PHASE TRANSITION AND SEMI-GLOBAL REDUCIBILITY

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ABSTRACT. We obtain a (weak) phase transition result for one-dimension continuous analytic quasi-periodic Schödinger operators, which is proved by semi-global reducibility theory of analytic quasi-periodic linear systems with two frequencies.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we are interested in the spectrum of analytic quasi-periodic Schrödinger equation defined on $L^2(\mathbb{R})$:

$$(H_{\lambda V,\alpha,\theta}y)(t) = -y^{''}(t) + \lambda V(\theta_1 + t, \theta_2 + \alpha t)y(t) = Ey(t), \tag{1.1}$$

where $V : \mathbb{T}^2 \to \mathbb{R}$ is an analytic potential. In it, $\theta \in \mathbb{T}^2$ is called the phase, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is called the frequency and $\lambda \in \mathbb{R}$ is called the coupling constant. The spectrum of $H_{\lambda V,\alpha,\theta}$, which we denote by $\Sigma(H_{\lambda V,\alpha})$, is independent of the phase θ. The operators come from the study of quasi-crystals and become a central research subject in the spectral theory of Schrödinger operators. The discrete version of the operator is

$$(L_{\lambda V,\alpha,\theta}u)_n = u_{n+1} + u_{n-1} + \lambda V(\theta + n\alpha)u_n. \tag{1.2}$$

An extensively studied example is *almost Mathieu operator*, where $V(\theta) = 2\cos 2\pi\theta$.

One remarkable feature of *almost Mathieu operator* is that it reveals a phase transition phenomenon. It has been proved that $L_{2\lambda\cos,\alpha,\theta}$ has purely absolute continuous spectrum for all α, θ if $\lambda < 1$ [1, 4, 6, 20], while $L_{2\lambda\cos,\alpha,\theta}$ has pure point for a.e. α, θ if $\lambda > 1$ [20]. Thus $\lambda = 1$ is a phase transition point and actually the spectrum is purely singular continuous for a.e. α, θ[7]. In physics, it means that when the coupling constant λ changes, it undergos a metal-insulator transition at $\lambda = 1$.

For the discrete Schrödinger operator $L_{\lambda V,\alpha,\theta}$ with general analytical potential, the picture is less clear, but the coupling constant λ still plays the role of the transition parameter. One knows that for small λ, the operator has purely absolutely continuous spectrum and good transport properties [6, 16], and for large λ, the operator has pure point spectrum and related localization type phenomenon [12]. Recently Bjerklöv and Krikorian [11]

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obtained the first example which has the coexistence of absolutely continuous spectrum and pure point spectrum. Related result can be found in [9]. Before that, people usually believe that in general the coexistence of the spectrum does not occur for a fixed λ in discrete case.

However the spectrum of the continuous Schrödinger operator $H_{\lambda V,\alpha,\theta}$ is totally different. For a fixed large coupling constant λ, the operator usually has mixed spectrum. In the bottom of the spectrum, the operator $H_{\lambda V,\alpha,\theta}$ often has point spectrum or positive Lyapunonv exponent. More precisely, Frölish-Spencer-Wittner [18] proved that if $V(\theta_1,\theta_2)=\cos(2\pi\theta_1)+\cos(2\pi\theta_2)$, then the spectrum in the interval

$$[\inf\Sigma(H_{\lambda V,\alpha}),\inf\Sigma(H_{\lambda V,\alpha})+const(\lambda,\alpha)]$$

is pure point for $a.e.$ θ, provide that λ is large enough and α is Diophantine. Recall that α is Diophantine (denote $\alpha\in DC(\gamma,\tau)$), if there exist $\gamma,\tau>1$ such that $\|k\alpha\|_{\mathbb{T}}\geq\frac{\gamma^{-1}}{|k|^{\tau}}, 0\neq k\in\mathbb{Z}$. Later, Bjerklöv [8] proved that if $\alpha\in DC(\tau,\gamma)$, and V attains its minimum value at most finitely many points in T^2(without lose of generality, we assume that $\min V=0$), then there is a constant $c_0=c_0(V)$ and a $\lambda_0=\lambda_0(V,\tau,\gamma)>0$, such that for all $\lambda>\lambda_0$, we have $L(E)\geq c_0\sqrt{\lambda}$ for all $[0,\lambda^{\frac{2}{3}}]$. On the other side, the operator has absolutely continuous spectrum in the upper part of the spectrum. From Eliasson's result [16], we know that there is $c_1=c_1(V,\tau,\gamma)$, such that the spectrum in $\Sigma(H_{\lambda V,\alpha})\cap[c_1(V,\tau,\gamma)\lambda^2,\infty)$ is purely absolutely continuous. So there are phase transitions somewhere in $E\in[\lambda^{\frac{2}{3}},c_1(V,\tau,\gamma)\lambda^2]$. In this paper, we will try to locate the transition energy more precisely. Intuitively, the transition occurs at the place where E and λ maintain a kind of balance. If we can take $\frac{\ln E}{\ln\lambda}$ as the transition parameter, then the transition happens roughly at $\frac{\ln E}{\ln\lambda}=1$. The precise result is the following:

Theorem 1.1. *Let $\alpha\in DC(\tau,\gamma)$, $V\in C^{\omega}(\mathbb{T}^2,\mathbb{R})$ which attains its minimum value at most finitely many points, then for arbitrary small $\varepsilon>0$,*

(1) *the spectrum in $\Sigma(H_{\lambda V,\alpha})\cap(-\infty,\lambda^{1-\varepsilon}]$ is singular,*

(2) *the spectrum in $\Sigma(H_{\lambda V,\alpha})\cap[\lambda^{1+\varepsilon},\infty)$ is purely absolutely continuous,*

provided that $\lambda>\lambda_1$ where $\lambda_1=\lambda_1(V,\tau,\gamma,\varepsilon)>0$ is a big constant.

Remark 1.1. *It is plausible that $\Sigma(H_{\lambda V,\alpha})\cap(-\infty,\lambda^{1-\varepsilon}]$ is actually pure point. A related result is due to Bjerklöv* [10]*, who proved that if V is C^2 and has a unique non-degenerate minimum, then for large λ, $H_{\lambda V,\alpha,\theta}$ has eigenvalue in the bottom of the spectrum for some $\theta\in\mathbb{T}^2$.*

Theorem 1.1 (1) was pointed to us by Bjerklöv, and the proof is essentially contained in [8]. The main contribution of this paper is Theorem 1.1 (2). Actually, we can prove a stronger result which covers some Liouvillean frequencies. We recall that the frequency α is not super-liouvillean, if

$$\widetilde{\beta}(\alpha):=\sup_{n\to\infty}\frac{\ln\ln q_{n+1}}{\ln q_n}<\infty,$$

then our result is the following:

Theorem 1.2. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\widetilde{\beta}(\alpha) < \infty$, $V \in C^\omega(\mathbb{T}^2, \mathbb{R})$,*

(1) *there exist $C_1 = C_1(V) > 0$, $\lambda_2 = \lambda_2(\widetilde{\beta}) > 0$, such that for all $\lambda > \lambda_2$, the spectrum in $\Sigma(H_{\lambda V,\alpha}) \cap [C_1(V)\lambda, \infty)$ is purely absolutely continuous.*
(2) *there exist $C_2 = C_2(V, \widetilde{\beta}) > 0$, such that for all $\lambda \in \mathbb{R}^+$, the spectrum in $\Sigma(H_{\lambda V,\alpha}) \cap [C_2(V, \widetilde{\beta})\lambda, \infty)$ is purely absolutely continuous.*

As in [16], we rewrite the system (1.1) as

$$\begin{cases} \dot{x} = (A(\sqrt{E}) + F(\theta))x \\ \dot{\theta} = \omega_0 = (1, \alpha), \end{cases} \tag{1.3}$$

where

$$A(\sqrt{E}) = \sqrt{E}J = \sqrt{E}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad F(\theta) = \frac{\lambda V(\theta)}{2\sqrt{E}}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

then the proof of Theorem 1.1 and Theorem 1.2 are based on reducibility theory of general analytic quasi-periodic $sl(2, \mathbb{R})$ system (ω, A):

$$\begin{cases} \dot{x} = A(\theta)x \\ \dot{\theta} = \omega, \end{cases} \tag{1.4}$$

since reducibility in the spectrum implies absolutely continuous spectrum. Recall that (ω, A) is *reducible (resp. rotations reducible)*, if there exist $B \in C^\omega(2\mathbb{T}^2, SL(2, \mathbb{R}))$ and $A_* \in sl(2, \mathbb{R})$(resp. $A_* \in C^\omega(\mathbb{T}^2, so(2, \mathbb{R}))$ such that B conjugates (ω, A) to (ω, A_*).

