Hölder Continuity of the Lyapunov Exponent for Analytic Quasiperiodic Schrödinger Cocycle with Weak Liouville Frequency

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Abstract

For analytic quasiperiodic Schrödinger cocycles, Goldstein and Schlag [13] proved that the Lyapunov Exponent is Hölder continuous provided that the base frequency ω satisfies strong Diophantine condition. In this paper, we give a refined large deviation theorem, which implies the Hölder continuity of the Lyapunov exponent for all Diophantine frequencies ω, even for weak Liouville ω, which improves the result of [13].

Keywords: Hölder continuity; Lyapunov exponents; quasiperiodic Schrödinger operators.

1 Introduction and the Main result

In this paper we study Hölder continuity of the Lyapunov exponent of cocycles associated with 1-D quasiperiodic operators on $l^2(\mathbb{Z})$

$$(H\phi)(n) = \phi(n-1) + \phi(n+1) + v(x+n\omega)\phi(n),\ n \in \mathbb{Z}, \tag{1.1}$$

where $x \in \mathbb{T}, \omega \in \mathbb{R}\backslash\mathbb{Q}$, and assume $v : \mathbb{T} \to \mathbb{R}$ is real analytic on $\mathbb{T}$. Consider $SL_2(\mathbb{R})$ valued matrixes

$$A(x,E) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix},\ E \in \mathbb{R}, x \in \mathbb{T}. \tag{1.2}$$

We call (ω, A) a Schrödinger cocycle associated with (1.1). For $n \in \mathbb{N}$, set

$$M_n(x,E) = \prod_{k=n}^{1} A(x+k\omega, E), \tag{1.3}$$

$$u_n(x) = \frac{1}{n}\|M_n(x,E)\|, \tag{1.4}$$

$$L_n(\omega, E) = \int_{\mathbb{T}} u_n(x)\mathrm{dx}.$$

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$\|\cdot\|$ in (1.4) can be any type of matrix norm on $SL_2(\mathbb{R})$. Notice that $\|M\| \geq 1$ if $M \in SL_2(\mathbb{R})$. M_n, u_n and L_n can be defined for $n \in \mathbb{Z}$. Actually for $n > 0$, let

$$M_{-n} = A^{-1}(x-(n-1)\omega)\cdots A^{-1}(x-\omega)A^{-1}(x).$$

Since $A \in SL_2(\mathbb{R})$ and the dynamic on the base space is uniquely ergodic, the approximate behavior of L_n in both cases $n \to +\infty$ and $n \to -\infty$ will be the same. Thus we focus on $n \in \mathbb{N}$ all through this paper.

The Lyapunov exponent associated with the cocycle (ω, A) is defined as:

Definition 1.1

$$L(\omega, E) = \lim_{n\to+\infty} L_n(\omega, E). \tag{1.5}$$

Remark 1.1 *The limit (1.5) exists and is equal to* $\lim_{n\to+\infty} u_n(x)$ *for a.e.* x *by subadditive Ergodic Theorem.*

We assume that the potential v has a bounded extension on the complex strip $|\Im Z| < \rho$ and define its norm as

$$\|v\|_\rho = \sup_{|\Im Z|<\rho} |v(Z)|. \tag{1.6}$$

If $E \in (-\infty, -2+\|v\|_\rho]\bigcup[2+\|v\|_\rho, +\infty)$, the cocycle is uniformly hyperbolic, thus the regularity of $L(E)$ is not an issue. So without loss of generality, we assume $|E| \leq 2+\|v\|_\rho$. Simple computation shows that $\sup_{|\Im Z|<\rho}\|A(Z)\|, \sup_{|\Im Z|<\rho}\|A^{-}(Z)\| \leq 3+2\|v\|_\rho$ and

$$\sup_{|\Im Z|<\rho} |u_n(Z)| \leq \log(3+2\|v\|_\rho) := C_v \tag{1.7}$$

Before coming to the main result of this paper, we have to introduce one last important concept.

Definition 1.2 (Deviation Set) *For* $\kappa > 0,\ n \in \mathbb{N}$*, let*

$$\Omega_n(\kappa) = \{x \in \mathbb{T} :| u_n(x) - L_n(E) |> \kappa\}.$$

Fix $\omega \in \mathbb{R}\backslash\mathbb{Q}$, consider the continued fraction expansion $\omega = [a_1, a_2, \cdots]$ with convergent $\frac{p_s}{q_s}$ for $s = 1, 2, \cdots$. Let

$$\beta = \beta(\omega) = \limsup_s \frac{\log q_{s+1}}{q_s}.$$

Our main result is the following theorem:

Theorem 1 (Refined Large Deviation Theorem) *For any* $\kappa > 0$*, let* C_v *be the constant in (1.7). There exist some absolute constants* c_0, c_1 *which are independent of* κ, C_v*, such that if* $\beta < c_0 \cdot \frac{\kappa}{C_v}$ *and* $n > N(\kappa, C_v)$*, then*

$$mes\Omega_n(\kappa) < e^{-\frac{c_1}{C_v^3}\kappa^3 n}. \tag{1.8}$$

Remark 1.2 *Goldstein and Schlag [13] established so-called* sharp large deviation theorem *saying that*

$$mes\Omega_n(\kappa) < e^{-c\kappa n} \tag{1.9}$$

for strong Diophantine frequencies ω [1]. *They also got*

$$mes\Omega_n(\kappa) < e^{-C(\kappa)n^\sigma},\ \sigma \in (0,1) \tag{1.10}$$

under the usual Diophantine condition. While for any irrational frequency ω*, Bourgain and Jitomirskaya [9] proved*

$$mes\Omega_n(\kappa) < e^{-c\kappa q},\ q < c\kappa^2 n. \tag{1.11}$$

(1.10),(1.11) account for the weaker regularity for general frequencies. The word 'Refined' in Theorem 1 indicates that we removed the exponent $\sigma \in (0,1)$ *as well as removed the restriction* $q < c\kappa^2 n$ *for* $0 \le \beta(\omega) \ll 1$*, which enables us to establish Hölder continuity of the Lyapunov Exponent for all Diophantine, and some weak Liouville frequencies.*

