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Invariant Tori and Lagrange Stability of Pendulum-Type Equations

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In this paper we prove that the pendulum-type equation $x'' + g(t, x) = 0$ possesses infinitely many invariant tori whenever $g(t, x)$ has zero mean value on the torus T^2, where $g(t, x)$ belongs to $C^\infty(T^2)$. This yields the boundedness for solutions of the considered pendulum-type equation and thus leads to an answer to J. Moser's boundedness problem (1973, *Ann. of Math. Stud.* 77). © 1990 Academic Press, Inc.

I. INTRODUCTION

In this paper, we discuss the periodically excited nonlinear equation of pendulum-type, with the form

$$x'' + g(t, x) = 0,$$

where $g(t+1, x) = g(t, x+1) = g(t, x)$. It is equivalent to the equation

$$x'' + G'_x(t, x) = p(t), \tag{1.1}$$

where $G(t+1, x) = G(t, x+1) = G(t, x)$, and $p(t) = p(t+1)$. The equation is equivalent to the system

$$x' = y, \qquad y' = -G'_x(t, x) + p(t), \tag{1.2}$$

which is a Hamiltonian system with Hamiltonian

$$H = \tfrac{1}{2} y^2 + G(t, x) - xp(t).$$

Since $G(t, x)$ is periodic in x, system (1.2) can be treated as a non-autonomous system on a cylinder. But it is not a Hamiltonian system on the cylinder whenever $p(t) \not\equiv 0$, since in this case $H(t, x, y)$ is not periodic in x.

There were several papers concerning this equation, J. Mawhin [2] and

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M. Willem [3] proved that (1.1) possesses at least two periodic solutions if $G(t, x)$ is independent of t and $p(t)$ belongs to $C^0(S^1)$ with zero mean value.

In 1973, J. Moser [1] proposed a problem of Lagrange stability for the pendulum-type equation; i.e., whether or not all the solutions of system (1.2) are bounded on cylinder. This paper deals with this problem, and proves the following main theorem by using the J. Moser's twist theorem.

MAIN THEOREM. *If $G(t, x)$ belongs to $C^\infty(T^2)$ and $p(t)$ belongs to $C^0(S^1)$, then system (1.2) is Lagrange stable if and only if $\int_0^1 p(t)\,dt = 0$. Moreover, system (1.2) admits of an infinite number of invariant tori, and thus an infinite number of almost periodic solutions, when $\int_0^1 p(t)\,dt = 0$; and no invariant torus when $\int_0^1 p(t)\,dt \neq 0$.*

Remark. We only need that $G'_t(t, x)$ is continuous, and that $G(t, x)$ has continuous derivatives with respect to x up to order l for some positive integer l.

II. SYSTEM (1.2) AND ITS DEFORMAL SYSTEM

In this section, we transform system (1.2) into a simple integrable Hamiltonian system with a perturbation for large y by using a symplectic transformation.

We first introduce a space of functions. Given a constant r, denote by $F(r)$ the set of C^∞-functions in $(t, \theta, \lambda) \in T^2 \times R^+$, which are defined in $\lambda \geqslant \lambda_0$ for some $\lambda_0 > 0$ and for which there is a sequence $\lambda_{ij} > 0$, such that

$$\sup |\lambda^{-r+i}(D_\lambda)^i (D_\theta)^j f(t, \theta, \lambda)| < \infty,$$

where $(t, \theta) \in T^2$, and $\lambda > \lambda_{ij}$.

We collect some properties of $F(r)$ in the following lemma, which can easily be verified from the definition (see [4]).

