Functinal Analysis II

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摘要

This course is a continuation study of the basic functional analysis. In this course, along with the theorems and principles in functional analysis, we will introduce their applications, particularly in differential equations. The materials of this lecture notes are mainly summarized from the book [8], [9]. Some of the concepts in [4], [10] and [2] are used. This lecture notes can only be used for nonprofitable purpose.

目录

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1 Banach spaces and fixed-point theorems

1.1 Topological spaces

We give the general definition of topological spaces, open sets and continuous mappings.

Definition 1.1. *(a) Let X be a set and 2^X be the collection of all subsets of X. A collection of X subsets $\tau \subset 2^X$ is said to be a* topology *in X if τ has the following properties:*

(i) $\emptyset \in \tau$ *and* $X \in \tau$.

(ii) If $V_j \in \tau,\ j = 1, \cdots, N$, *then* $V_1 \cap V_2 \cap \cdots \cap V_N \in \tau$.

(iii) If $\{V_\alpha\}_{\alpha \in I}$ *is an arbitrary collection of members of* τ *(finite, countable, or uncountable), then* $\bigcup_{\alpha \in I} V_\alpha \in \tau$.

(b) If τ *is a topology in* X, *then* (X, τ) *is called a topological space, and the members of* τ *are called the open sets in* X.

(c) A set $E \subset X$ *is closed if its complement* E^c *is open. Hence* $\emptyset$ *and* X *are closed, finite unions of closed sets are closed, and arbitrary intersections of closed sets are closed.*

(d) If X *and* Y *are topological spaces and if* f *is a mapping of* X *into* Y, *then* f *is said to be continuous provided that* $f^{-1}(V)$ *is an open set in* X *for every open set* V *in* Y. *It can be shown that* f *is continuous iff* $f^{-1}(B)$ *is a closed set in* X *for each closed set* B *in* Y.

(e) The closure $\bar{E}$ *of a set* $E \subset X$ *is the smallest closed set in* X *which contains* E. *(The following argument proves the existence of* E *: The collection* Q *of all closed subsets of* X *which contain* E *is not empty, since* $X \in Q$*; let* $\bar{E}$ *be the intersection of all members of* Q.*)*

(f) A set $K \subset X$ *is compact if every open cover of* K *contains a finite subcover. In particular, if* X *is itself compact, then* X *is called a compact space.*

(g) Let $p \in X$. *A neighborhood of* p *is an open set containing* p.

(h) X *is a* Hausdorff space *if the following is true : If* $p \in X$, $q \in X$, *and* $p \neq q$, *then* p *has a neighborhood* U *and* q *has a neighborhood* V *such that* $U \cap V = 0$.

(i) X *is* locally compact *if every point of* X *has a neighborhood whose closure is compact.*

(j) Heine-Borel Theorem: The compact subsets of the euclidean space $\mathbb{R}^d$ *are precisely those that are closed and bounded.*

(k) From this it follows easily that $\mathbb{R}^d$ *is a locally compact Hausdorff space. Also, every metric space is a Hausdorff space.*

Remark 1.2. *To avoid terminology complexisty, when there is not much confusion in the context, we often simply say* X *is a topological space without emphasizing its topology* τ.

1.2 Metric spaces

Definition 1.3. *(a) A metric space is an ordered pair* (M, d) *where* M *is a set and* d *is a metric on* M, *i.e., a function* $d \colon M \times M \to \mathbb{R}$ *such that for any* $x, y, z \in M$, *the following holds:*

- $d(x, y) \geq 0$ *non-negativity*
- $d(x, y) = 0 \Leftrightarrow x = y$ *identity of indiscernibles*
- $d(x, y) = d(y, x)$ *symmetry*
- $d(x, z) \leq d(x, y) + d(y, z)$ *subadditivity or triangle inequality*

(b) *Every metric space is a topological space in a natural manner. Let (M, d) be a metric space. For each $x \in M$, we define the open ball of radius $r > 0$ about x as the set*

$$B(x, r) = \{y \in M : d(x, y) < r\}.$$

The notation $B_r(x)$ for such a ball is also often used.

These open balls form the base for a topology on M, making it a topological space. Explicitly, a subset U of M is called open if for every x in U there exists an $r > 0$ such that $B(x, r)$ is contained in U. The complement of an open set is called closed.

(c) *A topological space which can arise in this way from a metric space is called a metrizable space.*

(d) *A sequence (x_n) in a metric space M is said to converge to the limit $x \in M$ iff for every $\varepsilon > 0$, there exists a natural number N such that $d(x_n, x) < \varepsilon$ for all $n > N$.*

(e) *A subset A of the metric space M is closed iff every sequence in A that converges to a limit in M has its limit in A.*

1.3 Banach spaces

We give the definition of Banach spaces and some examples. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, where $\mathbb{R}$ and $\mathbb{C}$ are the set of real and complex numbers respectively.

Definition 1.4. (a) *Let X be a vector space over $\mathbb{F}$. A seminorm on X is a function $p : X \to [0, +\infty)$ having the properties:*

(i) $p(x + y) \leq p(x) + p(y)$ *for all* $x, y \in X$.

(ii) $p(\alpha x) = |\alpha| p(x)$ *for all* $\alpha \in \mathbb{F}$ *and* $x \in X$.

(b) *It follows from (ii) that $p(0) = 0$. A norm is a seminorm p such that $x = 0$ if $p(x) = 0$. Usually a norm is denoted by $\| \cdot \|$.*

(c) *A normed space is a pair $(X, \| \cdot \|)$, where X is a vector space and $\| \cdot \|$ is a norm on X.*

(d) *A normed space $(X, \| \cdot \|)$ is a metric space with the natural metric defined by the norm: $d(x, y) := \|x - y\|$, for all $x, y \in X$.*

(e) *A Banach space is a normed space that is complete with respect to the metric defined by the norm.*

Example 1.5. *Banach spaces:* $C[a,b]$, $L^p(\Omega)$.

1.4 Open and closed sets in normed spaces

Definition 1.6. (a) *Let X be a normed space. For fixed $u_0 \in X$ and $r > 0$, denote the ball*

$$B(u_0, r) := \{u \in X : \|u - u_0\| < r\}.$$

(b) *A subset M of X is called open iff, for each point $u_0 \in M$, there is a ball $B(u_0, r)$ such that $B(u_0, r) \subset M$.*

(c) *A subset M of X is called closed iff $M^c = X \setminus M := \{u \in X : u \notin M\}$ is open.*

Proposition 1.7. *Let X be a normed space and $M \subset X$. Then M is closed iff for any sequence $\{u_n\} \subset M$ with $u_n \to u$ in X, there holds $u \in M$.*

证明. 略。练习。 □

1.5 The Banach fixed-point theorem—contraction principle

Definition 1.8. *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F}$ and M be a closed nonempty subset of X. A map $A : M \to M$ is said to be λ-contractive provided*

$$\|Au - Av\| \le \lambda \|u - v\|, \quad \textit{for all } u, v \in M.$$

Such a map A on M is said to be a λ-contraction or a λ-contractive operator.

Theorem 1.9. *Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{F}$ and M be a closed nonempty subset of X. Suppose $A : M \to M$ is λ-contractive with $0 \le \lambda < 1$, then the operator A admits a unique fixed point in M, i.e. there exists a unique $u \in M$ such that $Au = u$. Moreover, for any $u_0 \in M$, the sequence $\{u_n\}$ constructed by $u_{n+1} = Au_n$, $n = 0, 1, 2, \cdots$ converges to the unique fixed point u. In addition the iteration sequence $\{u_n\}$ satisfies the following estimates:*

- *Error estimates. For all $n = 0, 1, \cdots$ we have the so-called a priori estimate*

$$\|u_n - u\| \le \lambda^n (1 - \lambda)^{-1} \|u_1 - u_0\|, \tag{1.1}$$

and the so-called a posteriori estimate

$$\|u_{n+1} - u\| \le \lambda (1 - \lambda)^{-1} \|u_{n+1} - u_n\|. \tag{1.2}$$

- *Rate of convergence. For all all $n = 0, 1, \cdots$ we have*

$$\|u_{n+1} - u\| \le \lambda \|u_n - u\|. \tag{1.3}$$

This theorem was proved by Banach in 1920, and is called the Banach fixed-point theorem, and is also called the contraction principle. The phase *a priori* means *from the earlier* and *a posteriori* means *from the later*.

证明. For any $n \in \mathbb{Z}_+$,

$$\|u_{n+1} - u_n\| = \|Au_n - Au_{n-1}\| \le \lambda \|u_n - u_{n-1}\| \le \cdots \le \lambda^n \|u_1 - u_0\|. \tag{1.4}$$

Then for any $n, m \in \mathbb{Z}_+$,

$$\begin{aligned} \|u_n - u_{n+m}\| &\le \|u_n - u_{n+1}\| + \|u_{n+1} - u_{n+2}\| + \cdots + \|u_{n+m-1} - u_{n+m}\| \\ &\le (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+m-1}) \|u_1 - u_0\| \\ &\le \lambda^n \sum_{k=1}^{\infty} \lambda^k \|u_1 - u_0\| \\ &= \lambda^n (1-\lambda)^{-1} \|u_1 - u_0\|. \end{aligned} \tag{1.5}$$

It follows from $0 \le \lambda < 1$ that $\lambda^n \to 0$ as $n \to \infty$. Hence the sequence $\{u_n\}$ is Cauchy. Since X is a Banach space which is complete, then the Cauchy sequence $\{u_n\}$ converges, i.e.

$$\lim_{n \to \infty} u_n \to u \text{ in } X.$$

Since A maps M into M, we have $\{u_n\} \subset M$. Since M is closed, we have the limit $u \in M$. Moreover,

$$\|u_{n+1} - Au\| = \|Au_n - Au\| \le \lambda \|u_n - u\|. \tag{1.6}$$

Passing $n \to \infty$ in (1.6) implies

$$u = Au. \tag{1.7}$$

The rate of convergence (1.3) following directly from (1.6) and (1.7). Passing $m \to \infty$ in (1.5) implies the a priori estimate (1.1).

