

THE EXISTENCE OF INTEGRABLE INVARIANT MANIFOLDS OF HAMILTONIAN PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this note, it is shown that some Hamiltonian partial differential equations such as semi-linear Schrödinger equations, semi-linear wave equations and semi-linear beam equations are partial integrable, i.e., they possess integrable invariant manifolds foliated by invariant tori which carry periodic or quasi-periodic solutions. The linear stability of the obtained invariant manifolds is also concluded. The proofs are based on a special invariant property of the considered equations and a symplectic change of variables firstly observed in [26].

1. Introduction. Hamiltonian partial differential equations have strong backgrounds in mathematical physics. Many famous equations, such as Schrödinger equations, wave equations and KdV equations, have Hamiltonian structures and thus can be regarded as Hamiltonian systems. In some special cases, they are integrable and exhibit a rich structure of periodic, quasi-periodic, almost periodic or soliton solutions. Research along this line has formed a big branch in mathematical physics. Unfortunately, most of the equations are not integrable, for example the semi-linear Schrödinger equation with periodic boundary conditions

$$iu_t - \Delta u + mu + f(|u|^2)u = 0, \quad x \in \mathbb{T}^d, \quad (1.1)$$

if $f(x) \neq x$. However, it defines an infinite dimensional Hamiltonian dynamical system in some functional spaces (e.g., the Sobolev space $H^r(\mathbb{T}^d)$ on torus for some r). And in a neighborhood of the trivial solution $u = 0$, equation (1.1) can be regarded as a perturbation of the linear equation $iu_t - \Delta u + mu = 0$, thus it is a nearly integrable Hamiltonian system. In recent years, the local dynamics in a neighborhood of $u = 0$, especially the existence of periodic and quasi-periodic solutions of small amplitude for such kind of Hamiltonian PDEs, has received broad attention.

So far there are two main approaches to deal with the periodic and quasi-periodic solutions of Hamiltonian PDEs. The first one is the infinite dimensional KAM theory which is the extension of the classical KAM theory, see Wayne [25], Kuksin [20],

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Pöschel [22], Chierchia and You [12], Geng and You [14]. The second approach is the Craig-Wayne-Bourgain method. It is a generalization of the Liapunov-Schmidt reduction and the Newtonian method. The reader is referred to Craig-Wayne [13], Bourgain [6, 7, 8, 9].

Concerning the periodic solutions, there are also some recent results by Berti, and Bolle [4, 5], Gentile and Mastropietro [15, 16], Gentile and Procesi [17]. All the work follows the main line of Craig-Wayne-Bourgain's approach. In [4, 5], the Q -equations are solved by variational methods, while in [15, 16, 17] the P -equations are solved by the Lindstedt series method. We remark that the small divisor is the key obstacle in all the approaches mentioned above.

When constructing some special kind of periodic solutions, the small divisor problem does not appear. In [21], the existence of a two-dimensional disc foliated by periodic solutions of a one-dimensional Schrödinger equation was shown. In the proof (Appendix 3 of [21]), the special form of the nonlinear term $f(|u|^2)u$ in (1.1) plays an important role. When constructing some kind of periodic solutions of the semi-linear wave equations Bambusi [1] noticed that the Craig-Wayne-Bourgain's approach does not really involve the small divisor problem. Thus the periodic solutions can be easily constructed by the implicit function theorem. Using also the Craig-Wayne-Bourgain's approach, Yuan [27] further constructed some running wave-like quasi-periodic solutions for nonlinear Schrödinger equations. Also, in his proof, there is no small divisor problem.

In this note, we prove that some Hamiltonian PDEs such as the semi-linear Schrödinger equations mentioned above, semi-linear wave equations and semi-linear beam equations actually have a lot of $2d$ -dimensional invariant manifolds. And the partial differential equations restricted to such invariant manifolds are integrable. Moreover, the manifolds are foliated by invariant tori. The solutions on each torus are either periodic or quasi-periodic. Our proof is completely different from that of the above mentioned papers and much simpler. A special symmetry property of the considered equations and a symplectic change of variables firstly observed by Xu-You in [26] play key roles on our proof.

