EMBEDDING OF ANALYTIC QUASI-PERIODIC COCYCLES INTO ANALYTIC QUASI-PERIODIC LINEAR SYSTEMS AND ITS APPLICATIONS

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ABSTRACT. In this paper, we prove that any analytic quasi-periodic cocycle close to constant is the Poincaré map of an analytic quasi-periodic linear system close to constant. With this local embedding theorem, we get fruitful new results. We show that the almost reducibility of an analytic quasi-periodic linear system is equivalent to the almost reducibility of its corresponding Poincaré cocycle. By the local embedding theorem and the equivalence, we transfer the recent local almost reducibility results of quasi-periodic linear systems [21] to quasi-periodic cocycles, and the global reducibility results of quasi-periodic cocycles [4, 5] to quasi-periodic linear systems. Finally, we give a positive answer to a question of [5] and use it to prove Anderson localization results for long-range quasi-periodic operator with Liouvillean frequency, which gives a new proof of [6, 7, 12]. The method developed in our paper can also be used to prove some nonlinear local embedding results.

1. MOTIVATIONS AND MAIN RESULTS

We are concerned with smooth quasi-periodic linear systems

$$\begin{cases} \dot{x} = A(\theta)x \\ \dot{\theta} = \omega, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^2$, $\theta \in \mathbb{T}^d$, $\omega \in \mathbb{R}^d$ is rational independent and $A \in C^r(\mathbb{T}^d, sl(2,\mathbb{R}))$, $r \in N \cup \{\infty, \omega\}$, we denote it by (ω, A). Typical examples are Schrödinger systems where

$$A(\theta) = V_{E,q}(\theta) = \begin{pmatrix} 0 & 1 \\ q(\theta) - E & 0 \end{pmatrix} \in sl(2,\mathbb{R}).$$

The time discrete counterparts of the quasi-periodic linear systems are smooth quasi-periodic $SL(2,\mathbb{R})$ cocycles:

$$\begin{aligned} (\mu, \mathcal{A}): \quad & \mathbb{T}^{d-1} \times \mathbb{R}^2 \to \mathbb{T}^{d-1} \times \mathbb{R}^2 \\ & (\theta, v) \mapsto (\theta + \mu, \mathcal{A}(\theta) \cdot v), \end{aligned}$$

Date: February 17, 2012.

where $\mu \in \mathbb{T}^{d-1}$ with $(1,\mu)$ being rational independent, $\mathcal{A} \in C^r(\mathbb{T}^{d-1}, SL(2,\mathbb{R}))$. Typical examples are quasi-periodic Schrödinger cocycles where

$$\mathcal{A}(\theta) = S_E^V(\theta) = \begin{pmatrix} V(\theta) - E & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{R}).$$

They are related to one-dimensional quasi-periodic Schrödinger operators on $l^2(\mathbb{Z})$:

$$(H_{V,\mu,\phi}x)_n = x_{n+1} + x_{n-1} + V(n\mu + \phi)x_n = Ex_n. \tag{1.2}$$

We denote by $C_0^r(\mathbb{T}^{d-1}, SL(2,\mathbb{R}))$ the set of maps $\mathcal{A} \in C^r(\mathbb{T}^{d-1}, SL(2,\mathbb{R}))$ that are homotopic to the identity.

If $d = 2$, it is also interesting to consider the dual operator of (1.2), the Long-range quasi-periodic operator:

$$(L_{V,\alpha,\varphi}\psi)_n = \sum_{k\in\mathbb{Z}} V_k \psi_{n-k} + 2cos2\pi(\varphi + n\alpha)\psi_n, \tag{1.3}$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, V_k are fourier coefficients of $V(\theta) \in C^r(\mathbb{T},\mathbb{R})$. These two operators are closely related by Aubry duality (consult section 2.3 for more details).

We say that (ω, A) is C^r reducible, if there exist $B \in C^r(2\mathbb{T}^d, SL(2,\mathbb{R}))$ and $A_* \in sl(2,\mathbb{R})$ such that B conjugates (ω, A) to (ω, A_*). It is clear that the concept of reducibility defined above for Liouvillean frequencies is too restrictive since in general even a $\mathbb{R}$-valued cocycle is not reducible. So we need to introduce the weaker concept *almost reducibility.* A system (ω, A) is C^r *almost reducible (resp. almost rotations reducible)* if there exist sequences of $B_n \in C^r(2\mathbb{T}^d, SL(2,\mathbb{R}))$, $A_n \in sl(2,\mathbb{R})(resp. A_n \in C^r(\mathbb{T}^d, so(2,\mathbb{R})))$ and $F_n \in C^r(\mathbb{T}^d, sl(2,\mathbb{R}))$, such that B_n conjugate (ω, A) to $(\omega, A_n + F_n)$, where F_n is C^r converging to zero. Another useful concept is *rotations reducibility.* We say that (ω, A) is C^r *rotations reducible*, if there exist $B \in C^r(2\mathbb{T}^d, SL(2,\mathbb{R}))$ and $A_* \in C^r(\mathbb{T}^d, so(2,\mathbb{R}))$ such that B conjugates (ω, A) to (ω, A_*). These concepts can be defined similarly for cocycles $\mathcal{A} \in C_0^r(\mathbb{T}^{d-1}, SL(2,\mathbb{R}))$. We say that the operator $L_{V,\alpha,\varphi}$, (resp.$H_{V,\alpha,\phi}$) displays Anderson localization, if it has pure point spectrum with exponentially decaying eigenfunctions. We remark that reducibility, almost reducibility and Anderson localization are important issues in the study of quasi-periodic linear systems and the spectral theory of Schrödinger operators [4, 7].

1.1. **Classical results review.**

1.1.1. *Reducibility of analytic quasi-periodic linear systems.* The earliest result of local reducibility was due to Dinaburg and Sinai [15], who showed that $(\omega, V_{E,q}(\theta))$ is reducible for "most" sufficiently large E, where ω is assumed to satisfy the classical Diophantine condition:

$$|\langle k, \omega \rangle| \geq \frac{\gamma^{-1}}{|k|^\tau}, \quad 0 \neq k \in \mathbb{Z}^d,$$

and $\gamma, \tau > 1$ are fixed positive constants. Here (γ, τ) are called the Diophantine constants of ω. Later, Eliasson [16] obtained the *full measure reducibility* and the *almost reducibility* for $(\omega, V_{E,q}(\theta))$ with Diophantine ω. The proof is based on a crucial "resonance-cancellation" technique which was initially developed by Moser and Pöschel [30]. It should be noted that these results are *perturbative*. Stronger concept is *non-perburbative reducibility*, which means that the smallness of the perturbation does not depend on the Diophantine constants (γ, τ).

However, few results for quasi-periodic linear systems were obtained since [16] until [21]. Based on a unified approach (KAM theory and Floquet theory), Hou and You [21] proved that any two-frequencies quasi-periodic linear system close to constant is always *almost reducible* and *non-perturbative reducible*. They further proved that it is analytically *rotations reducible* if the rotation number is Diophantine w.r.t ω.

1.1.2. *Anderson localization.* Since non-perturbative reducibility results for quasi-periodic Schrödinger cocycles strongly depends on Anderson localization for the dual operator, we review the Anderson localization results first. For the almost Mathieu operator $H_{\lambda cos,\alpha,\phi}$, Jitomirskaya [22] proved that if α is Diophantine, $\lambda > 2$, then for a.e. ϕ, $H_{\lambda cos,\alpha,\phi}$ has Anderson localization. In the sequel, Avila and Jitomirskaya [6] further proved that if $\lambda > 2e^{16\beta/9}$, [1] then $H_{\lambda cos,\alpha,\phi}$ has Anderson localization. Similar results for long-range operator $L_{\lambda V,\alpha,\varphi}$ were considered by Bourgain and Jitomirskaya [12], Avila and Jitomirskaya [7], and we refer to [10, 11] for results on Schrödinger operators.

1.1.3. *Reducibility of analytic quasi-periodic cocycles.* Combining Aubry duality [1] with Anderson localization results [12], Puig [31] obtained a *non-perturbative* extension of Eliasson's results. Avila and Jitomirskaya [7] further developed a quantitative Aubry duality to prove that *almost localization* implies *almost reducibility* for the dual model.

Different from the continuous case, *global reducibility* of quasi-periodic cocycles (reducibility of a cocycle which is not necessarily close to a constant) is being developed rapidly. Based on Kotani theory [24, 33] and renormalization scheme [25], Avila and Krikorian [8] proved that: if α is recurrent Diophantine, $\mathcal{A} \in C_0^{\omega}(\mathbb{T}, SL(2,\mathbb{R}))$, then for Lebesgue $a.e.\varphi \in [0,1]$, $(\alpha, R_\varphi \mathcal{A})$[2] is either analytic reducible or non-uniformly hyperbolic. With respect to Liouvillean frequency, Fayad and Krikorian [18] first obtained that: if α is irrational, $\mathcal{A} \in C_0^{\infty}(\mathbb{T}, SL(2,\mathbb{R}))$, then for Lebesgue a.e.$\varphi \in [0,1]$, $(\alpha, R_\varphi \mathcal{A})$ is either C^∞ *almost rotations reducible* or non-uniformly hyperbolic. Their main techniques in the proof were "algebraic conjugacy trick" and renormalization scheme [8]. Later, Avila, Fayad and Krikorian [5] developed this method and obtained a local positive measure *rotations reducibility* result.

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[1] The definition of β can be found in section 2.1.

[2] $R_\varphi := \begin{pmatrix} cos2\pi\varphi & -sin2\pi\varphi \\ sin2\pi\varphi & cos2\pi\varphi \end{pmatrix}$.

Subsequently, they further proved that for Lebesgue $a.e.\varphi \in [0,1]$, $(\alpha, R_\varphi \mathcal{A})$ is either analytic *rotations reducible* or non-uniformly hyperbolic.

Recently, Avila [3] proposed an authentic "global theory" of one-frequency $SL(2,\mathbb{R})$ cocycles. Cocycles which are not uniformly hyperbolic are classified into three classes: supercritical, subcritical and critical. A central issue in the reducibility theory is his Almost Reducible Conjecture: "subcritical implies almost reducibility". If α is exponentially Liouvillean, Avila [4] proved the conjecture and obtained a corollary: any one-frequency analytic quasi-periodic cocycle close to constant is *almost reducible*.

1.1.4. *Comparison of continuous case and discrete case.* In the continuous case, local almost reducibility result is completely established recently by Hou and You [21], while there is no result for global reducibility. Since there is no Aubry duality [1] in the continuous case, reducibility can not be obtained by proving localization or almost localization of the dual system as in the discrete case [7, 31]. However, there is a uniform way to prove the local almost reducibility results directly in the continuous case. Unfortunately, the methods developed in [21] can not be applied to quasi-periodic cocycles directly. Global reducibility results are completely missing for continuous systems since there is no suitable renormalization scheme as in the discrete case. It is still an interesting question whether there is corresponding renormalization scheme for the quasi-periodic linear systems. We note that [14] also used renormalization technique to get reducibility results, however, the method can only be applied to the local situation and restricted to Brjuno frequency.

In the discrete case, although various global reducibility results [4, 5, 8, 18, 25] were obtained, local almost reducibility results are not satisfactory. Firstly, there is no unified and direct approach to deal with the almost reducibility problem for the quasi-periodic cocycles even in the local regime. The existing approach which highly depends on the localization results for the dual model, works only for Schrödinger cocycles. Secondly, as pointed by Avila, Fayad and Krikorian [5], the study of parabolic behavior was missing, also there was no generalization of Eliasson's results [16]. Readers may refer to the section 1.1 of [5] for more discussions.