For any $n, m \in \mathbb{Z}_+$,

$$\begin{aligned} \|u_{n+1} - u_{n+m+1}\| &\le \|u_{n+1} - u_{n+2}\| + \|u_{n+2} - u_{n+3}\| + \cdots + \|u_{n+m} - u_{n+m+1}\| \\ &\le (\lambda + \lambda^2 + \cdots + \lambda^m) \|u_n - u_{n+1}\|. \end{aligned} \tag{1.8}$$

Passing $m \to \infty$ in (1.8) implies the a posteriori estimate (1.2). □

1.6 Applications to ODEs

We want to solve the following initial-value problem

$$\begin{aligned} u'(x) &= F(x,u), \quad x \in [x_0 - h, x_0 + h], \\ u(x_0) &= u_0. \end{aligned} \tag{1.9}$$

Here x_0 and u_0 are given. Let $h > 0,\ r > 0$, and define

$$X_h = C[x_0 - h, x_0 + h], \quad M_{h,r} = \{u \in X_h : \|u - u_0\| \le r\}. \tag{1.10}$$

Clearly X_h is a Banach space and $M_{h,r}$ is a closed subset of X. It is straightforward to show that the initial value problem (1.9) of differential equation is equivalent to the following integral equation:

$$u(x) = u_0 + \int_{x_0}^{x} F(y, u(y))\, \mathrm{d}y, \quad x \in [x_0 - h, x_0 + h]. \tag{1.11}$$

Thus we turn to consider the integral equation (1.11) along with the iteration method:

$$u_{n+1}(x) = u_0 + \int_{x_0}^{x} F(y, u_n(y))\, \mathrm{d}y, \quad x \in [x_0 - h, x_0 + h]. \tag{1.12}$$

We have the following result:

Proposition 1.10. *[The Picard-Lindelöf Theorem] Assume that the function* $F : [x_0-h_0, x_0+h_0]\times[u_0-r_0, u_0+r_0] \to \mathbb{R}$ *is continuous and the partial derivative* $F_u : [x_0-h_0, x_0+h_0]\times[u_0-r_0, u_0+r_0] \to \mathbb{R}$ *is also continuous, where* $h_0 > 0,\ r_0 > 0$ *are fixed numbers. Choose* $0 < h \le h_0,\ 0 < r \le r_0$ *such that*

$$\begin{aligned} &h \max\{|F(x,u)| : x \in [x_0 - h, x_0 + h], u \in [u_0 - r, u_0 + r]\} \le r, \\ &h \max\{|F_u(x,u)| : x \in [x_0 - h, x_0 + h], u \in [u_0 - r, u_0 + r]\} < 1. \end{aligned} \tag{1.13}$$

Then

(i) The problem (1.11) *has a unique solution* $u \in M_{h,r}$*. This is also the unique solution to* (1.9)*.*

(ii) The sequence $\{u_n\}$ *constructed by* (1.12) *converges to this unique solution* u *in* X_h*.*

(iii) There holds the error estimates:

$$\|u_n - u\| \le \lambda^n (1-\lambda)^{-1} \|u_1 - u_0\|, \quad \|u_{n+1} - u\| \le \lambda (1-\lambda)^{-1} \|u_{n+1} - u_n\|, \tag{1.14}$$

where

$$\lambda := h \max\{|F_u(x,u)| : x \in [x_0 - h, x_0 + h], u \in [u_0 - r, u_0 + r]\} < 1. \tag{1.15}$$

证明. Under the assumptions in Proposition 1.10, it is not difficult to check that the operator A defined through

$$(Au)(x) := u_0 + \int_{x_0}^{x} F(y, u(y))\, \mathrm{d}y, \quad \text{for all } x \in [x_0 - h, x_0 + h] \tag{1.16}$$

maps $M_{h,r}$ into $M_{h,r}$, and satisfies

$$\|Au - Av\| \leq \lambda \|u - v\|, \quad \text{for all } u,\ v \in M_{h,r}, \tag{1.17}$$

where $0 \leq \lambda < 1$ is defined as (1.15). This means that $A : M_{h,r} \to M_{h,r}$ is a contractive map. By the contraction principle, A admits a unique fixed point $u \in M_{h,r}$, i.e. $Au = u$ which is exactly (1.11). The other results follow directly from Theorem 1.9.

□

1.7 Continuity

In the following three subsections, we will discuss the continuity, convexity and compactness in normed spaces.

In normed spaces, continuity is equivalent to sequential continuity:

Proposition 1.11. *Let X and Y be normed spaces over $\mathbb{F}$ and $A : X \to Y$ is an operator from X to Y. Then the following statements are equivalent:*

(i) A is continuous, i.e. $A^{-1}(V)$ is an open set in X for every open set V in Y, or equivalently, $A^{-1}(V)$ is a closed set in X for every closed set V in Y.

(ii) A is sequentially continuous, i.e. for each sequence $\{u_n\}$ that converges to u in X, there holds $Au_n \to Au$ in Y.

(iii) For each $u \in X$ and each $\varepsilon > 0$, there is a number $\delta(\varepsilon, u)$ such that for all $\tilde{u} \in X$ satisfying $\|\tilde{u} - u\| < \delta$, there holds $\|A\tilde{u} - Au\| < \varepsilon$.

证明. (i) ⇒ (ii). Suppose A is continuous, i.e. $A^{-1}(V)$ is a closed set in X for every closed set V in Y. Let $u_n \to u$ in X, we want to show $Au_n \to Au$ in Y. By contradiction we suppose Au_n does not converge to Au in Y, i.e. there exists $\varepsilon_0 > 0$ and there exists a subsequence Au_{n_k} such that $\|Au_{n_k} - Au\| \geq \varepsilon_0$. Define $V := \{v \in Y : \|v - Au\| \geq \varepsilon_0\}$. Clearly V is a closed set in Y and $\{Au_{n_k}\} \subset V$. Then $U := A^{-1}(V)$ is a closed set in X and $\{u_{n_k}\} \subset U$. Since $u_n \to u$ in X, and U is closed, we thus have $u \in U$. This means $Au \in V$ which implies a contradiction:

$$0 = \|Au - Au\| \geq \varepsilon_0.$$

(ii) ⇒ (iii). Suppose A is sequentially continuous. We want to prove statement (*iii*). By contradiction we suppose that there exists $u_0 \in X$ and $\varepsilon_0 > 0$ such that for each $\delta > 0$, there exists u_δ satisfying $\|u_\delta - u_0\| \leq \delta$ such that $\|Au_\delta - Au_0\| \geq \varepsilon_0$. By choosing $\delta = 1/n,\ n = 1, 2, \cdots$, we obtain a sequence $\{u_n\}$ which converges to u_0 and $\|Au_n - Au_0\| \geq \varepsilon_0$ for all n. A contradiction with the sequential continuity of A.

(iii) $\Rightarrow$ (i). Let $V \subset Y$ be open. We want to prove $A^{-1}(V)$ is open in X. Given $u \in A^{-1}(V)$, we have $Au \in V$. Since V is open, there exists $\varepsilon > 0$ such that $B(Au, \varepsilon) = \{v \in Y : \|v - Au\| < \varepsilon\} \subset Y$. By (ii), there exists $\delta > 0$, such that for all $\|\tilde{u} - u\| < \delta$ there holds $\|A\tilde{u} - Au\| < \varepsilon$. We thus have $A\tilde{u} \in B(Au, \varepsilon) \subset V$ for all $\|\tilde{u} - u\| < \delta$. This means $B(u, \varepsilon) \subset A^{-1}(V)$. Thus $A^{-1}(V)$ is an open set in X.

□

1.8 Convexity

Definition 1.12. *(a) Let X be a vector space (linear space), a subset $M \subset X$ is called convex iff*

$$\alpha u + (1 - \alpha)v \in M, \text{ for all } u,\ v \in M,\ 0 \le \alpha \le 1.$$

(b) Let M be a convex set. The function $f : M \to \mathbb{R}$ is called convex iff

$$f(\alpha u + (1 - \alpha)v) \le \alpha f(u) + (1 - \alpha)f(v), \text{ for all } u,\ v \in M,\ 0 \le \alpha \le 1.$$

Intuitively, the convexity of M means that the entire line segment joining two points in M is contained in M. The convexity of the real function $f : [a, b] \to \mathbb{R}$ means that the chords always lie above the graph of f.

Example 1.13.
- *The open and closed balls in a normed space are convex.*
- *The norm function $\|\cdot\|$ is continuous and convex.*

Definition 1.14. *Let X be a vector space over $\mathbb{F}$ and let M be subset of X. Define:*

(a) $\operatorname{span} M$:= *smallest linear subspace of X containing M.* $\operatorname{span} M$ *is called the linear hull of M.*

(b) $\operatorname{co} M$:= *smallest convex set of X containing M.* $\operatorname{co} M$ *is called the convex hull of M.*

If moreover X is a normed space, define:

(c) $\overline{M}$:= *smallest closed set of X containing M and is called to be the closure of M.*

(d) $\overline{\operatorname{co}}\, M$:= *smallest closed convex set of X containing M and is called the closed convex hull of M.*

(e) $\operatorname{int} M$:= *largest open set of X contained in M and is called the interior of M.*

(f) $\partial M := \overline{M} - \operatorname{int} M$ *is called the boundary of M.*

(f) $\operatorname{ext} M := \operatorname{int}(M^c)$ *is called the exterior of M.*

Proposition 1.15. *Let M be a nonempty subset of the normed space X over $\mathbb{F}$. Then the following hold true:*

(i) $u \in \operatorname{span} M$ *iff there exist* $u_1, \cdots, u_n \in M$ *and* $\alpha_1, \cdots, \alpha_n \in \mathbb{F}$ *such that* $u = \alpha_1 u_1 + \cdots \alpha_n u_n$.

(ii) $u \in \operatorname{co} M$ *iff there exist* $u_1, \cdots, u_n \in M$ *and* $0 \leq \alpha_1, \cdots \alpha_n \leq 1$ *satisfying* $\alpha_1 + \cdots + \alpha_n = 1$ *such that* $u = \alpha_1 u_1 + \cdots \alpha_n u_n$.

(iii) $u \in \overline{M}$ *iff there exist a sequence* $\{u_n\}$ *in* M *such that* $u_n \to u$ *in* X.

证明. 略。练习。 □

1.9 Compactness

It turns out that most of the statements on finite dimensional spaces have nice generalizations to a certain class of subsets or operators on infinite dimensional spaces, namely, to the compact sets and operators.

1.9.1 Compact sets

Definition 1.16. *Let X be a normed space over $\mathbb{F}$ and let M be subset of X.*

(a) M *is called* sequentially compact *iff each sequence in M admits a convergent subsequence with limit in M.*

(b) M *is called* relatively sequentially compact *iff each sequence in M admits a convergent subsequence with limit in X.*

*(c) Let $\varepsilon > 0$. A set $\{x_\alpha \in M : \alpha \in I\}$ is said to be an ε-*net *for M if*

$$M \subset \bigcup_{\alpha \in I} B(x_\alpha, \varepsilon).$$

(d) M *is said to be* totally bounded *if it has a finite ε-net for every $\varepsilon > 0$.*

(e) A closed subset M is said to have the finite intersection property for closed sets if every decreasing sequence of closed, nonempty sets in M has nonempty intersection.