To the authors' knowledge, there is no result on the partial integrability for non-integrable partial differential equations, i.e., on the existence of integrable manifolds.

2. The existence of invariant integrable manifolds. In this section, we will prove an existence theorem of invariant manifolds of infinite dimensional Hamiltonian systems. Moreover, the systems are integrable when restricted on the invariant manifold. This theorem can be applied to Schrödinger equations, wave equations and beam equations.

Let $\mathbb{Z}_1^d = \mathbb{Z}^d \setminus \{e_1, \dots, e_d\}$, where e_i 's are vectors in \mathbb{Z}^d with the i -th component 1 and other components zero. $z = (z_n)$ is a vector of infinite dimension indexed by $n \in \mathbb{Z}_1^d$. We introduce the following weighted norm for z ,

$$\|z\|_{a,\rho} = \sum_{n \in \mathbb{Z}_1^d} |z_n| |n|^a e^{|n|\rho},$$

$$\|\bar{z}\|_{a,\rho} = \sum_{n \in \mathbb{Z}_1^d} |\bar{z}_n| |n|^a e^{|n|\rho},$$

where $|n| = \sqrt{n_1^2 + \dots + n_d^2}$, $n = (n_1, \dots, n_d)$ and $a \geq 0$, $\rho \geq 0$. In order to get smooth solutions, we assume that a is sufficiently large when $\rho = 0$. Let $l_{a,\rho}$ be the space of (z, \bar{z}) with $\|z\|_{a,\rho}, \|\bar{z}\|_{a,\rho}$ finite, and denote the space $\mathbb{T}^d \times \mathbb{R}^d \times l_{a,\rho} \times l_{a,\rho}$

by $\mathcal{P}_{a,\rho}$, which is a Banach space and endowed with the symplectic structure $dI \wedge d\theta + i \sum_n dz_n \wedge d\bar{z}_n$, and thus a symplectic space.

Consider an infinite dimensional Hamiltonian system defined on $\mathcal{P}_{a,\rho}$ with Hamiltonian

$$H = \langle \omega, I \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n + P(\theta, I, z, \bar{z}). \quad (2.1)$$

We assume that $\nabla P \equiv (\frac{\partial P}{\partial \theta}, \frac{\partial P}{\partial I}, \frac{\partial P}{\partial z_n}, \frac{\partial P}{\partial \bar{z}_n})$ defines a map from $\mathcal{P}_{a,\rho}$ into itself. Moreover, we assume that P satisfies the *Symmetry Property*:

$$P(\theta, I - \sum_{n \in \mathbb{Z}_1^d} \langle n, \omega \rangle z_n \bar{z}_n, e^{-i\langle n, \theta \rangle} z_n, e^{i\langle n, \theta \rangle} \bar{z}_n)$$

is independent of θ . We denote $P(\theta, I - \sum_{n \in \mathbb{Z}_1^d} \langle n, \omega \rangle z_n \bar{z}_n, e^{-i\langle n, \theta \rangle} z_n, e^{i\langle n, \theta \rangle} \bar{z}_n)$ by

$$P_1(I, z, \bar{z}).$$

Remark Later, we will see that this symmetry property is satisfied by Hamiltonian PDEs that do not explicitly contain the space variables.

Theorem 1. *Suppose that for fixed $I = I_0$, $\sum_{n \in \mathbb{Z}_1^d} (\Omega_n - \langle n, \omega \rangle) z_n \bar{z}_n + P_1(I_0, z, \bar{z})$*

has a critical point, then (2.1) has an invariant torus which carries periodic or quasi-periodic solutions.

In order to prove Theorem 1, we need the following lemma.