We first review the local reducibility theory, i.e., $A(\theta)$ in (1.4) is close to constants. In this respect, the rotation number (consult 2.2 for its definition) is an important quantity. If $\alpha \in DC(\gamma, \tau)$ and the perturbation is small enough (smallness depend on (γ, τ)), Dinaburg-Sinai [15] proved that for a *positive measure* set of rotation numbers, the systems are reducible. Later, Eliasson [16] proved that actually for a *full measure* set of rotation numbers, the systems are reducible. We remark that what we are interested in is the region $E > c\lambda$. In this region $F(\theta)$ defined in (1.3) could be arbitrary large. So Eliasson's perturbative result can not be applied.

Similar problem arises in proving non-perturbative reducibility (the smallness of perturbation does not depend on the Diophantine constants) of system (1.3). If $\omega = \omega_0 = (1, \alpha)$ with α Diophantine, *full measure* non-perturbative reducibility was obtained by Hou-You [19]. One can also consult similar result for Schrödinger cocycles [6, 23]. If α is not Diophantine, and the systems (1.3) are close to constants, then Avila-Fayad-Krikorian [5] and Hou-You [19] proved that for *positive measure* rotation number, the systems are rotations reducible. *Full measure* rotations reducibility was obtained by Hou-You [19] and You-Zhou [25]. In fact, Hou-You proved that for *all* rotation numbers, the systems $(\omega_0, A + F(\theta))$ are almost reducible, provided that F is small enough, and the smallness doesn't depend on α.

Recall that (ω, A) is *almost reducible* if the closure of its analytical conjugacy class contains a constant.

We point out that the method developed in [19] highly depends on the fact that $\omega_0 = (1, \alpha)$. A natural question is that whether Hou-You's almost reducibility result still holds for $\omega = (\omega_1, \omega_2)$ with arbitrary small $|\omega|$? Especially, if ω is Diophantine, whether non-perturbative reducibility holds for two-frequencies quasi-periodic linear systems with frequency ω? Rewriting $\omega = \frac{1}{\lambda}(1, \alpha)$, it means that whether the size of the perturbation does not depend on λ when α is Diophantine. Unfortunately, the answers to both questions are negative due to the following reasons.

If A is zero matrix, then non-perturbative result doesn't hold for systems $(\omega, A + F(\theta))$ if no further assumption is added, since it could be non-uniformly hyperbolic [7]. If A is parabolic, non-perturbative result is also not always true, which is due to the following counter-example derived from [24], in it, Sorets and Spencer proved that if the potential $V(\theta) = \cos 2\pi\theta_1 + \cos 2\pi\theta_2$, then when λ is large enough, there is a set $\mathcal{E}$ composed of intervals of width $\sqrt{\lambda}$ separated by $\mathcal{O}(\lambda^{-1})$ such that for any $E \in \mathcal{E}$, $\gamma(E) \sim \sqrt{\lambda}$ and $\mathcal{E} \cap \Sigma(H_{\lambda V,\alpha}) \neq \varnothing$. The eigenvalue equation $H_{\lambda V,\alpha,\theta} y = Ey$ is equivalent to

$$\begin{cases} \dot{x} = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda^{-1}(\cos 2\pi\theta_1 + \cos 2\pi\theta_2) - \lambda^{-2}E & 0 \end{pmatrix} \right) x \\ \dot{\theta} = \frac{1}{\lambda}(1, \alpha), \end{cases}$$

which is non-uniformly hyperbolic and thus not reducible if $E \in \mathcal{E} \cap \Sigma(H_{\lambda V,\alpha})$ and λ is sufficiently large.

Although there are above counterexamples, we still hope to get some positive and interesting results. Based on the above discussion, it is natural to introduce the following concept: semi-global regime.[1] For a bounded analytic (possibly matrix valued) function F defined on $|Im\theta| < h$, let $\|F\|_h = \sup_{|Im\theta|<h} \|F\|$. We denote by $C_h^\omega(\mathbb{T}^2, *)$ the set of all these $*$-valued functions ($*$ will usually denote $\mathbb{R}$, $sl(2, \mathbb{R})$). Consider the quasi-periodic linear system $(\omega, A + F(\theta))$, where $A \in sl(2, \mathbb{R})$, $F(\theta) \in C_h^\omega(\mathbb{T}^2, sl(2, \mathbb{R}))$. We say that the system $(\omega, A + F(\theta))$ is *semi-global* (or we say the system $(\omega, A + F(\theta))$ is in the *semi-global* regime) if A is non-singular($detA \neq 0$) and $\|F\|_h \leq c\|A\|$ where c is a small constant independent of ω.

We only consider the elliptic case $A = \rho J$ since the hyperbolic case is trivial, then by time rescaling, the system we are interested in can be rewritten as

$$\begin{cases} \dot{x} = (\lambda\rho J + \lambda F(\theta))x \\ \dot{\theta} = \omega_0 = (1, \alpha) \end{cases} \tag{1.5}$$

where $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, $F \in C_h^\omega(\mathbb{T}^2, sl(2, \mathbb{R}))$. Let the sequences (q_n) be the denominators of best rational approximations of α. According to q_n, we define

[1] The authors would like to thank Haken Eliasson for useful discussions about this concept.

the forbidden intervals by

$$I_n = [(\frac{1}{2\rho}+1)e^{\frac{q_n h}{4(1+\chi)}}, (\frac{1}{\rho}+1)\frac{16\chi}{h}\ln q_{n+1}],$$

where h, ρ have been defined before and $\chi > 1$ is an absolute constant defined in Theorem 4.1 (one can also consult Theorem 1.1 of [19]). We remark that I_n might be empty if the jumping form q_n to q_{n+1} is not very big. The following is the main result of this paper.

Theorem 1.3. *Suppose that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\rho > 0$, $0 < h < 1$, $\lambda \in \mathbb{R}^+$. There exists $\varepsilon_0 = \varepsilon_0(h,\rho,\chi)$ such that if $\|F\|_h < \varepsilon_0$, then we have the following:*

(1) *If $\lambda \in \mathbb{R}\backslash\bigcup_{n=1}^{\infty} I_n$, then the system (1.5) is almost reducible.*
(2) *If $\lambda \in \bigcup_{n=1}^{\infty} I_n$, then $L(\omega_0, \lambda\rho J + \lambda F(\theta)) \leq \varepsilon_0 e^{-\frac{\lambda h(\rho+1)}{8\rho}}$.*

We give some comments to Theorem 1.3 and the concept of semi-global reducibility. Firstly we emphasize again that, for a fixed n, the forbidden interval I_n might be empty, it is interesting to investigate what happens in those forbidden intervals. We believe that they include some non-uniformly hyperbolic systems which are obviously not reducible. Secondly, the results with small λ has been covered by Hou-You [19]. The main contributions of this paper is to deal the large λ. Thirdly, by time rescaling, one can easily transform these results into almost reducibility of the system $(\frac{\omega_0}{\lambda}, A + F(\theta))$. We just point out that time rescaling is key part of renormalizaiton of quasi-periodic linear systems [14], we hope this result can be helpful for a better understanding of time rescaling. Finally, semi-global systems are good candidates to understand why non-perturbative reducibility doesn't hold for three dimensional systems. In three dimensional quasi-periodic linear systems, our semi-global results can also be used to construct almost reducible system with Liouvillean frequency. We will come back to this issue elsewhere.