Proposition 1.17. *Let M be a nonempty subset of the normed space X over $\mathbb{F}$. Then M is sequentially compact iff M is relatively sequentially compact and closed.*

证明. 略。练习。 □

Proposition 1.18. *Let M be a nonempty subset of the Banach space X over $\mathbb{F}$. Then M is relatively sequentially compact iff M is totally bounded.*

证明. We first suppose that M is relatively sequentially compact. By contradiction, we suppose that M is not totally bounded, i.e. there exists $\varepsilon_0 > 0$ such that there is no finite ε_0 net of M. Given $x_1 \in M$, since M has no finite ε_0 net, we have $M \not\subset B_{\varepsilon_0}(x_1)$. Thus, there exists $x_2 \in M$ and $x_2 \notin B_{\varepsilon_0}(x_1)$ which means $\|x_2 - x_1\| \geq \varepsilon_0$. By induction, we find a sequence $\{x_n\} \subset M$ satisfying

$$\|x_m - x_n\| \geq \varepsilon_0, \quad \text{for all } m,\ n \in \mathbb{Z}_+.$$

This implies that $\{x_n\}$ has no Cauchy subsequence and then no convergent subsequence. This contradicts with the fact that M is relatively sequentially compact.

We next suppose that M is totally bounded. Let $\{x_n\}$ be a sequence in M. Since M is totally bounded, it has a finite ε-net for every $\varepsilon > 0$. We first choose $\varepsilon = 1$, then there exists finite 1 net $\{y_1, \cdots, y_k\}$ of M. Since the sequence $\{x_n\} \subset \bigcup_{i=1}^k B_1(y_i)$, there is at least one ball $B_1(y_i)$ that contains infinite terms in the sequence $\{x_n\}$. These infinite terms form a subsequence $\{x_n^{(1)}\}$.

Now for the subsequence $\{x_n^{(1)}\}$, since there is a finite $1/2$ net of M, again denoted by $\{y_1, \cdots, y_k\}$, then $\{x_n^{(1)}\} \subset M \subset \bigcup_{i=1}^k B_{1/2}(y_i)$. This means there at least one ball say $B_{1/2}(y_i)$ that contains infinite terms in the sequence $\{x_n^{(1)}\}$. These infinite terms form a subsequence $\{x_n^{(2)}\}$.

Continuing this construction for $\varepsilon = 1/m,\ m = 1, 2, \cdots$, we obtain the subsequences

$$\begin{aligned} & x_1^{(1)}, x_2^{(1)}, \cdots x_n^{(1)}, \cdots, \\ & x_1^{(2)}, x_2^{(2)}, \cdots x_n^{(2)}, \cdots, \\ & \vdots \\ & x_1^{(m)}, x_2^{(m)}, \cdots x_n^{(m)}, \cdots, \\ & \vdots \end{aligned} \tag{1.18}$$

which have the following property: for each $m \in \mathbb{Z}_+$, $\{x_n^{(m+1)}\}$ is a subsequence of $\{x_n^{(m)}\}$, and

$$\|x_{n_1}^{(m)} - x_{n_2}^{(m)}\| \leq \|x_{n_1}^{(m)} - y_i\| + \|y_i - x_{n_2}^{(m)}\| \leq 1/m + 1/m = 2/m,\ \forall n_1, n_2 \in \mathbb{Z}_+. \tag{1.19}$$

Finally consider the diagonal subsequence $\{x_n^{(n)}\}$. It following from (1.19) that

$$\|x_{n+m}^{(n+m)} - x_n^{(n)}\| \leq \frac{2}{n},\ \text{for all } n, m \in \mathbb{Z}_+.$$

This implies that $\{x_n^{(n)}\}$ is a Cauchy sequence, and thus is a convergent sequence in Banach space X. This proves the relative sequential compactness of M.

□

Proposition 1.19. *Let M be a nonempty closed subset of the normed space X over $\mathbb{F}$. Then M is sequentially compact iff M has the finite intersection property for closed sets.*

证明. First suppose that M is sequentially compact. Given a decreasing sequence of nonempty closed sets $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ in M, choose $x_n \in F_n$ for each $n \in \mathbb{Z}_+$. Then $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x$ in M, as $k \to \infty$. Since $x_{n_k} \in F_n$ for all $n_k \geq n$ and F_n is closed, then $x \in F_n$ for every $n \in \mathbb{Z}_+$, so $x \in \bigcap_{n=1}^{\infty} F_n$. This implies $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

We next suppose that M has the finite intersection property for closed sets. Let $\{x_n\}$ be a sequence in M. Define a decreasing sequence of nonempty closed sets $F_n \subset M$ as

$$F_n := \overline{T_n}, \quad T_n := \{x_k : k > n\}.$$

Thus, by the finite intersection property of M, there exists

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

Choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows. For $k = 1$, since $x \in F_1 = \overline{T}_1$, there exists $x_{n_1} \in T_1$ such that $\|x_{n_1} - x\| < 1$. Similarly, since $x \in F_{n_1} = \overline{T}_{n_1}$, there exists $x_{n_2} \in T_{n_1}$ with $n_2 > n_1$ such that $\|x_{n_2} - x\| < 1/2$. By induction, given x_{n_k}, we choose $x_{n_{k+1}} \in T_{n_k}$, where $n_{k+1} > n_k$, such that $\|x_{n_{k+1}} - x\| < 1/(k+1)$. Then the subsequence $x_{n_k} \to x$ as $k \to \infty$. This proves the sequential compactness of M.

□

Lemma 1.20. *[Lebesgue Covering Lemma] Let M be a sequentially compact subset of the normed space X over $\mathbb{F}$. If $\{G_\alpha \subset X : \alpha \in I\}$ be an open cover of M, there exists $\delta > 0$ such that for every $x \in M$, there is some $\alpha \in I$ with $B_\delta(x) \subset G_\alpha$.*

证明. By contradiction we suppose that no such $\delta > 0$ exists. Then for every $n \in \mathbb{Z}_+$ there exists $x_n \in M$ such that $B_{1/n}(x_n)$ is not contained in G_α for any $\alpha \in I$. Since M is sequentially compact, the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in M. Let $x = \lim_{k\to\infty} x_{n_k} \in M$. Then $x \in G_{\alpha_0}$ for some $\alpha_0 \in I$. Since G_{α_0} is open, there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) \subset G_\alpha$. Since $x_{n_k} \to x$ and $1/n_k \to 0$ as $k \to \infty$, then there exists $N \in \mathbb{Z}_+$ such that for $k \geq N$ there holds $\|x_{n_k} - x\| < \varepsilon_0/2$ and $1/n_k < \varepsilon_0/2$. Hence,

$$B_{1/n_k}(x_{n_k}) \subset B_{\varepsilon_0}(x) \subset G_{\alpha_0}.$$

This contradicts to the choice of x_n.

□

Now we are ready to prove the following result:

Proposition 1.21. *Let M be a nonempty subset of the Banach space X over $\mathbb{F}$. Then M is sequentially compact iff M is compact.*

证明. Suppose that M is compact. Firstly M is then closed (why?). Let $\{F_n\}_{n=1}^{\infty}$ be a decreasing sequence of closed nonempty subsets of M. We want to show that $\bigcap_{n=1}^{\infty} F_n$ is not empty. By contradiction we assume that $\bigcap_{n=1}^{\infty} F_n$ is empty. Then, defining $G_n := F_n^c$, one has $\bigcup_{n=1}^{\infty} G_n = X \supset M$. This means then $\{G_n\}_{n=1}^{\infty}$ is an open cover of M, so it has a finite subcover $\{G_n\}_{n=1}^{N}$ of M. Thus,

$$F_N = \bigcap_{n=1}^{N} F_n = \left(\bigcup_{n=1}^{N} G_n \right)^c \subset M^c.$$

A contradiction to the choice of $\{F_n\}$. Thus the closed set M has the finite intersection property. It follows from Proposition 1.19 that M is sequentially compact.

We next suppose that M is sequentially compact. Let $\{G_\alpha : \alpha \in I\}$ be an open cover of M. By Lemma 1.20, there exists $\delta > 0$ such that for every $x \in M$, there is some $\alpha \in I$ with $B_\delta(x) \subset G_\alpha$.

Since M is sequentially compact, it is totally bounded. Then there exists a finite collection of balls $\{B_\delta(x_i) : i = 1, 2, \cdots, n\}$ of radius δ with $x_i \in M,\ i = 1, 2, \cdots, n$ that covers M, i.e. a finite δ net of M.

Choose $\alpha_i \in I$ such that $B_\delta(x_i) \subset G_{\alpha_i}$. Then $\{G_{\alpha_i} : i = 1, 2, \cdots, n\}$ is a finite subcover of M. This proves M is compact.

□

1.10 Compact operators

Definition 1.22. *Let X and Y be Banach spaces over $\mathbb{F}$, and $A : X \to Y$ be a continuous operator. $A : X \to Y$ is called compact iff $\overline{A(U)}$ is compact in Y (or $A(U)$ is relatively compact) for every bounded set $U \subset X$.*

Example 1.23. *Consider the integral operator*

$$Au(x) := \int_a^b F(x, y, u(y))\, \mathrm{d}y \quad \text{for all } x \in [a, b],$$

where $-\infty < a < b < +\infty$. Set

$$Q := \{(x, y, u) \in \mathbb{R}^3 : x, y \in [a, b], |u| \le r\}, \quad r > 0 \text{ is fixed.}$$

Set

$$X := C[a, b], \quad M := \{u \in X : \|u\| \le r\}.$$

Suppose $F : Q \to \mathbb{R}$ is continuous. Then the operator $A : M \to X$ is compact. (why?)

Proposition 1.24. *Let X and Y be Banach spaces over $\mathbb{F}$, and $A : X \to Y$ be a compact operator. Let $M \subset X$ be a bounded subset. Then there exists a sequence of continuous operators $\{A_n : M \to Y\}_{n=1}^{\infty}$ such that*

$$\sup_{u \in M} \|Au - A_n u\| \le 1/n, \quad \dim \operatorname{span} A_n(M) < \infty, \quad A_n(M) \subset \operatorname{co} A(M).$$

证明. Since M is bounded, then $A(M)$ is relatively compact. Thus, for any $n \in \mathbb{Z}_+$, there exists a finite $1/n$-net for $A(M)$. That is, there is a finite set $\{v_j : j = 1, 2, \cdots, J_n\}$ such that

$$\min_{1\le j\le J_n} \|Au - v_j\| < 1/n, \quad \text{for all } u \in M. \tag{1.20}$$

Define the Schauder operator $A_n : M \to M$:

$$A_n u := \frac{\sum_{j=1}^{J_n} a_j(u) v_j}{\sum_{j=1}^{J_n} a_j(u)}, \quad \text{for all } u \in M, \tag{1.21}$$

where

$$a_j(u) = \max\{1/n - \|Au - v_j\|, 0\}, \quad \text{for all } u \in M. \tag{1.22}$$

By (1.20), for each $u \in M$, there exists some $j \in \{1, 2, \cdots, J_n\}$ such that $\|Au - v_j\| < 1/n$. Thus

$$\sum_{j=1}^{J_n} a_j(u) > 0, \quad \text{for all } u \in M.$$

Clearly

$$\operatorname{span} A_n(M) \subset \operatorname{span}\{v_j : j = 1, \cdots, J_n\}$$

is finite dimensional.