LEMMA 2.1. (i) *If $r_1 < r_2$, then $F(r_1) \subset F(r_2)$;*

(ii) *if $f \in F(r)$, then $(D_\lambda)^i \in F(r - i)$;*

(iii) *if $f_1 \in F(r_1)$ and $f_2 \in F(r_2)$, then $f_1 f_2 \in F(r_1 + r_2)$;*

(iv) *if $f \in F(r)$ satisfies $f(\cdot, \lambda) \geqslant c\lambda^r$ for $\lambda = \lambda_0$, with a constant $c > 0$, then $f^{-1} \in F(-r)$.*

If $\lambda_0 > 0$, then we denote by $A_{\lambda_0} \in T^2 \times R^+$ the set

$$A_{\lambda_0} := \{(t, \theta, \lambda) \mid (t, \theta) \in T^2, \lambda = \lambda_0\}.$$

Now, we give the main result of this section.

PROPOSITION 2.1. *There is a symplectic diffeomorphism Ψ, depending periodically on t, of the form*

$$\Psi: x = u + U(t, u, v), \qquad y = v + V(t, u, v), \tag{2.1}$$

with $U(t, u, v) \in F(-2)$ and $V(t, u, v) \in F(-1)$, such that

$$A_{v_+} \subset \Psi(A_{v0}) \subset A_{v_-},$$

for large $v_- < v_0 < v_+$. Moreover, Ψ transforms (1.2) *into the system*

$$\begin{aligned} u' &= v + h_1(t, u, v), \\ v' &= p(t) + h_2(t, u, v), \end{aligned} \tag{2.2}$$

with $h_1(t, u, v) \in F(-2)$ and $h_2 \in F(-1)$.

Proof. We construct a symplectic transformation for sufficiently large v as

$$\begin{aligned} u &= x + g_1(t, x, v) := x + v^{-2} \int_0^x G(t, s)\, ds, \\ y &= v + g_2(t, x, v) := v - v^{-1} G(t, x). \end{aligned} \tag{2.3}$$

It is obvious from the implicit function theorem that we can solve the first equation of (2.3) and get $x = u + U(t, u, v)$ for large v. Then, substituting it into the second equation of (2.3), we get the desired diffeomorphism Ψ:

$$\begin{aligned} x &= u + U(t, u, v), \\ y &= v + V(t, u, v) := v - v^{-1} G(t, u - U). \end{aligned} \tag{2.4}$$

It follows that Ψ transforms system (1.2) into a new system for large $v > 0$:

$$\begin{aligned} u' &= v - v^{-2}\left(v^{-1} G^2(t, x) - \int_0^x G'_t(t, s)\, ds + M_1(t, x, v)\right), \\ v' &= p(t) - v^{-1}\left(-G'_t(t, x) + v^{-1} G(t, x)\, G'_x(t, x) + M_2(t, x, v)\right), \end{aligned} \tag{2.5}$$

where

$$M_1 = 2 \int_0^x G(t, s)\, ds\, (1 + v^{-2} G)^{-1} \left(p(t) - v^{-2} G \cdot G'_x - v^{-1} G'_t\right) \cdot v^{-1},$$

$$M_2 = v^{-1} G(t, x)\left(p(t) - v^{-2} G \cdot G'_x + v^{-1} G'_t\right) \int_0^1 (1 + \tau v^{-2} G)^{-2}\, d\tau.$$

We arrived at (2.2) by setting

$$h_1 = -v^{-2}\left(v^{-1}G^2(t,x) - \int_0^x G_t'(t,s)\,ds + M_1(t,x,v)\right),$$

$$h_2 = -v^{-1}(-G_t'(t,x) + v^{-1}G(t,x)\,G_x'(t,x) + M_2(t,x,v)),$$

with $x = u + U(t, u, v)$.

