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Quasiperiodic Solutions for a Class of Quasiperiodically Forced Differential Equations

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In this paper, the KAM theorem is used to prove the existence of quasiperiodic solutions of quasiperiodically forced second-order differential equations which do not contain any small parameters.

1. INTRODUCTION

The quasiperiodically forced second-order equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = p\left(\omega_1 t, \ldots, \omega_m t, x, \frac{dx}{dt}\right), \tag{1.1}$$

has been widely investigated because of its importance in applications and because it exhibits intrinsic nonlinear phenomena (see [H, GH, R]). If p is periodic in t, i.e., $m = 1$, there are many contributions to the existence problem of periodic solutions using topological degree theory and critical point theory [MW]. For the case $m > 1$, and if system (1.1) is dissipative, extensive methods have been developed to deal with the existence of quasiperiodic solutions [F]. However, if Eq. (1.1) is not dissipative, e.g., $f(-x) = -f(x)$, results obtained by these techniques are not enlightening. In pioneering work of Moser [M1], it was proved that equations of the form

$$\frac{d^2x}{dt^2} + a(\mu)x + bx^3 = \mu p\left(\omega_1 t, \ldots, \omega_m t, x, \frac{dx}{dt}\right), \tag{1.2}$$

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have a rich structure of quasiperiodic solutions provided the parameter μ is sufficiently small, $b > 0$, $a(\mu) \sim 1$, $p(-\omega_1 t, \ldots, -\omega_m t, -x, dx/dt) = p(\omega_1 t, \ldots, \omega_m t, x, dx/dt)$ is quasiperiodic in t, and the frequency vector $(\omega_1, \ldots, \omega_m)$ satisfies the Diophantine condition

$$|\langle, k, \omega\rangle| > \gamma|k|^{-\tau}, \qquad \gamma > 0, \tau > m-1, \tag{1.3}$$

for any $k \in Z^m\backslash\{0\}$. The method of the proof in [M1] is the famous KAM iteration. Recently Berger and Chen [BC] used variational methods to prove the existence of a uniformly almost periodic solution for the equation

$$\frac{d^2x}{dt^2} - x - x^3 = h(t), \tag{1.4}$$

where $h(t)$ is a uniformly almost periodic function. Although quasiperiodic functions are a special case of almost periodic functions, it is known that Eqs. (1.2) and (1.4) are quite different from the point of view of variational methods because $a(\mu)$, b are positive in (1.2) but negative in (1.4). Indeed, the method in [BC] cannot be applied to Eq. (1.2).

In this paper, we use the KAM theorem to provide the existence of quasiperiodic solutions for a class of reversible differential equations not containing any small parameters. We will carry this out for the following two typical examples:

In that follows let $\theta = (\theta_1, \ldots, \theta_m) = (\omega_1 t, \ldots, \omega_m t)$.

EXAMPLE 1. The reversible pendulum equation

$$\frac{d^2\phi}{dt^2} + F'_\phi(\theta, \phi)\frac{d\phi}{dt} + G'_\phi(\theta, \phi) = 0, \tag{1.5}$$

where $F(\theta, \phi)$, $G(\theta, \phi)$ are periodic in ϕ, θ with period 1, and F, G satisfy

$$F(-\theta, -\phi) = F(\theta, \phi), \qquad G(-\theta, -\phi) = G(\theta, \phi). \tag{1.6}$$

EXAMPLE 2. The reversible Duffing equation

$$\frac{d^2x}{dt^2} + x^{2n+1} + \sum_{i=1}^{2n} p_l(\theta_1, \ldots, \theta_m)x^l = 0, \tag{1.7}$$

where $p_l(\theta)$ are periodic in θ with period 1, and $p_l(\theta)$ is an odd function when l is even while p_l is an even function when l is odd.

The main results of this paper are the following.