Since A is continuous, the norm function $\|\cdot\|$ is continuous, together with the fact that the composition of continuous operators is continuous (why?), we know that $a_j : M \to \mathbb{R}$ is continuous and $A_n : M \to Y$ is continuous. Moreover, $A_n(M) \subset \operatorname{co} A(M)$, and for each $u \in M$,

$$\begin{aligned}
\|A_n u - Au\| &= \frac{\|\sum_{j=1}^{J_n} a_j(u)(v_j - Au)\|}{\sum_{j=1}^{J_n} a_j(u)} \le \frac{\sum_{j=1}^{J_n} a_j(u)\|(v_j - Au)\|}{\sum_{j=1}^{J_n} a_j(u)} \\
&= \frac{\sum_{a_j(u)\neq 0} a_j(u)\|(v_j - Au)\|}{\sum_{j=1}^{J_n} a_j(u)} \le \frac{\sum_{a_j(u)\neq 0} a_j(u) n^{-1}}{\sum_{j=1}^{J_n} a_j(u)} \\
&\le 1/n.
\end{aligned} \tag{1.23}$$

□

1.11 Finite-dimensional Banach spaces

Finite-dimensional normed spaces enjoy similar properties as classical $\mathbb{R}^d$.

Proposition 1.25.

- *If X is a finite dimensional normed space over $\mathbb{F}$, then any two norms on X are equivalent.*
- *Each finite dimensional normed space is complete, i.e. is a Banach space.*
- *Each finite-dimensional linear subspace of a normed space is closed.*

- *A subset of a finite-dimensional normed subspace is relatively compact iff it is bounded, and is compact iff it is bounded and closed.*

- *Any two finite dimensional normed spaces with the same dimension are homeomorphic.*

证明. 略。练习。 □

1.12 The Minkowski functional and homeomorphisms

Definition 1.26. *Let X and Y be normed spaces. A map $A : X \to Y$ is called* homeomorphism *(or* topological isomorphism*) provided:*

- *A is continuous.*

- *A is bijective.*

- *A^{-1} is also continuous.*

We then say X is homeomorphic (or isomorphic) to Y

Proposition 1.27. *Let M be a closed, bounded, convex, nonempty subset of a normed space X, and $\operatorname{int} M \neq \emptyset$. Then there exists a homeomorphism $A : X \to X$ such that $A(M) = B$ where B is the closed ball $B := \{u \in X : \|u\| \leq 1\}$. This means M is homeomorphic to the closed ball $B := \{u \in X : \|u\| \leq 1\}$.*

Before proving Proposition (1.27), we first recall the concept of Minkowski functional.

Definition 1.28 (Minkowski functional)**.** *Let M be a closed, bounded, convex, nonempty subset of a normed space X, and $0 \in \operatorname{int} M$. The Minkowski functional $p : X \to \mathbb{R}$ of the set M is defined as*

$$p(u) := \inf\{\lambda > 0 : \lambda^{-1} u \in M\}, \quad \text{for all } u \in X. \tag{1.24}$$

The Minkowski functional is well defined (why?). The intuitive meaning of $p(u)$ is that, the ray through the point u and the origin intersects the boundary ∂M of the set M at the point $p(u)^{-1}u$.

The Minkowski functional has the following properties:

Lemma 1.29. *The following are true:*

(i) *There exists $a, b > 0$ such that $a\|u\| \leq p(u) \leq b\|u\|$ for all $u \in X$.*

(ii) *For all $\alpha \geq 0$, $u \in X$, there holds $p(\alpha u) = \alpha p(u)$.*

(iii) *For all $u, v \in X$, there holds $p(u + v) \leq p(u) + p(v)$ (triangle inequality).*

(iv) *$p : X \to \mathbb{R}$ is continuous.*

(v) $M = \{u \in X : p(u) \leq 1\}$.

Proof of Lemma 1.29. (i). By definition, we have $p(u) \geq 0$ for all $u \in X$ and $p(0) = 0$. Given $u \neq 0$. Since $0 \in \operatorname{int} M$, there exists $r > 0$ such that

$$\{x \in X : \|u\| \leq r\} \subset M.$$

We then have $\|\lambda^{-1}u\| = r$ for $\lambda := r^{-1}\|u\|$. Hence $\lambda^{-1}u \in M$. Then the definition of $p(u)$ makes sense and

$$p(u) \leq r^{-1}\|u\|.$$

Since M is bounded, i.e. there exists $R > 0$ such that

$$\|v\| \leq R, \quad \text{for all } v \in M.$$

Thus, $\lambda^{-1}u \in M$ implies $\|\lambda^{-1}u\| \leq R$, i.e. $\lambda \geq R^{-1}\|u\|$. This implies

$$p(u) \geq R^{-1}\|u\|.$$

(ii) Firstly $p(0) = 0$. Let $\alpha > 0$. Observe that $\lambda^{-1}u \in M$ iff $(\alpha\lambda)^{-1}\alpha u \in M$.

(iii) Given $u, v \in X$. For any $\varepsilon > 0$, choosing α, β such that

$$p(u) < \alpha < p(u) + \varepsilon, \quad p(v) < \beta < p(v) + \varepsilon.$$

Then $\alpha^{-1}u, \beta^{-1}v \in M$. Let $\gamma = \alpha + \beta$. Since $\gamma^{-1}\alpha + \gamma^{-1}\beta = 1$ and M is convex, we have

$$\gamma^{-1}(u + v) = \gamma^{-1}\alpha(\alpha^{-1}u) + +\gamma^{-1}\beta(\beta^{-1}v) \in M.$$

Thus

$$p(u + v) \leq \gamma = \alpha + \beta < p(u) + p(v) + 2\varepsilon$$

Letting $\varepsilon \to 0$ implies (iii).

(iv). It follows from (iii) that

$$p(u) \leq p(v) + p(u - v), \quad p(v) \leq p(u) + p(v - u).$$

Using (i) implies

$$|p(u) - p(v)| \leq \max\{p(u - v), p(v - u)\} \leq b\|u - v\|, \quad \text{for all } u, v \in X.$$

This implies that p is continuous.

(v). Given $u \in M$. Since $0 \in M$ and M is convex, we have $\mu u \in M$ for all $0 \leq \mu \leq 1$. Hence $\lambda^{-1}u \in M$ for all $\gamma \geq 1$. This implies that $p(u) \leq 1$.

Conversely suppose that $p(u) \leq 1$ for some $u \in X$. We want to show that $u \in M$. If $u = 0$, then $u \in M$. Suppose now that $u \neq 0$. Then $p(u) > 0$ and by the definition of $p(u)$, there holds

$$\lambda^{-1}u \in M, \quad \text{for all } \lambda > p(u).$$

Passing $\lambda \to p(u)$ and using the fact that M is closed, we have

$$p(u)^{-1}u \in M, \quad \text{for all } u \in X. \tag{1.25}$$

Using $0 \in M$ and $p(u)^{-1} \geq 1$, the convexity of M implies $u \in M$.

□

Proof of Proposition (1.27). If $X = \{0\}$, then $M = B = \{0\}$, and the statement if trivial. We suppose $X \neq \{0\}$ and let $u_0 \in M$ be an interior point. Replacing u with $u - u_0$, we may assume that $u_0 = 0$.

Step 1. The homeomorphism. Define $A : X \to X$ as

$$Au := \frac{p(u)}{\|u\|}u, \ u \in X, \ u \neq 0; \quad A0 = 0. \tag{1.26}$$

By (i) in Lemma 1.29, we have

$$\|Au\| \leq b\|u\|, \text{ for all } u \in X. \tag{1.27}$$

Thus $A : X \to X$ is continuous.

The map $A : X \to X$ is bijective (why?) and its inverse is given as

$$A^{-1}v := \frac{\|v\|}{p(v)}v, \ v \in X, \ v \neq 0; \quad A^{-1}0 = 0. \tag{1.28}$$

Again by (i), there holds

$$\|A^{-1}v\| \leq a\|v\|, \text{ for all } v \in X. \tag{1.29}$$

Thus A^{-1} is continuous, and $A : X \to X$ is a homeomorphism.

Step 2. $A(M) = B$. Given $u \in M$, one has $p(u) \leq 1$. Thus

$$\|Au\| = p(u) \leq 1.$$

So $Au \in M$. This proves $A(M) \subset B$.

Given $v \in B$, $\|v\| \leq 1$. By (1.25) and the convexity of M, we have

$$A^{-1}v = \frac{\|v\|}{p(v)}v \in M.$$

This means $A^{-1}B \subset M$. So $B = AA^{-1}(B) \subset A(M)$. Thus $A(M) = B$.

□

1.13 The Brouwer fixed-point theorem

We recall the following classical result:

Theorem 1.30. *If* $1 \leq d < \infty$, $B =$ *the closed unit ball of* $\mathbb{R}^d$, *and* $f : B \to B$ *is a continuous map, then there is a point* x *in* B *such that* $f(x) = x$.

A more general version is the following, which can be seen as a corollary for the above theorem:

Theorem 1.31. *Let* M *be a compact, convex, nonempty set in a finite dimensional normed space over* $\mathbb{F}$. *Then the continuous operator*

$$A : M \to M$$

has a fixed point.

A direct corollary is the following:

Corollary 1.32. *Let* K *be a subset of a finite dimensional normed space over* $\mathbb{F}$. *If* K *is homeomorphic to a set* M *as considered in Theorem 1.31, the continuous operator*

$$A : K \to K$$

has a fixed point.

Proof of Corollary 1.32. Let $H : M \to K$ be a homeomorphism. Then the operator

$$\tilde{A} := H^{-1} \circ A \circ H : M \to M$$

is continuous. By Theorem 1.31, there exists $u \in M$ such that

$$u = \tilde{A}u = H^{-1} \circ A \circ Hu,$$

which is equivalent to

$$Hu = A \circ Hu$$

This means $v = Hu \in K$ is a fixed point of A.

□

To prove Theorem 1.31 by using Theorem 1.30, we need the following result, which says that a

Example 1.33. *Each continuous function* $A : [a, b] \to [a, b]$ *has a fixed point.*

The Brouwer fixed point theorem can be proved by using the Sperner simplex and the Sperner's lemma, which is a combinatorial analog of the Brouwer fixed point theorem, which is equivalent to it. At this moment we will not address this proof. One can find in Section 1.14 in [8]. Other proofs can be found in algebraic topology books, e.g. in the book Dugundji [5].