2. PRELIMINARIES

2.1. **Continued Fraction Expansion.**

Let $\alpha \in (0,1)$ be irrational. Define $a_0 = 0, \alpha_0 = \alpha$, and inductively for $k \geq 1$,

$$a_k = [\alpha_{k-1}^{-1}], \qquad \alpha_k = \alpha_{k-1}^{-1} - a_k = G(\alpha_{k-1}) = \{\frac{1}{\alpha_{k-1}}\},$$

Let $p_0 = 0, p_1 = 1, q_0 = 1, q_1 = a_1$, then we define inductively

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}.$$

The sequence (q_n) is the sequence of denominators of best rational approximations of α since we have

$$\forall 1 \leq k < q_n, \quad \|k\alpha\|_{\mathbb{T}} \geq \|q_{n-1}\alpha\|_{\mathbb{T}}, \tag{2.1}$$

and

$$\|q_n\alpha\|_{\mathbb{T}} \le \frac{1}{q_{n+1}}. \tag{2.2}$$

2.2. **Lyapunov exponent, rotation number.** Let $\Phi^t(\theta)$ be the basic matrix solution of quasi-periodic systems (1.3). We call that

$$L(\omega, A) = \lim_{t\to+\infty} \frac{1}{t}\int_{\mathbb{T}} \ln|\Phi^t(\theta)|d\theta$$

the *Lyapunov exponent* of the system (ω, A).

The *rotation number* of (1.3) is defined as

$$\rho(\omega, A) = \lim_{t\to+\infty} \frac{arg(\Phi^t(\theta)x)}{t},$$

where $0 \ne x \in \mathbb{R}^2$, and arg denote the angle. The rotation number ρ is well-defined and does not depend on θ and x [21]. ρ is said to be Diophantine w.r.t. ω with some constants $\gamma, \tau > 1$ if

$$|\langle k, \omega\rangle - 2\rho| \ge \frac{\gamma^{-1}}{|k|^\tau}, \quad k \in \mathbb{Z}^2\backslash\{0\}.$$

3. A basic proposition

The key part in the proof of the main results is to get rid of the non-abelian part of $\lambda F(\theta)$ roughly, and transform $(\omega_0, \lambda\rho J + \lambda F(\theta))$ to

$$\begin{cases} \dot{x} = (\lambda\rho J + \lambda\varphi(\theta)J + \widetilde{F}(\theta))x \\ \dot{\theta} = \omega_0 = (1, \alpha) \end{cases}$$

with $\|\widetilde{F}(\theta)\| \sim e^{-\lambda\sigma}$ and $\varphi(\theta) = O(\varepsilon_0)$.

First we give some notations which will be used in the sequel. Let $\xi = \frac{|\omega_0|}{2\rho} + 1$, and denote

$$(\mathcal{T}_K f)(\theta) = \sum_{k\in\mathbb{Z}^2, |k|<K} \widehat{f}(k) e^{2\pi i<k,\theta>},$$

$$(\mathcal{R}_K f)(\theta) = \sum_{k\in\mathbb{Z}^2, |k|\ge K} \widehat{f}(k) e^{2\pi i<k,\theta>}.$$

To avoid a flood of constants, in the following, we will use c to denote the numerical constant regardless of its quantity. With the above notations, our main proposition will be the following:

Proposition 3.1. *Suppose that* $\rho > 0$, $0 < h < 1$, $\lambda > 1$, $F \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$. *Then there exists* ε_0 *depending on* ρ, h *but not on* λ, *such that if* $\|F\|_h \le \varepsilon_0$, *then there exists* $U \in C_h^\omega(\mathbb{T}^2, SL(2,\mathbb{R}))$, *which conjugates the system* $(\omega_0, \lambda\rho J + \lambda F(\theta))$ *to*

$$\begin{cases} \dot{x} = (\lambda\rho J + \lambda\varphi(\theta)J + \widehat{F}(\theta))x \\ \dot{\theta} = \omega_0 = (1, \alpha) \end{cases} \tag{3.1}$$

with the estimates

$$\|\varphi(\theta)\|_h \leq 2\varepsilon_0, \tag{3.2}$$

$$\|\widehat{F}\|_{\frac{3h}{4}} \leq \varepsilon_0 \lambda e^{-[\frac{\lambda}{\xi}]\frac{h}{4}}. \tag{3.3}$$

Furthermore, we have $\mathcal{R}_{[\lambda/\xi]}\varphi(\theta) = 0$.

The proposition can be proved in several different ways. We will provide two different proofs. Both are interesting since they represent two different ways to understand the semi-global problem.

The basic observation of the first proof is that as a result of fast rotation $\lambda\rho J$, we can remove lower order terms of the non-abelian part of the perturbation. This can be done by Implicit Functional Theorem or by homotopy method. The key is to solve homological equations depending on θ

$$\partial_\omega y(\theta) + i\lambda(\rho + b(\theta))y(\theta) = \lambda f(\theta),$$

up to a very small error.

In the second proof, we use the "cheap trick" which was developed in [5, 17]. In this proof, we consider the equivalent system $(\frac{\omega_0}{\lambda}, \rho J + F(\theta))$. The idea is to take advantage of large λ. More precisely, we view $(\frac{\omega_0}{\lambda}, \rho J + F(\theta))$ as a perturbation of $(0, \rho J + F(\theta))$. The conjugation of the latter one is actually the diagonalization in usual algebraic sense:

$$B^{-1}(\theta)(\rho J + F(\theta))B(\theta) = \rho(\theta)J.$$

Thus B will conjugate $(\frac{\omega_0}{\lambda}, \rho J + F(\theta))$ to a rotation with a smaller perturbation,

$$B^{-1}(\theta)[(\rho J + F(\theta))B(\theta) - \partial_{\frac{\omega_0}{\lambda}} B(\theta)] = \rho(\theta)J - B(\theta)^{-1}\partial_{\frac{\omega_0}{\lambda}} B(\theta),$$

with the new perturbation of size $\frac{|\omega_0|}{\lambda}\varepsilon_0$. Using this trick iteratively, one can get the desired result.

3.1. **First approach: Homotopy method.**

We develop the homotopy method to eliminate all the non-resonant modes of the perturbation. Similar proof appeared in [14].

We first give some notations which will be used in the sequel. We recall $sl(2,\mathbb{R})$ is the set of 2 by 2 matrices with real coefficients of the form

$$\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$. It is isomorphic to $su(1,1)$, the group of matrices of the form

$$\begin{pmatrix} it & \nu \\ \bar{\nu} & -it \end{pmatrix}$$

with $t \in \mathbb{R}$, $\nu \in \mathbb{C}$. We simply denote such a matrix by $\{t, \nu\}$. The isomorphism between $sl(2,\mathbb{R})$ and $su(1,1)$ is given by $B \to MBM^{-1}$ where

$$M = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Direct calculation shows that

$$M\begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}M^{-1}=\begin{pmatrix} iz & x-iy \\ x+iy & -iz \end{pmatrix},$$

thus $H:=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}=M\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}M^{-1}$.

Define

$$\widetilde{\mathfrak{B}}_h^{(nre)}=\left\{\begin{pmatrix} 0 & \mathcal{T}_{[\lambda/\xi]}g(\theta) \\ \mathcal{T}_{[\lambda/\xi]}\bar{g}(\theta) & 0 \end{pmatrix}\Big| g\in C_h^{\omega}(\mathbb{T}^2,\mathbb{C})\right\},$$

$$\widetilde{\mathfrak{B}}_h^{(re)}=\left\{\begin{pmatrix} if(\theta) & \mathcal{R}_{[\lambda/\xi]}g(\theta) \\ \mathcal{R}_{[\lambda/\xi]}\bar{g}(\theta) & -if(\theta) \end{pmatrix}\Big| f\in C_h^{\omega}(\mathbb{T}^2,\mathbb{R}), g\in C_h^{\omega}(\mathbb{T}^2,\mathbb{C})\right\},$$

and define $\mathfrak{B}_h^{(nre)}=M^{-1}\widetilde{\mathfrak{B}}_h^{(nre)}M$, $\mathfrak{B}_h^{(re)}=M^{-1}\widetilde{\mathfrak{B}}_h^{(re)}M$. It follows that

$$C_h^{\omega}(\mathbb{T}^2,su(1,1))=\widetilde{\mathfrak{B}}_h^{(nre)}\oplus\widetilde{\mathfrak{B}}_h^{(re)},$$
$$C_h^{\omega}(\mathbb{T}^2,sl(2,\mathbb{R}))=\mathfrak{B}_h^{(nre)}\oplus\mathfrak{B}_h^{(re)}.$$

Let $\mathbb{I}^{nre}$ ($\mathbb{I}^{re}$) be the projection from $C_h^{\omega}(\mathbb{T}^2,sl(2,\mathbb{R}))$ onto $\mathfrak{B}_h^{(nre)}$ ($\mathfrak{B}_h^{(re)}$) correspondingly.