THEOREM 1. *Suppose that $F(\theta, \phi)$, $G(\theta, \phi)$ are real analytic functions satisfying the assumptions of Example* 1. *Then Eq.* (1.5) *possesses an invariant torus T^{m+1} in $(\theta, \phi, \dot{\phi})$-space composed of quasiperiodic solutions of the form*

$$\phi(t) = \lambda t + \Phi(\lambda t, \omega_1 t, \ldots, \omega_m t),$$

for sufficiently large λ and provided that the vector $(\lambda, \omega_1, \ldots, \omega_m)$ satisfies the Diophantine condition

$$|\langle k, \omega\rangle + l\lambda| > \gamma(|k| + |l|)^{-\tau}, \qquad \gamma > 0, \tau > m, \tag{1.8}$$

for any $(k, l) \in Z^{m+1} - \{0\}$. Moreover, all solutions of (1.5) *are bounded in the phase cylinder, i.e., all the solutions of Eq.* (1.5) *satisfy the condition*:

$$\sup_{t \in R^1} |\dot{\phi}(t, 0, \phi_0, \dot{\phi}_0)| < \infty.$$

THEOREM 2. *Suppose that $p_l(\theta)$ are real analytic functions satisfying the assumptions of Example* 2. *Then Eq.* (1.7) *possesses an invariant torus T^{m+1} in $(\theta, x, \dot{x})$-space which is composed of quasiperiodic solutions of the form $x(t) = x(\lambda t, \omega_1 t, \ldots, \omega_m t)$ for sufficiently large λ and provided that the vector $(\lambda, \omega_1, \ldots, \omega_m)$ satisfies the Diophantine condition* (1.8). *Moreover, all solutions of* (1.7) *are bounded, i.e.,*

$$\sup_{t \in R^1} (|x(t, 0, x_0, \dot{x}_0)| + |\dot{x}(t, 0, x_0, \dot{x}_0)|) < \infty$$

holds for all solutions of (1.7).

Remark. Actually, the KAM theorem yields stronger results; in place of analyticity of F, G, p_l, it is sufficient to assume that p_i, F, $G \in C^{2\tau+2}$.

The outline of this paper is as follows: In Section 2 we give a version of the KAM theorem which can be applied to Examples 1 and 2. The application is carried out in Section 3.

2. A KAM THEOREM FOR REVERSIBLE SYSTEMS

Consider the system of differential equations

$$\frac{dx}{dt} = f(x, y, \theta), \qquad \frac{dy}{dt} = g(x, y, \theta), \qquad \frac{d\theta}{dt} = \omega, \tag{2.1}$$

where $(x, y, \theta) \in T^n \times R^n \times T^m$ and $\omega = (\omega_1, \ldots, \omega_m)$. System (2.1) is defined in a domain which is preserved under the reflection Λ: $(x, y, \theta) \to (-x, y, -\theta)$. System (1.2) is called reversible with respect to Λ if

$$f(-x, y, -\theta) = f(x, y, \theta), \qquad g(-x, y, -\theta) = -g(x, y, \theta). \tag{2.2}$$

It is easy to see that there is no difference between system (2.1) and the forced quasiperiodic differential system,

$$\frac{dx}{dt} = f(x, y, \omega_1 t, \ldots, \omega_m t), \qquad \frac{dy}{dt} = g(x, y, \omega_1 t, \ldots, \omega_m t). \tag{2.3}$$

If system (2.1) is reversible with respect to Λ, it can be rewritten in the equivalent form

$$\frac{dx}{dt} = f_0(y) + f_1(x, y, \theta), \qquad \frac{dy}{dt} = g(x, y, \theta), \qquad \frac{d\theta}{dt} = \omega, \tag{2.4}$$

where $\int\int_{T^n \times T^m} f(x, y, \theta)\, dx\, d\theta = \int\int_{T^n \times T^m} g(x, y, \theta)\, dx\, d\theta = 0$. If $f_0(y)$ is nondegenerate, i.e., $\det(\partial f_0/\partial y) \neq 0$, a change of variables in system (2.4) yields the equivalent form,

$$\frac{dx}{dt} = y + f_1(x, y, \theta), \qquad \frac{dy}{dt} = f_2(x, y, \theta), \qquad \frac{d\theta}{dt} = \omega. \tag{2.5}$$

We now give a version of the KAM theorem for system (2.5) which can be applied to Eqs. (1.5) and (1.7).