1.14 The Schauder fixed point theorem

Theorem 1.34. *Let M be a closed, bounded, convex, nonempty subset in a normed space X over $\mathbb{F}$. If $A : M \to X$ is a compact operator and $A(M) \subset M$, then there exists $u \in M$ such that $Au = u$.*

This theorem was proved by Schauder in 1930. If X has finite dimension, the Schauder fixed point theorem coincides with the Brouwer fixed point theorem.

证明. Let $K = \overline{A(M)}$. Since A is a compact operator, then K is compact. Since $A(M) \subset M$ and M is closed, then $K \subset M$. Since K is compact, for each $n \in \mathbb{Z}_+$, there is a finite $1/n$-net $\{v_1, \cdots, v_{J_n}\} \subset K$ such that

$$K \subset \bigcup_{j=1}^{J_n} B_{1/n}(v_j).$$

Let $X_n := \operatorname{span}\{v_j : j = 1, \cdots, J_n\}$. For each n, let $A_n : M \to X$ be the Schauder operators define as in Proposition 1.24 associate with the above $1/n$-net. Thus

$$\sup_{u \in M} \|Au - A_n u\| \le 1/n, \quad \operatorname{span} A_n(M) \subset X_n, \quad A_n(M) \subset \operatorname{co} A(M). \tag{1.30}$$

Since $A(M) \subset M$ and M is convex, we have

$$A_n(M) \subset \operatorname{co} A(M) \subset M. \tag{1.31}$$

By (1.30) and (1.31), we have

$$A_n(M) \subset M_n := M \cap X_n. \tag{1.32}$$

Clearly, X_n is a finite dimensional normed space, and M_n is a closed, bounded, convex, nonempty subset of X_n, and $A_n : M_n \to M_n$ is continuous. By the Brouwer fixed point theorem, there exists $u_n \in M_n$ such that $A_n u_n = u_n$. Then, using (1.30) implies

$$\|Au_n - u_n\| = \|Au_n - A_n u_n\| \le 1/n. \tag{1.33}$$

Since $\{u_n\}_{n=1}^{\infty} \subset M_n \subset M$, the compactness of the operator $A : M \to M$ implies that there exists a subsequence $\{u_{n_k}\}$ such that

$$\lim_{k \to \infty} Au_{n_k} = v \in M, \tag{1.34}$$

where we used the property that M is closed.

By (1.33) and (1.34), we have

$$\|v - u_{n_k}\| \le \|v - Au_{n_k}\| + \|u_{n_k} - Au_{n_k}\| \to 0.$$

This implies $u_{n_k} \to v$. Since the map $A : M \to M$ is continuous, we finally obtain that

$$Av = \lim_{k \to \infty} Au_{n_k} = v.$$

□

1.15 Application to ODEs

We want to solve the following initial-value problem

$$\begin{aligned} u'(x) &= F(x,u), \quad x \in [x_0 - h, x_0 + h], \\ u(x_0) &= u_0. \end{aligned} \tag{1.35}$$

Here x_0 and u_0 are given. Let $h > 0,\ r > 0$, and define

$$X_h = C[x_0 - h, x_0 + h], \quad M_{h,r} = \{u \in X_h : \|u - u_0\| \le r\}. \tag{1.36}$$

Clearly X_h is a Banach space and $M_{h,r}$ is a closed subset of X. It is straightforward to show that the initial value problem (1.35) of differential equation is equivalent to the following integral equation:

$$u(x) = u_0 + \int_{x_0}^{x} F(y, u(y))\,\mathrm{d}y, \quad x \in [x_0 - h, x_0 + h]. \tag{1.37}$$

We have the following result:

Proposition 1.35. *[The Peano Theorem] Assume that the function* $F : [x_0 - h_0, x_0 + h_0] \times [u_0 - r_0, u_0 + r_0] \to \mathbb{R}$ *and the partial derivative* $F_u : [x_0 - h_0, x_0 + h_0] \times [u_0 - r_0, u_0 + r_0] \to \mathbb{R}$ *are continuous, where* $h_0 > 0,\ r_0 > 0$ *are fixed numbers. Choose* $0 < h \le h_0,\ 0 < r \le r_0$ *such that*

$$h \max\{|F(x,u)| : x \in [x_0 - h, x_0 + h], u \in [u_0 - r, u_0 + r]\} \le r. \tag{1.38}$$

Then the problem (1.37) *has a unique solution* $u \in M_{h,r}$. *This is also the unique solution to* (1.35).

证明. Define the operator $A : M_{h,r} \to X_h$ through

$$(Au)(x) := u_0 + \int_{x_0}^{x} F(y, u(y))\,\mathrm{d}y, \quad \text{for all } x \in [x_0 - h, x_0 + h]. \tag{1.39}$$

Clearly X_h is a Banach space, and $M_{h,r}$ is a bounded, closed, convex, nonempty subset in X_h. Under the assumptions in Proposition 1.35, we have that (why?)

(i) $A : M_{h,r} \to X_h$ is continuous.

(ii) $A(M_{h,r})$ is equicontinuous.

(iii) $A(M_{h,r}) \subset M_{h,r}$.

By the Arzelá-Ascoli theorem, above three properties imply that $A(M_{h,r})$ is relatively compact in X_h. This means the operator $A : M_{h,r} \to X_h$ is compact. Hence, by the Schauder fixed point theorem, A admits a unique fixed point $u \in M_{h,r}$, i.e. $Au = u$ which is exactly (1.37). Differentiating the integral equation (1.37) implies that u is also a solution to the original problem (1.35).

□

1.16 The Leray-Schauder principle and a priori estimates

Let X be a Banach space and $A : X \to X$ is a continuous operator. We want to solve the equation

$$u = Au, \quad u \in X \tag{1.40}$$

by using properties of the parametrized equation

$$u = tAu, \quad u \in X, \quad 0 \le t \le 1. \tag{1.41}$$

For $t = 0$, equation (1.41) has the trivial solution $u = 0$, whereas (1.41) coincides with (1.40) if $t = 1$. The following condition is crucial:

(A). A priori estimate. There is a number $r > 0$ such that if u is a solution to (1.41), then

$$\|u\| \le r, \quad \text{for all } 0 \le t \le 1. \tag{1.42}$$

Theorem 1.36. *Let X be a Banach space over $\mathbb{F}$. Suppose that the operator $A : X \to X$ is compact and satisfies condition (A). Then the original equation* (1.40) *has a solution.*

This theorem was proved by Leray and Schauder in 1934. Roughly speaking, Theorem 1.36 corresponds to the following important principle in mathematics:

A priori estimates yield existence.

证明. Set $M := \{u \in X : \|u\| \le 2r\}$. We define the operator

$$Bu := \begin{cases} Au, & \|Au\| \le 2r, \\ \dfrac{2rAu}{\|Au\|}, & \|Au\| > 2r. \end{cases}$$

Obviously, $\|Bu\| \le 2r$ for all $u \in X$, i.e. $B(M) \subset M$.

We claim that $B : M \to M$ is compact. Firstly, we show that $B : M \to M$ is continuous. Let $u_0 \in M$. If $\|Au_0\| < 2r$ or $\|Au_0\| > 2r$, the continuity of A implies that B is continuous at u_0. Indeed, for example if $\|Au_0\| < 2r$, the continuity of A implies that there exists $\delta_0 > 0$ such that for all $\|u - u_0\| < \delta_0$ there holds $\|Au\| < 2r$. Then the continuity of A implies that for any $\varepsilon > 0$, there exists $0 < \delta < \delta_0$ such that for all $\|u - u_0\| < \delta$ there holds $\|Bu - Bu_0\| = \|Au - Au_0\| < \varepsilon$.

If $\|Au_0\| = 2r$. Then $Bu_0 = Au_0 = \frac{2rAu}{\|Au\|}$. For any $0 < \varepsilon < r$, there exists $\delta > 0$ such that for all $\|u - u_0\| < \delta_0$ there holds $\|Au - Au_0\| < \varepsilon$. This gives

$$r < 2r - \varepsilon = \|Au_0\| - \varepsilon < \|Au\| < \|Au_0\| + \varepsilon = 2r + \varepsilon < 3r. \tag{1.43}$$

Then

$$
\begin{aligned}
\|Bu - Bu_0\| = \|Bu - Au_0\| &\le \max\left\{\|Au - Au_0\|, \left\|\frac{2rAu}{\|Au\|} - Au_0\right\|\right\} \\
&\le \|Au - Au_0\| + \left\|\frac{2rAu}{\|Au\|} - Au_0\right\| \\
&\le \|Au - Au_0\| + \left\|\frac{2r}{\|Au\|}(Au - Au_0)\right\| + \left\|\left(\frac{2r}{\|Au\|} - 1\right) Au_0\right\| \\
&\le \|Au - Au_0\| + \frac{2r}{\|Au\|}\|Au - Au_0\| + \frac{\|Au_0\|}{\|Au\|}|2r - \|Au\|| \\
&\le 5\varepsilon.
\end{aligned}
\tag{1.44}
$$

Now we show that the compactness of B. Let $\{u_n\}$ be a sequence in M. Then there there exists a subsequence, still denoted by $\{u_n\}$ such that either $\|Au_n\| \le 2r$ for all n, or $\|Au_n\| > 2r$ for all n. (why?)

If $\|Au_n\| \le 2r$ for all n, then $Bu_n = Au_n$ for all n. Then the compactness of A implies that there exists a convergent subsequence of $\{Bu_n\}$ in M.

We now consider the case $\|Au_n\| > 2r$ for all n. Since $A : X \to X$ is compact, and $\{u_n\}$ is a bounded sequence, then there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that

$$Av_n \to z \quad \text{in } X.$$

Since $\|Av_n\| > 2r$ for all n, then $\frac{1}{\|Av_n\|} \le 1/(2r)$ is bounded. So there exists a subsequence $\{w_n\}$ such that

$$\frac{1}{\|Aw_n\|} \to \alpha.$$

Hence

$$Bw_n = \frac{2rw_n}{\|Aw_n\|} \to 2r\alpha z.$$

We obtain a convergent subsequence of $\{Bu_n\}$. So $B : M \to M$ is compact.

We apply the Schauder fixed point theorem to the compact operator $B : M \to M$ to obtain a fixed point $u \in M$ of B such that

$$u = Bu.$$

If $\|Au\| \le 2r$, then $Bu = u$, and hence $u = Au$.

The case $\|Au\| > 2r$ is impossible by the a priori estimate (A). Otherwise if $\|Au\| > 2r$, we have $\|u\| = \|Bu\| = 2r$, and

$$u = Bu = tAu, \quad t := \frac{2r}{\|Au\|} < 1.$$

The a priori estimate (A) implies that $\|u\| \le r$, a contradiction.