Then we want to prove that

$$U(t,u,v) \in F(-2), \qquad V(t,u,v) \in F(-1).$$

Notice that $g_1(t, x, v) \in F(-2)$, $g_2(t, x, v) \in F(-1)$, and

$$u = x + g_1(t, x, v), \tag{2.6}$$

with the inverse $x = u + U(t, u, v)$. Then we find out the equation for U,

$$U = -g_1(t, u+U, v). \tag{2.7}$$

It follows from (2.3) and (2.7) that if v is large enough, then U is uniquely determined by the contraction principle. Moreover, the implicit function theorem yields that $U \in C^\infty(A_{v_0})$, for some large v_0. We claim

$$U \in F(-2). \tag{2.8}$$

Indeed, apply the operator $(D_v)^n$ to Eq. (2.7), which transforms the right side into a sum of the terms

$$(D_u^s D_v^k g_1)\, D_v^{j_1}U\, D_v^{j_2}U \cdots D_v^{j_s}U, \tag{2.9}$$

with $1 \leqslant s+k \leqslant n$, and $\sum_{i=1}^s j_i = n-k$. The highest order term is the one with $s=1$ and $k=0$, namely $(D_u g_1)(D_v^n U)$, which we put to the left side of the equation. Inductively, assuming that for $j = n-1$ the estimates $|D_v^j U| \leqslant Cv^{-2-j}$ hold true, we conclude the same estimate for $j = n$. In fact, since $g_1 \in F(-2)$ and thus

$$|D_u^s D_v^k g_1| \leqslant cv^{-2-k},$$

we have

$$|(1 - D_u g_1)\, D^n U| \leqslant Cv^{-2-k} v^{-2-j_1} \cdots v^{-2-j_s} \leqslant Cv^{-2-n}.$$

Then the claim (2.8) follows.

Inserting $x = u + U(t, u, v)$ into the second equation of (2.3) yields

$$V(t,u,v) = g_2(t, u+U, v) = -v^{-1}G(t, u+U).$$

Since $g_2 \in F(-1)$ and $U \in F(-2)$, one concludes by using (2.9) that $V(t, u, v) \in F(-1)$.

Finally, we will prove that $h_1 \in F(-3)$, $h_2 \in F(-1)$. Set

$$\bar{h}_1(t, x, v) = -v^{-2}\left(v^{-1}G^2(t, x) - \int_0^x G'_t(t, s)\, ds + M_1(t, x, v)\right),$$

$$\bar{h}_2(t, x, v) = -v^{-1}(-G'_t(t, x) + v^{-1}G(t, x) \cdot G'_x(t, x) + M_2(t, x, v)).$$

Since $\bar{h}_1(t, x, v) \in F(-2)$, $\bar{h}_2(t, x, v) \in F(-1)$, and $U(t, u, v) \in F(-2)$, we can easily prove that

$$h_1(t, u, v) = \bar{h}_1(t, u + U, v) \in F(-2)$$

$$h_2(t, u, v) = \bar{h}_2(t, u + U, v) \in F(-1).$$

The proof of Proposition 2.1 is thus finished.

III. Some Estimates

The Poincaré mapping P of system (2.2) for large v_0 is of the form

$$u_1 = u + v + a^* + \int_0^1 [\bar{v}(t) - v - a^* + h_1(t, \bar{u}(t), \bar{v}(t))]\, dt,$$

$$v_1 = v + \int_0^1 p(t)\, dt + \int_0^1 h_2(t, \bar{u}(t), \bar{v}(t))\, dt,$$

where $a^* = \int_0^1 dt \int_0^t p(s)\, ds$, and $(\bar{u}(t), \bar{v}(t)) = (\bar{u}(t, u, v), \bar{v}(t, u, v))$ is the solution of (2.2) with $\bar{u}(0) = u$, $\bar{v}(0) = v$. Set

$$\begin{aligned} F_1 &= \int_0^1 [\bar{v}(t) - v - a^* + h_1(t, \bar{u}(t), \bar{v}(t))]\, dt, \\ F_2 &= \int_0^1 h_2(t, \bar{u}(t), \bar{v}(t))\, dt. \end{aligned} \tag{3.1}$$

Then the Poincaré mapping P can be written in the form

$$\begin{aligned} u_1 &= u + v + a^* + F_1(u, v), \\ v_1 &= v + \int_0^1 p(t)\, dt + F_2(u, v). \end{aligned} \tag{3.2}$$

In order to use Moser's twist theorem to mapping (3.2), we need some estimates of F_i $(i = 1, 2)$ and their derivatives.