THEOREM 2.1. *Suppose that system* (2.5) *is reversible and the functions* $f_1(x, y, \theta), f_2(x, y, \theta)$ *in system* (2.5) *are real analytic. Assume that* f_1, f_2 *can be extended to complex analytic functions defined in the domain*

$$D := \{(x, y, \theta): |\mathrm{Im}\, x|, |\mathrm{Im}\, \theta| < \rho_0, |\mathrm{Re}\, y| > R, |\mathrm{Im}\, y| < \delta_0\}$$

for some positive number δ_0, ρ_0, R. *Moreover,*

$$\lim_{|y| \to +\infty} (|f_1(x, y, \theta)| + |f_2(x, y, \theta|) = 0. \tag{2.6}$$

Then for any ω *satisfying the Diophantine condition* (1.3), *there exists a constant* y_0 *depending on* γ, n, m, τ, *but not on* D, *such that if* $|\lambda| > y_0$ *satisfies*

$$|(k, \omega) + (l, \lambda)| > \gamma(|k| + |l|)^{-\tau}, \qquad \tau > m + n - 1, \tag{2.7}$$

for $(k, l) \in Z^{m+n}\backslash\{0\}$, *the system* (2.5) *possesses a* $n + m$ *dimensional invariant torus composed of quasiperiodic solutions with frequency vector* (λ, ω); *i.e.*, (2.5) *has solutions of the form*

$$x(t) = \lambda t + x_0 + u(\lambda_1 t, \ldots, \lambda_n t, \omega_1 t, \ldots, \omega_m t),$$

$$y(t) = \lambda + v(\lambda_1 t, \ldots, \lambda_n t, \omega_1 t, \ldots, \omega_m t),$$

$$\theta(t) = \omega t,$$

where $u(\cdot), v(\cdot) \in C(T^{n+m})$.

Remarks. 1. The set of λ satisfying condition (2.7) has positive Lebsegue measure.

2. The formulation of Theorem 2.1 is slightly different from that of the standard KAM theorem.

3. The conclusion in Theorem 2.1 remains true if we replace the analyticity of f_1, f_2 and (2.6) by $f_1, f_2 \in C^l$ $(l > 2\tau + 2)$ and $\lim_{|y| \to \infty} (|f_1|_{C^l} + |f_2|_{C^l}) = 0$.

Since the KAM theorem is local, it is sufficient to consider system (2.5) in a small neighborhood of $T^n \times \{\lambda\} \times T^m$ for large enough λ. For this purpose, we introduce the new variables $(\tilde{x}, \tilde{y}, \tilde{\theta})$ by $\tilde{x} = x$, $\tilde{y} = y - \lambda$, $\tilde{\theta} = \theta$. Without confusion, we still denote by (x, y, θ) the coordinates of the following transformed system,

$$\frac{dx}{dt} = \lambda + y + f_{1\lambda}(x, y, \theta),$$

$$\frac{dy}{dt} = f_{2\lambda}(x, y, \theta), \tag{2.8}$$

$$\frac{d\theta}{dt} = \omega.$$

Theorem 2.1 is an immediate consequence of the following result for system (2.8).

THEOREM 2.2. *Let* $0 < \alpha < 1$ *and*

$$0 < \beta < \min\left(\frac{\alpha}{4n + 4m + 4\tau + 1}, 1 - \alpha\right) \tag{2.9}$$

and γ, τ as given by (2.7). *Then there exists a constant $\delta_0 = \delta_0(n, m, \alpha, \beta, \gamma, \tau)$ such that if system* (2.8) *is analytic in*

$$D := \{|y| < \delta, |\operatorname{Im} x_k| < \delta^\beta, |\operatorname{Im} \theta_l| < \delta^\beta, k = 1, \ldots, n, l = 1, \ldots, m\},$$

where $f_{1\lambda}$, $f_{2\lambda}$ satisfy

$$|f_{1\lambda}| + |f_{2\lambda}| < \delta^{1+\alpha}, \tag{2.10}$$

in D for some δ in $0 < \delta < \delta_0$, then system (2.8) *possesses an invariant torus T^{n+m} composed of the solutions of* (2.8) *of the form*

$$\begin{aligned} x(t) &= \lambda t + \xi_0 + u^0(\lambda t + \xi_0, \omega t, \\ y(t) &= v^0(\lambda t, \omega t) \\ \theta(t) &= \omega t, \end{aligned} \tag{2.11}$$

where u^0, v^0 are analytic functions defined on $C^\omega(T^{n+m})$.