In this paper, we shall establish a local embedding theorem, which serves as a bridge between analytic quasi-periodic linear systems and quasi-periodic cocycles. With this powerful tool, we can deduce fruitful new results. For example, one can exchange the almost reducibility results of quasi-periodic linear systems and quasi-periodic cocycles for free and then get many missing results both for the continuous systems and discrete cocycles. Furthermore, combining Aubry duality, we deduce Anderson localization results for the long-range operator with Liouvillean frequency. The proof of the local embedding theorem is interesting in itself and has further generalizations.

1.2. **Embedding theorem.**

1.2.1. *A local embedding theorem.* Let $\mathcal{G} = sl(n,\mathbb{R}), sp(2n,\mathbb{R}), o(n), u(n), so(n)$, and let G be the corresponding Lie groups. We consider the following C^r quasi-periodic linear system:

$$\left\{\begin{array}{l}\dot{x} = A(\theta)x \\ \dot{\theta} = \omega = (1,\mu),\end{array}\right. \tag{1.4}$$

where $A \in C^r(\mathbb{T}^d, \mathcal{G})$, $\mu \in \mathbb{T}^{d-1}$ with $(1,\mu)$ being rational independent, $\theta = (\theta_1, \widetilde{\theta})$, $\widetilde{\theta} = (\theta_2, \cdots, \theta_d)$. Denote by Φ^t the flow of (1.4) defined on $\mathbb{T}^d \times \mathbb{R}^n$ which is of the form: $\Phi^t(\theta_1, \widetilde{\theta}, y) = (\theta_1 + t, \widetilde{\theta} + t\mu, \Phi^t(\theta_1, \widetilde{\theta})y)$. We introduce $\mathcal{A}(\cdot) = \Phi^1(0,\cdot)$ which is clearly defined on $\mathbb{T}^{d-1}$ and called the corresponding Poincaré cocycle defined by (1.4). What we are interested in is the converse, whether we can embed a given C^r quasi-periodic cocycle into a C^r quasi-periodic flow? Apparently such a cocycle must be homotopy to the identity.

If $r \neq \omega$, using the method of suspension flow, Chavaudret [13] proved that any C^r smooth $G-$exponential cocycle [3] can be C^r embedded into the flow. Rychlik [32] proved that any C^r smooth $SU(2)$ cocycle can be C^r embedded into a quasi-periodic flow, the proof strongly depends on the fact that $SU(2)$ is simply connected. Both methods can't be applied to the analytic case.

The aim of this paper is to provide a new method to prove the local embedding of an analytic quasi-periodic $G-$ valued cocycle into an analytic quasi-periodic linear system. For a bounded analytic function F defined on $|Im\theta| < h$, let $\|F\|_h = \sup_{|Im\theta|<h} \|F\|$. We denote by $C^\omega_h(\mathbb{T}^d, *)$ the set of all these $*$-valued functions ($*$ will usually denote $\mathbb{R}$, $\mathcal{G}$). The main theorem is the following:

Theorem 1.1. *Let $h > 0$, $\mu \in \mathbb{T}^{d-1}$ with $(1,\mu)$ being rational independent, $A \in \mathcal{G}$, $G \in C^\omega_h(\mathbb{T}^{d-1}, \mathcal{G})$. There exist $\epsilon = \epsilon(A, h, |\mu|) > 0$, $c = c(A, h, |\mu|) > 0$ such that the quasi-periodic cocycle $(\mu, e^A e^{G(\cdot)})$ can be analytically embedded into a quasi-periodic linear system provided that $\|G\|_h = \varepsilon < \epsilon$. More precisely, there exist $\widetilde{A} \in \mathcal{G}$, $F \in C^\omega_{h/1+|\mu|}(\mathbb{T}^d, \mathcal{G})$ with $\|F\|_{h/1+|\mu|} \leq c\varepsilon^{1/2}$, such that $(\mu, e^A e^{G(\cdot)})$ is the Poincaré map of*

$$\left\{\begin{array}{l}\dot{x} = (\widetilde{A} + F(\theta))x \\ \dot{\theta} = (1,\mu).\end{array}\right.$$

Remark 1.1. *The selection of $\widetilde{A}$ and precise estimate on F in Theorems 1.1 will be given explicitly in the proof. We emphasize that $\widetilde{A} = A$, $\|F\|_{h/1+|\mu|} \leq c\varepsilon$ if A is diagonalizable.*

Remark 1.2. *The C^r $(r \neq \omega)$ version of the local embedding theorem is also true. Different from* [13], *the local structure is preserved with precise estimates.*

[3] It means that the cocycle can be written as $e^{A(\cdot)}$.

Remark 1.3. *The local embedding theorem is also true when* $(1,\mu)$ *is rational dependent. Although the result is trivial by Floquet theory, we will prove these results in a unified approach.*

Remark 1.4. *If* $d \geq 3$*, an example by Bourgain* [9] *shows that Eliasson's perturbative reducibility result is optimal. Since the embedding theorem does not depend on any Diophantine condition of the base dynamics, we do embed some non-uniformly hyperbolic cocycles into quasi-periodic liner systems.*

1.2.2. *Global embedding results.* We have partial results for global embedding of analytic quasi-periodic cocycles into quasi-periodic linear systems, which are restricted to one-frequency cocycle. The following observation is crucial:

Proposition 1.1. *Let* $\mu \in \mathbb{T}^{d-1}$ *with* $(1,\mu)$ *being rational independent,* $\mathcal{A} \in C^\omega(\mathbb{T}^{d-1}, SL(2,\mathbb{R}))$*. Suppose that* $(\mu,\mathcal{A})$ *is conjugated to* $(\mu,\tilde{\mathcal{A}})$*, and* $(\mu,\tilde{\mathcal{A}})$ *can be embedded into an analytic quasi-periodic linear system, then* $(\mu,\mathcal{A})$ *can also be embedded into an analytic quasi-periodic linear system.*

With the help of Proposition 1.1, we get the embedding result of uniformly hyperbolic and almost reducible cocycles:

Corollary 1.1. *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$*,* $\mathcal{A} \in C_0^\omega(\mathbb{T}, SL(2,\mathbb{R}))$*, then we have the following:*

1. *Any uniformly hyperbolic cocycle can be analytically embedded into a quasi-periodic linear system.*
2. *If* $(\alpha,\mathcal{A})$ *is almost reducible, then* $(\alpha,\mathcal{A})$ *can be analytically embedded into a quasi-periodic linear system.*

As a result of Corollary 1.1, we obtain that if a global cocycle is reduced to the local regime, then it can be analytically embedded into a quasi-periodic linear system, if we recall results of [4, 5], then we have:

Corollary 1.2. *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$*,* $\mathcal{A} \in C_0^\omega(\mathbb{T}, SL(2,\mathbb{R}))$*, then we have the following:*

1. *If* $E = \{\varphi \in [0,1] | L(\alpha, R_\varphi\mathcal{A}) = 0\}$*, then for almost every* $\varphi \in E$*,* $(\alpha, R_\varphi\mathcal{A})$ *can be analytically embedded into a quasi-periodic linear system.*
2. *Assume furthermore* $\beta(\alpha) > 0$*. If* $(\alpha,\mathcal{A})$ *is subcritical, then* $(\alpha,\mathcal{A})$ *can be analytically embedded into a quasi-periodic linear system.*

Question: For any $(\alpha,\mathcal{A}) \in \mathbb{R}\backslash\mathbb{Q} \times C_0^\omega(\mathbb{T}, SL(2,\mathbb{R}))$, whether it can be embedded into an analytical quasi-periodical linear system?

1.2.3. *Nonlinear local embedding results.* We point out that our proof of local embedding theorem is not restricted to linear case. For example, we can prove the following nonlinear local embedding result.

Theorem 1.2. *Let* $\rho > 0$*,* $r > 0$*,* $s > 0$*, and* $\mu \in \mathbb{T}^{d-1}$ *with* $(1,\mu)$ *being rational independent,* $f \in C_{r,s}^\omega(\mathbb{T} \times \mathbb{T}^{d-1}, \mathbb{R})$*. There exist* $c = c(\rho, r, s) > 0$*,*

$\varepsilon = \varepsilon(\rho, r, s) > 0$ such that if $\|f\|_{r,s} \leq \varepsilon$, then the quasi-periodically forced circle diffeomorphism:

$$\begin{cases} \theta \to \theta + \rho + f(\theta, \varphi) \\ \varphi \to \varphi + \mu \end{cases} \tag{1.5}$$

can be analytically embedded into a quasi-periodically forced circle flow

$$\begin{cases} \dot{\theta} = \rho + g(\theta, \varphi) \\ \dot{\varphi} = \omega = (1, \mu) \end{cases} \tag{1.6}$$

with the estimate $\|g\|_{\frac{r}{1+\rho}, \frac{s}{1+|\mu|}} \leq c\|f\|_{r,s}$.

Remark 1.5. *Analytic embedding of nearly integrable symplectic maps into Hamiltonian system is given by Kuksin and Pöschel* [28].

Since the result can be proved with the same method as in our paper, we will omit the proof. Consult Remark 3.4 for more discussions. Readers can find more interesting results on quasi-periodically forced circle diffeomorphism in [27].

1.3. Applications of the embedding theorem: from dynamical side.

1.3.1. *Equivalence.* By applying the local embedding theorem, we prove that almost reducibility of an analytic quasi-periodic system is equivalent to almost reducibility of its corresponding Poincaré cocycle.

Theorem 1.3. *An analytic quasi-periodic linear system (ω, A) is almost reducible (resp. rotations reducible) if and only if its corresponding Poincaré cocycle $(\mu, \mathcal{A})$ is almost reducible (resp. rotations reducible).*

Remark 1.6. *In* [26]*, Krikorian proved that a quasi-periodic linear system (ω, A) is reducible if and only if its corresponding Poincaré cocycle $(\mu, \mathcal{A})$ is reducible.*

As applications of local embedding theorem and the equivalence of almost reducibility, we get many missing results both for the continuous systems and discrete cocycles.

1.3.2. *Local reducibility of analytic quasi-periodic cocycles.* Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{A} \in C^\omega(\mathbb{T}, SL(2, \mathbb{R}))$. Is there a full Lebesgue measure subset $\Lambda(\alpha)$, which is explicitly given by some Diophantine condition, such that if $\mathcal{A}$ is sufficiently close to constant and the rotation number $\mathrm{rot}_f(\alpha, \mathcal{A}) \in \Lambda(\alpha)$, then $(\alpha, \mathcal{A})$ is rotations reducible? This question was asked in [5], it can be seen as a generalization of Eliasson's result [16]. We would like to give an even stronger result to the question:

Theorem 1.4. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $h > 0$, $\mathcal{A} \in C^\omega_h(\mathbb{T}, SL(2, \mathbb{R}))$. If the rotation number $\mathrm{rot}_f(\alpha, \mathcal{A})$ is Diophantine w.r.t ω,*

$$\|\mathcal{A} - R\|_h < \widetilde{C} min\{h^{2\chi}, 1\} e^{-\frac{12\pi h}{1+\alpha}},$$

for some constant matrix R, and $\widetilde{C}, \chi$ are numerical constants. Then we have the following:

(1) *The cocycle $(\alpha, \mathcal{A})$ is analytically rotations reducible.*
(2) *If $\beta(\alpha) = 0$, then $(\alpha, \mathcal{A})$ is analytically reducible.*
(3) *Furthermore, if $2\pi h > (1+\alpha)\beta > 0$, then $(\alpha, \mathcal{A})$ is analytically reducible.*

Remark 1.7. *Combining results of* [4] *and* [5]*, Avila gives an answer to the question. Our results will be more natural, and have nice spectral applications.*

In fact, the following stronger result on the local Almost Reducibility Conjecture is also a consequence of local embedding theorem and results of [21].