We now finish the first proof. Let

$$B_\delta=\{U\in C_h^{\omega}(\mathbb{T}^2,SL(2,\mathbb{R}))|\|U-id\|_h<\delta\},\qquad \text{for } \delta=\varepsilon_0^{\frac{1}{2}},$$

We define the operator

$$\mathcal{F}:\quad B_\delta\to\mathfrak{B}_h^{(nre)},$$
$$\mathcal{F}(U)=\mathbb{I}^{nre}(Ad_U(\lambda\rho J+\lambda F)-U^{-1}\partial_\omega U),$$

where $F\in C_h^{\omega}(\mathbb{T}^2,sl(2,\mathbb{R}))$ with $\|F\|_h<\varepsilon_0$. We want to construct the solution of

$$\mathcal{F}(U_t)=(1-t)\mathcal{F}(U_0)\tag{3.4}$$

with $0\le t\le 1$ and initial condition $U_0=id$.

Direct computation shows that the derivative of $\mathcal{F}$ at U is the linear map from $\mathfrak{B}_h^{(nre)}$ to $\mathfrak{B}_h^{(nre)}$ given by

$$D\mathcal{F}(U)Y=\mathbb{I}^{nre}[U^{-1}((\lambda\rho J+\lambda F)Y-Y Ad_U(\lambda\rho J+\lambda F)+\partial_{\omega_0}Y-YU^{-1}\partial_{\omega_0}U)],$$

and thus

$$D\mathcal{F}(id)Y=\mathbb{I}^{nre}(\partial_{\omega_0}Y-[\lambda\rho J+\lambda F,Y]).$$

Let $\widetilde{D\mathcal{F}}(U)=MD\mathcal{F}(U)M^{-1}$, $\widetilde{F}=\{F_1,F_2\}\in C_h^{\omega}(\mathbb{T}^2,su(1,1))$, $\widetilde{Y}=\{0,g\}\in\widetilde{\mathfrak{B}}_h^{(nre)}$, then we have

$$\begin{aligned}&\widetilde{D\mathcal{F}}(id)\widetilde{Y}=\mathbb{I}^{nre}(\partial_{\omega_0}\widetilde{Y}-[\lambda\rho H+\lambda\widetilde{F},\widetilde{Y}])\\ =&\begin{pmatrix} 0 & \mathcal{T}_{[\lambda/\xi]}(\partial_{\omega_0}g-2i\lambda(\rho+F_1)g) \\ \mathcal{T}_{[\lambda/\xi]}(\partial_{\omega_0}\bar{g}+2i\lambda(\rho+F_1)\bar{g}) & 0 \end{pmatrix}.\end{aligned}$$

Next we prove the existence and boundedness of $\widetilde{D\mathcal{F}}(id)^{-1}$:

Lemma 3.1. *Suppose that* $\rho > 0$, $\lambda > 1$, $0 < h < 1$, $b \in C_h^\omega(\mathbb{T}^2, \mathbb{R})$, $f \in C_h^\omega(\mathbb{T}^2, \mathbb{C})$. *If* $\|b\|_h < \varepsilon_0 < \min\{\rho, 1\}(\frac{1}{\xi})^2$, *then there exists* $y \in C_h^\omega(\mathbb{T}^2, \mathbb{C})$ *such that*

$$\partial_{\omega_0} y(\theta) + 2i\lambda\rho y(\theta) + 2i\lambda\mathcal{T}_{[\lambda/\xi]}(b(\theta)y(\theta)) = \lambda\mathcal{T}_{[\lambda/\xi]}f(\theta) \tag{3.5}$$

with $\mathcal{R}_{[\lambda/\xi]}y(\theta) = 0$ *and* $\|y\|_h \leq \frac{c\xi}{2\rho}\|\mathcal{T}_{[\lambda/\xi]}f\|_h$.

Proof. Comparing the fourier coefficients of (3.5), we have

$$(i\langle k, \omega_0\rangle + 2i\lambda\rho)\hat{y}(k) + 2i\lambda \sum_{q,l:q+l=k} \hat{b}(l)\hat{y}(q) = \lambda\hat{f}(k) \qquad |l|, |q|, |k| < [\lambda/\xi]. \tag{3.6}$$

Then (3.6) can be seen as a matrix equation:

$$(\overline{A} + \overline{B})\overline{Y} = \lambda\overline{F},$$

where

$$\begin{aligned}
&\overline{A} = diag(\ldots, i\langle k, \omega_0\rangle + 2i\lambda\rho, \ldots)_{|k|<[\lambda/\xi]},\\
&\overline{B} = (2i\lambda\hat{b}(k-l))_{|k|,|l|<[\lambda/\xi], k\neq l},\\
&\overline{Y} = (\hat{y}(k))^T_{|k|<[\lambda/\xi]}, \quad \overline{F} = (\hat{f}(k))^T_{|k|<[\lambda/\xi]}.
\end{aligned}$$

If we denote $\Omega_h = diag(\ldots e^{|k|h}\ldots)_{|k|<[\lambda/\xi]}$, then

$$\Omega_h(\overline{A} + \overline{B})\Omega_h^{-1}\Omega_h\overline{Y} = \lambda\Omega_h\overline{F}.$$

Rewrite it as

$$(\widetilde{A}_h + \widetilde{B}_h)\widetilde{Y}_h = \lambda\widetilde{F}_h,$$

where $\widetilde{A}_h = \Omega_h\overline{A}\Omega_h^{-1}$, $\widetilde{B}_h = \Omega_h\overline{B}\Omega_h^{-1}$, $\widetilde{Y}_h = \Omega_h\overline{Y}$, $\widetilde{F}_h = \Omega_h\overline{F}$.

Since $\|b\|_h \leq \min\{\rho, 1\}(\frac{1}{\xi})^2$, we have

$$|i\langle k, w_0\rangle + 2i\lambda\rho| \geq 2\lambda\rho - \lambda/\xi|\omega_0| \geq 2\lambda\rho(1 - \frac{|\omega_0|}{|\omega_0| + 2\rho}) \gg 2\lambda\|b\|_h, \tag{3.7}$$

which implies that $\widetilde{A}_h + \widetilde{B}_h$ is diagonal dominated, thus the matrix $\widetilde{A}_h + \widetilde{B}_h$ has a bounded inverse. Moreover,

$$\begin{aligned}
\|(\widetilde{A}_h + \widetilde{B}_h)^{-1}\| &= \|(I + \widetilde{A}_h^{-1}\widetilde{B}_h)^{-1}\widetilde{A}_h^{-1}\|\\
&\leq \|\widetilde{A}_h^{-1}\|\frac{1}{1 - \|\widetilde{A}_h^{-1}\|\|\widetilde{B}_h\|} \leq \frac{c\xi}{2\lambda\rho}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y\|_h = \|\widetilde{Y}_h\| &\leq \|(\widetilde{A}_h + \widetilde{B}_h)^{-1}\|\|\lambda\widetilde{F}_h\|\\
&\leq \|(\widetilde{A}_h + \widetilde{B}_h)^{-1}\|\|\lambda\mathcal{T}_{[\lambda/\xi]}f\|_h \leq \frac{c\xi}{2\rho}\|\mathcal{T}_{[\lambda/\xi]}f\|_h.
\end{aligned}$$