LEMMA 3.1. *If v is sufficiently large, then*

$$|F_i(u, v)| = O(v^{-1}), \qquad \text{for} \quad i = 1, 2.$$

Proof. We first claim

$$\int_0^1 [\bar{v}(t, u, v) - v - a^*]\, dt = O(v^{-1}). \tag{3.3}$$

In fact, from (2.2) we know that

$$\bar{v}(t, u, v) = v + \int_0^t p(s)\, ds + \int_0^t h_2(s, \bar{u}(s), \bar{v}(s))\, ds.$$

Notice that $h_2(t, u, v) \in F(-1)$. Then we have

$$\left| \int_0^1 [\bar{v}(t, u, v) - v - a^*]\, dt \right| \leqslant C \max_{t \in [0,1]} |(\bar{v}(t, u, v))^{-1}|.$$

One verifies easily that if $\bar{v}(0) = v$ is sufficiently large, the solution $(\bar{u}(t), \bar{v}(t)) = (\bar{u}(t, u, v), \bar{v}(t, u, v))$ exists for $t \in [0, 1]$ and satisfies

$$|\bar{v}(t, u, v)| \geqslant v - M,$$

for some constant $M > 0$, independent of v. It concludes that (3.3) holds. Then, from (3.1), (3.3), and $h_1(t, u, v) \in F(-2)$, it follows that $|F_1(u, v)| = O(v^{-1})$. Similarly, we can get that $|F_2(u, v)| = O(v^{-1})$. Thus, the proof of Lemma 3.1 is completed.

The next lemma gives the estimates of derivatives of F_1 and F_2 of high order.

LEMMA 3.2. *For $r + s \geqslant 1$, $v > v^*$ (large), we have*

$$|D_v^r D_u^s F_i(u, v)| < C v^{-1}, \qquad i = 1, 2.$$

Proof. Let X_H be the vector field of (2.2). For its flow $\Phi^t(u, v) = (\bar{u}(t, u, v), \bar{v}(t, u, v))$ with $\Phi^0 = \text{id}$, set

$$\bar{u}(t, u, v) = u + vt + \int_0^t ds \int_0^s p(\tau)\, d\tau + A(t, u, v),$$

$$\bar{v}(t, u, v) = v + \int_0^t p(s)\, ds \qquad\qquad + B(t, u, v).$$

Then the flow Φ^t satisfies the integral equation

$$\Phi^t(u, v) = \Phi^0(u, v) + \int_0^t X_H \circ \Phi^s.$$

This is equivalent to the following equations for A and B:

$$\begin{aligned} A &= \int_0^t H_1(s, u, v, A, B)\, ds \int_0^t ds \int_0^s H_2(\tau, u, v, A, B)\, d\tau, \\ B &= \int_0^t H_2(s, u, v, A, B)\, ds, \end{aligned} \tag{3.4}$$

where

$$H_i = h_i\left(t, u + vt + \int_0^t ds \int_0^s p(\tau)\, d\tau + A, v + \int_0^t p(s)\, ds + B\right) \qquad (i = 1, 2).$$

Using the contraction principle, one verifies easily that for $v > v^*$, (3.4) has a unique solution (A, B) in the closed domain $|A| \leqslant 1$, $|B| \leqslant 1$. Moreover, A and B are smooth.

Note that $D^n f(A, B)$ is a sum of terms

$$(D_1^s D_2^k f)(D^{i_1} A) \cdots (D^{i_k} A)(D^{j_1} B) \cdots (D^{j_s} B) \tag{3.5}$$

with $1 \leqslant s + k \leqslant n$ and $\sum_{r=1}^k i_r = n - s$, $\sum_{r=1}^s j_r = n - k$. The required estimates can inductively be verified from (3.4) and (3.5) in view of $h_1 \in F(-2)$ and $h_2 \in F(-1)$. The proof of Lemma 3.2 is thus completed.