Lemma 2.1. *For any $\{k_n, n \in \mathbb{Z}_1^d\} \subset \mathbb{Z}^d$ with¹ $|k_n| < C|n|$, the map $\Psi : (\theta, I, z, \bar{z}) \in \mathcal{P}_{a,\rho} \rightarrow (x, y, w, \bar{w}) \in \mathcal{P}_{a,\rho}$ defined by*

$$\left\{ \begin{array}{l} x = \theta \\ y = I - \sum_{n \in \mathbb{Z}_1^d} z_n \bar{z}_n k_n \\ w_n = e^{-i\langle k_n, \theta \rangle} z_n \\ \bar{w}_n = e^{i\langle k_n, \theta \rangle} \bar{z}_n \end{array} \right. \quad (2.2)$$

is symplectic, where i stands for $\sqrt{-1}$.

The proof of the lemma is straightforward. The reader is referred to Xu-You [26].

Proof of Theorem 1. Due to the symmetry property, the symplectic change of variables Ψ with $k_n = n$, transforms (2.1) into the following

$$H_1 = \langle \omega, I \rangle + \sum_{n \in \mathbb{Z}_1^d} (\Omega_n - \langle n, \omega \rangle) z_n \bar{z}_n + P_1(I, z, \bar{z}). \quad (2.3)$$

¹ $|k_n| < C|n|$ is assumed to guarantee the convergence of $\sum_{n \in \mathbb{Z}_1^d} z_n \bar{z}_n k_n$. It is not essential.

The corresponding Hamiltonian system is

$$\begin{aligned} \frac{d\theta}{dt} &= \omega + \frac{\partial P_1(I_0, z, \bar{z})}{\partial I} \\ \frac{dI}{dt} &= 0, \\ \frac{dz_n}{dt} &= i[(\Omega_n - \langle n, \omega \rangle)z_n + \frac{\partial P_1(I_0, z, \bar{z})}{\partial \bar{z}_n}], \quad n \neq e_1, \dots, e_d, \\ \frac{d\bar{z}_n}{dt} &= -i[(\Omega_n - \langle n, \omega \rangle)\bar{z}_n + \frac{\partial P_1(I_0, z, \bar{z})}{\partial z_n}], \quad n \neq e_1, \dots, e_d. \end{aligned} \quad (2.4)$$

If, for $I = I_0$, $P(I_0, z, \bar{z})$ has a critical point $z = z(I_0)$, $\bar{z} = \bar{z}(I_0)$, then the Hamiltonian system (2.4) has an invariant torus $\mathbb{T} = \mathbb{T}^d \times \{I_0\} \times \{z(I_0)\} \times \{\bar{z}(I_0)\}$. The frequency vector of the solutions on the invariant torus is $\tilde{\omega} = \omega + \frac{\partial P_1}{\partial I}|_{I=I_0, z=z(I_0), \bar{z}=\bar{z}(I_0)}$. The solutions are periodic if all components of $\tilde{\omega}$ are rationally dependent. Otherwise, the solutions are quasi-periodic. If all $\Omega_n - \langle n, \omega \rangle$ are positive, then (2.3) is positive definite while treating I as a parameter. It follows that the invariant manifolds are stable in the sense of Lyapunov. It is obvious that $\Psi(\mathbb{T})$ is an invariant torus of (2.1).

The following result is an immediate consequence of Theorem 1.

Theorem 2. *Suppose that $|\Omega_n - \langle n, \omega \rangle|$ has a lower bound $C > 0$ (independent of n) for all $n \in \mathbb{Z}_1^d$. Then (2.3) is integrable on a $2d$ -dimensional invariant manifold \mathcal{M}_{2d} , i.e., \mathcal{M}_{2d} is foliated by d -dimensional invariant tori and solutions on each torus are periodic or quasi-periodic. If all $\Omega_n - \langle n, \omega \rangle$ are positive, then the invariant manifolds are stable in the sense of Lyapunov.*

3. Applications. Theorem 2 can be applied to various x, t independent Hamiltonian PDEs. Since the proofs are similar, we only give the full proof in the case of Schrödinger equations in this note. The analogous results for beam equations and wave equations are presented without proof.