□

By Lemma 3.1, if $\|F\|_h \leq \varepsilon_0 < \min\{\rho, 1\}(\frac{1}{\xi})^2$, then we have $\|\widetilde{D\mathcal{F}}(id)^{-1}\| \leq \frac{c\xi}{2\lambda\rho}$, therefore $\|D\mathcal{F}(id)^{-1}\| \leq \frac{c\xi}{2\lambda\rho}$ holds. We also need the following:

Lemma 3.2. *Given $U \in B_\delta$, the linear operator $D\mathcal{F}(U) - D\mathcal{F}(id)$ mapping $\mathfrak{B}_h^{(nre)}$ into $\mathfrak{B}_h^{(nre)}$ is bounded and*

$$\begin{aligned} & \|D\mathcal{F}(U) - D\mathcal{F}(id)\| \\ \leq \quad & 2\|U\|_h \|U - id\|_h [|\omega_0|(1 + 2\|U\|_h) + 2\|\lambda F\|_h (1 + \|U\|_h + \|U\|_h^2)]. \end{aligned}$$

Proof. It is exactly Lemma 4.3 of [14]. □

Consequently, we have

$$\begin{aligned} \|D\mathcal{F}(U)^{-1}\| &\leq \|(I + D\mathcal{F}(id)^{-1}(D\mathcal{F}(U) - D\mathcal{F}(id)))^{-1}\| \|D\mathcal{F}(id)^{-1}\| \\ &\leq \|D\mathcal{F}(id)^{-1}\| \frac{1}{1 - \|D\mathcal{F}(id)^{-1}\| \|D\mathcal{F}(U) - D\mathcal{F}(id)\|} \\ &\leq \frac{c\xi}{2\lambda\rho}, \end{aligned}$$

thus the solution of (3.4) exists and is given by

$$U_t = id - \int_0^t D\mathcal{F}(U_s)^{-1} \mathcal{F}(id) ds.$$

Moreover, U_1 conjugate the system $(\omega_0, \lambda\rho J + \lambda F(\theta))$ to

$$\begin{cases} \dot{x} = (\lambda\rho J + \lambda \widetilde{F}^{(re)})x \\ \dot{\theta} = \omega_0 = (1, \alpha), \end{cases}$$

where $\lambda \widetilde{F}^{(re)} = \mathbb{I}^{re}(Ad_{U_1}(\lambda\rho J + \lambda F) - U_1^{-1}\partial_{\omega_0} U_1)$. It then follows from section 4.1 of [14] that $\|\widetilde{F}^{re}\|_h \leq 2\|F^{nre}\|_h \leq 2\|F\|_h$.

By the definition of $\mathfrak{B}_h^{(re)}$, there exist $\widetilde{f}, \widetilde{g}_1, \widetilde{g}_2 \in C_h^\omega(\mathbb{T}^2, \mathbb{R})$ such that $\widetilde{F}^{(re)}$ can be written as

$$\begin{aligned} \widetilde{F}^{(re)}(\theta) &= \begin{pmatrix} 0 & \mathcal{T}_{[\lambda/\xi]}\widetilde{f}(\theta) \\ -\mathcal{T}_{[\lambda/\xi]}\widetilde{f}(\theta) & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{R}_{[\lambda/\xi]}\widetilde{g}_1(\theta) & \mathcal{R}_{[\lambda/\xi]}\widetilde{g}_2(\theta) + \mathcal{R}_{[\lambda/\xi]}\widetilde{f}(\theta) \\ \mathcal{R}_{[\lambda/\xi]}\widetilde{g}_2(\theta) - \mathcal{R}_{[\lambda/\xi]}\widetilde{f}(\theta) & -\mathcal{R}_{[\lambda/\xi]}\widetilde{g}_1(\theta) \end{pmatrix} \\ &= \varphi(\theta)J + \widehat{F}(\theta), \end{aligned}$$

then estimates (3.2) and (3.3) can be proved easily. □

3.2. **Second approach: Cheap trick.** In this section, we play "cheap trick" developed in [5, 17] to prove Proposition 3.1, the basis is the following lemma.

Lemma 3.3. *Suppose that $h > \sigma > 0$, $\lambda \gg 1$. We consider*

$$\begin{cases} \dot{x} = (\rho(\theta)J + P(\theta))x \\ \dot{\theta} = \frac{\omega_0}{\lambda}. \end{cases} \tag{3.8}$$

where $\rho \in C_h^\omega(\mathbb{T}^2, \mathbb{R})$, $P \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$, *and denote* $\widetilde{\rho} = \inf_{|\Im\theta|<h} |\rho(\theta)|$. *Then there exists* $\varepsilon > 0$, *such that if*

$$\|P\|_h < \varepsilon < \min\{1, \widetilde{\rho}\}, \tag{3.9}$$

then there exist $Y \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$, $\rho_+ \in C_h^\omega(\mathbb{T}^2, \mathbb{R})$, $P_+ \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$ *such that* e^Y *conjugates (3.8) to*

$$\begin{cases} \dot{x} = (\rho_+(\theta)J + P_+(\theta))x \\ \dot{\theta} = \frac{\omega_0}{\lambda}. \end{cases}$$

Moreover, we have the following estimates:

$$\|Y\|_h \le \frac{1}{2\widetilde{\rho}}\|P\|_h,$$

$$\|P_+\|_{h-\sigma} \le \frac{c|\omega|}{2\lambda\widetilde{\rho}\sigma}\|P\|_h,$$

$$\|\rho_+ - \rho\|_h \le \|P\|_h.$$

Proof. We approximate the system (3.8) by

$$\begin{cases} \dot{x} = (\rho(\theta)J + P(\theta))x \\ \dot{\theta} = 0. \end{cases}$$

Once we have that $\varepsilon < \min\{1, \widetilde{\rho}\}$, where $\widetilde{\rho} = \inf_{|\Im\theta|<h} |\rho(\theta)|$, then the fiber can be seen as a family of perturbation of elliptic matrix, by simple algebra, there exist $Y \in C_h^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$, $\rho_+ \in C_h^\omega(\mathbb{T}^2, \mathbb{R})$, such that

$$e^{-Y(\theta)}(\rho(\theta)J + P(\theta))e^{Y(\theta)} = \rho_+(\theta)J.$$

It is a standard fact that $\|Y\|_h \le \frac{1}{2\widetilde{\rho}}\|P\|_h$, $\|\rho_+ - \rho\|_h \le \|P\|_h$.

Then $B(\theta) = e^{Y(\theta)}$ conjugates (3.8) to

$$\begin{cases} \dot{x} = (\rho_+(\theta)J + P_+(\theta))x \\ \dot{\theta} = \frac{\omega_0}{\lambda} \end{cases}$$

with $P_+ = -B^{-1}\partial_{\frac{\omega_0}{\lambda}} B$. By Cauchy estimate, we have

$$\|P_+\|_{h-\sigma} \le \frac{|\omega_0|}{\lambda\sigma}\|Y\|_h \le \frac{|\omega_0|}{2\lambda\widetilde{\rho}\sigma}\|P\|_h.$$

□

Proof of Proposition 3.1: Proposition 3.1 follows inductively from Lemma 3.3. We begin with $(\frac{\omega_0}{\lambda}, \rho J + F(\theta))$. Let $M = [\frac{\lambda h}{4e\xi}]$, $h_0 = h$,

$$h_j = h(1 - \frac{j}{4M}), \qquad j = 1, 2, \dots M,$$

and let $\rho_0(\theta) = \rho$, $P_0(\theta) = F(\theta)$. Assume that for $j = 1, 2, \cdots, n$, one find $Y_j(\theta), P_j(\theta) \in C_{h_j}^\omega(\mathbb{T}, sl(2,\mathbb{R})), \rho_j(\theta) \in C_{h_j}^\omega(\mathbb{T}, \mathbb{R})$ such that e^{Y_j} conjugate the system

$$\begin{cases} \dot{x} = (\rho_{j-1}(\theta)J + P_{j-1}(\theta))x \\ \dot{\theta} = \frac{\omega_0}{\lambda} \end{cases}$$

to

$$\left\{ \begin{array}{l} \dot{x} = (\rho_j(\theta)J + P_j(\theta))x \\ \dot{\theta} = \frac{\omega_0}{\lambda} \end{array} \right. \tag{3.10}$$

with the estimates

$$\begin{aligned}
\|Y_j\|_{h_j} &\le \frac{1}{2\widetilde{\rho}_{j-1}}\|P_{j-1}\|_{h_{j-1}},\\
\|P_j\|_{h_j} &\le \frac{4M\xi}{\lambda h}\|P_{j-1}\|_{h_{j-1}} \le (\frac{1}{e})^j\|P_0\|_h,\\
\|\rho_j - \rho_{j-1}\|_{h_j} &\le \|P_{j-1}\|_{h_{j-1}},
\end{aligned}$$

where $\widetilde{\rho}_j = \inf_{|\Im\theta|<h_j}|\rho_j(\theta)|$.