We denote $|f|_{c^l}$ the C^l norm of the function f, defined as

$$|f|_{c^l} = \sum_{k=0}^{l} \sup_{u \in s^1, v \in [a, b]} \left| \left(\frac{\partial}{\partial u}\right)^{k_1} \left(\frac{\partial}{\partial v}\right)^{k_2} f \right|, \qquad \text{with } k_1 + k_2 = k.$$

From Lemma 3.1 and Lemma 3.2, we know that for any real number $\delta > 0$, there exists a constant v_0, such that $|f|_{c^l} < \delta$, for $a > v_0$.

IV. Intersection Property

Definition 4.1. Assume that $P: S^1 \times R^1 \to S^1 \times R^1$ is an area-preserving mapping, and γ is a circle on $S^1 \times R^1$ homotopic to a circle $S^1 \times \{\lambda\}$, $\lambda =$ const. If $P(\gamma) \cap \gamma \neq \varnothing$ for any γ, then we say that P has intersection property.

In order to prove the intersection property of the Poincaré mapping P of system (2.2), which is needed in Moser's twist theorem, we first introduce the concept of Calabi invariant.

Consider an area preserving mapping P on the cylinder, and P is termed an end-preserving mapping if the points sufficiently far up (or down) the cylinder remain far up (or down) under P.

DEFINITION 4.2. Assume P is an area-preserving and end-preserving mapping on the cylinder. The Calabi invariant of P is defined by the formula

$$C(P) = \text{measure}\,(P(N)\backslash N) - \text{measure}(N\backslash P(N)),$$

where $N = N_h = \{(\theta, \lambda) \mid \lambda < h(\theta)\}$, for any function $h \in C^0(S^1)$.

It is easy to show that $C(P)$ is independent of N. It measures the average upward drift. If $P \in C^1(S^1 \times R^1)$, then

$$C(P) = \int_{P\gamma} \lambda\, d\theta - \int_\gamma \lambda\, d\theta = \int_\gamma (P^*(\lambda\, d\theta) - \lambda\, d\theta),$$

where γ is a circle in $S^1 \times R^1$, which is homotopic to a circle $S^1 \times \{\lambda\}$.

Now we compute the Calabi invariant of the Poincaré mapping of system (2.2).

PROPOSITION 4.1. *If $G(t, x) \in C^\infty(T^2)$, then the Calabi invariant of the Poincaré mapping P of system* (2.2) *is*

$$C(P) = \int_0^1 p(t)\, dt.$$

Proof. Set $\gamma = S^1 \times \{v^*\}$, with sufficiently large v^*. Then we have

$$\begin{aligned} C(P) &= \int_\gamma (P^*(v\, du) - v\, du) \\ &= \int_\gamma \left(v + \int_0^1 p(t)\, dt + F_2(u, v)\right) \\ &\quad \times d(u + v + a^* + F_1(u, v)) - \int_\gamma v\, du \\ &= \int_0^1 p(t)\, dt + O(v^{*-1}). \end{aligned}$$

We know that $C(P)$ is independent of v^*. It follows that $C(P) = \int_0^1 p(t)\, dt$. Thus the proof of Proposition 4.1 is completed.

COROLLARY 4.1. *The Poincaré mapping P of system* (2.2) *has intersection property if* $C(P)=0$.

Proof. Assume that $P(\gamma)\cap\gamma=\varnothing$ for a circle γ which is homotopic to $S^1\times\{\lambda\}$. Then we have

$$C(P)=\int_{P(\gamma)} v\,du-\int_{\gamma} v\,du=\iint_{D} dv\,du\neq 0,$$

where D is the region bounded by $P(\gamma)$ and γ. However, it is in conflict with $C(P)=0$. The proof is thus finished.