Corollary 1.3. *Any one-frequency analytic quasi-periodic $SL(2,\mathbb{R})$ cocycle close to constant is almost reducible.*

Remark 1.8. *It is a new proof of Avila-Jitomirskaya's theorems* [4, 7]*. When $\beta(\alpha) = 0$, their proof is based on almost localization results for Long-range operator* [7]*; when $\beta(\alpha) > 0$, his proof depends on periodic approximation* [4]*.*

1.3.3. *Global reducibility of analytic quasi-periodic systems.* As immediate corollaries of Theorem 1.3 and the results of [4, 5], we get some global reducibility results for analytic quasi-periodic linear systems:

Corollary 1.4. *Let $\omega = (1, \alpha)$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $A \in C^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$. Then we have the following:*

(1) *For almost every rotational number $\mathrm{rot}_f(\omega, A)$, (ω, A) is either non-uniformly hyperbolic or (analytically) rotations reducible.*
(2) *Assume further more $\beta(\alpha) > 0$. Then (ω, A) is almost reducible if it is subcritical.*

Remark 1.9. *By Corollary 1.4, we obtain that the Schrödinger conjecture* [29] *(in the essential support of the absolutely continuous spectrum, the generalized eigenfunctions are almost surely bounded) is true for continuous quasi-periodic Schrödinger operator. This result was first verified in a Liouvillean content for the discrete case* [5]*.*

1.3.4. *Density of positive Lyapunov exponents.* As an application of Corollary 1.4, we have the following result:

Corollary 1.5. *Let $\omega = (1, \alpha)$ with $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then there is a dense set of $A \in C^\omega(\mathbb{T}^2, sl(2,\mathbb{R}))$ (in the usual inductive limit topology) such that the linear quasi-periodic system (ω, A) has positive Lyapunov exponent.*

The result has been proved in the discrete case [2, 18]. If $\alpha \in \mathbb{R}$ is recurrent Diophantine, the result was proved in [26]. With the help of Corollary 1.4, the proof can be carried over for arbitrary irrational α without change. It is still open whether uniformly hyperbolic is dense in the category of analytic quasi-periodic $SL(2,\mathbb{R})$-cocycles which are homotopic to the identity.

1.4. **Applications of the embedding theorem: from spectral side.** We consider the following long-range quasi-periodic operator:

$$(L_{\lambda V,\alpha,\varphi}\psi)_n = \lambda \sum_{k\in\mathbb{Z}} V_k \psi_{n-k} + 2cos2\pi(\varphi + n\alpha)\psi_n.$$

When α is Liouvilean, due to Gordon's lemma [19], one often expect that the operator has singular continuous spectrum. Our result will be if $\beta(\alpha)$ is positive and finite, then for a suitable range of λ Anderson Localization occurs. This result gives a new proof of previous work [6, 7, 12]. What is interesting is that we obtain these Liouvillean results from the reducibility side, without any localization method.

Theorem 1.5. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $\beta(\alpha) < \infty$, $2\pi h > (1+\alpha)\beta$, $V \in C^{\omega}_{h}(\mathbb{T},\mathbb{R})$. Then there exists a set $\Phi \subset \mathbb{T}$ of full (Lebesgue) measure, such that if $\phi \in \Phi$,*

$$\lambda < \widetilde{C} min\{h^{2\chi}, 1\} e^{-\frac{12\pi h}{1+\alpha}} \|V\|_{(1+\alpha)\beta/2\pi}{}^{-1},$$

where $\widetilde{C}, \chi$ are numerical constants, then the long-range operator $L_{\lambda V,\alpha,\phi}$ has Anderson Localization.

Remark 1.10. *It is obvious that there exist constant $c_1 = c_1(V)$, $c_2 = c_2(h)$, such that if $\lambda < e^{-c_1\beta - c_2}$, then $L_{\lambda V,\alpha,\phi}$ has Anderson Localization.*

In case α is Diophantine, the result is due to Bourgain-Jitomirskaya [12]. Avila and Jitomirskaya [7] proved that if α is not super-liouvillean: $\xi = \sup_{n>0} \frac{\ln q_{n+1}}{q_n} < \infty$, then for $\lambda < \lambda_0(h,V)$, $L_{\lambda V,\alpha,\phi}$ is almost localized for all ϕ, and has Anderson Localization for a.e.ϕ. Actually, though Aubry duality, we can obtain almost localization by almost reducibility. We don't pursue this way in this paper.

If we are restricted to almost Mathieu operator, we can obtain better estimate:

Theorem 1.6. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ be such that $\beta(\alpha) < \infty$. If $\lambda < ce^{-\beta}$, with small constant c, then the almost Mathieu operator $L_{\lambda cos,\alpha,\phi}$ displays Anderson localization for almost every ϕ.*

Remark 1.11. *In* [6]*, the authors proved that if $\lambda < \frac{1}{2}e^{-16\beta/9}$, then if β is large, our result improves theirs. It is still open whether the optimal condition is $\lambda < \frac{1}{2}e^{-\beta}$* [6].

Outline of the paper. We first include some preliminaries in section 2. In section 3, we give the proof of the local embedding theorem and show some global embedding results in section 4. As applications, we prove the equivalence of almost reducibility between the continuous flow and discrete Poincaré cocycle in Section 5. The proof of Theorem 1.5 is shown in Section 6.

2. Preliminaries

2.1. **Continued Fraction Expansion.** Let $\alpha \in (0,1)$ be irrational. Define $a_0 = 0, \alpha_0 = \alpha$, and inductively for $k \geq 1$,

$$a_k = [\alpha_{k-1}^{-1}], \qquad \alpha_k = \alpha_{k-1}^{-1} - a_k = G(\alpha_{k-1}) = \{\frac{1}{\alpha_{k-1}}\},$$

We define

$$p_0 = 0, \qquad p_1 = 1$$

$$q_0 = 1, \qquad q_1 = a_1$$

and inductively,

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}.$$

It is easy to verify that

$$\forall 1 \leq k < q_n, \quad \|k\alpha\|_{\mathbb{T}} \geq \|q_{n-1}\alpha\|_{\mathbb{T}},$$

and

$$\|q_n\alpha\|_{\mathbb{T}} \leq \frac{1}{q_{n+1}}.$$

thus (q_n) is the sequence of denominators of the best rational approximations of α.

We also denote

$$\beta(\alpha) := \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n},$$

which means $\beta(\alpha)$ measures how Liouvillean α is.

2.2. **The rotation number.** Denote the flow of (1.1) by $\Phi^t(\theta)$, then we define the rotation number of (1.1) by

$$\mathrm{rot}_f(\omega, A) = \lim_{t\to+\infty} \frac{arg(\Phi^t(\theta)x)}{t},$$

where $0 \neq x \in \mathbb{R}^2$, arg denote the angle. It is well-defined and independent of (θ, x) [23]. The rotation number can be defined similarly for quasi-periodic cocyles $(\alpha, \mathcal{A}) \in \mathbb{R} \setminus \mathbb{Q} \times C_0^{\omega}(\mathbb{T}, SL(2,\mathbb{R}))$ [26]. The rotation number rot_f is said to be rational w.r.t. α if $\mathrm{rot}_f = \frac{1}{2}\langle k_0, \alpha\rangle$ for some $k_0 \in \mathbb{Z}$. It is said to be Diophantine w.r.t. α with some constants $\gamma, \tau > 0$, if

$$\|\langle k, \alpha\rangle - 2\mathrm{rot}_f\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{|k|^{\tau}}, \quad 0 \neq k \in \mathbb{Z},$$

and we use $DC_\alpha(\gamma, \tau)$ to denote the set of all such rot_f. The rotation number is not invariant under conjugation, but one has the following, the proof can be found in [26].

Proposition 2.1. *Let* $(\alpha, \mathcal{A}_1), (\alpha, \mathcal{A}_2) \in \mathbb{R} \setminus \mathbb{Q} \times C_0^r(\mathbb{T}, SL(2,\mathbb{R}))$ *be two conjugated quasi-periodic cocycles. If the conjugacy* $B \in C^r(\mathbb{T}, SL(2,\mathbb{R})$ *and has degree* k*, then*

$$\mathrm{rot}_f(\alpha, \mathcal{A}_1) = \mathrm{rot}_f(\alpha, \mathcal{A}_2) + \langle k, \alpha \rangle \quad mod\ 1,$$

If the conjugacy $B \in C^r(2\mathbb{T}, SL(2,\mathbb{R})$ *and has degree* k*, then*

$$\mathrm{rot}_f(\alpha, \mathcal{A}_1) = \mathrm{rot}_f(\alpha, \mathcal{A}_2) + \frac{1}{2}\langle k, \alpha \rangle \quad mod\ 1,$$

2.3. **Aubry Duality.** Suppose that the eigenvalue equation $H_{V,\alpha,\phi}x = Ex$ has an analytic quasi-periodic Bloch wave, which means there exist $\overline{\psi} \in C^\omega(\mathbb{T}, \mathbb{C})$ and $\varphi \in [0, 1)$ such that

$$x_n = e^{2\pi i n\varphi}\overline{\psi}\,(n\alpha + \phi)\,. \tag{2.1}$$

We call φ the Floquet exponent. If we write $\overline{\psi} = \sum_{n\in\mathbb{Z}} \psi_n e^{2\pi i n\theta}$, then direct computation shows that

$$\sum_{k\in\mathbb{Z}} V_k \psi_{n-k} + 2\cos 2\pi\,(n\alpha + \varphi)\,\psi_n = E\psi_n, \qquad n \in \mathbb{Z},$$

which means $(L_{V,\alpha,\varphi}\psi)_n = E\psi_n$. If α is irrational, then there exists $\sigma^L(\lambda V, \alpha) \subseteq \mathbb{R}$, such that

$$\sigma^L(\lambda V, \alpha) = Spec(L_{\lambda V,\alpha,\varphi}), \quad \forall \varphi.$$

The rigorous version of the Aubry duality can be found in [12, 20, 31].

Let $\sigma^L_{pp}(V, \alpha, \varphi)$ be the set of point eigenvalues of $L_{V,\alpha,\varphi}$ which has exponentially decaying eigenfunctions, and let $B_{V,\alpha,\varphi}$ be the set of spectrum of $H_{V,\alpha,\phi}$ which has quasi-periodic Bloch wave with Floquet exponent φ.

Lemma 2.1. *The following facts hold:*

- $\sigma^L_{pp}(V, \alpha, \varphi) = B_{V,\alpha,\varphi}$,
- $\sigma^H(V, \alpha) = \sigma^L(V, \alpha)$.

The proof can be found in [31].

3. Proof of Theorem 1.1

The local embedding theorem will be proved by Implicit Function Theorem. The crucial points are the solution of the homological equation and the construction of suitable Banach spaces. We remark that our proof doesn't use the method of suspension flow or the typical property of the Lie group. The method can also be used to prove the nonlinear version of the local embedding theorem.

3.1. **Resonance sites.** For any $\mathbf{k} = (k_1, k_2, \cdots, k_d) \in \mathbb{Z}^d$, we define the norm of $\mathbf{k}$ by

$$|\mathbf{k}| = |k_1| + |k_2| + \cdots + |k_d|,$$

and for any $\mu = (\mu_1, \mu_2, \cdots, \mu_d) \in \mathbb{T}^d$, we define its norm by

$$|\mu| = |\mu_1| + |\mu_2| + \cdots + |\mu_d|.$$

If $f(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{f}(\mathbf{k}) e^{2\pi i\langle \mathbf{k},\theta\rangle} \in C_h^\omega(\mathbb{T}^d, \mathbb{C})$, we use the weighted norm

$$\|f\|_h := \sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{f}(\mathbf{k})| e^{2\pi|\mathbf{k}|h} < \infty,$$

where $\theta = (\theta_1, \cdots, \theta_d)$ and $\langle \mathbf{k}, \theta\rangle = k_1\theta_1 + \cdots + k_d\theta_d$.