Since

$$\|P_0\|_h \le \|F\|_h \le \varepsilon_0 < \min\{1,\rho\}(\frac{h}{4})^4,$$

and then

$$\begin{aligned}
\widetilde{\rho}_n &\ge \rho - \sum_{j=1}^{n}\|\rho_j(\theta) - \rho_{j-1}(\theta)\|_{h_j}\\
&\ge \rho - \sum_{j=1}^{n}(\frac{1}{e})^{j-1}\varepsilon_0 > \rho - 2\varepsilon_0,
\end{aligned}$$

therefore we have

$$\|P_n\|_{h_n} < \min\{1, \widetilde{\rho}_n\}.$$

Apply Lemma 3.3 again, we get $e^{Y_{n+1}}$ which conjugates the system further to

$$\left\{ \begin{array}{l} \dot{x} = (\rho_{n+1}(\theta)J + P_{n+1}(\theta))x \\ \dot{\theta} = \frac{\omega_0}{\lambda} \end{array} \right.$$

with the following estimates

$$\begin{aligned}
\|Y_{n+1}\|_{h_{n+1}} &\le \frac{1}{2\widetilde{\rho}_n}\|P_n\|_{h_n},\\
\|P_{n+1}\|_{h_{n+1}} &\le \frac{4M\xi}{\lambda h}\|P_n\|_{h_n} \le (\frac{1}{e})^{n+1}\|P_0\|_h,\\
\|\rho_{n+1} - \rho_n\|_{h_{n+1}} &\le \|P_n\|_{h_n}.
\end{aligned}$$

Let $e^Y = \prod_{j=M}^{0} e^{Y_j}$, then e^Y conjugate the system $(\frac{\omega_0}{\lambda}, \rho J + F(\theta))$ to

$$\left\{ \begin{array}{l} \dot{x} = (\rho_M(\theta)J + P_M(\theta))x \\ \dot{\theta} = \frac{1}{\lambda}\omega_0. \end{array} \right. \tag{3.11}$$

Let $\varphi(\theta) = \mathcal{T}_{[\lambda/\xi]}\rho_M(\theta) - \rho$, $\widehat{F} = \lambda\mathcal{R}_{[\lambda/\xi]}\rho_M(\theta)J + \lambda P_M(\theta)$, and scale the system, then (3.11) becomes

$$\left\{ \begin{array}{l} \dot{x} = (\lambda J + \varphi(\theta)J + \widehat{F}(\theta))x \\ \dot{\theta} = \omega_0. \end{array} \right.$$

Furthermore, we have estimates $\|\varphi(\theta)\|_{3h/4} \le 2\varepsilon_0$ and

$$\|\widehat{F}\|_{\frac{3h}{4}} \le \|\lambda \mathcal{R}_{[\lambda/\xi]}\rho_M(\theta)\|_{\frac{3h}{4}} + \|\lambda P_M(\theta)\|_{\frac{3h}{4}} \le \varepsilon_0 \lambda e^{-\frac{\lambda h}{4\xi}}.$$

□

4. Proof of Theorem 1.3

By Proposition 3.1, one can reduce $(\omega_0, \lambda\rho J + F(\theta))$ to $(\omega_0, (\lambda\rho + \lambda\varphi(\theta))J + \widehat{F}(\theta))$ with

$$\|\varphi(\theta)\|_h \le 2\varepsilon_0, \quad \|\widehat{F}\|_{\frac{3h}{4}} \le \varepsilon_0 \lambda e^{-[\frac{\lambda}{\xi}]\frac{h}{4}}.$$

Moreover, $\mathcal{R}_{[\lambda/\xi]}\varphi(\theta) = 0$.

Let q_n be the sequence of denominators of the best rational approximations of α, then for any fixed λ, there exists N such that $q_N \le [\frac{\lambda}{\xi}] < q_{N+1}$, where $\xi = \frac{|\omega_0|}{2\rho} + 1$.

We now remove the θ dependent terms in $\varphi(\theta)$ by

$$\partial_{\omega_0}\psi(\theta) = \lambda\varphi(\theta) - \lambda\hat{\varphi}(0), \tag{4.1}$$

which always has a solution since $\varphi(\theta)$ is a polynomial. Although $\|\psi\|$ maybe very large, we will prove that $\|Im\psi(\theta)\|$ can be well controlled at the cost of reducing the analytic radius greatly.

Lemma 4.1. *Let $h > 0$, $\varphi(\theta) \in \mathcal{T}_{[\lambda/\xi]}C^{\omega}_{\frac{3h}{4}}(\mathbb{T}^2, \mathbb{R})$. If $\|\varphi(\theta)\|_{\frac{3h}{4}} \le \varepsilon_0 < (\frac{h}{4})^4$, then we have the following:*

1. *If $q_{N+1} < e^{\frac{q_N h}{4}}$, then (4.1) has a solution with*

$$\|Im\psi(\theta)\|_{\frac{h}{2q_N}} \le \lambda\varepsilon_0^{\frac{1}{2}}.$$

2. *Otherwise, (4.1) has a solution with*

$$\|Im\psi(\theta)\|_{\frac{3h}{4q_{N+1}}} \le \lambda\varepsilon_0^{\frac{1}{2}}.$$

Proof. We write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1(\theta) = \mathcal{T}_{q_N}\varphi(\theta), \varphi_2(\theta) = \varphi(\theta) - \varphi_1(\theta)$. Then we have

$$\|\varphi_2(\theta)\|_{\frac{h}{2}} \le \varepsilon_0 e^{-\frac{q_N h}{4}} \le \frac{\varepsilon_0}{q_{N+1}}.$$

Now we need a small trick to estimate the image part of $\psi(\theta)$.

Suppose that $\theta = \theta_1 + i\theta_2$, then the solution of (4.1) can be written as

$$\begin{aligned}\psi(\theta) &= \lambda\sum \frac{\widehat{\varphi}(k)}{i\langle k,\omega_0\rangle}e^{ik\theta_1 - k\theta_2} - \lambda\sum \frac{\widehat{\varphi}(k)}{i\langle k,\omega_0\rangle}e^{ik\theta_1} + \lambda\sum \frac{\widehat{\varphi}(k)}{i\langle k,\omega_0\rangle}e^{ik\theta_1}\\ &= \lambda\sum \frac{\widehat{\varphi}(k)}{i\langle k,\omega_0\rangle}e^{ik\theta_1}\big(1 - e^{-k\theta_2}\big) + \lambda\sum \frac{\widehat{\varphi}(k)}{i\langle k,\omega_0\rangle}e^{ik\theta_1},\end{aligned}$$

since $\varphi(\theta)$ is real analytic, we have that $Im\Big(\sum \frac{\widehat{\varphi}(k)}{i\langle k,\omega_0\rangle}e^{ik\theta_1}\Big) = 0$.