V. EXISTENCE OF INVARIANT TORI

In this section we prove that system (1.2) possesses an infinite number of invariant tori. For this aim, we first state a simple form of Moser's twist theorem [5].

THEOREM (J. Moser). *Suppose that* $F_1(u,v)\in C^l$ *and* $F_2(u,v)\in C^l$ $(l=333)$ *have period* 1 *in u, and suppose that*

$$P\colon\begin{cases} u_1=u+\alpha(v)+F_1(u,v),\\ v\qquad\qquad\quad +F_2(u,v)\end{cases}\tag{5.1}$$

is a mapping from the annulus $S^1\times[a,b]$ *to* $S^1\times R^1$, *with the properties that*

(i) $\alpha'(v)>0$;

(ii) *P has intersection property.*

Then, for every irrational number in $[\alpha(a)+\beta,\ \alpha(b)-\beta]$ $(\beta>0)$ *satisfying the Diophantine condition*

$$|n\lambda-m|>\beta n^{-3/2}\tag{5.2}$$

for all integers m and n, with $n>0$, *there is a real number* $\delta(\beta)>0$, *such that when*

$$|F_1|_{c^l}+|F_2|_{c^l}<\delta,$$

there is a closed C^1*-curve* Γ, *with the following properties*:

(i) Γ *is invariant under the mapping* P;

(ii) *The mapping P restricted on* Γ *has rotation number* λ.

The main aim of this paper is to prove the following theorem by using Moser's twist theorem for the Poincaré mapping P of system (2.2).

THEOREM 5.1. *If $G(t, x) \in C^{\infty}(T^2)$, then system* (1.2) *has an invariant torus if and only if $\int_0^1 p(t)\,dt = 0$. Moreover, when $\int_0^1 p(t)\,dt = 0$, there is a (large) $\lambda^* > 0$, such that for every irrational number $\lambda > \lambda^*$, with* (5.2), *system* (1.2) *has an invariant torus with the rotation number* $(1, \lambda)$.

Proof. In view of the results proved in Sections 2–4, we conclude that all the assumptions of Moser's twist theorem are met for the Poincaré mapping P of system (2.2) when $\int_0^1 p(t)\,dt = 0$ and v is large enough. It follows that for $\lambda > \lambda^* \gg 1$, satisfying (5.2), there is a C^1-closed curve Γ which is close to the circle $S^1 \times \{\lambda\}$ and invariant under the Poincaré mapping P. The solutions of (2.2) starting at time $t = 0$ on this invariant curve Γ determine a torus T_λ^2 in the (t, u, v)-space $T^2 \times R^1$, which is thus invariant under the flow of (2.2). It follows that every solution of (2.2) on T_λ^2 is almost periodic with frequency $(1, \lambda)$. Then, $\Psi(T_\lambda^2)$ is an invariant torus of (1.2) with rotation number $(1, \lambda)$.

Now, assume P possesses an invariant closed curve which is homotopic to $S^1 \times \{\lambda\}$. Then it is obvious that $C(P) = 0$. It follows that $\int_0^1 p(t)\,dt = 0$ because $\int_0^1 p(t)\,dt \neq 0$ implies $C(P) \neq 0$. Therefore, we have proved Theorem 5.1.

VI. LAGRANGE STABILITY

In this section, we discuss the Lagrange stability of system (1.2) by using Theorem 5.1.

THEOREM 6.1. *If $G(t, x) \in C^{\infty}(T^2)$, then system* (1.2) *is Lagrange stable if and only if $\int_0^1 p(t)\,dt = 0$.*

Proof. Suppose that $\int_0^1 p(t)\,dt = 0$. Then from Theorem 5.1 we know that system (1.2) has an invariant torus with the rotation $(1, \lambda)$, where $\lambda > \lambda^* \gg 1$, satisfying (5.2). Every solution on this invariant torus is almost periodic with frequency $(1, \lambda)$.