For any $\rho \in \mathbb{R}$, $\mu \in \mathbb{T}^{d-1}$, $\mathbf{k} \in \mathbb{Z}^{d-1}$, we define $\tilde{k}(\mathbf{k}) \in \mathbb{Z}$ by

$$|\langle \mathbf{k}, \mu\rangle + 2\rho - \tilde{k}| = \inf_{k\in\mathbb{Z}} |\langle \mathbf{k}, \mu\rangle + 2\rho - k|. \tag{3.1}$$

Thus $\tilde{k}(\mathbf{k})$ is uniquely defined if $\inf_{k\in\mathbb{Z}} |\langle \mathbf{k}, \mu\rangle + 2\rho - k| \neq \frac{1}{2}$. In case that $\inf_{k\in\mathbb{Z}} |\langle \mathbf{k}, \mu\rangle + 2\rho - k| = \frac{1}{2}$, we choose $\tilde{k}(\mathbf{k})$ to be the smaller one which satisfies (3.1).

By the construction, $\tilde{k}(\mathbf{k})$ is uniquely defined and

$$\tilde{k}(\mathbf{k}) \in \{[\langle \mathbf{k}, \mu\rangle + 2\rho] - 1, [\langle \mathbf{k}, \mu\rangle + 2\rho], [\langle \mathbf{k}, \mu\rangle + 2\rho] + 1\},$$

where $[\cdot]$ denotes the integer part.

Define the resonance sites $\mathcal{S}_\rho^\mu \subset \mathbb{Z}^d$ as follows

$$\mathcal{S}_\rho^\mu := \{(\tilde{k}(\mathbf{k}), \mathbf{k}), \mathbf{k} \in \mathbb{Z}^{d-1}\}. \tag{3.2}$$

For any $f \in C_h^\omega(\mathbb{T}^d, \mathbb{C})$ supported on $\mathcal{S}_\rho^\mu$, we define its weighted norm by

$$\|f\|_{\rho,h}^\mu := \sum_{\mathbf{k}\in\mathbb{Z}^{d-1}} |\widehat{f}(\tilde{k}(\mathbf{k}), \mathbf{k})| e^{2\pi|\mathbf{k}|(1+|\mu|)h},$$

therefore we define the linear sub-space $\mathcal{B}_{\rho,h}^\mu(\mathbb{T}^d, \mathbb{C})$ of $C_h^\omega(\mathbb{T}^d, \mathbb{C})$:

$$\mathcal{B}_{\rho,h}^\mu(\mathbb{T}^d, \mathbb{C}) = \{f \in C_h^\omega(\mathbb{T}^d, \mathbb{C}) | Supp\widehat{f}(k_1, \mathbf{k}) \subset \mathcal{S}_\rho^\mu\}.$$

The sub-space $\mathcal{B}_{\rho,h}^\mu(\mathbb{T}^d, \mathbb{R})$ of $C_h^\omega(\mathbb{T}^d, \mathbb{R})$ is defined similarly.

Remark 3.1. *In case that $d = 2$, we have*

$$e^{-2\pi h(2-2|\rho|)} \|f\|_{\rho,h}^\mu \leq \|f\|_h \leq e^{2\pi h(2|\rho|+1)} \|f\|_{\rho,h}^\mu,$$

which means that the norms $\|\cdot\|_{\rho,h}^\mu$ and $\|\cdot\|_h$ in $\mathcal{B}_{\rho,h}^\mu$ are equivalent. In case that $d \geq 3$, we only have

$$\|f\|_h \leq e^{2\pi h(2|\rho|+1)} \|f\|_{\rho,h}^\mu. \tag{3.3}$$

In the following, we will show that $\mathcal{B}^\mu_{\rho,\frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{C})$ is actually isomorphic to $C_h^\omega(\mathbb{T}^{d-1}, \mathbb{C})$, hence a Banach space. The spaces will be used to construct the embedded linear system.

3.2. **Embedding operator.** For any $f \in C_h^\omega(\mathbb{T}^d, \mathbb{C})$, $\lambda \in \mathbb{R}$, $\rho \in \mathbb{R}$, we define the linear operator

$$T_{\lambda+i\rho} : \mathcal{B}^\mu_{\rho, \frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{C}) \to C_h^\omega(\mathbb{T}^{d-1}, \mathbb{C})$$

by

$$T_{\lambda+i\rho} f(\theta) = \int_0^1 f(t, \theta + t\mu) e^{4\pi(\lambda+i\rho)t} dt.$$

If $\lambda \neq 0$, we have

$$T_{\lambda+i\rho} f(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^{d-1}} \widehat{f}(\tilde{k}(\mathbf{k}), \mathbf{k}) \frac{e^{4\pi\lambda + 2\pi i(\tilde{k}(\mathbf{k}) + \langle \mathbf{k}, \mu\rangle + 2\rho)} - 1}{4\pi\lambda + 2\pi i(\tilde{k}(\mathbf{k}) + \langle \mathbf{k}, \mu\rangle + 2\rho)} e^{2\pi i \langle \mathbf{k}, \theta\rangle},$$

where $(\tilde{k}, \mathbf{k}) \in \mathcal{S}^\mu_\rho$, consequently,

$$\|T_{\lambda+i\rho} f\|_h = \sum_{\mathbf{k}\in\mathbb{Z}^{d-1}} |\widehat{T_{\lambda+i\rho} f}(\mathbf{k})| e^{2\pi|\mathbf{k}|h} \leq \frac{e^{4\pi\lambda}+1}{4\pi\lambda} \|f\|^\mu_{\rho, \frac{h}{1+|\mu|}}.$$

If $\lambda = 0$ and ρ is not rational with respect to μ, which means for any $(\tilde{k}, \mathbf{k}) \in \mathcal{S}^\mu_\rho$, $\tilde{k}(\mathbf{k}) + \langle \mathbf{k}, \mu\rangle + 2\rho \neq 0$, then we have

$$T_{i\rho} f(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^{d-1}} \widehat{f}(\tilde{k}(\mathbf{k}), \mathbf{k}) \frac{e^{2\pi i(\tilde{k}(\mathbf{k}) + \langle \mathbf{k}, \mu\rangle + 2\rho)} - 1}{2\pi i(\tilde{k}(\mathbf{k}) + \langle \mathbf{k}, \mu\rangle + 2\rho)} e^{2\pi i \langle \mathbf{k}, \theta\rangle},$$

consequently,

$$\|T_{i\rho} f\|_h = \sum_{\mathbf{k}\in\mathbb{Z}^{d-1}} |\widehat{T_{i\rho} f}(\mathbf{k})| e^{2\pi|\mathbf{k}|h} \leq \|f\|^\mu_{\rho, \frac{h}{1+|\mu|}}. \tag{3.4}$$

If $\lambda = 0$ and ρ is rational with respect to μ, or in particular, $(1, \mu)$ is rational dependent, then there exist $\tilde{\mathbf{k}} \in \mathbb{Z}^{d-1}$, $\tilde{k}(\tilde{\mathbf{k}}) \in \mathbb{Z}$, such that $\tilde{k}(\tilde{\mathbf{k}}) + \langle \tilde{\mathbf{k}}, \mu\rangle + 2\rho = 0$, we have

$$\begin{aligned} T_{i\rho} f(\theta) &= \widehat{f}(\tilde{k}(\tilde{\mathbf{k}}), \tilde{\mathbf{k}}) e^{2\pi i \langle \tilde{\mathbf{k}}, \theta\rangle} \\ &\quad + \sum_{\mathbf{k}\in\mathbb{Z}^{d-1}, \mathbf{k}\neq\tilde{\mathbf{k}}} \widehat{f}(\tilde{k}(\mathbf{k}), \mathbf{k}) \frac{e^{2\pi i(\tilde{k}(\mathbf{k}) + \langle \mathbf{k}, \mu\rangle + 2\rho)} - 1}{2\pi i(\tilde{k}(\mathbf{k}) + \langle \mathbf{k}, \mu\rangle + 2\rho)} e^{2\pi i \langle \mathbf{k}, \theta\rangle}, \end{aligned}$$

then (3.4) still holds. Hence in any case, $T_{\lambda+i\rho}$ is a bounded linear operator.

We just point out that when $\rho = 0$,

$$T_\lambda : \mathcal{B}^\mu_{\rho, \frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{R}) \to C_h^\omega(\mathbb{T}^{d-1}, \mathbb{R})$$

is a bounded linear operator which maps real functions to real functions.

We say that $T_{\lambda+i\rho}$ is an embedding operator if $T^{-1}_{\lambda+i\rho}$ is a bounded linear operator. In the following, we prove that $T_{\lambda+i\rho}$ is a linear operator which does have a bounded inverse.

Proposition 3.1. *For any $\lambda \in \mathbb{R}$, $\rho \in \mathbb{R}$, $h > 0$, $\mu \in \mathbb{T}^{d-1}$, we have*

$$T_{\lambda+i\rho}^{-1} : C_h^{\omega}(\mathbb{T}^{d-1}, \mathbb{C}) \to \mathcal{B}^{\mu}_{\rho, \frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{C})$$

is a bounded linear operator. When $\rho = 0$, we have

$$T_{\lambda}^{-1} : C_h^{\omega}(\mathbb{T}^{d-1}, \mathbb{R}) \to \mathcal{B}^{\mu}_{\rho, \frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{R}).$$

Proof. For any $\varphi \in C_h^{\omega}(\mathbb{T}^{d-1}, \mathbb{C})$, we write $\varphi(\theta) = \sum_{\mathbf{k} \in \mathbb{Z}^{d-1}} \hat{\varphi}(\mathbf{k}) e^{2\pi i \langle \mathbf{k}, \theta \rangle}$. We first consider the case that $(1, \mu)$ is rational independent, three cases are distinguished.

Case 1 If $\lambda \neq 0$, then we define

$$\widehat{f}(k_1, \mathbf{k}) = \begin{cases} \frac{4\pi\lambda + 2\pi i(k_1 + \langle \mathbf{k}, \mu \rangle + 2\rho)}{e^{4\pi\lambda + 2\pi i(k_1 + \langle \mathbf{k}, \mu \rangle + 2\rho)} - 1} \hat{\varphi}(\mathbf{k}) & k_1 = -\tilde{k} \\ 0 & k_1 \neq -\tilde{k} \end{cases}$$

where $(\tilde{k}, \mathbf{k}) \in \mathcal{S}_{\rho}^{\mu}$.

Case 2 If $\lambda = 0$ and ρ is not rational with respect to μ. In this case, we define $\widehat{f}(k_1, \mathbf{k})$ by

$$\widehat{f}(k_1, \mathbf{k}) = \begin{cases} \frac{2\pi i(k_1 + \langle \mathbf{k}, \mu \rangle + 2\rho)}{e^{2\pi i(k_1 + \langle \mathbf{k}, \mu \rangle + 2\rho)} - 1} \hat{\varphi}(\mathbf{k}) & k_1 = -\tilde{k} \\ 0 & k_1 \neq -\tilde{k} \end{cases} \tag{3.5}$$

where $(\tilde{k}, \mathbf{k}) \in \mathcal{S}_{\rho}^{\mu}$.

Case 3 If $\lambda = 0$ and ρ is rational with respect to μ, which means that there exist $\widetilde{k}_1 \in \mathbb{Z}$, $\widetilde{\mathbf{k}}_2 \in \mathbb{Z}^{d-1}$, such that $2\rho = -\widetilde{k}_1 - \widetilde{\mathbf{k}}_2 \mu$. For $\mathbf{k} = \widetilde{\mathbf{k}}_2$, we define

$$\widehat{f}(k_1, \mathbf{k}) = \begin{cases} \hat{\varphi}(\mathbf{k}) & k_1 = \widetilde{k}_1 \\ 0 & k_1 \neq \widetilde{k}_1. \end{cases}$$

Otherwise, for $\mathbf{k} \neq \widetilde{\mathbf{k}}_2$, we define $\widehat{f}(k_1, \mathbf{k})$ by (3.5).