In order to prove Hölder continuity of the Lyapunov Exponent, we also need to assume that the Lyapunov Exponent is positive. When $L(E) = 0, E \in \sigma(H)$, the regularity of a dual quantity(*integrated density of states*, I.D.S.) is more often studied. We refer to Avila, Jitomirskaya [3] for more results on the regularity of the I.D.S.

We focus on regularity of the Lyapunov Exponent under the following conditions.

Condition 1.1 (Positivity of $L(E)$)

$$L(E) > \gamma > 0,\ \text{for } E \in [E_1, E_2]. \tag{1.12}$$

Now take $\kappa = \frac{1}{100}\gamma$ in Theorem 1. Combine Theorem 1 with the *avalanche principle*, the Hölder continuity of $L(E)$ can be proved easily following the iteration steps developed by Goldstein and Schlag in [13]. More precisely, we have:

Theorem 2 *Suppose that Condition 1.1 holds. Let* C_v *be the constant in (1.7) and let* c_0 *be the constant in Theorem 1. Assume that* $\beta < \frac{c_0}{100} \cdot \frac{\gamma}{C_v}$*. Then*

$$|L(E) - L(E')| \le C|E - E'|^\tau,\ E, E' \in [E_1, E_2],$$

where the Hölder exponent $\tau = c_2 \cdot 2^{-\frac{200C_v}{\gamma}} \cdot \frac{\gamma^3}{C_v^4}$*. The constant* C *depends on* C_v, γ *and the length of the interval* $[E_1, E_2]$*, while* c_2 *is an absolute constant independent of* C_v, γ*.*

Remark 1.3 *Based on (1.9), Goldstein and Schlag in [13] proved that* $L(E)$ *is a Hölder continuous function on a neighborhood of any point where it is positive provided that the base frequency* ω *is strong Diophantine. We improve their result by not only removing the strong Diophantine condition restriction but also extending the Hölder continuity to the weak Liouville case(*$\beta(\omega) \ll 1$*). Meanwhile, since* $L(E)$ *is known to be discontinuous at rational* ω*, it might escape any modulus of continuity for Baire generic* ω*(this argument appears, albeit briefly, in the paper of Avila and Jitomirskaya [3]). This fact indicates that we can not expect the Hölder continuity to hold for all* $\beta > 0$ *and our restriction on* β *is reasonable, although might not be optimal.*

[1]Strong Diophantine means that $\omega \in (0,1)$ satisfies $\|n\omega\|_1 := \inf_{m\in\mathbb{Z}} |n\omega - m| \ge \frac{C_\omega}{|n|(\log|n|)^a}$ for all $n \ne 0$ and some fixed $a > 1$, which is stronger than the usual Diophantine condition $\|n\omega\|_1 \ge \frac{C_\omega}{|n|^b}$ for all $n \ne 0$ and some fixed $b > 1$. Notice that $\beta(\omega) = 0$ if ω is Diophantine.

Remark 1.4 *Though in Theorem 2 we focus on Schrödinger cocycle, the conclusion holds for any quasi-periodic $SL_2(\mathbb{R})$ cocycles $(\omega, A(x,E))$ which are analytic in both x and E.*

Similar results hold for quasi-periodic linear equation. Consider Hill's equation

$$-y''(t) + q_\theta(t)y(t) = Ey(t), \quad t \in \mathbb{R}, \tag{1.13}$$

where $q_\theta(t) = Q(\theta_1 + t, \theta_2 + \omega t)$. Q is an real analytic function from $\mathbb{T}^2$ to $\mathbb{R}$. We can prove that the Lyapunov exponent $\gamma(E)$ associated with this equation is Hölder continuous w.r.t. E provided $\beta(\omega) \ll 1$ and $\gamma(E) > 0$. Actually, let $sl_2(\mathbb{R})$ be the set of 2×2 real matrices of zero trace, let $a_\theta(t) = \begin{pmatrix} 0 & 1 \\ q_\theta(t) - E & 0 \end{pmatrix} : \mathbb{T}^2 \to sl_2(R)$ and $z = \begin{pmatrix} y \\ y' \end{pmatrix}$. Write (1.13) as the family of differential systems

$$z' = a_\theta z. \tag{1.14}$$

For each $\theta \in \mathbb{T}^2$, let $\Phi_{\theta,E}(t)$ be the fundamental matrix solution of equation (1.14). The family of maps $\Phi_{\theta,E}(t) : \mathbb{T}^2 \times \mathbb{R} \to SL_2(\mathbb{R})$ is called a continuous cocycle.

The (maximal) Lyapunov exponent of the family (1.13) or (1.14) is defined as

$$\gamma(E) = \lim_{|t| \to \infty} \frac{1}{|t|} \log \|\Phi_{\theta,E}(t)\|, \tag{1.15}$$

the limit exists and is constant θ-a.e.($\omega \in \mathbb{R}\backslash\mathbb{Q}$).