Proof. Theorem 2.2 has been proved in [M2] in the case that the function $f_{1\lambda}$, $f_{2\lambda}$ do not depend on θ variables. To take care of the θ dependence, following the footsteps of [M2], one can prove Theorem 2.2 without additional difficulties. Theorem 2.2 also can be derived from Theorem 5.1 of [M2] in following way: Consider the augmented system

$$\begin{aligned} \frac{dx}{dt} &= \lambda + y + f_{1\lambda}(x, y, \theta), \\ \frac{dy}{dt} &= f_{2\lambda}(x, y, \theta), \\ \frac{d\theta}{dt} &= \omega + \bar{y}, \qquad \bar{y} = (\bar{y}_1, \ldots, \bar{y}_m), \\ \frac{d\bar{y}}{dt} &= 0, \end{aligned} \tag{2.12}$$

defined in $D \times \{|\bar{y}| < \delta\}$. System (2.12) is a special case of Theorem 5.1 in [M2], since the perturbations here do not depend on $\bar{y}$. From Theorem 5.1 of [M2] and also its proof, we know that, if the assumptions of Theorem 2.2 hold, there is a change of variables Ψ of the form

$$\begin{aligned} x &= \bar{u}(\xi, \eta, \theta) = \xi + u^0(\xi, \eta, \theta) + u^1(\xi, \eta, \theta)\eta, \\ y &= \bar{v}(\xi, \eta, \theta) = \eta + v^0(\xi, \eta, \theta) + v^1(\xi, \eta, \theta)\eta, \\ \theta &= \theta, \\ \bar{y} &= \bar{y}, \end{aligned} \tag{2.13}$$

which transforms (2.12) into a system

$$\frac{d\xi}{dt} = \phi(\xi, \eta), \qquad \frac{d\eta}{dt} = \psi(\xi, \eta), \qquad \frac{d\theta}{dt} = \omega + \bar{y}, \qquad \frac{d\bar{y}}{dt} = 0, \quad (2.14)$$

with

$$\phi = \lambda + \eta + O(\eta^2), \qquad \psi = O(\eta^2).$$

Moreover, the $\bar{u}$, $\bar{v}$ in (2.13) satisfy

$$|\bar{u} - \xi| + |\bar{v}| < \delta, \qquad \text{in } |\operatorname{Im} \xi_k| < \frac{1}{2}\delta^\beta, |\eta_k| < \frac{\delta}{2}.$$

It is easy to see that (2.14) has an invariant torus $T^1 \times \{0\} \times T^m \times \{0\}$ composed of solutions of the form

$$\xi(t) = \lambda t + \xi_0, \eta(t) = 0, \qquad \theta(t) = \omega t, \qquad \bar{y}(t) = 0. \qquad (2.15)$$

Inserting (2.15) into (2.13) we see that

$$x(t) = \lambda t + \xi_0 + u^0(\lambda t + \xi_0, 0, \omega t), \qquad y(t) = v^0(\lambda t + \xi_0, 0, \omega t),$$
$$\theta(t) = \omega t, \qquad \bar{y}(t) = 0, \qquad (2.16)$$

is a solution of (2.12). It is also easy to see that $(x(t), y(t), \theta(t))$ in (2.16) is a solution of (2.8). This kind of solution forms an invariant torus of (2.8). This implies the proof of Theorem 2.2.

It is easy to see $x(t) = \lambda t + x_0 + u(\lambda t + x_0, 0, \omega t)$, $y(t) = \lambda + v(\lambda t + x_0, 0, \omega t)$, $\theta(t) = \omega t$ are solutions of system (2.5) which form an invariant torus of (2.5). This implies Theorem 2.1.