If $(1, \mu)$ is rational dependent, the construction is included in case 3.

In any cases, by our construction,

$$f(\theta_1, \theta) = \sum_{\mathbf{k} \in \mathbb{Z}^{d-1}} \widehat{f}(k_1, \mathbf{k}) e^{2\pi i (k_1 \theta_1 + \langle \mathbf{k}, \theta \rangle)}$$

is uniquely defined and it satisfies $T_{\lambda+i\rho} f(0, \theta) = \varphi(\theta)$. Also from the construction, one sees that $\widehat{f}(k_1, \mathbf{k})$ is supported on the resonance sites $\mathcal{S}_{\rho}^{\mu}$. We now show $T_{\lambda+i\rho}^{-1}$ is bounded.

If $\lambda \neq 0$, we have

$$\|f\|^{\mu}_{\rho, \frac{h}{1+|\mu|}} \leq \frac{\pi\sqrt{16\lambda^2 + 1}}{e^{4\pi\lambda} - 1} \|\varphi\|_h,$$

which follows by $(k_1, \mathbf{k}) \in \mathcal{S}_{\rho}^{\mu}$ and the estimate

$$\Big| \frac{4\pi\lambda + 2\pi i(k_1 + \langle \mathbf{k}, \mu \rangle + 2\rho)}{e^{4\pi\lambda + 2\pi i(k_1 + \langle \mathbf{k}, \mu \rangle + 2\rho)} - 1} \Big| \leq \frac{\pi\sqrt{16\lambda^2 + 1}}{e^{4\pi\lambda} - 1}.$$

Otherwise, if $\lambda = 0$, by our selection that $(k_1, \mathbf{k}) \in \mathcal{S}_\rho^\mu$, then we have

$$\Big|\frac{2\pi i(k_1 + \langle \mathbf{k}, \mu\rangle + 2\rho)}{e^{2\pi i(k_1+\langle \mathbf{k},\mu\rangle+2\rho)} - 1}\Big| < \frac{\pi}{2},$$

consequently, we have

$$\|f\|^\mu_{\rho, \frac{h}{1+|\mu|}} \leq \frac{\pi}{2}\|\varphi\|_h.$$

We conclude that $T^{-1}_{\lambda+i\rho}$ is a bounded linear operator, and hence $\mathcal{B}^\mu_{\rho,h}$ is a Banach space.

When $\rho = 0$, φ is real analytic, from the formula for $\widehat{f}(\tilde{k}(\mathbf{k}), \mathbf{k})$, one has

$$\widehat{f}(\tilde{k}(-\mathbf{k}), -\mathbf{k}) = \widehat{f}(-\tilde{k}(\mathbf{k}), -\mathbf{k}) = \overline{\widehat{f}(\tilde{k}(\mathbf{k}), \mathbf{k})}.$$

This proves that f is real analytic. □

Remark 3.2. *Similar construction was used by Fayad-Katok-Windor in* [17].

Remark 3.3. *The C^r ($r \neq \omega$) version of the proposition is also true (just check the decay of Fourier coeffients), which can be used to prove the C^r embedding of quasi-periodic cocycles into quasi-periodic linear systems.*

Remark 3.4. *One can also prove that for any $\varphi \in C^\omega(\mathbb{T} \times \mathbb{T}^{d-1}, \mathbb{R})$, there exists $f \in C^\omega(\mathbb{T} \times \mathbb{T}^d, \mathbb{R})$, such that*

$$\int_0^1 f(\theta + \rho t, t, \phi + t\mu)dt = \varphi(\theta, \phi).$$

This fact can be used to prove the nonlinear local embedding of analytic quasi-periodically forced circle diffeomorphism into quasi-periodically forced circle flow(c.f. Theorem 1.2).

For any $A \in sl(2, \mathbb{R})$, let $L : C^\omega_h(\mathbb{T}^d, sl(2, \mathbb{R})) \to C^\omega_h(\mathbb{T}^{d-1}, sl(2, \mathbb{R}))$ be the operator

$$LF = \int_0^1 e^{-As}F(s, \theta + s\mu)e^{As}ds. \tag{3.6}$$

In the following, we shall prove that there is a Banach sub-space $\mathcal{B}$ of $C^\omega_h(\mathbb{T}^d, sl(2, \mathbb{R}))$, depending on A, such that $L : \mathcal{B} \to C^\omega_h(\mathbb{T}^{d-1}, sl(2, \mathbb{R}))$ is a linear operator with bounded inverse.

We recall $sl(2, \mathbb{R})$ is the set of 2 by 2 matrices with real coefficients of the form

$$\begin{pmatrix} x & y + z \\ y - z & -x \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$. It is isomorphic to $su(1, 1)$, matrices of the form

$$\begin{pmatrix} it & \nu \\ \bar{\nu} & -it \end{pmatrix}$$

with $t \in \mathbb{R}$, $\nu \in \mathbb{C}$. We denote such a matrix by $\{t, \nu\}$. The isomorphism between $sl(2,\mathbb{R})$ and $su(1,1)$ is given by $B \to MBM^{-1}$ where

$$M = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Direct calculation shows that

$$M \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} M^{-1} = \begin{pmatrix} iz & x-iy \\ x+iy & -iz \end{pmatrix}.$$

Denote $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then we have $R := MJM^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

For any $\rho \in \mathbb{R}$, $h > 0$, $\mu \in \mathbb{T}^{d-1}$, we define Banach spaces

$$(3.7) \quad \overline{\mathcal{B}} = \left\{ \begin{pmatrix} if & g \\ \bar{g} & -if \end{pmatrix} \Big| f \in \mathcal{B}^{\mu}_{0,\frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{R}), g \in \mathcal{B}^{\mu}_{-\rho,\frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{C}) \right\},$$

and

$$\mathcal{B} = M^{-1}\overline{\mathcal{B}}M. \tag{3.8}$$

We point out that $\overline{\mathcal{B}} \subset C^{\omega}_h(\mathbb{T}^d, su(1,1))$ and thus $\mathcal{B} \subset C^{\omega}_h(\mathbb{T}^d, sl(2,\mathbb{R}))$, since by (3.3), we have

$$\|f\|_{\frac{h}{1+|\mu|}} \leq e^{\frac{2\pi h(2|\rho|+1)}{1+|\mu|}} \|f\|^{\mu}_{\rho,\frac{h}{1+|\mu|}}.$$

As a corollary of Proposition 3.1, we have the following:

Corollary 3.1. *For any $\rho \in \mathbb{R}$, $h > 0$, $\mu \in T^{d-1}$, the linear operator*

$$\overline{L} : \overline{\mathcal{B}} \to C^{\omega}_h(\mathbb{T}^{d-1}, su(1,1))$$

defined by

$$\overline{L}F = \int_0^1 e^{-2\pi\rho Rs} F(s, \theta + s\mu) e^{2\pi\rho Rs} ds, \tag{3.9}$$

is bounded. Moreover, there exists $C(\rho, h, |\mu|) > 0$ such that

$$\overline{L}^{-1} : C^{\omega}_h(\mathbb{T}^{d-1}, su(1,1)) \to \overline{\mathcal{B}}$$

is bounded with $\|\overline{L}^{-1}\| \leq C(\rho, h, |\mu|)$.

Proof. For any $F = \{f_1, f_2\} \in \overline{\mathcal{B}}$, we have

$$\begin{aligned} \overline{L}F &= \int_0^1 \begin{pmatrix} if_1(s,\theta+s\mu) & f_2(s,\theta+s\mu)e^{-4\pi i\rho s} \\ \bar{f}_2(s,\theta+s\mu)e^{4\pi i\rho s} & -if_1(s,\theta+s\mu) \end{pmatrix} ds \\ &= \begin{pmatrix} iT_0 f_1 & T_{-i\rho} f_2 \\ T_{i\rho}\bar{f}_2 & -iT_0 f_1 \end{pmatrix}. \end{aligned}$$

Therefore, $\overline{L}$ is a bounded linear operator.

For any $G = \{g_1, g_2\} \in C_h^\omega(\mathbb{T}^{d-1}, su(1,1))$, by Proposition 3.1 and (3.3), there exists a unique $f_1 \in \mathcal{B}^\mu_{0,\frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{R})$, such that $T_0 f_1 = g_1$, with the estimate

$$\|f_1\|_h \leq e^{\frac{2\pi h}{1+|\mu|}} \|f_1\|^\mu_{0,\frac{h}{1+|\mu|}} \leq \frac{\pi}{2} e^{\frac{2\pi h}{1+|\mu|}} \|g_1\|_h.$$

Again by Proposition 3.1 and (3.3), there exists a unique $f_2 \in \mathcal{B}^\mu_{-\rho,\frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{C})$, such that $T_{-i\rho} f_2 = g_2$, with the estimate

$$\|f_2\|_h \leq e^{\frac{2\pi h(2|\rho|+1)}{1+|\mu|}} \|f_2\|^\mu_{-\rho,\frac{h}{1+|\mu|}} \leq \frac{\pi}{2} e^{\frac{2\pi h(2|\rho|+1)}{1+|\mu|}} \|g_2\|_h.$$

We remark that $T_{i\rho} \bar{f}_2 = \bar{g}_2$.

So there exists a unique $F \in \overline{\mathcal{B}}$ such that $\overline{L} F = G$ with the estimate

$$\|F\|_h \leq C(\rho, h, |\mu|) \|G\|_h,$$

where $C(\rho, h, |\mu|) = \frac{\pi}{2} e^{\frac{2\pi h(2|\rho|+1)}{1+|\mu|}}$. It follows that $\overline{L}^{-1}$ exists and is bounded with $\|\overline{L}^{-1}\| \leq C(\rho, h, |\mu|)$. □

Corollary 3.2. *For any $\rho \in \mathbb{R}$, $h > 0$, $\mu \in \mathbb{T}^{d-1}$, let $A = 2\pi\rho J$, then the linear operator*

$$L : \mathcal{B} \to C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R}))$$

defined by (3.6) *is bounded. Moreover, there exists constant $C(\rho, h, |\mu|) > 0$ such that*

$$L^{-1} : C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R})) \to \mathcal{B}$$

is bounded with $\|L^{-1}\| \leq C(\rho, h, |\mu|)$.

Proof. It is an immediate corollary of Corollary 3.1, since the Banach spaces $C_h^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$ and $C_h^\omega(\mathbb{T}^d, su(1,1))$ are isomorphic by $B \to MBM^{-1}$. □

If A is hyperbolic, the operator defined by (3.6) is still bounded, which is the following:

Corollary 3.3. *For any $\lambda \in \mathbb{R}$, $h > 0$, $\mu \in \mathbb{T}^{d-1}$, let $A = 2\pi\lambda H$,*

$$\widetilde{\mathcal{B}} = \left\{ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & -f_{11} \end{pmatrix} \Big| f_{11}, f_{12}, f_{21} \in \mathcal{B}^\mu_{0,\frac{h}{1+|\mu|}}(\mathbb{T}^d, \mathbb{R}) \right\},$$

then the linear operator

$$L : \widetilde{\mathcal{B}} \to C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R}))$$

defined by (3.6) *is bounded. Moreover $L^{-1} : C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R})) \to \widetilde{\mathcal{B}}$ is bounded with $\|L^{-1}\| \leq \frac{\pi\sqrt{16\lambda^2+1}}{e^{4\pi\lambda}-1} e^{\frac{2\pi h}{1+|\mu|}}$.*

Proof. We omit the proof since it is similar with the proof of Corollary 3.1. □

3.3. **Proof of Theorem 1.1.** In order to make the ideas of proof clearly, we will only prove the theorem in the group $SL(2,\mathbb{R})$, and just give an outline of the proof in other Lie groups. The following embedding theorem also includes the case that $(1,\mu)$ is rational dependent.