We take the Poincaré map of (1.14), then the discrete version of (1.15) would be

$$\gamma_d(E) = \lim_{|n| \to \infty} \frac{1}{|n|} \log \|\Phi_{\theta,E}(n)\|,$$

and of course $\gamma(E) = \gamma_d(E)$. Notice that the base dynamic is now reduced to the one-frequency map $\theta_2 \mapsto \theta_2 + \omega$ and $\Phi_{\theta,E}(n)$ has the form of matrices product just as M_n in (1.3). Moreover, the fundamental matrix solution $\Phi_{\theta,E}(t)$ is analytic in θ, E. According to Remark 1.4, $\gamma(E)$ is Hölder continuous at E with $\gamma(E) > 0$ if $\beta(\omega) \ll 1$.

Now, we consider regularity of $L(E)$ associated with (1.1) with potentials of the form $v = \lambda v_0$, where $v_0 : \mathbb{T} \mapsto \mathbb{R}$ is a nonconstant real analytic function and $\lambda > 0$ is called the coupling constant. In general, the positivity of $L(E)$ is hard to predict. However, if we assume that the potential has the form λv_0, there are plenty of results to guarantee the positivity of $L(E)$ by assuming that λ is large enough. The operators associated with the one parameter family of potential λv_0 are extensively studied. The most thoroughly studied example is the almost Mathieu operator when taking $v_0(x) = 2\cos(2\pi x)$. In the following, we state the results of Theorem 1,2 for this family of potentials.

Sorets and Spencer [17] proved that (1.12) holds for $\lambda > \lambda_0$. Bourgain, Goldstein [8] gave an alternative proof of Sorets and Spencer's result [17]. We cite the results as follows:

Proposition 1 (Sorets and Spencer [17]) *For any nonconstant real analytic potential v_0 with an analytic extension on $|\Im Z| < \rho$, there is a λ_0 depends on $\|v_0\|_\infty$ and ρ, such that for all E and $\lambda > \lambda_0$, the Lyapunov exponent of $v = \lambda v_0$ satisfying:*

$$L(\omega, E) > c_{v_0} \log \lambda,$$

where c_{v_0} depends only on $\|v_0\|_\infty$ and ρ.

If we take λ_0 large enough, it can also be proved that $L(\omega, E) > \frac{1}{2}\log\lambda := \gamma$. Without loss of generality, we assume that $\|v_0\|_\rho \leq 1$. Notice that in this case, $C_v = 2\log\lambda$ and the quotient $\frac{\gamma}{C_v}$ satisfies

$$1 > \frac{\gamma}{C_v} \geq \frac{1}{4},$$

which is uniformly bounded from below(w.r.t. E).

Follow the proof of Theorem 1 and 2, we have in this case:

Theorem 3 *There exist constants λ_0, N_0 which depend only on v_0 and some absolute constants c_0', c_1', such that if $\lambda > \lambda_0$, $\beta(\omega) < c_0'$ and $n > N_0$, then*

$$mes\{x \in \mathbb{T} :| u_n(x) - L_n(E) |> \frac{1}{100}\log\lambda\} < e^{-c_1' n}.$$

Theorem 4 *Let λ_0, c_0' be the constant in Theorem 3. There exist constants C', τ' which depend only on v_0, such that if $\lambda > \lambda_0$, $\beta < c_0'$, then*

$$|L(E) - L(E')| \leq C'\lambda|E - E'|^{\frac{\tau'}{\log\lambda}},\ \ E, E' \in [-2+\lambda, 2+\lambda].$$

Remark 1.5 *According to the proof(see Section 3), the constant C in Theorem 2 is locally independent of C_v . Correspondingly, if we require $|E-E'| \ll 1$, the conclusion of Theorem 4 will be*

$$|L(E) - L(E')| \leq C'|E - E'|^{\frac{\tau'}{\log\lambda}}.$$

Background and Related Results. In recent years, lots of progresses in cocyles have been made in order to understand the spectral properties of the Schrödinger operators (1.1). Besides the Lyapunov exponent, another important subject is the I.D.S. $N(E)$, which is connected with the Lyapunov exponent via the Thouless formula:

$$L(E) = \int \log|E - E'| \mathrm{d}N(E').$$

Lots of work has been done concerning the regularity of $L(E)$ and $N(E)$, see [3, 9, 13, 18]. It is known that the I.D.S. is a continuous increasing function and it is locally constant outside the spectrum, see [4]. And the continuity of $N(E)$ is not enough to conclude the continuity of the Lyapunov exponent. However, by Hilbert transform and the theory of singular integral operator, the Hölder continuity of $L(E)$ and $N(E)$ do pass from each other by Thouless formula(see details in [13]).

We focus on the regularity of Lyapunov exponent. Under the assumption that $v(x)$ possesses a certain degree of regularity and that ω is a generic irrational number, the recent regularity results of $L(E)$ and $N(E)$ are as follows:

For real analytic potential v on $\mathbb{T}^1$, Bourgain and Jitomirskaya proved in [9] that $L(\omega, E)$ is continuous in E for any ω and jointly continuous in (ω, E) in $\mathbb{R}\backslash\mathbb{Q}\times\mathbb{R}$. Bourgain [7] extended the continuity result of [9] to real analytic potential v on $\mathbb{T}^d, d > 1$. Consider $L(\omega, A)$ as a function on $\mathbb{T} \times C^l(\mathbb{T}, SL_2(\mathbb{R}))(l = 0, 1, \cdots, \infty, \omega)$, Jitokirskaya, Koslover and Schulteis [16] proved that the $L : \mathbb{T} \times C^\omega(\mathbb{T}, SL_2(\mathbb{R})) \to [0, \infty)$ is jointly continuous in (ω, A)(actually, they study more general $M_2(\mathbb{C})$ cocycles). When regularity of the cocycles is weakened, things may be different. Furman [12] proved that for irrational rotation, the Lyapunov exponent is discontinuous at every non-uniformly hyperbolic cocycle in C^0 topology. Bochi [10] proved that cocycles which are either hyperbolic or have zero Lyapunov exponent are dense in $C^0(\mathbb{T}, SL_2(\mathbb{R}))$. Wang and You [18] proved that for any $l = 0, 1, \cdots, \infty$ and any fixed ω of bounded type, there is $D_l \in C^l(\mathbb{T}, SL_2(\mathbb{R}))$ such that the Lyapunov exponent is discontinuous at (ω, D_l) in C^l-topology.