3. Applications

EXAMPLE 1. Consider the reversible pendulum equation

$$\frac{d^2\phi}{dt^2} + F'_\phi(\theta, \phi)\frac{d\phi}{dt} + G'_\phi(\theta, \phi) = 0, \qquad (3.1)$$

where $F(\theta, \phi)$, $G(\theta, \phi)$ are real, analytical, and periodic in ϕ, θ period 1 and satisfy (1.6). Let Ψ_1: $\phi = \phi$, $\theta = \theta$, $v = \dot{\phi} + F(\theta, \phi)$, system (3.1) becomes

$$\frac{d\phi}{dt} = v - F(\theta, \phi), \qquad \frac{dv}{dt} = -\overline{G}'_\phi(\theta, \phi), \qquad \frac{d\theta}{dt} = \omega, \qquad (3.2)$$

where $\overline{G}(\theta, \phi) = G(\theta, \phi) - \int_0^\phi F_i'(\theta, s)\, ds$. It is easy to see that $\overline{G}(-\theta, -\phi) = \overline{G}(\theta, \phi)$.

Let $y^* > \sup |F|$. Denote by $A_{y^*} \subset T^{m+1} \times R^1$ the set

$$A_{y^*} := \{(t, x, y) | (t, x) \in T^{m+1}, |y| > y^*\}.$$

LEMMA 3.1. *There is a change of variables Ψ_2, defined in A_{y^*}, of the form*

$$\phi = x + X(\theta, x, y), \qquad v = y + Y(\theta, x, y),$$

where X, Y are real analytical and close to identity if y is sufficiently large, such that Φ_2 transforms (3.2) *into the form*

$$\frac{dx}{dt} = y + h_1(\theta, x, y), \qquad \frac{dy}{dt} = h_2(\theta, x, y), \qquad \frac{d\theta}{dt} = \omega, \tag{3.3}$$

which is defined in A_{2y^*}. Moreover, (1.6) *implies that* (3.3) *is a reversible system with respect to Λ, i.e.,*

$$h_1(-\theta, -x, y) = h_1(\theta, x, y), \qquad h_2(-\theta, -x, y) = -h_2(\theta, x, y).$$

Proof. We construct the change of variables implicitly for large y by

$$x = \phi + \frac{1}{y}\int_0^\phi F(\theta, s)\, ds, \qquad v = y - \frac{1}{y}\overline{G}(\theta, x). \tag{3.4}$$

Without loss of generality, we assume that the mean value of $F(\theta, \phi)$ is zero and $\int_0^\phi F(\theta, s)\, ds$ is a periodic function in ϕ. It is obvious from the implicit function theorem that we can solve the first equation of (3.4) and get $\phi = x + X(\theta, x, y)$ for large y. Then substituting it into the second equation of (3.4), we get the desired transformation,

$$\phi = x + X(\theta, x, y), \qquad v = y + Y(\theta, x, y) := y - \frac{1}{y}\overline{G}(\theta, x + X). \tag{3.5}$$

It follows that Ψ_2 transforms system (3.2) into a new system for a large y of the form

$$\frac{dx}{dt} = y + h_1(\theta, x, y), \qquad \frac{dy}{dt} = h_2(\theta, x, y),$$

where

$$h_1(\theta, x, y) = \frac{1}{y}\left[\int_0^{\phi} F_t'(\theta, s)\, ds - F^2(\theta, \phi) - \overline{G}(\theta, \phi) - \frac{1}{v}(F(\theta, \phi)\overline{G}(\theta, \phi) - \int_0^{\phi} F(\theta, s)\cdot h_2(\theta, x, y))\right],$$

$$h_2(\theta, x, y) = \frac{1}{y}\left(1 + \frac{1}{y^2}\overline{G}(\theta, \phi)\right)^{-1}\left(\overline{G}_t'(\theta, \phi) - \overline{G}_\phi'(\theta, \phi)F(\theta, \phi) - \frac{\overline{G}_\phi'(\theta, \phi)\overline{G}(\theta, \phi)}{y}\right), \tag{3.6}$$

with $\phi = x + X(\theta, x, y)$.

