theta,E,\alpha_k).\tag{11}$$

Now we can estimate the approximation error $\|M_n-M_n^{(k)}\|$, we are going to show the following lemma:

Lemma 3.2. *Suppose that V is Lipschitz continuous with Lipschitz constant C_1, then the following conclusion holds: for any $k\in\mathbb{N}$,*

$$\sup_{|n|\leq 2Dq_k}\sup_{\theta\in\mathbb{T}}\|M_n-M_n^{(k)}\|\leq 4D^2C_1\cdot\frac{q_k}{q_{k+1}}\Big(\frac{2\|V\|_\infty+\|f\|_1}{|f_D|}\Big)^{2Dq_k}.$$

Proof. As a result of the Lipschitz continuity of V and explicit form of $A(\theta, E, \alpha)$ in (9), we have for any q_k,

$$\sup_{|i|\le 2Dq_k}\sup_{\theta\in\mathbb{T}}\|A_i - A_i^{(k)}\| < \frac{2DC_1}{|f_D|q_{k+1}}. \tag{12}$$

On the other hand, notice that

$$\sigma(H^D_{V,\theta,\alpha}) \subset [-\|V\|_\infty - \|f\|_1, \|V\|_\infty + \|f\|_1],$$

then for $|E| \le \|V\|_\infty + \|f\|_1$, the maximal norm of $A(\theta, E, \alpha)$ satisfies

$$\|A(\theta, E, \alpha)\| \le \frac{2\|V\|_\infty + \|f\|_1}{|f_D|}, \tag{13}$$

for any $\theta \in \mathbb{T}, \alpha \in \mathbb{R}$.

Therefore, when $0 < n \le 2Dq_k$, according to estimates (12) and (13), we have

$$\begin{aligned}
& \|M_n - M_n^{(k)}\| \\
= \ & \Big\| \sum_{i=1}^{n} A_n \cdots A_{i+1}\Big(A_i - A_i^{(k)}\Big)A_{i-1}^{(k)} \cdots A_1^{(k)}\Big\| \\
\le \ & \sum_{i=1}^{n} \Big(\frac{2\|V\|_\infty + \|f\|_1}{|f_D|}\Big)^{n-i} \cdot \frac{2DC_1}{|f_D|q_{k+1}} \cdot \Big(\frac{2\|V\|_\infty + \|f\|_1}{|f_D|}\Big)^{i-1} \\
\le \ & \frac{2nDC_1}{|f_D|q_{k+1}}\Big(\frac{2\|V\|_\infty + \|f\|_1}{|f_D|}\Big)^{n-1} \\
\le \ & 4D^2C_1\frac{q_k}{q_{k+1}}\Big(\frac{2\|V\|_\infty + \|f\|_1}{|f_D|}\Big)^{2Dq_k}.
\end{aligned}$$

The above proof holds true for $-2Dq_k \le n < 0$. □

Proof of Theorem 1.2: Take

$$\varepsilon = \beta - 2D\ln\Big(\frac{2\|V\|_\infty + \|f\|_1}{|f_D|}\Big) > 0.$$

By the definition (3), there exist subsequence $\{q_{k_i}\}$ and K such that for all $k_i > K$

$$\frac{\ln q_{k_i+1}}{q_{k_i}} > \beta - \frac{\varepsilon}{2}.$$

By Lemma 3.2, for any $k_i > K$, and for all $\theta \in \mathbb{T}$ and $|n| \le 2Dq_{k_i}$,

$$\begin{aligned} & \|M_n - M_n^{(k_i)}\| \\ \le\ & 4D^2 C_1 \frac{q_{k_i}}{q_{k_i+1}} \Big(\frac{2\|V\|_\infty + \|f\|_1}{|f_D|} \Big)^{2Dq_{k_i}} \\ =\ & 4D^2 C_1 q_{k_i} \exp\Big\{ q_{k_i} \Big(2D \ln \big(\frac{2\|V\|_\infty + \|f\|_1}{|f_D|} \big) - \frac{\ln q_{k_i+1}}{q_{k_i}} \Big) \Big\} \\ \le\ & 4D^2 C_1 q_{k_i} \exp\{ -\frac{1}{2} q_{k_i} \varepsilon \}. \end{aligned}$$

Letting $q_{k_i} \to \infty$, we have:

$$\sup_{|n| \le 2Dq_{k_i}} \sup_{\theta \in \mathbb{T}} \|M_n - M_n^{(k_i)}\| \longrightarrow 0.$$