□

A typical application of the Leray-Schauder fixed point theorem is the existence theory of generalized solutions (finite energy weak solutions) to the Navier-Stokes equations. See Section 5.17 in [9].

1.17 Subsolutions and supersolutions, the iteration method in ordered Banach spaces

The idea of ordered Banach spaces is to introduce a relation $u \le v$ in Banach spaces, which generalizes the corresponding relation for real numbers.

Definition 1.37. *A subset X_+ of a normed space X is called an* order cone *provided*

(i) X_+ is closed, convex, nonempty, and $X_+ \neq \{0\}$.

(ii) If $u \in X_+$ and $\alpha \ge 0$, then $\alpha u \in X_+$.

(iii) If $u \in X_+$ and $-u \in X_+$, then $u = 0$.

Given $u, v \in X$. We define the relation $\le$ by

$$u \le v \quad \text{iff} \quad v - u \in X_+.$$

By an ordered normed space *(ordered Banach space), we understand a normed space (Banach space) together with an order cone.*

If $u \le v$, we define the order interval

$$[u, v] := \{w \in X : u \le w \le v\}.$$

The order cone X_+ is called normal iff there a number $C > 0$ such that

$$0 \le u \le v \quad \Longrightarrow \quad \|u\| \le C\|v\|.$$

Example 1.38.

- *$X = \mathbb{R}$, $X_+ = \mathbb{R}_{\ge 0} := \{u \in X : u \ge 0\}$. X_+ is normal.*
- *$X = \mathbb{R}^d$, $X_+ = \mathbb{R}^d_{\ge 0} := \{u = (u_1, \cdots, u_d) \in X : u_j \ge 0,\ j = 1, \cdots, d\}$. X_+ is normal.*
- *$X = C[a, b]$, $X_+ := \{u \in X : u(x) \ge 0,\ x \in [a, b]\}$. X_+ is normal.*

The following proposition shows that the relation $u \le v$ has the usual properties.

Proposition 1.39. *Let (X, X_+) be an ordered Banach space. Let $u, v, w, u_n, v_n \in X_+$, $\alpha \ge 0$. Then*

(i) $u \le v$ and $v \le w$ imply $u \le w$.

(ii) $u \le v$ and $v \le u$ imply $u = v$.

(iii) $u \le v$ implies $u + w \le v + w$ and $\alpha u \le \alpha v$.

(iv) $u_n \le v_n$ and $u_n \to u$, $v_n \to v$ imply $u \le v$.

(v) If the order cone X_+ is normal, then $u \leq v \leq w$ implies

$$\|v-u\| \leq C\|w-u\|, \quad \|w-v\| \leq C\|w-u\|.$$

证明. 练习。 □

We want to solve the equation

$$u = Au, \quad u_0 \leq u \leq v_0, \quad u \in X, \tag{1.45}$$

by means of the two-iteration method:

$$u_{n+1} = Au_n, \quad v_{n+1} = Av_n, \quad n = 0, 1, \cdots, \tag{1.46}$$

where $u_0 \leq v_0$ are given in the ordered Banach space X.

We have the following theorem:

Theorem 1.40. *Let (X, X_+) is an ordered Banach space with normal order cone X_+. Suppose*

- *$A : [u_0, v_0] \subset X :\to X$ is compact.*
- *A is monotone increasing, i.e. $u \leq v$ implies $Au \leq Av$.*
- *u_0 is a subsolution of (1.45), i.e. $u_0 \leq Au_0$.*
- *v_0 is a supersolution of (1.45), i.e. $v_0 \geq Av_0$.*

  Then the iteration sequences $\{u_n\}$ and $\{v_n\}$ constructed in (1.46) converge to u and v which are solutions of the original equation (1.45), respectively. In addition, we have the error estimates:

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq u \leq v \leq v_n \leq v_{n-1} \leq \cdots \leq v_0, \text{ for all } n. \tag{1.47}$$

This theorem corresponds to the following general existence principle in mathematics:

The existence of both a subsolution and a supersolution yields the existence of a solution.

证明. By induction and the monotonicity of A, we have

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq v_n \leq v_{n-1} \leq \cdots \leq v_0, \text{ for all } n. \tag{1.48}$$

Since the order cone X_+ is normal, by Proposition 1.39, we have

$$\|v_0 - u_n\| \leq C\|v_0 - u_0\|, \quad \|v_0 - v_n\| \leq C\|v_0 - u_0\|, \text{ for all } n.$$

Thus $\{u_n\}$ and $\{v_n\}$ are both bounded. By the compactness of A, there exists a subsequence $\{u_{n_k}\}$ and a subsequence $\{v_{n_k}\}$ such that

$$Au_{n_k} \to u, \quad Av_{n_k} \to v.$$

Thus, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{Z}_+$ such that

$$\|u_{n_\varepsilon} - u\| = \|Au_{n_\varepsilon - 1} - u\| \le \varepsilon, \quad \|v_{n_\varepsilon} - v\| = \|Av_{n_\varepsilon - 1} - v\| \le \varepsilon.$$

Passing $n_k \to \infty$ in (1.48) implies

$$u_{n_\varepsilon} \le u_n \le u, \; v \le v_n \le v_{n_\varepsilon}, \quad \text{for all } n \ge n_\varepsilon,$$

Again by Proposition 1.39, there holds

$$\|u - u_n\| \le C\|u - u_{n_\varepsilon}\| \le C\varepsilon, \quad \|v_n - v\| \le C\|v_{n_\varepsilon} - v\| \le C\varepsilon, \quad \text{for all } n \ge n_\varepsilon.$$

This means

$$u_n \to u, \quad v_n \to v.$$

Finally, the continuity of A and passing $n \to \infty$ in (1.46) implies that u and v are solutions to (1.45).

□

1.18 Linear operators

Definition 1.41. *Let X and Y be linear spaces over $\mathbb{F}$. The operator $A : X \to Y$ is called linear iff*

$$A(\alpha u + \beta v) = \alpha Au + \beta Av, \quad \text{for all } u, v \in X, \; \alpha, \beta \in \mathbb{F}.$$

We introduce the range space $\text{Range}(A) := A(X)$ and the kernel of A (or the null space of A): $N(A) = \ker A := \{u \in X : Au = 0\}$. A linear operator is injective iff its kernel is $\{0\}$.

The following proposition says that the continuity and boundedness of a linear operator are equivalent:

Proposition 1.42. *Let X and Y be normed spaces over $\mathbb{F}$, and let $A : X \to Y$ be a linear operator, then the following statements are equivalent:*

- *$A : X \to Y$ is continuous.*
- *$A : X \to Y$ is continuous at some point $u_0 \in X$.*
- *$A : X \to Y$ is continuous at $\{0\}$.*
- *$A : X \to Y$ is bounded: there exists $C > 0$ such that $\|Au\| \le C\|u\|$ for all $u \in X$.*

证明. 练习。 □

Proposition 1.43. *Let $L(X,Y)$ denote the space of linear continuous operators $A : X \to Y$ where X is a normed space and Y is a Banach space over $\mathbb{F}$. Then $L(X,Y)$ is a Banach space over $\mathbb{F}$ with respect to the operator norm:*

$$\|A\| := \sup_{u\neq 0} \frac{\|Au\|}{\|u\|} = \sup_{\|u\|=1} \|Au\|. \tag{1.49}$$

Remark that we do not require X to be complete.

证明. It is straightforward to show that $L(X,Y)$ is a linear space and (1.49) defines a norm on it.

Now we show that $L(X,Y)$ is complete. Let $\{A_n\}$ be a Cauchy sequence in $L(X,Y)$. This means, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{Z}_+$ such that

$$\|A_n - A_m\| \leq \varepsilon, \quad \text{for all } n, m \geq n_\varepsilon. \tag{1.50}$$

Then for each $u \in X$, there holds

$$\|A_n u - A_m u\| \leq \varepsilon\|u\|, \quad \text{for all } n, m \geq n_\varepsilon. \tag{1.51}$$

This implies that $\{A_n u\}$ is Cauchy in Y. Since Y is a Banach space, the sequence $\{A_n u\}$ converges, and we denote its limit as Au:

$$Au := \lim_{n\to\infty} A_n u, \quad \text{for all } u \in X.$$

It is direct to show that $A : X \to Y$ is a linear and bounded operator. Moreover, passing $m \to \infty$ in (1.51) implies that

$$\|A_n u - Au\| \leq \varepsilon\|u\|, \quad \text{for all } n \geq n_\varepsilon, \tag{1.52}$$

for all $u \in X$. Thus $\|A_n - A\| \leq \varepsilon$ for all $n \geq n_\varepsilon$, i.e. $A_n \to A$ in $L(X,Y)$.

This proves each Cauchy sequence in $L(X,Y)$ is convergent, i.e. $L(X,Y)$ is a Banach space.

□

1.19 The dual space

Definition 1.44. *Let X be a normed space over $\mathbb{F}$. By a linear continuous functional on X we understand a linear continuous operator $f : X \to \mathbb{F}$. The collection of all continuous functionals on X is called the dual space of X, and is denoted by X^*.*

Clearly $X^* = L(X,\mathbb{F})$. Since $\mathbb{F}$ is a Banach, we have X^* armed with the operator norm

$$\|f\| := \sup_{\|v\|=1} |f(v)|$$

is a Banach space. We often use the following notation:

$$\langle f, u\rangle = f(u), \quad \text{for all } u \in X,\ f \in X^*.$$

Example 1.45. *Let (Ω, μ) be a measure space. If $1 < p < \infty$ Then $L^p(\Omega, \mu)^* = L^{p'}(\Omega, \mu)$ with $1/p + 1/p' = 1$.*

If moreover (Ω, μ) is a σ-finite measure space, then $L^1(\Omega, \mu)^* = L^\infty(\Omega, \mu)$

What about $L^\infty(\Omega, \mu)^*$? Let (Ω, μ) be a complete σ-finite space. Then $(L^\infty(\Omega, \mu))^*$ is the collection of all finitely additive finite signed (complex) measures which are absolutely continuous with respect to μ, equipped with the total variation norm. See Theorem IV.8.16 in [6], page 296. See Chapter 6 in [7] for some related definitions.

1.20 The Hahn-Banach Theorem

The Hahn-Banach Theorem is the most important result about the structure of linear continuous functionals on normed spaces. In terms of geometry, the Hahn-Banach theorem guarantees the separation of convex sets in normed spaces by hyperplanes.

Theorem 1.46 (The Hahn-Banach Theorem for the linear spaces)**.** *We assume that*

(i) L is a linear subspace of the real linear space X.