Besides the result mentioned in Remark 1.3, Goldstein and Schlag [13] also proved certain weaker Hölder regularity for analytic v on $\mathbb{T}^d, d \geq 1$ in the regime $L(E) > 0$ for all Diophantine frequcneies. We would like to emphasize that in [8],[13], etc, Bourgain, Goldstein and Schlag developed a series of powerful methods to study analytic Schrödinger operators, e.g., large deviation theorem, avalanche principle, estimates of semi-algebra sets and Green's function. Lack of space forbids further discussion about all their results. As we will see later, our paper strongly rely on large deviation theorem and avalanche principle.

For analytic potential v on $\mathbb{T}^1$, there are some sharp estimates about the Hölder exponent of $L(E)$ and $N(E)$ with Diophantine frequency. An almost precise estimate ($\frac{1}{2} - \epsilon$ Hölder regularity) for the almost Mathieu operator at high coupling is contained in [6]. The Hölder exponent can be estimated by $\frac{1}{2k} - \epsilon$ for v being small C^∞ neighborhood of a trigonometric polynomial of degree k [14]. Avila and Jitomirskaya [3] get the sharp $\frac{1}{2}$ Hölder exponent of the I.D.S. for analytic potential v and Diophantine ω in the reducibility regime($\lambda < \lambda(v)$). We would like to emphasize that all the regularity results above(beyond continuity) require the base frequency ω to be Diophantine.

Craig and Simon [11] proved the log-Hölder continuity of $N(E)$ associated with a general bounded ergodic stationary process. That is the only issue in which the regularity of $N(E)$(beyond continuity) with Liouville frequency was once dealt with. However, the log-Hölder continuity in [11] is obvious from the nonnegativity of $L(E)$ and the Thouless formula. And it is much weaker than the Hölder continuity. Our contribution in this paper is to prove the Hölder continuity of $L(E)$ with Liouville frequency for the first time, though still not for all frequencies. We would like to refer the reader to [5, 15] for more background and discussions.

We would like to list the following questions which are also very interesting. The first one is about the restriction on the size of β. We believe the requirement $\beta \ll 1$ in Theorem 1,2 is technical and the Hölder continuity should hold for some larger β. Meanwhile, as mentioned in Remark 1.3, $L(E)$ might escape any modulus of continuity for Baire generic ω(see [1],[3]). Thus, it is expected that things will deteriorate for β extremely large. It is reasonable to ask whether there is $L_0(\beta)$ such that if $L(E) > L_0(\beta)$, then $L(E)$ is Hölder continuous in E?

Secondly, except [3],[14], little is known about the sharp Hölder exponent of the Lyapunov exponent for general v when $L(E) > 0$. Both in our present paper(Theorem 4) as well as in Goldstein and Schlag [13], the Hölder exponent would become worse as λ gets larger, which contradicts the intuition that the Hölder exponent should be improved with an increase of λ. This is probably due to some shortcoming of the method.

We also want to know more about the Anderson Localization(A.L.) of the quasiperiodic operator with Liouville frequencies. It is believed that Anderson Localization should also hold for Liouville frequencies provided condition such as $\lambda > Ce^{c\beta}$. Avila and Jitomirskaya [2] proved A.L. for almost Mathieu operator when $\lambda > e^{\frac{16}{9}\beta}$. You and Zhou [19] also got A.L. for long range operator with cosine potential when $\lambda > Ce^{\beta}$ using duality. These results strongly rely on the cosine potential. For general potentials, according to [8], A.L. follows from large deviation theorem of type $mes\Omega_n(n^{-\sigma}) < e^{-cn^{\sigma}}$ and semi-algebraic set theorem. Our present refinement of LDT for Liouville frequencies seems unable to estimate $mes\Omega_n(n^{-\sigma})$. Moreover, the method in [8] need to remove a zero-measure set from the Diophantine set, which is not applicable for Liouville frequencies. In general, proving A.L. for Liouville frequencies seems to be a real challenge and new ideas are required.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of the Refined Large Deviation Theorem(Theorem 1). In Section 3, the Refined LDT will be combined with *avalanche principle* to obtain estimates on convergence of $L_n(E)$. The proof of the Hölder continuity of $L(E)$(Theorem 2) is also completed there.

Our proof builds on some ideas and techniques in [9, 13], see also in [5], Chap.VII. Some precise estimates on the continued fraction expansion of ω appear in the proof of Theorem 1, which are new. The proof in Section 3 is standard, which is essentially contained in [13]. Still, we sketch the proof associated with our notations in order to get an exact expression of the Hölder exponent in Theorem 2,4 and leave it to Appendix 1(Section 3).