3.1. Higher dimensional Schrödinger equations. We consider the d -dimensional Schrödinger equations with periodic boundary condition

$$iu_t + Au + f(|u|^2)u = 0, \quad Au = -\Delta u + mu, \quad x \in \mathbb{T}^d, \quad t \in \mathbb{R}. \quad (3.1)$$

For simplicity, we assume that f is analytic in a neighborhood of $0 \in \mathbb{C}$ and vanishes at zero. Actually, f only needs to be C^2 in this paper.

Equation (3.1) can be rewritten as Hamiltonian equation

$$u_t = i \frac{\partial H}{\partial \bar{u}} \quad (3.2)$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}(Au, u) + \int_{\mathbb{T}^d} g(|u|^2) dx, \quad (3.3)$$

where (\cdot, \cdot) denotes the inner product in L^2 and g is a primitive of f .

The eigenvalues of A in $L^2(\mathbb{T}^d)$ are $\{\mu_n = |n|^2 + m\}$. The corresponding eigenfunctions $\phi_n(x) = \sqrt{\frac{1}{(2\pi)^d}} e^{i\langle n, x \rangle}$ form a complete orthogonal basis of $L^2(\mathbb{T}^d)$.

Lemma 3.1. $u(t, x) = \sum_{n \in \mathbb{Z}^d} q_n(t) \phi_n(x)$ is a formal solution of (3.1) if and only if $q = (q_n(t))$ is a solution of the lattice Hamiltonian equations

$$\dot{q}_n = i(\mu_n q_n + \frac{\partial G}{\partial \bar{q}_n}), \quad (3.4)$$

with the Hamiltonian

$$H = \sum_{n \in \mathbb{Z}^d} \mu_n q_n \bar{q}_n + G = \sum_{n \in \mathbb{Z}^d} \mu_n q_n \bar{q}_n + \int_{\mathbb{T}^d} g(|u|^2) dx. \quad (3.5)$$

Moreover, the solution $q = (q_n(t))$ of (3.4) with $\sum_{n \in \mathbb{Z}^d} |q_n(t)| |n|^r < \infty$ corresponds to a solution of (3.1) in $H^r(\mathbb{T}^d)$.

Let Λ be the set of vectors of infinite dimension indexed by $n \in \mathbb{Z}_+^d$ with finite number of nonzero components in the set of nonnegative integers \mathbb{Z}_+ . For $\alpha = (\alpha_n) \in \Lambda$, we denote $\prod_{n \in \mathbb{Z}_+^d} q_n^{\alpha_n}$ by q^α .

Lemma 3.2. Let $G \equiv \int_{\mathbb{T}^d} g(|u|^2) dx = \sum_{\alpha, \beta \in \Lambda} G_{\alpha\beta} q^\alpha \bar{q}^\beta$, then the coefficients $G_{\alpha\beta}$ satisfy

$$G_{\alpha\beta} = 0 \quad \text{if} \quad \sum_{n \in \mathbb{Z}^d} (\alpha_n - \beta_n) n \neq 0, \quad (3.6)$$

where α_n and β_n are components of α and β respectively.

To prove this lemma, firstly, we observe that $g(q, \bar{q})$ is real analytic in q, \bar{q} since $f(u)$ is real analytic in u . Then making use of $u(t, x) = \sum_{n \in \mathbb{Z}^d} q_n \phi_n(x)$, we may rewrite g as follows

$$g(|u|^2) = \sum_{\alpha, \beta \in \Lambda} g_{\alpha\beta} q^\alpha \bar{q}^\beta \phi^\alpha \bar{\phi}^\beta,$$

hence

$$\begin{aligned} G(q, \bar{q}) &\equiv \int_{\mathbb{T}^d} g(|\sum_{n \in \mathbb{Z}^d} q_n \phi_n|^2) dx = \sum_{\alpha, \beta \in \Lambda} G_{\alpha\beta} q^\alpha \bar{q}^\beta, \\ \text{so } G_{\alpha\beta} &= 0, \quad \text{if} \quad \sum_{n \in \mathbb{Z}^d} (\alpha_n - \beta_n) n \neq 0. \end{aligned} \quad (3.7)$$

As in [21, 22, 12, 14], the perturbation G in (3.5) has the following regularity property.