From (3.4) it is easy to see that

$$X(-\theta, -x, y) = X(\theta, x, y), \qquad Y(-\theta, -x, y) = Y(\theta, x, y). \tag{3.7}$$

In view of (3.6) and (3.7), we know that (1.6) implies that

$$h_1(-\theta, -x, y) = h_1(\theta, x, y), \qquad h_2(-\theta, -x, y) = -h_2(\theta, x, y).$$

LEMMA 3.2. *$h_1(\theta, x, y)$ and $h_2(\theta, x, y)$ can be treated as complex analytical functions defined in domain*

$$D_{\rho_1 R_1 \delta_1} := \{(x, \theta, y)\colon |\mathrm{Im}\, x|, |\mathrm{Im}\, \theta| < \rho_1, |\mathrm{Re}\, y| > R_1, |\mathrm{Im}\, y| < \delta_1,\}$$

for some numbers δ_1, R_1, ρ_1. Moreover, $\lim_{|y| \to \infty}(|h_1|(\theta, x, y)| + |h_2(\theta, x, y)|) = 0$, *where δ_1, R_1, ρ_1 are some positive numbers.*

Proof. If we treat ϕ, v as complex variables, (3.4) is an analytical equation in some complex domain $D_{\rho R \delta}$. And thus $X(\theta, x, y)$, $Y(\theta, x, y)$ are analytical in a slightly smaller domain $D_{\rho_1 R_1 \delta_1}$. In view of (3.6), we know that $h_1(\theta, x, y)$, $h_2(\theta, x, y)$ are analytical in $D_{\rho_1 R_1 \delta_1}$. The estimates for $h_1(\theta, x, y)$, $h_2(\theta, x, y)$ are obvious from (3.6).

Proof of Theorem 1. In view of Lemma 3.1 and Lemma 3.2, it follows that there is a R^*, ρ^*, δ^*, such that system (3.3) satisfies all the assumptions of Theorem 2.1 in $D_{\rho^* R^* \delta^*}$. It follows that (3.3) possesses an invariant torus T^{1+m} composed of quasiperiodic solutions of (3.3) with frequencies $(\lambda, \omega_1, \ldots, \omega_m)$ satisfying the Diophantine condition (2.7). The solution on T^{1+m} has the form $x(t) = \lambda t + U(\lambda t, \omega_1 t, \ldots, \omega_m t)$, $y(t) = \lambda + V(\lambda t, \omega_1 t, \ldots, \omega_m t)$, $\theta(t) = \omega t$. Then $\Psi_{-1} \circ \Psi_2 \circ T^{1+m} \circ \Psi_2^{-1} \circ \psi_1$ is an invariant torus of (1.5) in $(\theta, \phi, \dot{\phi})$-space which carries quasiperiodic solutions of (1.5) with the same frequency.

The boundedness of solutions is proved as the following: We know that the autonomous system (3.2) possesses a series of invariant tori of the form $\Psi_2 \circ T_\lambda^{1+m} \circ \Psi_2^{-1}$ in phase cylinder $(\theta, \phi, v) \in T^m \times T^1 \times R^1$, which tend to both ends of the cylinder. It yields a bound of the interior solutions. For a detailed proof, we refer to [DZ, Y].

EXAMPLE 2. Consider the reversible Duffing equation

$$\frac{d^2x}{dt^2} + x^{2n+1} + \sum_{l=1}^{2n} p_l(\theta_1, \dots, \theta_m)x^l = 0, \tag{3.8}$$

where $p_l(\theta)$ are periodic in θ with period 1, and $p_l(\theta)$ are odd functions when l are even while p_l are even functions when l are odd.

Equation (3.8) is equivalent to the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x^{2n+1} - \sum_{l=1}^{2n} p_l(\theta_1, \dots, \theta_m)x^l, \qquad \frac{d\theta}{dt} = \omega, \tag{3.9}$$

which has been considered by Dieckerhoff and Zehnder [DZ], Liu [Li], and Laederich and Levi [LL] for the case $m = 1$. The main idea is to introduce an iterate sequence of finitely many transformations of $R^+ \times S^1$ such that the system (3.9) is transformed into an integrable Hamiltonian system with small reversible perturbations, and thus the KAM theorem can be used.