2 Proof of the Refined Large Deviation Theorem

The proof of Theorem 1 follows the outlines of [9]. In [9](see Lemma 4), Bourgain and Jitomirskaya proved a large deviation theorem in 'weak sense' to prove the continuity of $L(E)$ for any irrational frequency. With more complicated and precise analysis on the continued fraction expansion and the approximant of the frequency, we get a refined Large Deviation Theorem, which implies the desired Hölder continuity immediately.

First, we introduce some notations needed in the proof.

Notation 1 *For $n, R, K \in \mathbb{N}$, let $u_n(x)$ be as in (1.4).*

1. *Expand $u_n(x)$ into its Fourier series and denote the Fourier coefficient as $\hat{u}(k)$, i.e.,*

$$u_n(x) = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{2\pi \mathrm{i} kx},\ \ \hat{u}(k) = \int_{x\in\mathbb{T}} u_n(x)e^{-2\pi \mathrm{i} kx}\mathrm{dx}$$

2. *Consider the Féjèr average $u_R(x)$ of $u_n(x)$ along the orbit and denote the Féjèr kernel as $F_R(k)$,*

$$u_R(x) = \sum_{|j|<R} \frac{R-|j|}{R^2} u_n(x+j\omega),\ \ F_R(k) = \sum_{|j|<R} \frac{R-|j|}{R^2} e^{2\pi \mathrm{i} kj\omega}.$$

With 1,2, we have

$$u_R(x) = \sum_{|j|<R} \frac{R-|j|}{R^2} \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{2\pi \mathrm{i} k(x+j\omega)} = \sum_{k\in\mathbb{Z}} \hat{u}(k)F_R(k)e^{2\pi \mathrm{i} kx}$$

3. Truncate $u_R(x)$ into two parts:

$$u_I(x) = \sum_{0<|k|<K} \hat{u}(k)F_R(k)e^{2\pi i k x},\ \ u_{II}(x) = \sum_{|k|\geq K} \hat{u}(k)F_R(k)e^{2\pi i k x}.$$

With 2,3, we have

$$u_R(x) = L_n + u_I(x) + u_{II}(x).$$

With these notations, we are ready to show for fixed κ and appropriate choices of $n, R, K, \beta(\omega)$:

Lemma 2.1

1.

$$|u_n(x) - u_R| < \frac{\kappa}{3}$$

2.

$$|u_I| < \frac{\kappa}{3}$$

3.

$$\|u_{II}\|^2_{L^2(\mathbb{T})} \leq (6C_1C_v)^2 \frac{2}{K}$$

where $K > \exp\{\frac{\kappa^3}{3000C_3^3C_v^3}n\}$ and the constants C_1, C_3 (independent of γ) will be specified later, C_v is in (1.7).

Once Lemma 2.1 is established, we come to the

Proof of Theorem 1: It is clear that

$$\begin{aligned}
mes\Omega_n &= mes\{x \in \mathbb{T} :| u_n(x) - L_n(E) |> \kappa\} \\
&\leq mes\{x \in \mathbb{T} :| u_n(x) - u_R(x) |> \frac{\kappa}{3}\} + mes\{x \in \mathbb{T} :| u_I |> \frac{\kappa}{3}\} + mes\{x \in \mathbb{T} :| u_{II} |> \frac{\kappa}{3}\} \\
&\leq \frac{1}{(\frac{\kappa}{3})^2}\|u_{II}\|_2^2 \\
&\leq \frac{9}{\kappa^2} \cdot (6C_1C_v)^2 \frac{2}{K} \\
&\leq \frac{18}{\kappa^2}(6C_1C_v)^2 \cdot \exp\{-\frac{\kappa^3}{3000C_3^3C_v^3}n\} \\
&\leq \exp\{-\frac{1}{6000C_3^3C_v^3} \cdot \kappa^3 \cdot n\} \qquad \square
\end{aligned}$$

Now we turn back to the proof of Lemma 2.1, we need the following lemmas:

Lemma 2.2 *For any $n, R \in \mathbb{N},\ k \in \mathbb{Z}$,*

1.

$$|u_n(x) - u_R| \leq 2C_v \cdot \frac{R}{n} \tag{2.1}$$

2.
$$|\hat{u}(k)| \leq C_1 \frac{\sup_{|\Im Z|<\rho} |u_n(Z)|}{|k|} \leq \frac{C_1 C_v}{|k|} \tag{2.2}$$

3.
$$|F_R(k)| \leq \frac{6}{1+R^2\|k\omega\|^2} \tag{2.3}$$

Lemma 2.3 *Let $\frac{p}{q}$ be the approximants of ω, then $\forall |k| < \frac{q}{2}$, $\|kw\| \geq \frac{1}{2q}$. Also we have*

1.
$$\sum_{1\leq|k|<\frac{q}{4}} \frac{1}{1+R^2\|k\omega\|^2} \leq C_2 \frac{q}{R}$$

2. $\forall l \geq 1$
$$\sum_{|k|\in[\frac{q}{4}l,\frac{q}{4}(l+1))} \frac{1}{1+R^2\|k\omega\|^2} \leq C_2(1+\frac{q}{R})$$

The proof of Lemma 2.2,2.3 is essentially contained in [5, 9], while some details in [5] are missing and the constant C_v has not been specified in those papers. In order to get a precise expression of the Hölder exponent, we rewrite the details of the proof in Appendix 2 with precise estimates of C_v.