(ii) $p : X \to \mathbb{R}$ is a sublinear functional, that is for all $u, v \in X$ and all $\alpha \geq 0$, there holds

$$p(u + v) \leq p(u) + p(v), \quad p(\alpha u) = \alpha p(u).$$

(iii) $f : L \to \mathbb{R}$ is a linear functional such that

$$f(u) \leq p(u) \quad \text{for all } u \in L.$$

Then f can be extended to a linear functional $F : X \to \mathbb{R}$ such that

$$F(u) = f(u) \quad \text{for all } u \in L; \quad F(u) \leq p(u) \quad \text{for all } u \in X.$$

Note that the substance of the theorem is not that the extension exists but that an extension can be found that remains dominated by the same sublinear functional p.

Theorem 1.47 (The Hahn-Banach Theorem for the normed spaces)**.** *We assume that*

(i) L is a linear subspace of the normed space X over $\mathbb{F}$.

(ii) $f : L \to \mathbb{F}$ is a linear functional such that

$$|f(u)| \leq \alpha \|u\| \quad \text{for all } \ u \in L \text{ and some fixed } \alpha \geq 0.$$

Then f can be extended to a linear functional $F : X \to \mathbb{F}$ such that

$$F(u) = f(u) \quad \text{for all } u \in L; \quad |F(u)| \leq \alpha \|u\| \quad \text{for all } u \in X.$$

The proof of the Hahn-Banach theorems can be found in each textbook on functional analysis.

Corollary 1.48. *Let X be a normed space over $\mathbb{F}$. Then for each nonzero $u_0 \in X$, there exists a functional $F \in X^*$ such that*

$$F(u_0) = \|u_0\|, \quad \|F\| = 1.$$

证明. Indeed, we set $L := \text{span}\,\{u_0\}$ and

$$f(u) := \lambda \|u_0\|, \quad \text{for all } u = \lambda u_0 \in L.$$

Obviously, $|f(u)| = \|u\|$ for all $u \in L$. By the Hahn-Banach theorem, there exists a linear continuous functional $F \in X^*$ such that

$$F(u) = f(u) \quad \text{for all } u \in L; \quad |F(u)| \le \|u\| \quad \text{for all } u \in X.$$

Clearly $\|F\| = 1$.

□

Two direct consequences of the above example are the following:

Corollary 1.49. *Let X be a normed space over $\mathbb{F}$. Then for each $u \in X$,*

$$\|u\| = \max_{F \in X^*, \|F\| \le 1} |F(u)|.$$

Corollary 1.50. *Let X be a normed space over $\mathbb{F}$ and $u \in X$. Then $u = 0$ iff*

$$F(u) = 0 \quad \textit{for all } F \in X^*.$$

1.21 The dual space of $C[a, b]$

We first recall some concepts about BV functions.

Definition 1.51 (Functions of bounded variation).

- *Let $-\infty < a < b < \infty$. The function $g : [a, b] \to \mathbb{R}$ is called to be of bounded variation, a BV function for short, iff*

$$V_a^b(g) := \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |g(x_{i+1}) - g(x_i)| < +\infty, \tag{1.53}$$

where the supremum is taken over the set

$$\mathcal{P} := \{P = \{x_0, \cdots, x_{n_P}\} : P \textit{ is a partition of } [a, b] \textit{ satisfying } a = x_0 < x_1 \cdots < x_{n_P} = b\ \}.$$

- *The functional $V_a^b(g)$ is called the total variation of g on interval $[a, b]$.*
- *A function $g : [a, b] \to \mathbb{C}$ is called BV, iff its real part and imaginary part are both BV.*

We introduce some properties of BV functions, and the proofs are left to the students.

Theorem 1.52 (Jordan decomposition of a BV function). *Let $-\infty < a < b < \infty$. The function $g : [a, b] \to \mathbb{R}$ is of bounded variation iff it can be written as the difference $g = g_1 - g_2$ of two non-decreasing functions on $[a, b]$.*

This result is known as the Jordan decomposition of a function and it is related to the Jordan decomposition of a measure.

Proposition 1.53.

- *If g is differentiable and its derivative g' is Riemann-integrable on $[a, b]$, then $g \in BV[a, b]$ and its total variation is $V_a^b(g) = \int_a^b |g'(x)| \, \mathrm{d}x$.*
- *A BV function is differentiable almost everywhere.*

Proposition 1.54 (The Stieltjes integral). *Let $-\infty < a < b < \infty$. Let $f : [a, b] \to \mathbb{C}$ be continuous, and let Let $g : [a, b] \to \mathbb{C}$ be of BV. We assume that g is normalized such that it is right-continuous. Then the approximating sum*

$$S(P, f, g) = \sum_{i=0}^{n-1} f(x_i)(g(x_{i+1}) - g(x_i))$$

converges as the norm of the partition (i.e. the length of the longest subinterval)

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$ tends to zero. This limit is called the Stieltjes integral (or the Riemann-Stieltjes integral) and is denoted by

$$\int_a^b f(x) \, dg(x).$$

Moreover, there holds the estimate

$$\left| \int_a^b f(x) \, dg(x) \right| \le \max_{[a,b]} |f(x)| V_a^b(g).$$

The Stieltjes integral is a generalization of the classical Riemann integral. Indeed, by taking $g(x) = x$, the Stieltjes integral becomes the Riemann integral.

If $f : \mathbb{R} \to \mathbb{C}$ is continuous, and let Let $g : \mathbb{R} \to \mathbb{C}$ is BV on each compact interval, we then set

$$\int_{-\infty}^{+\infty} f(x) \, dg(x) := \lim_{b \to +\infty, a \to -\infty} \int_a^b f(x) \, dg(x)$$

provided the limit exists.

Now we are ready to state the result about the dual space of $C[a, b]$.

Proposition 1.55. *Let $-\infty < a < b < \infty$. Then $f \in C[a,b]^*$ iff there exists a BV function $\rho : [a,b] \to \mathbb{R}$ such that*

$$f(u) = \int_a^b u(x)\, d\rho(x), \quad \text{for all } u \in C[a,b], \tag{1.54}$$

where the integral represents a Stieltjes integral. Moreover,

$$\|f\| = V_a^b(\rho).$$

证明. We know that $C[a,b]$ is a Banach space with norm $\|u\| = \sup_{[a,b]} |u(x)|$. Let f be defined as (1.54). By Proposition 1.54, one has

$$|f(u)| \le \|u\| V_a^b(\rho), \quad \text{for all } u \in C[a,b].$$

Hence $f \in C[a,b]^*$.

Given $f \in C[a,b]^*$, now we prove that f has a representation of the form (1.54).

Let Y denote the space of all bounded functions $u : [a,b] \to \mathbb{R}$. Then Y is a normed space with the same norm $\|\cdot\|$. Since $C[a,b]$ is a subspace of Y, it follows from the Hahn-Banach theorem that f can be extended to a linear continuous functional

$$F : Y \to \mathbb{R} \quad \text{with } \|F\| = \|f\|.$$

Set $\rho(t) := F(v_t)$ for all $t \in [a,b]$ where

$$v_t(x) := \begin{cases} 1, & a \le x \le t, \\ 0, & t < x \le b. \end{cases} \tag{1.55}$$

We claim that ρ is a BV function on $[a,b]$ and $V_a^b(\rho) \le \|f\|$. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$. Define $s_i := \operatorname{sgn}(\rho(x_{i+1}) - \rho(x_i))$. Then

$$\begin{aligned} \Delta_n &:= \sum_{i=0}^{n-1} |\rho(x_{i+1}) - \rho(x_i)| = \sum_{i=0}^{n-1} s_i \left(\rho(x_{i+1}) - \rho(x_i)\right) \\ &= \sum_{i=0}^{n-1} s_i \left(F(v_{x_{i+1}}) - F(v_{x_i})\right) = F\left(\sum_{i=0}^{n-1} s_i (v_{x_{i+1}} - v_{x_i})\right) \\ &\le \|F\| \left\| \sum_{i=0}^{n-1} s_i (v_{x_{i+1}} - v_{x_i}) \right\|. \end{aligned} \tag{1.56}$$

By the definition in (1.55),

$$(v_{x_{i+1}} - v_{x_i})(x) = \begin{cases} 1, & x_i < x \le x_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for all } i = 0, 1, \cdots, n-1. \tag{1.57}$$

Thus, for each $x_0 \in [a,b]$, there exists a unique i_0 such that $x_{i_0} < x \le x_{i_0+1}$. Thus

$$\left|\sum_{i=0}^{n-1} s_i(v_{x_{i+1}} - v_{x_i})(x_0)\right| = \left|s_i(v_{x_{i_0+1}} - v_{x_{i_0}})(x_0)\right| = 1.$$

This implies that

$$\left\|\sum_{i=0}^{n-1} s_i(v_{x_{i+1}} - v_{x_i})\right\| = 1.$$

Hence,

$$\Delta_n \le \|F\| = \|f\|.$$

By taking the supreme of the partitions to Δ_n, we obtain $V_a^b(\rho) \le \|f\|$.

In the last step, we show that ρ is the BV function such that the representation formula (1.54) holds. Again let $a = x_0 < x_1 < \cdots < x_n = b$ be the uniform partition of $[a,b]$: $x_i := a + \frac{i}{n}(b-a)$. Given $u \in C[a,b]$, we consider a sequence of step functions in Y defined as

$$u_n := \sum_{i=0}^{n-1} u(x_i)(v_{x_{i+1}} - v_{x_i})(x)$$

Since u is continuous on closed interval $[a,b]$, it is uniform continuous. Then it is straightforward to show that

$$u_n \to u \text{ in } Y, \quad \text{as } n \to \infty. \tag{1.58}$$

Since

$$F(u_n) = \sum_{i=0}^{n-1} u(x_i)\big(F(v_{x_{i+1}}) - F(v_{x_i})\big) = \sum_{i=0}^{n-1} u(x_i)\big(\rho_{x_{i+1}} - \rho_{x_i}\big),$$

and $\rho \in BV[a,b]$, by Proposition 1.54, we have

$$\lim_{n\to\infty} F(u_n) = \lim_{n\to\infty} \sum_{i=0}^{n-1} u(x_i)\big(\rho_{x_{i+1}} - \rho_{x_i}\big) = \int_a^b u(x)\,d\rho(x).$$

On the other hand, by the continuity of F and (1.58), we have

$$f(u) = F(u) = \lim_{n\to\infty} F(u_n) = \int_a^b u(x)\,d\rho(x).$$

We thus complete the proof.