Following [DZ, Li], we first consider an autonomous system

$$\dot{x} = y, \qquad \dot{y} = -x^{2n+1}, \tag{3.10}$$

which is a time-independent Hamiltonian system with Hamiltonian

$$h(x, y) = \frac{1}{2}y^2 + \frac{1}{2n+2}x^{2n+2}.$$

Clearly, h is positive on R^2 except at the unique equilibrium point $(x, y) = (0, 0)$ where $h = 0$. Note that $h(x, y) = E$ is a first integral of the system (2.3); hence, all the solutions of (3.10) are periodic with period tending to zero as E tends to infinity.

Suppose that $(S(t), C(t))$ is the solution of (3.10) satisfying the initial condition:

$$(S(0), C(0)) = (0, 1).$$

Let $T_0 > 0$ be its minimal period.

From the system (3.10), we can find that $S(t)$ and $C(t)$ satisfy

(i) $S(t), C(t) \in C^\infty(R^1)$, $S(t + T_0) = S(t)$, $C(t + T_0) = C(t)$ with $S(0) = 0$, $C(0) = 1$;

(ii) $(d/dt)S(t) = C(t)$, $(d/dt)C(t) = -S^{2n+1}(t)$;

(iii) $\frac{1}{2} C^2(t) + (1/(2n + 2))S^{2n+2}(t) = \frac{1}{2}$;

(iv) $C(-t) = C(t)$, $S(-t) = -S(t)$.

The action and angle variables are now defined by the mapping Ψ: $R^+ \times S^1 \to R^2/\{0\}$; $(\rho, \theta) \mapsto (x, y)$, where $\rho > 0$ and θ (mod2π) is given by

$$\Psi: x = a^\alpha \rho^\alpha S\left(\frac{\phi T_0}{2\pi}\right), \qquad y = a^\beta \rho^\beta C\left(\frac{\phi T_0}{2\pi}\right),$$

where

$$\alpha = \frac{1}{n+2}, \qquad \beta = 1 - \alpha, \qquad a = \frac{2\pi}{\beta T_0}.$$

In the new coordinates, the system (3.9) becomes

$$\frac{d\rho}{dt} = \frac{\partial h}{\partial \phi}(\theta, \phi, \rho), \qquad \frac{d\phi}{dt} = \beta a^{2\beta}\rho^{2\beta-1} + \frac{\partial h}{\partial \rho}(\theta, \phi, \rho), \qquad \frac{d\theta}{dt} = \omega, \quad (3.11)$$

where $h(\theta, \phi, \rho) = \sum_{j=1}^{2n+1} a^{\alpha j} S^j(\phi T_0/2\pi) p_{j-1}(\theta)\rho^{\alpha j}$. It is easy to see that (3.11) is a reversible system with respect to Λ.

LEMMA 3.3. *There is a diffeomorphism Ψ_1 in (θ, ϕ, ρ) space $T^{m+1} \times R^+$, which transforms Eq.* (3.11) *for large ρ into*

$$\frac{d\psi}{dt} = \mu + h_1(\theta, \psi, \mu), \qquad \frac{d\mu}{dt} = h_2(\theta, \psi, \mu), \qquad \frac{d\theta}{dt} = \omega, \qquad (3.12)$$

where h_1, h_2 satisfy

$$\lim_{\mu \to +\infty} (|f_1|_{C^\infty} + |f_2|_{C^\infty}) = 0.$$

Moreover, $h_1(-\theta, -\psi, \mu) = -h_1(\theta, \psi, \mu)$, $h_2(-\theta, -\psi, \mu) = -h_2(\theta, \psi, \mu)$.

Proof. See [DZ, Li].

Theorem 2 is proved similarly as Theorem 1 by applying Theorem 2.1. We omit the details here.

We remark here that Theorem 2.1 can also be established similarly for the Hamiltonian system; the oddness and evenness restriction on $p_l(\theta)$ in Example 2 is not essential.

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