Lemma 3.3. For any fixed $a \geq 0, \rho > 0$, the gradient $G_{\bar{q}}$ is real analytic as a map in a neighborhood of the origin with

$$\|G_{\bar{q}}\|_{a, \rho} \leq c \|q\|_{a, \rho}^2. \quad (3.8)$$

Next we introduce the standard action-angle variables $(\theta, I) = ((\theta_1, \dots, \theta_d), (I_1, \dots, I_d))$ in the $(q_{e_1}, \dots, q_{e_d}, \bar{q}_{e_1}, \dots, \bar{q}_{e_d})$ -space for (3.5) by setting ²

$$q_{e_j} = \sqrt{2I_j} e^{i\theta_j} = \sqrt{2I_j} e^{i\langle e_j, \theta \rangle}, \quad \bar{q}_{e_j} = \sqrt{2I_j} e^{-i\theta_j} = \sqrt{2I_j} e^{-i\langle e_j, \theta \rangle}, \quad j = 1, \dots, d,$$

²At this stage, one can see that the tangential sites are not necessary to be fixed as $\{e_1, \dots, e_d\}$, they could be any d independent vectors $\{v_1, \dots, v_d\}$. In this case, we let $q_i = f_i(I) e^{i\langle v_i, \theta \rangle}$, $i = 1, \dots, d$, where f_i is chosen so that the change of variables is symplectic.

and setting $q_n = z_n$, $\bar{q}_n = \bar{z}_n$, $n \neq e_1, \dots, e_d$, let $\mathbb{Z}_1^d = \mathbb{Z}^d \setminus \{e_1, \dots, e_d\}$, so that the system (3.4) becomes

$$\begin{aligned} \frac{d\theta_j}{dt} &= \omega_j + \frac{\partial P}{\partial I_j}, & \frac{dI_j}{dt} &= -\frac{\partial P}{\partial \theta_j}, & j &= 1, \dots, d, \\ \frac{dz_n}{dt} &= -i(\Omega_n z_n + \frac{\partial P}{\partial \bar{z}_n}), & \frac{d\bar{z}_n}{dt} &= i(\Omega_n \bar{z}_n + \frac{\partial P}{\partial z_n}), & n &\in \mathbb{Z}_1^d, \end{aligned} \quad (3.9)$$

where P is just G with the $(q_{e_1}, \dots, q_{e_d}, \bar{q}_{e_1}, \dots, \bar{q}_{e_d}, q_n, \bar{q}_n)$ -variables expressed in terms of the $(\theta, I, z_n, \bar{z}_n)$ variables. The Hamiltonian associated to (3.9) (with respect to the symplectic structure $dI \wedge d\theta + i \sum_{n \in \mathbb{Z}_1^d} dz_n \wedge d\bar{z}_n$) is given by

$$H = \langle \omega(m), I \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(m) z_n \bar{z}_n + P(\theta, I, z, \bar{z}), \quad (3.10)$$

with $\omega(m) = (1 + m, \dots, 1 + m)$, $\Omega_n(m) = |n|^2 + m$, $n \in \mathbb{Z}^d \setminus \{e_1, \dots, e_d\}$. Let

$$P(\theta, I, z, \bar{z}) = \sum_{k \in \mathbb{Z}^d, l \in \mathbb{Z}_+^d, \alpha, \beta \in \Lambda} P_{kl\alpha\beta} I^l e^{i\langle k, \theta \rangle} z^\alpha \bar{z}^\beta.$$

By (3.7), it is easy to verify that

$$P_{kl\alpha\beta} = 0, \quad \text{if } k + \sum_{n \in \mathbb{Z}_1^d} (\alpha_n - \beta_n) n \neq 0. \quad (3.11)$$

It follows that

$$P(\theta, I - \sum_{n \in \mathbb{Z}_1^d} \langle n, \omega \rangle z_n \bar{z}_n, e^{i\langle n, \theta \rangle} z_n, e^{-i\langle n, \theta \rangle} \bar{z}_n)$$

is independent of θ . We denote it by $P_1(I, z, \bar{z}, m)$.