Now take $R = [\frac{\kappa}{6C_v}n]$, $C_3 = 50C_1C_2$, $K = [\exp\{\frac{\kappa^2}{200C_3^2C_v^2}R\}]$ and $\beta < \frac{\kappa}{40C_3C_v}$. Since $q_k \to +\infty$ monotonously , there is $s \in \mathbb{N}$ such that $\frac{\kappa}{10C_3C_v}R \in [q_s, q_{s+1})$. With these settings, we have

Proof of Lemma 2.1:

Part 1: By (2.1) and the choice of n, R,
$$\begin{aligned} |u_n(x) - u_R| &\leq 2C_v \cdot \frac{R}{n} \\ &\leq 2C_v \cdot \frac{\frac{\kappa}{6C_v}n}{n} \leq \frac{\kappa}{3}. \end{aligned}$$
□

Part 2:
$$u_I = \sum_{1\leq|k|<\frac{q_s}{4}} \hat{u}(k)F_R(k)e^{2\pi i kx} + \tag{2.4}$$
$$\sum_{\frac{q_s}{4}\leq|k|<\frac{q_{s+1}}{4}} \hat{u}(k)F_R(k)e^{2\pi i kx} + \tag{2.5}$$
$$\sum_{\frac{q_{s+1}}{4}\leq|k|<K} \hat{u}(k)F_R(k)e^{2\pi i kx}. \tag{2.6}$$

With Lemma 2.2,2.3, we have
$$|\ (2.4)\ | \leq \sum_{1\leq|k|<\frac{q_s}{4}} C_1C_v \cdot \frac{6}{1+R^2\|k\omega\|^2} \leq 6C_1C_2C_v\frac{q_s}{R} \leq C_3C_v\frac{q_s}{R},$$

and

$$
\begin{aligned}
|\,(2.5)\,| &\le \sum_{l=1}^{[q_{s+1}/q_s]+1} \sum_{|k|\in[\frac{q_s}{4}l,\frac{q_s}{4}(l+1))} \frac{C_1C_v}{|k|}\frac{6}{1+R^2\|k\omega\|^2} \\
&\le \sum_{l=1}^{q_{s+1}} \sum_{|k|\in[\frac{q_s}{4}l,\frac{q_s}{4}(l+1))} \frac{C_1C_v}{\frac{q_s}{4}l}\frac{6}{1+R^2\|k\omega\|^2} \\
&\le \sum_{l=1}^{q_{s+1}} \frac{C_1C_v}{\frac{q_s}{4}l}\cdot 6C_2(1+\frac{q_s}{R}) \\
&\le \frac{C_1C_v}{\frac{q_s}{4}} 6C_2(1+\frac{q_s}{R})2\log q_{s+1} \\
&\le C_3C_v(\frac{\log q_{s+1}}{q_s}+\frac{\log q_{s+1}}{R}).
\end{aligned}
$$

(2.6) can be estimated in the same way as (2.5), i.e.,

$$
\begin{aligned}
|\,(2.6)\,| &\le \sum_{l=1}^{K} \sum_{|k|\in[\frac{q_{s+1}}{4}l,\frac{q_{s+1}}{4}(l+1))} \frac{C_1C_v}{|k|}\frac{6}{1+R^2\|k\omega\|^2} \\
&\le C_3C_v(\frac{\log K}{q_{s+1}}+\frac{\log K}{R}).
\end{aligned}
$$

All together with the above estimates, we have

$$
|u_I|\le C_3C_v(\frac{q_s}{R}+\frac{\log q_{s+1}}{q_s}+\frac{\log q_{s+1}}{R}+\frac{\log K}{q_{s+1}}+\frac{\log K}{R}).
$$

Recall the choice of n, R, K(for n sufficiently large), we see

$$
\begin{aligned}
&q_s\le\frac{\kappa}{10C_3C_v}R<R,\quad \frac{\kappa}{10C_3C_v}R\le q_{s+1},\quad \frac{\log q_{s+1}}{q_s}\le 2\beta\le\frac{\kappa}{20C_3C_v},\\
&q_{s+1}\le\exp\{\beta\frac{\kappa}{10C_3C_v}R\}\le\exp\{\frac{\kappa^2}{400C_3^2C_v^2}R\}\le K\le\exp\{\frac{\kappa^2}{200C_3^2C_v^2}R\},
\end{aligned}
$$

thus

$$
\begin{aligned}
|u_I| &\le C_3C_v\Big(\frac{\kappa}{10C_3C_v}+2\frac{\log q_{s+1}}{q_s}+\frac{\log K}{\frac{\kappa R}{10C_3C_v}}+\frac{\log K}{R}\Big) \\
&\le C_3C_v\Big(\frac{\kappa}{10C_3C_v}+4\beta+\frac{20C_3C_v}{\kappa}\frac{\log K}{R}\Big) \\
&\le \frac{\kappa}{10}+\frac{\kappa}{10}+\frac{\kappa}{10} \\
&\le \frac{\kappa}{3}.
\end{aligned}
$$

□

Part 3:

$$
\|u_{II}\|_2^2 = \|\sum_{|k|\ge K}\hat{u}(k)F_R(k)e^{2\pi \mathrm{i}kx}\|_2^2
$$

$$\begin{aligned}
&\leq \sum_{|k|\geq K} |\hat{u}(k)F_R(k)|^2 \\
&\leq \sum_{|k|\geq K} \Big|\frac{C_1C_v}{|k|}\cdot 6\Big|^2 \\
&\leq (6C_1C_v)^2\frac{2}{K}.
\end{aligned}$$

□

3 Appendix 1: Proof of Theorem 2

When establishing the large deviation theorem, the Hölder continuity directly follows from the method in [13]. We rewrite the proof here associated with the notations in Theorem 1 because we want to indicate the exact expression of the Hölder exponent τ, which is not so explicit in [13]. Especially the relation between τ and γ, C_v has not been specified in [13]. With the expression we get in this part, the Hölder exponent in Theorem 4 would be $\frac{\tau'}{\log\lambda}$, which obviously gets worse as λ goes larger. Actually, this phenomenon also appears in [13](but was not indicated there) and it contradicts the intuition that the Hölder exponent should be improved with an increase of λ, which leaves an open question about the sharp Hölder exponent.