Set $X(0) = \begin{pmatrix} u_{D-1} \\ u_{D-2} \\ \cdots \\ u_{-D} \end{pmatrix} \neq 0$, and

$$X(n) = M_n(\theta, E, \alpha) X(0),\ X^{(k)}(n) = M_n(\theta, E, \frac{p_k}{q_k}) X(0).$$

By Lemma 3.1, for all $\theta \in \mathbb{T}^1, E \in \mathbb{R}$, and any $k \in \mathbb{N}$,

$$\max_{0 < |i| \le 2D, i \in \mathbb{Z}} \{ \|X(iq_k)\| \} \ge \frac{1}{2D} \|X(0)\|.$$

Then we have for $i = \pm 1, \cdots, \pm 2D$ and for any $\theta \in \mathbb{T}$,

$$\lim_{q_{k_i} \to \infty} \|X(iq_{k_i}) - X^{(k_i)}(iq_{k_i})\| \le \lim_{q_{k_i} \to \infty} \sup_{|n| \le 2Dq_{k_i}} \sup_{\theta \in \mathbb{T}} \|M_n - M_n^{(k_i)}\| \|X(0)\| = 0.$$

i.e., there is a sequence n_k with $|n_k| \to \infty$ such that

$$\sqrt{|u_{n_k}|^2 + |u_{n_k-1}|^2 + \cdots + |u_{n_k-2D+1}|^2} \ge \frac{1}{4D} \sqrt{|u_{D-1}|^2 + \cdots + |u_{-D}|^2}$$

Therefore, $H^D_{V,\theta,\alpha} u = Eu$ has no l^2 solution for any $\theta \in \mathbb{T}$ if $\beta(\alpha)$ satisfies (8). □

Proof of Theorem 1.1: If $D < \infty$, we apply Theorem 1.2 to the operator $L_{\lambda^{-1} f, \phi, \alpha} = H^D_{V,\phi,\alpha}$ with $V(\phi) = 2\cos 2\pi\phi$. Clearly, (8) gives the estimate

$$\beta > 2D \ln \frac{\lambda^{-1}\|f\|_1 + 2\|V\|_\infty}{\lambda^{-1}|f_D|},$$

which immediately implies

$$\lambda < \frac{1}{4} \big(|f_D| e^{\frac{\beta}{2D}} - \|f\|_{l^1} \big) \tag{14}$$

since $\|V\|_\infty = 2$.

Notice that $\lambda > 0$, thus the above estimate makes sense only when the right hand side of (14) is positive. □

4. Long-range case

In this section, we show that Gordon's lemma is not always true for quasi-periodic long-range operators. The basis are the newly developed reducible theory for Liouvillean frequency [15] and the duality argument. Recall that (α, A) is *reducible*, if it can be conjugated to a constant cocycle. A cocycle (α, A^1) is said to be *rotations reducible* if it is conjugated to a cocycle $(\alpha, A^2) \in \mathbb{R}/\mathbb{Q} \times C^\omega(\mathbb{T}, SO(2,\mathbb{R}))$. Readers can consult [15] for classical results on reducibility and rotations reducibility and the reference therein.

Proof of Theorem 1.3 (a)**:** We first recall the following rotations reducible result for $SL(2,\mathbb{R})$ cocycle with Liouvillean frequency, which was first proved in the continuous case [11], and then was proved in the cocycle case by local embedding theorem [15]. Similar result can be found in [1].

Theorem 4.1. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $h > 0$, $R \in SL(2,\mathbb{R})$, $A \in C^\omega_h(\mathbb{T}, SL(2,\mathbb{R}))$. Then there exist ε_0 which depends on R, h such that if $\|A - R\|_h < \varepsilon_0$, and the rotation number $rot_f(\alpha, A)$ is Diophantine w.r.t. α, then (α, A) is analytically rotations reducible.*

We rewrite the Schrödinger operator

$$(H_{V,\theta,\alpha}u)_n = u_{n+1} + u_{n-1} + V(\theta + n\alpha)u_n$$

as a Schrödinger cocycle (α, S^V_E), where

$$S^V_E(\theta) = \begin{pmatrix} V(\theta) - E & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{R}).$$