Theorem 3.1. *Suppose that* $\mu \in \mathbb{T}^{d-1}$, $h>0$, $G \in C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R}))$, $A \in sl(2,\mathbb{R})$ *being constant. Then there exist* $\epsilon = \epsilon(A,h,|\mu|)>0$, $c = c(A,h,|\mu|)>0$, $\widetilde{A} \in sl(2,\mathbb{R})$ *and* $F \in C^\omega_{h/1+|\mu|}(\mathbb{T}^d, sl(2,\mathbb{R}))$ *such that the cocycle* $(\mu, e^A e^{G(\cdot)})$ *is the Poincaré map of*

$$\begin{cases} \dot{x} = (\widetilde{A}+F(\theta))x \\ \dot{\theta} = \omega = (1,\mu) \end{cases} \tag{3.10}$$

provided that $\|G\|_h = \varepsilon < \epsilon$. *Moreover, we have the following*

(1) *If* A *is in the real normal forms* $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ *or* $\begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}$, *then* $\widetilde{A} = A$, $\|F\|_{\frac{h}{1+|\mu|}} \leq c\varepsilon$.

(2) *If* A *is in the real normal form* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, *then* $\widetilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\|F\|_{\frac{h}{1+|\mu|}} \leq c\varepsilon^{\frac{1}{2}}$.

Proof. **Case 1.** A is elliptic. Without lose of generality, we assume $A = 2\pi\rho J$, and define $\mathcal{B}$ as in (3.8). Suppose that $\Phi^t(\theta)$ is the flow of (3.10),

$$\Phi^t(\theta) = e^{At}\Big(I + \int_0^t e^{-As}F(\theta+s\omega)\Phi^s(\theta)ds\Big),$$

where I denotes the identity matrix. The cocycle $(\mu, e^A e^{G(\widetilde{\theta})})$ can be embedded into the linear system (3.10), which means $\Phi^1(0,\widetilde{\theta}) = e^A e^{G(\widetilde{\theta})}$, i.e.,

$$e^A\Big(I + \int_0^1 e^{-As}F(s,\widetilde{\theta}+s\mu)\Phi^s(0,\widetilde{\theta})ds\Big) = e^A e^{G(\widetilde{\theta})}. \tag{3.11}$$

We construct the nonlinear functional

$$\Psi : \mathcal{B} \times C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R})) \to C_h^\omega(\mathbb{T}^{d-1}, gl(2,\mathbb{R}))$$

by

$$\Psi(F,G) = I + \int_0^1 e^{-As}F(s,\widetilde{\theta}+s\mu)\Phi^s(0,\widetilde{\theta})ds - e^{G(\widetilde{\theta})}.$$

Immediate check shows that $\Psi(0,0)=0$, and

$$\begin{aligned} D_F\Psi(F,G)(\widetilde{F}) \quad &= \int_0^1 e^{-As}\widetilde{F}(s,\widetilde{\theta}+s\omega)\Phi^s(0,\widetilde{\theta})ds \\ &+ \int_0^1 e^{-As}F(s,\widetilde{\theta}+s\mu)D_F\Phi^s(0,\widetilde{\theta})\widetilde{F}(s,\widetilde{\theta}+s\mu)ds. \end{aligned}$$

Consequently, we have

$$D_F\Psi(0,0)(\widetilde{F}) = \int_0^1 e^{-As}\widetilde{F}(s,\widetilde{\theta}+s\mu)e^{As}ds.$$

By Corollary 3.2, $D_F\Psi(0,0) : \mathcal{B} \to C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R}))$ is a bounded linear operator with bounded inverse. Moreover, we have the estimate $\|D_F\Psi(0,0)^{-1}\| \leq C(\rho,h,|\mu|)$, where $C(\rho,h,|\mu|)$ is defined in Corollary 3.1.

By Implicit Function Theorem, when

$$\|G\|_h \leq \varepsilon < \frac{1}{C(\rho,h,|\mu|)^2},$$

then there exists $F \in \mathcal{B} \subset C^\omega_{\frac{h}{1+|\mu|}}(\mathbb{T}^d, sl(2,\mathbb{R}))$ with $\|F\|_{\frac{h}{1+|\mu|}} \leq C(\rho,h,|\mu|)\varepsilon$, such that the nonlinear functional $\Psi(F,G)=0$ has a solution. That is to say $(\mu, e^A e^{G(\cdot)})$ is the Poincaré map of (3.10).

Case 2 A is hyperbolic. Without lose of generality, we assume $A = 2\pi\lambda H$. In this case, the proof goes along the same line as in case 1. We only need to substitute $\mathcal{B}$ by $\widetilde{\mathcal{B}}$, and Corollary 3.2 by Corollary 3.3.

Case 3 A is parabolic. Without lose of generality, we assume $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. For any $G \in C_h^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R}))$ with $\|G\|_h \leq \varepsilon$, we set $B = \begin{pmatrix} \varepsilon^{\frac{1}{4}} & 0 \\ 0 & \varepsilon^{-\frac{1}{4}} \end{pmatrix}$, then we have

$$Be^A e^{G(\cdot)} B^{-1} = e^{\begin{pmatrix} 0 & \varepsilon^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}} e^{\widetilde{G}(\cdot)},$$

with $\|\widetilde{G}\|_h \leq \varepsilon^{\frac{1}{2}}$. This means the cocycle $(\mu, e^A e^{G(\cdot)})$ can be seen as a perturbation of (μ, I), then we apply case 1 to finish the proof. □

Remark 3.5. *One can generalize the proof to other Lie groups without difficulty. Suppose the constant matrix A is diagonalizable, the proof follows case* 1 *above. Suppose the constant part A has Jordan blocks, the proof is similar to the case* 3 *above.*

4. Global embedding results.

In this section, we first prove that embedding is conjugacy invariant (Proposition 1.1), and then apply it to prove some global embedding results.

Proof of Proposition 1.1.

By the assumption, there exist $B \in C^\omega(2\mathbb{T}^d, SL(2,\mathbb{R}))$ and the flow $\tilde{\Phi}^t(\theta_1,\widetilde{\theta}) \in C^\omega(2\mathbb{T}^d, SL(2,\mathbb{R}))$, such that

$$B(\widetilde{\theta}+\mu)\mathcal{A}(\widetilde{\theta})B(\widetilde{\theta})^{-1} = \tilde{\mathcal{A}}(\widetilde{\theta}),$$

and

$$\tilde{\mathcal{A}}(\widetilde{\theta}) = \tilde{\Phi}^1(0, \widetilde{\theta}).$$

For $(x_1, \widetilde{x}) \in \mathbb{R}^d$,

$$\Phi^{x_1}(0, \widetilde{x} - x_1\mu) = B(\widetilde{x} - x_1\mu + \mu)^{-1}\tilde{\Phi}^{x_1}(0, \widetilde{x} - x_1\mu)B(\widetilde{x} - x_1\mu)$$

is well-defined, 2-periodic in $\widetilde{x}$ and analytic.

Once $\Phi^{x_1}(0, \widetilde{x})$ is given, we define

$$\Phi^t(x_1, \widetilde{x}) = \Phi^{x_1+t}(0, \widetilde{x} - x_1\mu)\big(\Phi^{x_1}(0, \widetilde{x} - x_1\mu)\big)^{-1}. \tag{4.1}$$

Immediate check shows that $\Phi^t(x_1, \widetilde{x})$ is also 2-periodic in x_1 and analytic, hence $\Phi^t \in C^\omega(2\mathbb{T}^d, SL(2, \mathbb{R}))$. Next we show (4.1) indeed defines a flow. To prove this, let $t \to t + s$, then we have

$$\begin{aligned}
\Phi^{t+s}(\theta_1, \widetilde{\theta}) &= \Phi^{\theta_1+t+s}(0, \widetilde{\theta} - \theta_1\mu)\big(\Phi^{\theta_1}(0, \widetilde{\theta} - \theta_1\mu)\big)^{-1} \\
&= \Phi^t(\theta_1 + s, \widetilde{\theta} + s\mu)\Phi^{\theta_1+s}(0, \widetilde{\theta} - \theta_1\mu)\big(\Phi^{\theta_1}(0, \widetilde{\theta} - \theta_1\mu)\big)^{-1} \\
&= \Phi^t(\theta_1 + s, \widetilde{\theta} + s\mu)\Phi^s(\theta_1, \widetilde{\theta}).
\end{aligned}$$

The second equality follows from the substitution $\theta_1 \to \theta_1 + s$, $\widetilde{\theta} \to \widetilde{\theta} + s\mu$.

It is clearly that

$$\Phi^1(0, \widetilde{\theta}) = B(\widetilde{\theta} + \mu)^{-1}\tilde{\Phi}^1(0, \widetilde{\theta})B(\widetilde{\theta}) = B(\widetilde{\theta} + \mu)^{-1}\tilde{\mathcal{A}}(\widetilde{\theta})B(\widetilde{\theta}) = \mathcal{A}(\widetilde{\theta}),$$

which means $(\mu, \mathcal{A})$ can be embedded into the flow $\Phi^t(\theta_1, \widetilde{\theta})$. □

As an immediate consequence of Proposition 3.1 and Proposition 1.1, we have the following embedding result for uniformly hyperbolic cocycle.

Corollary 4.1. *Let $\mu \in \mathbb{T}^{d-1}$, $\omega = (1, \mu)$ is rational independent, $\mathcal{A} \in C_0^\omega(\mathbb{T}^{d-1}, SL(2, \mathbb{R}))$. If $(\mu, \mathcal{A})$ is uniformly hyperbolic, then it can be analytically embedded into a quasi-periodic linear system.*

Proof. If $(\mu, \mathcal{A})$ is uniformly hyperbolic, then there exist $\varphi \in C^\omega(\mathbb{T}^{d-1}, \mathbb{R})$, $B \in C^\omega(2\mathbb{T}^{d-1}, SL(2, \mathbb{R}))$ such that

$$B(\cdot + \mu)\mathcal{A}(\cdot)B(\cdot)^{-1} = \widetilde{\mathcal{A}}(\cdot) = \begin{pmatrix} e^{\varphi(\cdot)} & 0 \\ 0 & e^{-\varphi(\cdot)} \end{pmatrix}.$$

By Proposition 3.1, there exists $f \in C^\omega(\mathbb{T}^d, \mathbb{R})$ such that $T_0 f = \varphi$, which means the quasi-periodic cocycle $(\mu, \widetilde{\mathcal{A}})$ can be embedded into

$$\begin{cases} \dot{x} = \begin{pmatrix} f(\theta) & 0 \\ 0 & -f(\theta) \end{pmatrix} x \\ \dot{\theta} = \omega = (1, \mu). \end{cases}$$

Hence the result follows from Proposition 1.1. □

Remark 4.1. *By Corollary 4.1, we get another proof of Theorem 3.1 in the case A is hyperbolic, since uniformly hyperbolic is an open condition, if $\|G\|_h$ is small, then $(\mu, e^A e^{G(\cdot)})$ is uniformly hyperbolic.*

Corollary 4.2. *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\mathcal{A} \in C_0^\omega(\mathbb{T}, SL(2,\mathbb{R}))$. If $(\alpha, \mathcal{A})$ is almost reducible, then $(\alpha, \mathcal{A})$ can be analytically embedded into a quasi-periodic linear system.*

Proof. If $(\alpha, \mathcal{A})$ is almost reducible, then there exist $B_n \in C_{h_n}^\omega(2\mathbb{T}, SL(2,\mathbb{R}))$, $A_n \in sl(2,\mathbb{R})$, $F_n \in C_{h_n}^\omega(\mathbb{T}, sl(2,\mathbb{R}))$ with $\|F_n\|_{h_n} \leq \varepsilon_n \to 0$, such that

$$\mathcal{A}(\cdot)B_n(\cdot) = B_n(\cdot+\alpha)e^{A_n}e^{F_n(\cdot)}.$$

When n is large enough, $\varepsilon_n \leq \varepsilon$, where $\varepsilon = \varepsilon(A_n, h_n, \alpha)$ is defined in Theorem 1.1. By Theorem 1.1, the quasi-periodic cocycle $(\alpha, e^{A_n}e^{F_n(\cdot)})$ can be analytically embedded into a quasi-periodical linear system. Therefore the result follows from Proposition 1.1. □

5. Equivalence of almost reducibility results

We only give the proof in the $SL(2,\mathbb{R})$ cocycle case, it can be generalized to other Lie groups without difficulty. In the following, we let $\mu \in \mathbb{T}^{d-1}$, $\omega = (1, \mu)$ is rational independent.