Recall that in Condition 1.1., $L(E) > \gamma > 0$ for $E \in [E_1, E_2]$, now take $\kappa = \frac{1}{100}\gamma$ in Theorem 1, we actually have

Theorem 5 *Assume that Condition 1.12 holds. Let C_v be as in (1.7). If $\beta < \frac{\gamma}{4000C_3C_v}$ and $n > N(\gamma, C_v)$, then*

$$mes\{x \in \mathbb{T} :| u_n(x) - L_n(E) |> \frac{1}{100}\gamma\} < e^{-cn}, \tag{3.1}$$

where $c = \frac{1}{6000\cdot 100^3C_3^3}\cdot\frac{\gamma^3}{C_v^3}$, the constant C_3 is independent of γ, C_v.

Denote $c' = \frac{1}{6\cdot 10^9C_3^3}$ below.

We need the following Theorem:

Theorem 6 (Avalanche Principle,Goldstein and Schlag [13]) *Let $B_1, \cdots, B_m$ be a sequence of unimodular 2×2-matrices. Suppose that*

$$\min_{1\leq j\leq m}\|B_j\| \geq \mu > m \quad \text{and} \tag{3.2}$$

$$\max_{1\leq j<m}[\log\|B_{j+1}\| + \log\|B_j\| - \log\|B_{j+1}B_j\|] < \frac{1}{2}\log\mu. \tag{3.3}$$

(3.4)

Then

$$|\log\|B_m\cdots B_1\| + \sum_{j=2}^{m-1}\log\|B_j\| - \sum_{j=1}^{m-1}\log\|B_{j+1}B_j\| |< C_A\frac{m}{\mu}, \tag{3.5}$$

where C_A is an absolute constant.

The following proposition can also be found in the first paragraph of the proof of Lemma 4.2 in [13]:

Proposition 2 *For any $\tilde{n}$, γ , let $t = [\frac{100C_v}{\gamma}] + 2 < \frac{200C_v}{\gamma} := t_0$, then is $n \in [\tilde{n}, 2^{t_0}\tilde{n}]$ such that*

$$0 \leq L_n(E) - L_{2n}(E) < \frac{1}{100}\gamma. \tag{3.6}$$

Now based on the standard iteration approach(see Section 5,6 in [13] and Chapter VII in [5]), let $n_0 = n$ be given by Proposition 2 and set $n_{s+1} = n_s[\frac{e^{\frac{c}{2}n_s}}{n_s}]$, we have for $s \geq 1$

Proposition 3 (Iteration of $L_n(E)$)

1^s

$$\begin{aligned} \mid L_{n_{s+1}}(E) + L_{n_s}(E) - 2L_{2n_s}(E) \mid &< 3e^{-\frac{c}{4}n_s}, \\ \mid L_{2n_{s+1}}(E) + L_{n_s}(E) - 2L_{2n_s}(E) \mid &< 3e^{-\frac{c}{4}n_s}. \end{aligned} \tag{3.7}$$

2^s

$$\mid L_{n_{s+1}}(E) - L_{2n_{s+1}}(E) \mid < 6e^{-\frac{c}{4}n_s} < \frac{1}{100}\gamma. \tag{3.8}$$

3^s

$$\mid L_{n_{s+1}}(E) - L_{n_s}(E) \mid < 15e^{-\frac{c}{4}n_{s-1}}, \quad n_0 = n. \tag{3.9}$$

Note that 1^0 follows from Theorem 5,6 and Proposition 2, and 2^0 follows from 1^0. For $s \geq 0$, Theorem 5,6 and 2^s implies 1^{s+1}, and 1^{s+1} implies 2^{s+1}. Then the iteration carries on. For $s \geq 1$, 3^s follows from 1^s and 2^{s-1}. At each step, one may use the fact that

$$\gamma \leq L_n \leq C_v, c' \ll 1 \Longrightarrow \frac{99}{100}\gamma \gg \frac{1}{2}c'\frac{\gamma}{C_v} \geq \frac{1}{2}c'\frac{\gamma^3}{C_v^3} = \frac{c}{2},$$

and $(C_A + 8C_v)e^{-\frac{c}{2}n} < 2e^{-\frac{c}{4}n}$.

When the iteration is established for all $s \geq 1$, $\{3^s\}_{s\geq 1}$ implies

$$\mid L(E) - L_{n_1}(E) \mid \leq 20e^{-\frac{c}{4}n_0}, \tag{3.10}$$

and finally from 1^0 and (3.10)

$$\mid L(E) + L_n(E) - L_{2n}(E) \mid < 30e^{-\frac{c}{4}n}. \tag{3.11}$$

For E' satisfying $|E - E'| \ll 1$, we can also get

$$\mid L(E') + L_n(E') - L_{2n}(E') \mid < 30e^{-\frac{c}{4}n}. \tag{3.12}$$

Now we can turn to the proof of Theorem 2. Obviously, for any n

$$\|\partial_E M_n(E)\| \leq n(e^{C_v})^{n-1},$$

then

$$|L_n(E) - L_n(E')| \leq e^{nC_v}|E - E'|. \tag{3.13}$$

From (3.11,3.12) and (3.13),

$$|L(E) - L(E')| \leq 60e^{-\frac{c}{4}n} + 2e^{nC_v}|E - E'|, \tag{3.14}$$

where c is in (3.1).