Now suppose that $\|V\|_h \le \varepsilon_0$, and $rot_f(\alpha, S^V_E)$ is Diophantine w.r.t. α, then by Theorem 4.1, (α, S^V_E) is rotations reducible, which means there exist $B_E \in C^\omega_{h_*}(2\mathbb{T}, SL(2,\mathbb{R}))$, $\phi_E \in C^\omega_{h_*}(\mathbb{T},\mathbb{R})$ such that

$$B_E(\theta + \alpha)S^V_E(\theta)B_E(\theta)^{-1} = R_{\phi_E(\theta)},$$

by Proposition 2.1, we have

$$[\phi_E] = rot_f(\alpha, S^V_E) + \frac{deg B_E(\theta)\alpha}{2}.$$

Notice that (α, S^V_E) can be accumulated by reducible cocycles. To see this, we only need to consider the reference systems $(\alpha, R_{\mathcal{T}_{q_n}\phi_E(\theta)})$,

which are reducible since $\mathcal{T}_{q_n}\phi_E(\theta) = \sum_{|k|\le q_n}\widehat{\phi}_E(k)e^{2\pi ik\theta}$ are polynomials and

$$\psi_E(x+\alpha) - \psi_E(x) + \mathcal{T}_{q_n}\phi_E(x)) = [\phi_E]$$

always has a solution with $\psi_E \in C^\omega_{h_*}(\mathbb{T},\mathbb{R})$. As a consequence, the cocycles

$$(\alpha, A_k(\cdot,\lambda)) = Ad_{B_E^{-1}}(\alpha, R_{\mathcal{T}_{q_k}\phi_E(\theta)})$$

are reducible since reducibility is conjugacy invariant. Furthermore, we have for $k\to\infty$

$$\begin{aligned}\|A_k - S_E^V\|_{h_*} &= \|B_E(\theta+\alpha)^{-1}(R_{\phi_E(\theta)} - R_{\mathcal{T}_{q_k}\phi_E(\theta)})B_E(\theta)\|_{h_*}\\ &\le \|B_E^{-1}\|^2_{h_*}\|\mathcal{R}_{q_k}\phi_E(\theta)\|_{h_*}\to 0.\end{aligned}$$

To this stage, we need the following theorem, which states that a non-Schrödinger perturbations of Schrödinger cocycles can be converted to a Schrödinger cocyle. The proof can be found in [4].

Theorem 4.2. *Let $V\in C^\omega_h(\mathbb{T},\mathbb{R})$ be non-identically zero. There exists $\varepsilon>0$, such that if $A\in C^\omega_h(\mathbb{T},SL(2,\mathbb{R}))$ satisfies $\|A-S_E^V\|_h<\varepsilon$, then there exists $\widetilde{V}\in C^\omega_h(\mathbb{T},\mathbb{R})$, and $B\in C^\omega_h(\mathbb{T},SL(2,\mathbb{R}))$, such that*

$$B(\theta+\alpha)A(\theta)B(\theta)^{-1} = S_E^{\widetilde{V}}(\theta).$$

Therefore, when k is large enough, there exists $\widetilde{V}_k\in C^\omega_{h_*}(\mathbb{T},\mathbb{R})$, and $B_k\in C^\omega_{h_*}(\mathbb{T},SL(2,\mathbb{R}))$, such that

$$B_k(\theta+\alpha)A_k(\theta)B_k(\theta)^{-1} = S_E^{\widetilde{V}_k}(\theta).$$

Then $(\alpha, S_E^{\widetilde{V}_k})$ can be reduced to $(\alpha, R_{[\phi_E]})$, consequently the Schrödinger operator

$$(H_{\widetilde{V}_k,\alpha,\theta}u)_n = u_{n+1}+u_{n-1}+\widetilde{V}_k(n\alpha+\theta)u_n = Eu_n, \tag{15}$$

has Bloch waves with Floquet exponent

$$[\phi_E] = rot_f(\alpha, S_E^V) + \frac{degB_E(\theta)\alpha}{2} + degB_k(\theta)\alpha.$$

Expand $\widetilde{V}_k(\theta)$ into its Fourier series $\widetilde{V}_k(\theta) = \sum_m \widetilde{V}_k(m)e^{2\pi im\theta}$, then by Aubry duality,

$$(L_{\widetilde{V}_k,\varphi,\alpha}\psi)_n = \sum_{m\in\mathbb{Z}}\widetilde{V}_k(m)\psi_{n-m} + 2cos2\pi(\varphi+n\alpha)\psi_n = E\psi_n, \tag{16}$$

has an exponentially decay solution for $\varphi = \pm[\phi_E]$. Thus E is a eigenvalue.

Proof of Theorem 1.3 (b)**:** According to Theorem 1.2, if $\beta(\alpha)=\infty$, then for any finite degree trigonometric polynomials f, the finite range

operator $L_{f,\phi,\alpha}$ has no point spectrum for any ϕ. Since trigonometric polynomials are dense in $C_h^\omega(\mathbb{T},\mathbb{R})$, we conclude that if $\beta=\infty$, then there is a dense set of $f\in C_h^\omega(\mathbb{T},\mathbb{R})$, such that the operator $L_{f,\phi,\alpha}$ has no point spectrum for any $\phi\in\mathbb{T}$.

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