By the change of variables Ψ in Lemma 2.1, we arrive at a Hamiltonian as follows

$$H = \langle \omega(m), I \rangle + \sum_{n \in \mathbb{Z}_1^d} \tilde{\Omega}_n(m) z_n \bar{z}_n + P_1(I, z, \bar{z}, m), \quad (3.12)$$

where $\omega = (1 + m, \dots, 1 + m)$, $\tilde{\Omega}_n(m) = |n|^2 + m - (1 + m)(\sum_{i=1}^d n_i)$. For m in a bounded interval, $\tilde{\Omega}_n$ is positive if n is sufficiently large. So there are only finite number of m_i 's such that $\tilde{\Omega}_n = 0$. Excluding the set of such m 's from the interval, we have $|\tilde{\Omega}_n| > C(m) > 0$.

Now we are ready to apply Theorem 2 to get the following result.

Theorem 3. *Except for a finite set of m 's, (3.1) has a $2d$ -dimensional invariant manifold \mathcal{M} foliated by invariant tori. The solutions of each torus are periodic or quasi-periodic of the form $u(x_1 - \omega_1 t, \dots, x_d - \omega_d t)$. If all $\tilde{\Omega}_n$ are positive, the obtained torus is linearly stable, otherwise it is unstable.*

Remark The invariant manifold we constructed is local. The size of the manifolds are not uniform, but dependent on m .

Remark The nonlinear term could be as more general as $f(u, \bar{u})$.

Remark Since e_1, \dots, e_d can be replaced by any independent vectors v_1, \dots, v_d in \mathbb{Z}^d , one can prove that the equation can be integrable on infinitely many $2d$ -dimensional invariant manifolds by this method. We don't know whether or not the equation is integrable on any higher dimensional manifolds.

Remark With a result of Bourgain on periodic solutions of Schrodinger equations, one can further prove the existence of quasi-periodic solutions of (1.1) with $d + 1$ frequencies by constructing periodic solutions of (3.5).

3.2. Beam equations and wave equations. In this subsection, we state similar results for beam equations and wave equations. The proofs are omitted since they proceed along the same line as that of the Schrodinger equations.

Consider dD beam equations

$$u_{tt} + (-\Delta + m)^2 u + f(u) = 0, \quad x \in \mathbb{T}^d, t \in \mathbb{R}, \quad (3.13)$$

where $f(u)$ is a real-analytic function near $u = 0$ with $f(0) = f'(0) = 0$.

Theorem 4. *Except for a finite set of m 's, there is a 2d-dimensional manifold in $H^r(\mathbb{T}^d)$ such that the restriction of the beam equation on this manifold is integrable. The solutions on the manifold are periodic or quasi-periodic of the form $u(x_1 - \omega_1 t, \dots, x_d - \omega_d t)$.*

For wave equations

$$u_{tt} - \Delta u + mu + f(u) = 0, \quad x \in \mathbb{T}^d, t \in \mathbb{R}, \quad (3.14)$$

where $f(u)$ is a real-analytic function near $u = 0$ with $f(0) = f'(0) = 0$, we have the following result.

Theorem 5. *Suppose that the parameter $m \in (-1, 0)$. Then there is a 2d-dimensional manifold in $H^r(\mathbb{T}^d)$ such that the restriction of the wave equation on this manifold is integrable. The solutions on the manifold are periodic or quasi-periodic of the form $u(x_1 - \omega_1 t, \dots, x_d - \omega_d t)$.*

Remark The restriction on m is to guarantee the assumptions of Theorem 2 to be satisfied. This restriction is not essential for the existence of periodic or quasi-periodic solutions, but we do not know if it is essential for the existence of integrable sub-manifolds.

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