Denote $\sigma = \frac{h}{2q_N}$ for simplicity and suppose that $q_{N+1} < e^{\frac{q_N h}{4}}$.

$$\begin{aligned}
\|Im\psi(\theta)\|_\sigma &\leq \sum_{0<|k|<q_N} \frac{\lambda|\widehat{\varphi}(k)|}{|i\langle k,\omega_0\rangle|}|e^{-k\theta_2}-1| + \sum_{q_N\leq|k|<[\lambda/\xi]} \frac{\lambda|\widehat{\varphi}(k)|}{|i\langle k,\omega_0\rangle|}|e^{-k\theta_2}-1| \\
&\leq \sum_{0<|k|<q_N} \frac{\lambda|\widehat{\varphi}(k)|}{|i\langle k,\omega_0\rangle|}e^{|k|\sigma}|k|\sigma + \sum_{q_N\leq|k|<[\lambda/\xi]} \frac{\lambda|\widehat{\varphi}(k)|}{|i\langle k,\omega_0\rangle|}e^{|k|\sigma}|k|\sigma \\
&\leq \lambda q_N\varepsilon_0 \frac{\sigma}{(h/2-\sigma)^2} + \lambda q_{N+1}\varepsilon_0 e^{-\frac{q_N h}{4}} \frac{\sigma}{(h/2-\sigma)^2} \\
&\leq \lambda\varepsilon_0^{\frac{1}{2}},
\end{aligned}$$

the last inequality is possible, since $\varepsilon_0 < (\frac{h}{4})^4$. This finishes the proof of the first statement.

The proof of the second statement is straightforward.

$$\begin{aligned}
\|Im\psi(\theta)\|_{\frac{3h}{4q_{N+1}}} &\leq \sum_{0<|k|<[\lambda/\xi]} \frac{\lambda|\widehat{\varphi}(k)|}{|i\langle k,\omega_0\rangle|}|e^{-k\theta_2}-1| \\
&\leq \sum_{0<|k|<[\lambda/\xi]} \frac{\lambda|\widehat{\varphi}(k)|}{|i\langle k,\omega_0\rangle|}e^{|k|\sigma}|k|\sigma \\
&\leq \lambda q_{N+1}\varepsilon_0 \frac{3h}{4q_{N+1}(3h/4-3h/4q_{N+1})^2} \\
&\leq \lambda\varepsilon_0^{\frac{1}{2}}.
\end{aligned}$$

□

Remark 4.1. *The estimates of this lemma are optimal. One may understand this result by Hadamard's three-lines theorem for subharmonic function.*

If we only eliminate the lower order terms of $\varphi(\theta)$ up to $q_N - 1$, then we have the following:

Lemma 4.2. *Let $h > 0$, $\varphi(\theta) \in C^\omega_{\frac{3h}{4}}(\mathbb{T}^2, \mathbb{R})$. If $\|\varphi(\theta)\|_{\frac{3h}{4}} \leq \varepsilon_0 < (\frac{h}{4})^4$, then the homological equation*

$$\partial_{\omega_0}\psi(\theta) = \lambda\mathcal{T}_{q_N}\varphi(\theta) - \lambda\hat{\varphi}(0)$$

has a solution with

$$\|Im\psi(\theta)\|_{\frac{h}{2\lambda}} \leq q_N\varepsilon_0^{\frac{1}{2}}. \tag{4.2}$$

Proof. The proof is essentially included in Lemma 4.1. □

The strategy of the proof is to transform semi-global system into local system, then apply Hou-You's result [19] to finish the proof. Before giving the final proof of Theorem 1.3, we first state Hou-You's results precisely.

Theorem 4.1. [19] *Let $\omega_0=(1,\alpha)$, $\alpha\in\mathbb{R}\backslash\mathbb{Q}$, $0<h<1$, $A\in sl(2,\mathbb{R})$ and $F\in C_h^\omega(\mathbb{T}^2,sl(2,\mathbb{R}))$. Then there exists $\epsilon>0$ depending on A,h but not on α, such that if $\|F\|_h<\epsilon$, then the system $(\omega_0,A+F(\theta))$ is almost reducible. In fact, we can select $\epsilon=Ch^\chi$, where C,χ are absolute constants.*

Now we finish the whole proof. According to the growth of q_N, we distinguish the proof into two main cases: $q_{N+1}\le e^{\frac{q_Nh}{4}}$ and $q_{N+1}>e^{\frac{q_Nh}{4}}$.

If $q_{N+1}\le e^{\frac{q_Nh}{4}}$ (Diophantine side), we denote $\overline{\lambda}=\lambda\rho+\lambda\widehat{\varphi}(0)$ and let $\partial_{\omega_0}\psi(\theta)=\lambda\varphi(\theta)-\lambda\widehat{\varphi}(0)$, then we have

$$\partial_{\omega_0}e^{\psi(\theta)J}=(\lambda J+\varphi(\theta)J+\widehat{F}(\theta))e^{\psi(\theta)J}-e^{\psi(\theta)J}(\overline{\lambda}J+\widetilde{F}(\theta)),$$

where $\widetilde{F}(\theta)=e^{-\psi(\theta)J}\widehat{F}(\theta)e^{\psi(\theta)J}$. It means that (3.1) is conjugated to

$$\begin{cases}\dot{x}=(\overline{\lambda}J+\widetilde{F}(\theta))x\\ \dot{\theta}=\omega_0.\end{cases}\tag{4.3}$$

Note that $q_{N+1}<e^{\frac{q_Nh}{4}}$, then by Lemma 4.1 (1), we have the estimate

$$\|\widetilde{F}\|_{\frac{h}{2q_N}}\le\varepsilon_0e^{-\frac{\lambda h}{8\xi}}e^{2\|Im\psi\|_{h/2q_N}}\le\varepsilon_0e^{-\frac{\lambda h}{16\xi}},$$

since $\xi q_N\le\lambda$, we further have

$$\|\widetilde{F}\|_{\frac{h}{2q_N}}\le\varepsilon_0e^{-\frac{\lambda h}{16\xi}}\le C(\frac{h}{2q_N})^\chi.$$

Applying Theorem 4.1, we prove that the system (4.3) is almost reducible.

If $q_{N+1}>e^{\frac{q_Nh}{4}}$ (Liouvillean side), we further divided it into the following three sub-cases.

Case 1: $\frac{16\chi}{h}\ln q_{N+1}\le[\frac{\lambda}{\xi}]<q_{N+1}$. In this case the proof is similar to the above. Again let $\partial_{\omega_0}\psi(\theta)=\lambda\varphi(\theta)-\lambda\widehat{\varphi}(0)$, then $e^{\psi(\theta)J}$ conjugate (3.1) to (4.3). By Lemma 4.1 (2), we have

$$\|\widetilde{F}\|_{\frac{3h}{4q_{N+1}}}\le\varepsilon_0e^{-\frac{\lambda h}{8\xi}}e^{2\|Im\psi\|_{3h/4q_{N+1}}}\le\varepsilon_0e^{-\frac{\lambda h}{16\xi}}.$$

Since $\frac{16\chi}{h}\ln q_{N+1}\le[\frac{\lambda}{\xi}]$, we have

$$\|\widetilde{F}\|_{\frac{3h}{4q_{N+1}}}\le\varepsilon_0e^{-\frac{\lambda h}{16\xi}}\le C(\frac{3h}{4q_{N+1}})^\chi.$$

Again, Theorem 1.3 follows from Theorem 4.1.

Case 2: $q_N\le[\frac{\lambda}{\xi}]\le e^{\frac{q_Nh}{4(1+\chi)}}$. In this case, we let $\partial_{\omega_0}\psi(\theta)=\lambda\mathcal{T}_{q_N}\varphi(\theta)-\lambda\widehat{\varphi}(0)$. Then $e^{\psi(\theta)J}$ conjugate (3.1) to (4.3) with

$$\widetilde{F}=e^{-\psi(\theta)J}\widehat{F}(\theta)e^{\psi(\theta)J}+\lambda\mathcal{R}_{q_N}\varphi(\theta)J.$$

By Lemma 4.2, we have

$$\begin{aligned}\|\widetilde{F}\|_{\frac{h}{2\lambda}} &\leq \varepsilon_0 e^{-\frac{\lambda h}{8\xi}} e^{2\|Im\psi\|_{h/2\lambda}} + \lambda\varepsilon_0 e^{-q_N h/4} \\ &\leq 2\lambda\varepsilon_0 e^{-q_N h/4}.\end{aligned}$$

Since $[\frac{\lambda}{\xi}] \leq e^{\frac{q_N h}{4(1+\chi)}}$, then we have

$$\|\widetilde{F}\|_{\frac{h}{2\lambda}} \leq 2\lambda\varepsilon_0 e^{-q_N h/4} \leq C(\frac{h}{2\lambda})^{\chi}.$$

Now in this case, Theorem 1.3 follows from Theorem 4.1.