For any $|E - E'| \ll 1, E, E' \in [E_1, E_2]$, let $B = 3t_0C_v$, $\tau = \frac{c}{8B}$, where $t_0 = \frac{200C_v}{\gamma}$ is the constant in Proposition 3.6. Let $\tilde{n} = [\frac{1}{B}\log\frac{1}{|E-E'|}]$, there is $n \in [\tilde{n}, t_0\tilde{n}]$ satisfying Proposition 3.6, thus (3.14) holds for such n. Direct computation shows that

$$|E - E'|^{-\frac{1}{2B}} < e^n < |E - E'|^{-\frac{t_0}{B}}.$$

Then

$$|L(E) - L(E')| \leq 60|E - E'|^{\frac{c}{8B}} + 2|E - E'|^{-\frac{2t_0C_v}{B}+1} \leq 62|E - E'|^{\frac{c}{8B}},$$

the Hölder exponent

$$\tau = \frac{c}{8B} = \frac{1}{6000 \cdot 100^3C_3^3} \cdot \frac{\gamma^3}{C_v^3} \cdot \frac{1}{24 \cdot 2^{\frac{200C_v}{\gamma}} \cdot C_v}. \qquad \square$$

4 Appendix 2: Proof of Lemma 2.2,2.3

Proof of Lemma 2.2:

Proof of (2.1): because of (1.7), it is easy to show that

$$\Big|u_n(x + j\omega) - u_n(x)\Big| \leq 2C_v\frac{|j|}{n}, \ \forall j \in \mathbb{Z}.$$

Then

$$|u_n(x) - u_R| = \Big|\sum_{|j|<R} \frac{R - |j|}{R^2}\Big(u_n(x) - u_n(x + j\omega)\Big)\Big| \leq 2C_v\frac{R}{n}.$$

(2.2) is shown in ref. [8] (see also ref. [5], Chap. IV).

(2.3) is mentioned in ref.[5](Chap. V, Page 26) without proof. Compute directly, we see

$$F_R(k) = \frac{\sin^2(\pi Rk\omega)}{R^2\sin^2(\pi k\omega)} \leq \frac{\sin^2(\pi R\|k\omega\|)}{4R^2\|k\omega\|^2}.$$

(2.3) follows from dividing the situations into $R\|k\omega\| \geq 1$ and $R\|k\omega\| < 1$. And it is obvious that coupling constant has nothing to do with C_v.

Proof of Lemma 2.3:

Original idea can be found in [9], Lemma 4. However, they bound the second part in Lemma 2.3 by $1 + C(\frac{q}{R})^2$. Since we need to consider the case $q > R$, we rewrite the proof here to remove the squre and also indicate that the constant is independent of C_v.

Since $\frac{p}{q}$ is the approximant of ω, $|\omega - \frac{p}{q}| < \frac{1}{q^2}$. Then for $|k| < \frac{q}{2}$, $|k\omega - \frac{kp}{q}| < \frac{k}{q^2} < \frac{1}{2q}$ and hence $\|k\omega\| \geq \frac{1}{2q}$. Then for any $k_1, k_2 \in (0, \frac{q}{4}]$, $|k_1 \pm k_2| < \frac{1}{2q}$. Thus $\Big|\|k_1\omega\| - \|k_2\omega\|\Big| = \|(k_1 \pm k_2)\omega\| > \frac{1}{2q}$. We see that $\|k_i\omega\|, i = 1, \cdots, [\frac{q}{4}]$ are $\frac{1}{2q}$ departed and the smallest one is more than $\frac{1}{2q}$. We rearrange them increasingly as $\|k_{i_1}\omega\| < \|k_{i_2}\omega\| < \cdots$. Thus $\|k_{i_s}\omega\| \geq \frac{s}{2q}$.
Hence

$$\sum_{1 \leq |k| < \frac{q}{4}} \frac{1}{1 + R^2\|k\omega\|^2} \leq \sum_{s=1}^{q/4} \frac{1}{1 + R^2(\frac{s}{2q})^2} \leq \frac{2q}{R} \int \frac{\mathrm{d}x}{1 + x^2} \leq \pi \frac{q}{R}.$$

Moreover, if $I_l = [\frac{q}{4}l, \frac{q}{4}(l+1)), l \geq 1$, we divide I_l into two sets, $S_1 = \{k \in I_l, |k\omega - [k\omega]| < 0.5\}$, $S_2 = \{k \in I_l, |k\omega - [k\omega]| > 0.5\}$. Then for $k_1, k_2 \in I_l$ belong to the same subset(S_1 or S_2), $|k_1 - k_2| < \frac{q}{2}$ and $\Big|\|k_1\omega\| - \|k_2\omega\|\Big| = \|(k_1 - k_2)\omega\| > \frac{1}{2q}$. Thus, the increasing rearrangement still carries on. However, the smallest one of $\{\|k\omega\|, k \in I_l\}$ might be less than $\frac{1}{2q}$. Hence

$$\sum_{|k| \in [\frac{q}{4}l, \frac{q}{4}(l+1))} \frac{1}{1 + R^2\|k\omega\|^2} \leq \frac{1}{1 + R^2\|k_{i_1}\omega\|^2} + 2\sum_{s \geq 1} \frac{1}{1 + R^2(\frac{s}{2q})^2} \leq 1 + 2\pi \frac{q}{R}. \quad \square$$

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