Now, we set $\xi = -x$, $\eta = -y$. Then (1.2) becomes

$$\xi' = \eta, \qquad \eta' = -\bar{G}'_x(t, \xi) + \bar{p}(t) := G'_x(t, -\xi) + p(t). \tag{1.2$'$}$$

Similarly, we can prove that (1.2)′ has an invariant torus with rotation number λ, where λ is sufficiently large and satisfies (5.2). Thus, system (1.2) has an invariant torus with rotation number $-\lambda$. Two such invariant tori confine the solutions in its interior, which therefore yields a bound for those solutions. In fact for any $(x_0, y_0) \in S^1 \times R^1$, there is a $\lambda > y_0$ such that (1.2) possesses two invariant tori $T^2_{\pm\lambda}$. Thus, for the solution $(x(t, x_0, y_0), y(t, x_0, y_0))$ of (1.2) with $x(0) = x_0$, $y(0) = y_0$, one has

$$\sup_{t \in R'} |y(t, x_0, y_0)| < \lambda + K_0,$$

where

$$K_0 = \sup_{(t,x) \in T^2} |-G'_x(t, x) + p(t)| + 1.$$

This proves that $\int_0^1 p(t)\, dt = 0$ is a sufficient condition for the Lagrange stability of (1.2).

Now let us prove that $\int_0^1 p(t)\, dt = 0$ is also necessary. We need only to prove that system (1.2) has an unbounded solution if the mean value of $p(t)$ is not zero. Without loss of generality, we can assume that $\int_0^1 p(t)\, dt = p^* > 0$. First, we claim that (2.2) possesses an unbounded solution. Note that $h_1 \in F(-2)$ and $h_2 \in F(-1)$. Then, we can choose a sufficiently large constant v^* such that $|h_1| \leqslant \frac{1}{2} p^*$, $|h_2| \leqslant \frac{1}{2} p^*$, for $v > v^*$. Since p^* is the mean value of $p(t)$, we can take a constant $M > 0$, such that

$$\left| \int_0^1 (p(s) - p^*)\, ds \right| < M, \qquad \text{for all } t \in R^1.$$

Denote by $(u(t, u_0, v_0), v(t, u_0, v_0))$ the solution of (2.2) with $u(0) = u_0$, $v(0) = v_0$, $v_0 > v^* + M$. Then, the second equation of (2.2) yields

$$\begin{aligned} v(t, u_0, v_0) &\geqslant v_0 + \frac{1}{2} p^* t + \int_0^t (p(s) - p^*)\, ds \\ &\geqslant v^* + \frac{1}{2} p^* t, \end{aligned}$$

when $t > 0$. Therefore, $v(t, u_0, v_0)$ must tend to $+\infty$ when t tends to $+\infty$. It follows that $\Psi \circ \Phi'(u_0, v_0)$ is an unbounded solution of (1.2).

Combining Theorem 5.1 and Theorem 6.1, we get the main theorem given in the Introduction.

Remark. I am glad to know from the referee that Mark Levi [6] and J. Moser [7] have independently obtained similar results for the case $p(t) = 0$.

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REFERENCES

1. J. MOSER, Stable and random motions in dynamical systems, *Ann. of Math. Stud.* **77** (1973).
2. J. MAWHIN AND M. WILLEM, Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, *J. Differential Equations* **52** (1984), 264–287.
3. M. WILLEM, Oscillations forcées de systèmes hamiltoniens, Publications Séminaires Analyse non linéaire de l'Univ. de Besançon, 1981.
4. E. ZEHNDER, An "a priori" estimate for oscillatory equations, *in* "Lecture Notes in Mathematics, Vol. 1125," Springer-Verlag, New York/Berlin, 1983.
5. J. MOSER, On invariant curves of area-preserving mappings of an annulus, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, (1962), 1–20.
6. M. LEVI, KAM theory for particles in periodic potential, *Ergodic Theory Dynamical Systems*, in press.
7. J. MOSER, Minimal foliations on a torus, *Ergodic Theory Dynamical Systems* **8** (1988).