Lemma 5.1. *Let $h > 0$, $A \in C_h^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$. If (ω, A) is almost reducible, then the corresponding Poincaré cocycle $(\mu, \mathcal{A})$ is almost reducible.*

Proof. If (ω, A) is almost reducible, then there exist $B_n \in C_{h_n}^\omega(2\mathbb{T}^d, SL(2,\mathbb{R}))$, $A_n \in sl(2,\mathbb{R})$, $F_n \in C_{h_n}^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$ such that B_n conjugate (ω, A) to

$$\begin{cases} \dot{x} = (A_n + F_n(\theta))x \\ \dot{\theta} = \omega = (1, \mu) \end{cases} \tag{5.1}$$

with $\|F_n\|_{h_n} \leq \varepsilon_n \to 0$. Denote by $\Phi^t(\theta)$ the flow induced by (ω, A). Now we fix n which is large enough.

Case 1 : A_n is hyperbolic. In this case, (5.1) is uniformly hyperbolic, and then (ω, A) is uniformly hyperbolic. Consequently, $(\mu, \mathcal{A})$ is almost reducible since it is uniformly hyperbolic.

Case 2 : A_n is elliptic. We write $A_n = 2\pi\rho_n J$, suppose that $\tilde{\Phi}^t(\theta)$ is the corresponding flow of (5.1), then we have

$$\tilde{\Phi}^t(\theta) = e^{2\pi t\rho_n J}\Big(I + \int_0^t e^{-2\pi s\rho_n J}F_n(\theta+s\omega)\tilde{\Phi}^s(\theta)ds\Big). \tag{5.2}$$

Denote by $G^t(\theta) = e^{-2\pi t\rho_n J}\tilde{\Phi}^t(\theta)$, then

$$G^t(\theta) = I + \int_0^t e^{-2\pi s\rho_n J}F_n(\theta+\omega s)e^{2\pi s\rho_n J}G^s(\theta)ds,$$

let $g(t) = \|G^t(\theta)\|_{h_n}$, then

$$g(t) \leq 1 + \int_0^t \|F_n\|_{h_n} g(s)ds.$$

By Gronwall's inequality, we have $g(t) \le e^{\varepsilon_n t}$.

In the equation (5.2), let $t=1$, then we have $\tilde{\Phi}^1(0,\widetilde{\theta}) = e^{2\pi\rho_n J}(I+\widetilde{F}_n(\widetilde{\theta}))$, with the estimate $\|\widetilde{F}_n\|_{h_n} \le \int_0^1 \varepsilon_n g(t)dt \le 2\varepsilon_n$. Since B_n conjugates (ω, A) to (5.1) :

$$\Phi^t(0,\widetilde{\theta}) = B_n(t, \widetilde{\theta}+t\mu)\tilde{\Phi}^t(0,\widetilde{\theta})B_n(0,\widetilde{\theta})^{-1}.$$

Let $t=1$, we have

$$B_n(0,\widetilde{\theta}+\mu)^{-1}\mathcal{A}(\widetilde{\theta})B_n(0,\widetilde{\theta}) = e^{2\pi\rho_n J}(I+\widetilde{F}_n(\widetilde{\theta})),$$

which means that the Poincaré cocycle $(\mu,\mathcal{A})$ is almost reducible.

Case 3 : A_n is parabolic. Without lose of generality, we assume $A_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, let $B = \begin{pmatrix} \varepsilon_n^{\frac{1}{4}} & 0 \\ 0 & \varepsilon_n^{-\frac{1}{4}} \end{pmatrix}$, then $x = By$ transformation (5.1) to

$$\begin{cases} \dot{x} = \left(\begin{pmatrix} 0 & \varepsilon_n^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} + \overline{F}_n(\theta) \right) x \\ \dot{\theta} = \omega = (1,\mu) \end{cases}$$

with $\|\overline{F}_n\|_{h_n} \le \varepsilon_n^{\frac{1}{2}}$. It is then reduced to Case 2. □

The converse of Lemma 5.1 is also true:

Lemma 5.2. *Let $h>0$, $A \in C_h^\omega(\mathbb{T}^d, sl(2,\mathbb{R}))$, $(\mu,\mathcal{A})$ is the corresponding Poincaré cocycle of (ω, A). If $(\mu,\mathcal{A})$ is almost reducible, then (ω, A) is almost reducible.*

Proof. If $(\mu,\mathcal{A})$ is almost reducible, then there exist $B_n \in C_{h_n}^\omega(2\mathbb{T}^{d-1}, SL(2,\mathbb{R}))$, $\widetilde{A}_n \in sl(2,\mathbb{R})$, $\widetilde{F}_n \in C_{h_n}^\omega(\mathbb{T}^{d-1}, sl(2,\mathbb{R}))$ with $\|\widetilde{F}_n\|_{h_n} \le \varepsilon_n$, such that

$$\mathcal{A}(\cdot)B_n(\cdot) = B_n(\cdot+\mu)e^{\widetilde{A}_n}e^{\widetilde{F}_n(\cdot)}. \tag{5.3}$$

When n is large enough, by Theorem 1.1, there exists $\overline{F}_n \in C^\omega_{\frac{h_n}{1+|\mu|}}(\mathbb{T}^d, sl(2,\mathbb{R}))$, such that the quasi-periodic cocycle $(\mu, e^{\widetilde{A}_n}e^{\widetilde{F}_n(\cdot)})$ can be embedded into

$$\begin{cases} \dot{x} = (\widetilde{A}_n + \overline{F}_n(\theta))x \\ \dot{\theta} = (1,\mu). \end{cases} \tag{5.4}$$

Suppose that $\overline{\Phi}^t(\theta)$ is the corresponding flow of (5.4), we thus extend $B_n(\cdot)$ to the torus $2\mathbb{T}^d$ in the following way: for $(x_1,\widetilde{x}) \in \mathbb{R}^d$, we define:

$$\overline{B}_n(x_1,\widetilde{x})\overline{\Phi}^{x_1}(0,\widetilde{x}-x_1\mu) = \Phi^{x_1}(0,\widetilde{x}-x_1\mu)B_n(\widetilde{x}-x_1\mu). \tag{5.5}$$

Clearly, $\overline{B}_n(x_1,\widetilde{x})$ are analytic and 2-periodic in $\widetilde{x}$. We now check that they are also 2-periodic in x_1. By (5.3) and (5.5), we have

$$\begin{aligned}
& \overline{B}_n(x_1+2,\widetilde{x})\overline{\Phi}^{x_1}(2,\widetilde{x}-x_1\mu)\overline{\Phi}^2(0,\widetilde{x}-x_1\mu-2\mu) \\
= & \overline{B}_n(x_1+2,\widetilde{x})\overline{\Phi}^{x_1+2}(0,\widetilde{x}-x_1\mu-2\mu) \\
= & \Phi^{x_1+2}(0,\widetilde{x}-x_1\mu-2\mu)B_n(\widetilde{x}-x_1\mu-2\mu) \\
= & \Phi^{x_1}(2,\widetilde{x}-x_1\mu)\Phi^1(0,\widetilde{x}-x_1\mu-2\mu)B_n(\widetilde{x}-x_1\mu-2\mu) \\
= & \Phi^{x_1}(0,\widetilde{x}-x_1\mu)B_n(\widetilde{x}-x_1\mu)\overline{\Phi}^2(0,\widetilde{x}-x_1\mu-2\mu) \\
= & \overline{B}_n(x_1,\widetilde{x})\overline{\Phi}^{x_1}(0,\widetilde{x}-x_1\mu)\overline{\Phi}^2(0,\widetilde{x}-x_1\mu-2\mu),
\end{aligned}$$

which means $\overline{B}_n(x_1+2,\widetilde{x})=\overline{B}_n(x_1,\widetilde{x})$, and then $\overline{B}_n\in C^{\omega}_{\frac{h_n}{1+|\mu|}}(2\mathbb{T}^d,SL(2,\mathbb{R}))$. By similar reasoning as above, we have

$$\overline{B}_n(\theta_1+t,\widetilde{\theta}+t\mu)\overline{\Phi}^t(\theta_1,\widetilde{\theta})=\Phi^t(\theta_1,\widetilde{\theta})\overline{B}_n(\theta_1,\widetilde{\theta}).$$

This means $\overline{B}_n$ conjugate (ω,A) to (5.4), it concludes that (ω,A) is almost reducible. □

Lemma 5.3. *Let $h>0$, $A\in C^{\omega}_h(\mathbb{T}^d,sl(2,\mathbb{R}))$, $(\mu,\mathcal{A})$ is the corresponding Poincaré cocycle of (ω,A), then $(\mu,\mathcal{A})$ is rotations reducible, if and only if (ω,A) is rotations reducible.*

Proof. If (ω,A) is rotations reducible, the corresponding Poincaré cocycle is clearly rotations reducible by definition.

Now we prove the converse part. If $(\mu,\mathcal{A})$ is rotations reducible, then there exist $h_*<h$, $B\in C^{\omega}_{h_*}(2\mathbb{T}^{d-1},SL(2,\mathbb{R}))$, $\varphi\in C^{\omega}_{h_*}(\mathbb{T}^{d-1},\mathbb{R})$ such that

$$\mathcal{A}(\cdot)B(\cdot)=B(\cdot+\mu)R_{\varphi(\theta)}.$$

By Proposition 3.1, there exists $\rho\in C^{\omega}_{\frac{h_*}{1+|\mu|}}(\mathbb{T}^d,\mathbb{R})$ such that it satisfies

$$\int_0^1\rho(t,\cdot+t\mu)dt=\varphi(\cdot),$$

which means the quasi-periodic cocycle $(\mu,R_{\varphi(\cdot)})$ can be embedded into the quasi-periodic linear system $(\omega,\rho(\theta)J)$.