Case 3: $e^{\frac{q_N h}{4(1+\chi)}} \leq [\frac{\lambda}{\xi}] \leq \frac{16\chi}{h} \ln q_{N+1}$.

It is this case that we cann't prove the almost reducibility. However, we can give an estimate for their Lyapunov exponents. What we need is the following result on the continuity of the Lyapunov exponent by Bourgain-Jitomirskaya [13].

Theorem 4.2. [13] *If $\omega_0 = (1, \alpha)$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $A \in C^\omega(\mathbb{T}^2, sl(2, \mathbb{R}))$, then $A \mapsto L(\omega_0, A)$ is continuous.*

Remark 4.2. *The above Bourgain-Jitomirskaya's result* [13] *was restricted to one-frequency Schrödinger cocycles (in particular, $SL(2, \mathbb{R})$ valued) cocycles. But this result can be generalized to the two-frequencies $sl(2, \mathbb{R})$ valued quasi-periodic linear systems, by standard Poincare map argument.*

By Proposition 3.1, the system $(\omega_0, \lambda\rho J + \lambda F(\theta))$ is reduced to $(\omega_0, (\lambda\rho + \lambda\varphi(\theta))J + \widehat{F}(\theta))$ with $\|\widehat{F}\|_{\frac{3h}{4}} \leq \varepsilon_0 e^{-[\frac{\lambda}{\xi}]\frac{h}{4}}$. Since $L(\omega_0, (\lambda\rho + \lambda\varphi(\theta))J) = 0$, we have

$$L(\omega_0, (\lambda\rho + \lambda\varphi(\theta))J + \widehat{F}(\theta)) < \varepsilon_0 e^{-[\frac{\lambda}{\xi}]\frac{h}{8}}$$

by Theorem 4.2. The invariance of Lyapunov exponent under conjugacy leads to

$$L(\omega_0, \lambda\rho J + \lambda F(\theta)) < \varepsilon_0 e^{-[\frac{\lambda}{\xi}]\frac{h}{8}}.$$

□

5. Applications to the spectrum

In this section, we apply semi-global reducibility results to the spectrum of continuous quasi-periodic Schödinger operators. When the frequency is Diophantine, we obtain a (weak) phase transition result.

Proof of Theorem 1.1:

As we have mentioned before, Bjerklöv's result [8] actually implies that for arbitrary small $\varepsilon > 0$, then there exists $\lambda_1 = \lambda_1(V, \tau, \gamma, \varepsilon) > 0$, such that if $\lambda > \lambda_1$, $E \in \Sigma(H_{\lambda V, \alpha}) \cap (-\infty, \lambda^{1-\varepsilon}]$, then $L(E) > c_0\sqrt{\lambda}$. By applying Kotani theory [22], Theorem 1.1 (1) follows immediately, since absolutely continuous spectrum is only supported on the spectrum with zero Lyapunov exponent.

For the second part of Theorem 1.1, we only need to prove the following:

Theorem 5.1. *Let $\alpha \in DC(\tau,\gamma)$, $V \in C^\omega(\mathbb{T}^2,\mathbb{R})$, there exists $C_1 = C_1(V) > 0$, $\lambda_2 = \lambda_2(\gamma,\tau) > 0$, such that for all $\lambda > \lambda_2$, the spectrum in $\Sigma(H_{\lambda V,\alpha}) \cap [C_1(V)\lambda,\infty)$ is purely absolutely continuous.*

Proof. Consider the following Schödinger equation:

$$(H_{\lambda V,\alpha,\theta}y)(t) = -y^{''}(t) + \lambda V(\theta_1+t,\theta_2+\alpha t)y(t) = \lambda E y(t), \tag{5.1}$$

It is easy to see that (5.1) is equivalent to system:

$$\begin{cases} \dot{x} = (\sqrt{\lambda E}J + F(\theta))x \\ \dot{\theta} = \omega_0 = (1,\alpha), \end{cases} \tag{5.2}$$

where

$$F(\theta) = \frac{\sqrt{\lambda}V(\theta)}{2\sqrt{E}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

In order to prove Theorem 5.1 by applying Theorem 1.3, it is sufficient to prove that there are no forbidden zones if n is large enough. To see this fact, we note that $q_{n+1} \le \frac{q_n^\tau}{\gamma}$ since $\alpha \in DC(\gamma,\tau)$. Thus there exists $K = K(\gamma,\tau)$ such that, when $n \ge K$,

$$\frac{16\chi}{h}\ln q_{n+1} \le \frac{16\chi}{h}(\tau \ln q_n - \ln\gamma) \le e^{\frac{q_n h}{4(1+\chi)}}.$$

This implies that $I_n = \emptyset$.

Once this is proved, we can finish the proof of Theorem 5.1 easily. In fact, we only need to consider the case $E \gg 1$, since we are only interested in the spectrum with high energy. By Theorem 1.3, if $\|\frac{V(\theta)}{2\sqrt{E}}\|_h \le \varepsilon_0(h)$ and

$$\lambda \ge 2q_K \ge q_K(\frac{|\omega_0|}{2\sqrt{E}}+1) = q_K\xi,$$

the systems (5.2) are almost reducible. Since all systems are conjugated to systems in Eliasson's perturbative regime [16], the spectral result of Theorem 5.1 follows from [16] immediately. □

Proof of Theorem 1.2:

The proof of Theorem 1.2 follows the same line as Theorem 5.1. For the proof of the first part, we recall $q_{n+1} \le e^{q_n^{\widetilde{\beta}}}$ since $\widetilde{\beta}(\alpha) < \infty$. Thus there exists $K = K(\widetilde{\beta})$ such that, when $n \ge K$,

$$\frac{16\chi}{h}\ln q_{n+1} \le \frac{16\chi}{h} q_n^{\widetilde{\beta}} \le e^{\frac{q_n h}{4(1+\chi)}}.$$

The rest part of the proof is same as Theorem 5.1, except invoking Avila's result (almost reducibility implies purely absolutely continuous spectrum in Liouville case) [2, 3] instead of Eliasson's result [16].

For the proof of the second part, we only need to consider the case $q_N \leq [\frac{\lambda}{\xi}] \leq \frac{16\chi}{h}\ln q_{N+1}$ in the proof of Theorem 1.3.

Let $\partial_{\omega_0}\psi(\theta) = \lambda\mathcal{T}_{q_N}\varphi(\theta) - \lambda\hat{\varphi}(0)$, then $e^{\psi(\theta)J}$ conjugate (3.1) to (4.3) with

$$\widetilde{F} = e^{-\psi(\theta)J}\widehat{F}(\theta)e^{\psi(\theta)J} + \lambda\mathcal{R}_{q_N}\varphi(\theta)J.$$

By Lemma 4.2, we have $\|\widetilde{F}\|_{\frac{h}{2\lambda}} \leq 2\lambda\varepsilon_0 e^{-q_N h/4}$. In view of

$$[\frac{\lambda}{\xi}] \leq \frac{16\chi}{h}\ln q_{N+1} \leq \frac{16\chi}{h}q_N^{\widetilde{\beta}},$$

there exists $\varepsilon_0 = \varepsilon_0(h, \widetilde{\beta})$ such that the estimate

$$\|\widetilde{F}\|_{\frac{h}{2\lambda}} \leq 2\lambda\varepsilon_0 e^{-q_N h/4} \leq C(\frac{h}{2\lambda})^{\chi}$$

holds for any N. In fact, we only need to choose

$$\varepsilon_0 \leq \min_N q_N^{-\widetilde{\beta}(\chi+1)}e^{q_N h/4}.$$

In this case, there is no forbidden zones. It concludes that the systems (5.2) are almost reducible for any $\lambda \in \mathbb{R}^+$ if $\|F\|_h < \varepsilon_0(h, \widetilde{\beta})$. The absolute continuity of the spectrum follows from the almost reducibility. □

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