We now extend $B(\cdot)$ to the torus $2\mathbb{T}^d$ in the following way: for $(x_1,\widetilde{x})\in\mathbb{R}^d$, we define:

$$\overline{B}(x_1,\widetilde{x})e^{2\pi\int_0^{x_1}\rho(t,\widetilde{x}-x_1\mu+t\mu)dtJ}=\Phi^{x_1}(0,\widetilde{x}-x_1\mu)B_n(\widetilde{x}-x_1\mu).$$

By the same reasoning as in Lemma 5.2, it can be checked that

$$\overline{B}\in C^{\omega}_{\frac{h_*}{1+|\mu|}}(2\mathbb{T}^d,SL(2,\mathbb{R})),$$

and

$$\overline{B}(\theta_1+t,\widetilde{\theta}+t\mu)e^{2\pi\int_0^t\rho(\theta_1+s,\widetilde{\theta}+s\mu)dsJ}=\Phi^t(\theta_1,\widetilde{\theta})\overline{B}_n(\theta_1,\widetilde{\theta}),$$

which means $\overline{B}$ conjugate (ω,A) to $(\omega,\rho(\theta)J)$, it concludes that (ω,A) is rotations reducible. □

Proof of Theorem 1.4.
First we recall results of [21]:

Theorem 5.1. *Let* $\omega = (1,\alpha)$*,* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$*,* $h > 0$*,* $A \in sl(2,\mathbb{R})$ *and* $F \in C^{\omega}_{h}(\mathbb{T}^2, sl(2,\mathbb{R}))$*. If*

$$\|F\|_h < C_0 min\{h^{\chi}, 1\},$$

where C_0, χ *are numerical constants, then the following results hold:*

1. *The system* $(\omega, A+F(\theta))$ *is almost reducible.*
2. *If the rotation number is Diophantine w.r.t* ω*, then it is analytically rotations reducible.*
3. *Assume furthermore that* $\beta(\alpha) = 0$*, then it is analytically reducible.*

In [21], the transformation converges on analyticity strips of width going to zero. As remarked in [21], with minor modification of the proof, one obtains a strong version of the almost reducibility below: one can get convergence of the perturbation on strips of fixed width[4]. If we use such result, one can easily prove that in Theorem 5.1, if $2\pi h > \beta > 0$, the rotation number is Diophantine w.r.t ω, then the system is reducible.

We now finish the proof of Theorem 1.4. Suppose $\|\mathcal{A}-R\|_h = \varepsilon < e^{-\frac{12\pi h}{1+\alpha}}$, then by Theorem 3.1, there exist $\widetilde{R} \in sl(2,\mathbb{R})$ and $\widetilde{F} \in C^{\omega}_{h/1+\alpha}(\mathbb{T}^2, sl(2,\mathbb{R}))$ such that the cocycle $(\alpha, \mathcal{A})$ can be embedded into the quasi-periodic linear system $(\omega, \widetilde{R}+\widetilde{F})$ with estimate $\|\widetilde{F}\|_{h/1+\alpha} \leq e^{\frac{6\pi h}{1+\alpha}}\varepsilon^{\frac{1}{2}}$. Thus if

$$e^{\frac{6\pi h}{1+\alpha}}\varepsilon^{\frac{1}{2}} < C_0 min\{(\frac{h}{1+\alpha})^{\chi}, 1\},$$

which means

$$\varepsilon < \widetilde{C} min\{h^{2\chi}, 1\} e^{-\frac{12\pi h}{1+\alpha}},$$

then we can apply Theorem 5.1 and Theorem 1.3 to finish the proof. □

6. Proof of Theorem 1.5

By Aubry duality, we know that the dual operator of $L_{\lambda V,\alpha,\varphi}$ is $H_{\lambda V,\alpha,\phi}$. The eigenfunction equation $H_{\lambda V,\alpha,\phi}x = Ex$ corresponds to the Schrödinger cocycle $(\alpha, S_E^{\lambda V})$. We assume that

$$\|\lambda V(\theta)\|_h < \varepsilon < \widetilde{C} min\{h^{2\chi}, 1\} e^{-\frac{12\pi h}{1+\alpha}},$$

and denote by

$$R_{\lambda V,\alpha,\varphi} = \{E | \mathrm{rot}_f(\alpha, S_E^{\lambda V}) = \varphi + \frac{1}{2}\langle k, \alpha\rangle \mod 1\},$$

$$\Phi = \{\varphi | \varphi \quad \text{is Diophantine w.r.t} \quad \alpha\},$$

and recall that $\sigma^L_{pp}(\lambda V, \alpha, \varphi)$ and $B_{\lambda V,\alpha,\varphi}$ have been defined in section 2.3.

The proof of Theorem 1.5 is distinguished into two steps. First we prove the following:

[4] In deed, in order to prove such strong version of the almost reducibility, one only need to re-estimate Lemma 5.2 of [21].

Lemma 6.1. *If* $\varphi \in \Phi$, *then* $R_{\lambda V,\alpha,\varphi} = \sigma^L_{pp}(\lambda V, \alpha, \varphi)$.

Proof. By Lemma 2.1, it is sufficient for us to prove

$$R_{\lambda V,\alpha,\varphi} = B_{\lambda V,\alpha,\varphi} \quad \text{if} \quad \varphi \in \Phi. \tag{6.1}$$

First we prove:

$$R_{\lambda V,\alpha,\varphi} \supset B_{\lambda V,\alpha,\varphi}. \tag{6.2}$$

If $E \in B_{\lambda V,\alpha,\varphi}$, then by Lemma 16 of [31], there exists $B \in C^\omega(2\mathbb{T}, SL(2,\mathbb{R}))$ such that

$$B(\theta+\alpha)S_E^{\lambda v}(\theta)B(\theta)^{-1} = \begin{pmatrix} cos\varphi & sin\varphi \\ -sin\varphi & cos\varphi \end{pmatrix},$$

thus by Proposition 2.1, we have $\mathrm{rot}_f(\alpha, S_E^{\lambda V}) = \varphi + \frac{1}{2}\langle k, \alpha\rangle \mod 1$, which proves (6.2).

Then we prove

$$R_{\lambda V,\alpha,\varphi} \subset B_{\lambda V,\alpha,\varphi}. \tag{6.3}$$

If $\varphi \in \Phi$ is Diophantine w.r.t α, then $\mathrm{rot}_f(\alpha, S_E^{\lambda V})$ is also Diophantine w.r.t α. In fact,

$$\|2\varphi - k'\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{\gamma}{|k'|^\tau},$$

implies that

$$\begin{aligned} \|2\mathrm{rot}_f(\alpha, S_E^{\lambda V}) - k'\alpha\|_{\mathbb{R}/\mathbb{Z}} &= \|2\varphi + k\alpha - k'\alpha\|_{\mathbb{R}/\mathbb{Z}} \\ &\geq \frac{\gamma}{|k-k'|^\tau} \geq \frac{(1+|k|)^{-\tau}\gamma}{|k'|^\tau}. \end{aligned}$$

Then by Corollary 1.4, $(\alpha, S_E^{\lambda V})$ is rotations reducible, since we assume $2\pi h > (1+\alpha)\beta$, in fact we have $(\alpha, S_E^{\lambda V})$ is reducible. So there exists analytic $B : \mathbb{T} \to SL(2,\mathbb{R})$, $k \in \mathbb{Z}$, such that

$$B(\theta+\alpha)S_E^{\lambda V}(\theta)B(\theta)^{-1} = \begin{pmatrix} e^{2\pi i(\varphi+k\alpha)} & 0 \\ 0 & e^{-2\pi i(\varphi+k\alpha)} \end{pmatrix}.$$

It follows that the solution of $H_{\lambda V,\alpha,\phi}x = Ex$ has quasi-periodic Bloch waves with Floquet exponent $\varphi + k\alpha$. Therefore (6.3) holds. □

Then we prove the following:

Lemma 6.2. *If* $\varphi \in \Phi$, *then* $\overline{R_{\lambda V,\alpha,\varphi}} = \sigma^H(\lambda V, \alpha)$.

Proof. By (6.1), we have

$$\overline{R_{\lambda V,\alpha,\varphi}} = \overline{B_{\lambda V,\alpha,\varphi}} \subset \sigma^H(\lambda V, \alpha),$$

it is sufficient for us to prove that

$$\sigma^H(\lambda V, \alpha) \subset \overline{R_{\lambda V,\alpha,\varphi}}.$$

The crucial observation is that when restricted to the spectrum, the rotation number is strictly monotonic. To simplify the notation, we let $\mathrm{rot}_f(\alpha, S_E^{\lambda V}) = \mathrm{rot}_f(E)$.

Since for any $\varphi \in \mathbb{T}$, the orbit of $\{\varphi + k\alpha \mod 1\}$ is dense in $[0,1]$, we can find $E_n \in R_{\lambda V,\alpha,\varphi}$ for any $E_0 \in \sigma^H(\lambda V,\alpha)$, such that

$$\mathrm{rot}_f(E_n) \to \mathrm{rot}_f(E_0), \quad n \to \infty.$$

Moreover, we can assume $(\mathrm{rot}_f(E_n))_{n\in\mathbb{Z}}$ is monotonic, since the rotation number is monotonic, we then have E_n is monotonic and bounded (the boundness follows from the compactness of $\sigma^H(\lambda V,\alpha)$). Thus there exists $\widetilde{E} \in \sigma^H(\lambda V,\alpha)$, such that $E_n \to \widetilde{E}$. By the continuity of the rotational number, we have

$$\mathrm{rot}_f(E_n) \to \mathrm{rot}_f(\widetilde{E}), \quad n \to \infty.$$

Since the rotation number is strictly monotonic when it restricted to the spectrum $\sigma^H(\lambda V,\alpha)$, then $\widetilde{E} = E_0$, it follows that $E_0 \in \overline{R_{\lambda V,\alpha,\varphi}}$. □

By the last two steps and Lemma 2.1, we have for any $\varphi \in \Phi$,

$$\overline{\sigma^L_{pp}(\lambda V,\alpha,\varphi)} = \overline{R_{\lambda V,\alpha,\varphi}} = \sigma^H(\lambda V,\alpha) = \sigma^L(\lambda V,\alpha),$$

which means that the long-range operator $L_{\lambda V,\alpha,\varphi}$ has Anderson Localization. □

Proof of Theorem 1.6: If we apply Theorem 1.5 to almost Mathieu operator directly, we can not get the best estimate, since by the local embedding theorem, we lose some analyticity, in order to get finer estimate, we need the following:

Theorem 6.1. *For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $h > 0$, $\tau > 0$, $\gamma > 0$, let $\mathcal{A} \in SL(2,\mathbb{R})$ be such that $\|\mathcal{A} - R\|_h < c(h\gamma)^\tau$ for some constant R, and $\mathrm{rot}_f(\alpha,\mathcal{A}) \in DC_\alpha(\tau,\gamma)$, then the system $(\omega, A + F(\theta))$ is analytically rotations reducible.*

Remark 6.1. *The continuous version Theorem of 6.1 appear in* [21]*. In fact, the proof of this theorem in* [21] *applies essentially unchanged to the discrete case, thus the same result holds.*

By Aubry duality, we know that the almost Mathieu operator is self-dual, and the dual of $L_{\lambda cos,\alpha,\varphi}$ is $H_{\lambda cos,\alpha,\phi}$. Suppose

$$\lambda e^{2\pi h} = \|\lambda cos(2\pi\theta)\|_h \leq \varepsilon(h) \ll 1$$

is small enough, since in this case, $\beta(\alpha) > 0$, and it lies in the subcritical regime, then by Avila's theorem [4], one can provide (without any restrictions on the fibered rotation number) a sequence of conjugacies which put the cocycle arbitrarily close to constants, (we only lose arbitrary small analyticity strips of width), so that Theorem 6.1 eventually can be applied. Thus if the rotation number is Diophantine w.r.t ω, $2\pi h > \beta(\alpha) > 0$, then the cocycle is rotations reducible and consequently reducible: if $f \in C^\omega_h(\mathbb{T},\mathbb{R})$, $2\pi h > \beta(\alpha)$, then

$$\varphi(\theta+\alpha) - \varphi(\theta+\alpha) = f(\theta) - \widehat{f}(0)$$

has an analytic solution $\varphi \in C^\omega_{h-\beta/2\pi}(\mathbb{T},\mathbb{R})$. Finally, we can replay the proof of Theorem 1.5 to finish the proof. □

Acknowledgements

The work was supported by NNSF of China (Grant 10531050), NNSF of China (Grant 11031003) and a project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. The authors would like to thank Xuanji Hou, Haken Eliasson, David Damanik and Svetlana Jitomirskaya for useful discussions.

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