□

1.22 Banach algebras and operator functions

Definition 1.56. *By a Banach algebra $\mathcal{B}$ over $\mathbb{F}$ we understand a Banach space over $\mathbb{F}$ where an additional multiplication AB is defined such that*

- $$AB \in \mathcal{B}, \quad \textit{for all } A, B \in \mathcal{B}. \tag{1.59}$$

- *For all* $A, B, C \in \mathcal{B}$ *and* $\alpha \in \mathbb{F}$,

$$(AB)C = A(BC),\ A(B+C) = AB + AC,\ (B+C)A = BA + CA,\ \alpha(AB) = (\alpha A)B = A(\alpha B) \tag{1.60}$$

- *For all* $A, B \in \mathcal{B}$,

$$\|AB\| \le \|A\|\|B\|. \tag{1.61}$$

- e *is called an identity of Banach algebra* $\mathcal{B}$ *provided*

$$eA = Ae = A \quad \textit{for all } A \in \mathcal{B}, \quad \|e\| = 1. \tag{1.62}$$

We remark that the condition (1.61) is not essential. If $\mathcal{B}$ is an algebra and has a norm $\|\cdot\|$ relative to which $\mathcal{B}$ is a Banach space and is such that the map of $\mathcal{B} \times \mathcal{B} \to \mathcal{B} : (A, B) \to AB$ is bounded, then there is an equivalent norm on $\mathcal{B}$ that satisfies (1.61). (why?)

If $\mathcal{B}$ has an identity e, then the map $\alpha \to \alpha e$ is an isomorphism of $\mathbb{F}$ into $\mathcal{B}$ and $\|\alpha e\| = |\alpha|$. So it will be assumed that $\mathbb{F} \subset \mathcal{B}$ via this identification. Thus the identity will be denoted by 1.

The content of the next proposition is that if $\mathcal{B}$ does not have an identity, it is possible to find a Banach algebra $\mathcal{B}_1$ that contains $\mathcal{B}$, that has an identity, and is such that $\dim \mathcal{B}_1/\mathcal{B} = 1$.

Proposition 1.57. *If Banach algebra* $\mathcal{B}$ *over* $\mathbb{F}$ *does not have an identity, let* $\mathcal{B}_1 := \mathcal{B} \times \mathbb{F}$. *Define algebraic operations on* $\mathcal{B}_1$ *by*

(i) $(A, \alpha) + (B, \beta) = (A + B, \alpha + \beta)$,

(ii) $\beta(A, \alpha) = (\beta A, \beta\alpha)$,

(iii) $(A, \alpha)(B, \beta) = (AB + \alpha B + \beta A, \alpha\beta)$,

for all $A, B \in \mathcal{B}$, *all* $\alpha, \beta \in \mathbb{F}$.

Define $\|(A, \alpha)\| = \|A\| + |\alpha|$. *Then* $\mathcal{B}_1$ *with this norm and the algebraic operations defined in (i), (ii), and (iii) is a Banach algebra with identity* $(0, 1)$ *and* $A \to (A, 0)$ *is an isometric isomorphism of* $\mathcal{B}$ *into* $\mathcal{B}_1$.

证明. Exercise. □

Now we come back the statement:

If $\mathcal{B}$ *is an algebra and has a norm* $\|\cdot\|$ *relative to which* $\mathcal{B}$ *is a Banach space and is such that the map of* $\mathcal{B} \times \mathcal{B} \to \mathcal{B} : (A, B) \to AB$ *is bounded, then there is an equivalent norm on* $\mathcal{B}$ *that satisfies* (1.61).

To prove this statement, we consider the Banach algebra $\mathcal{B}_1$ defined as in Proposition 1.57. For each $A \in \mathcal{B}$, we consider the linear operator

$$L_A : \mathcal{B}_1 \to \mathcal{B}_1; \quad L_A(B, \beta) = (AB + \beta A, 0) \text{ for each } (B, \beta) \in \mathcal{B}_1.$$

We claim that $\|A\|' := \|L_A\|$ is a equivalent to the norm of $\mathcal{B}$, and $\mathcal{B}$ is a Banach algebra with respect to the norm $\|\cdot\|'$. Indeed, on one hand, since the map of $\mathcal{B} \times \mathcal{B} \to \mathcal{B} : (A, B) \to AB$ is continuous, there holds

$$\|L_A(B, \beta)\|_{\mathcal{B}_1} = \|(AB + \beta A, 0)\|_{\mathcal{B}_1} \le C\|A\|\|B\| + |\beta|\|A\| \le C\|A\|(\|B\| + |\beta|) = C\|A\|\|(B, \beta)\|_{\mathcal{B}_1}.$$

This implies that

$$\|A\|' = \|L_A\| \le C\|A\|.$$

On the other hand, direct calculation gives

$$\|L_A(0, 1)\|_{\mathcal{B}_1} = \|(A, 0)\|_{\mathcal{B}_1} = \|A\| = \|A\|\|(0, 1)\|_{\mathcal{B}_1}$$

This implies that the operator norm of L_A satisfies

$$\|A\|' = \|L_A\| \ge \|A\|.$$

We thus have that the new norm $\|A\|' = \|L_A\|$ is a equivalent to the norm of $\mathcal{B}$. Since $L_{AB} = L_A \circ L_B$, we thus have

$$\|AB\|' = \|L_{AB}\| = \|L_A \circ L_B\| \le \|L_A\|\|L_B\| = \|A\|'\|B\|'$$

and we finally show that $\mathcal{B}$ is a Banach algebra with respect to the norm $\|\cdot\|'$.

Example 1.58.

- *Let X be a compact space, then $C(X)$ is a Banach algebra with the pointwise multiplication: $(fg)(x) = f(x)g(x)$. Note that $C(X)$ is albelian and has an identity: the constant* 1*.*
- *If X is a locally compact space, $C_0(X)$ is a Banach algebra with the multiplication defined pointwisely as in the preceding example. $C_0(X)$ is abelian, but if X is not compact, $C_0(X)$ does not have an identity: $1 \notin C_0(X)$.*
- *If (Ω, μ) is a σ-finite measure space, then $L^\infty(\Omega, \mu)$ is an abelian Banach algebra with identity.*
- *Let X be a Banach space. Then $L(X, X)$ with multiplication defined by composition is a Banach algebra with identity: the identity mapping. If $\dim X \ge 2$, $L(X, X)$ is not abelian.*

We refer to Chapter 5 of Conway [4] for more details about examples.

Proposition 1.59. *Let $\mathcal{B}$ be a Banach algebra with identity e, and let $A, B, A_n, B_n \in \mathcal{B}$ for all n. Then*

- $\|A^k\| \le \|A\|^k$ *for all* $k = 0, 1, \cdots$, *where we set* $A^0 = e$.
- *If* $A_n \to A$ *and* $B_n \to B$ *in* $\mathcal{B}$, *then* $A_n B_n \to AB$ *in* $\mathcal{B}$

证明. Exercise. □

1.23 Infinite series in normed spaces

Definition 1.60. *Let X be a normed space over $\mathbb{F}$. Let $\{u_j\}_{j=0}^{\infty}$ be a sequence in X. We set*

$$\sum_{j=0}^{\infty} u_j := \lim_{n\to\infty} \sum_{j=0}^{n} u_j$$

provided the limit exists in X. This infinite series is called absolutely convergent iff

$$\sum_{j=0}^{\infty} \|u_j\| < \infty.$$

Proposition 1.61. *Each absolutely convergent infinite series in a Banach space is convergent.*

证明. Exercise. □

1.24 Operator functions in Banach algebra

Proposition 1.62. *Let $z \in \mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Define*

$$F(z) := \sum_{j=0}^{\infty} a_j z^j$$

where

$$\sum_{j=0}^{\infty} |a_j||z|^j < \infty \quad \text{for all } |z| < r \text{ with a fixed } r > 0.$$

Let $\mathcal{B}$ is a Banach algebra and let $A \in \mathcal{B}$ such that $\|A\| < r$. Then the following infinite series

$$F(A) := \sum_{j=0}^{\infty} a_j A^j$$

converges in $\mathcal{B}$.

证明. Let $A \in \mathcal{B}$ with $\|A\| < r$. Then the positive series

$$\sum_{j=0}^{\infty} a_j \|A^j\| \le \sum_{j=0}^{\infty} a_j \|A\|^j$$

converges. This means the series

$$\sum_{j=0}^{\infty} a_j A^j$$

is absolutely convergent. Hence,

$$F(A) := \sum_{j=0}^{\infty} a_j A^j$$

converges in $\mathcal{B}$.

□

In particular, the result in Proposition 1.62 holds for linear operators in $\mathcal{B} = L(X, X)$ where X is a Banach space. In the following, we give more properties of the infinite series on Banach algebras. The results are given for the special case $\mathcal{B} = L(X, X)$ where X is a Banach space, while they hold for all Banach algebras.

Proposition 1.63 (The exponential function)**.** *Let X be a Banach space over $\mathbb{F}$. Then*

(i) The infinite series

$$e^A := \sum_{j=0}^{\infty} \frac{1}{j!} A^j \tag{1.63}$$

converges absolutely for all $A \in L(X, X)$.

(ii) For each $A \in L(X, X)$ and all $t, s \in \mathbb{F}$,

$$e^{tA} e^{sA} = e^{(t+s)A}. \tag{1.64}$$

(iii) Let $A, B \in L(X, X)$ satisfying $AB = BA$, then

$$e^A e^B = e^{A+B}. \tag{1.65}$$

证明. (i). This follows from the fact that

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \tag{1.66}$$

converges absolutely for all $z \in \mathbb{C}$.

(ii). This is a direct consequence of (iii).

(iii) Let $A, B \in L(X, X)$ satisfying $AB = BA$. We denote the partial sum

$$S_n(A) := \sum_{j=0}^{n} \frac{A^j}{j!}, \; S_n(B) := \sum_{j=0}^{n} \frac{B^j}{j!}, \; S_n(A+B) := \sum_{j=0}^{n} \frac{(A+B)^j}{j!}. \tag{1.67}$$

Since $AB = BA$, direct calculation implies

$$\begin{aligned}
S_n(A)S_n(B) - S_{2n}(A+B) &= \sum_{j=0}^{n} \frac{A^j}{j!} \sum_{k=0}^{n} \frac{B^k}{k!} - \sum_{l=0}^{2n} \frac{(A+B)^l}{l!} \\
&= \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{A^jB^k}{j!k!} - \sum_{l=0}^{2n}\sum_{j=0}^{l} \frac{1}{l!}\frac{l!}{j!(l-j)!}A^jB^{l-j} \\
&= \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{A^jB^k}{j!k!} - \sum_{l=0}^{2n}\sum_{j+k=l} \frac{1}{j!k!}A^jB^k \\
&= \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{A^jB^k}{j!k!} - \sum_{0\le j+k\le 2n} \frac{1}{j!k!}A^jB^k \\
&= \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{A^jB^k}{j!k!} - \sum_{j=0}^{n}\sum_{k=0}^{n} \frac{A^jB^k}{j!k!} - \sum_{j=n+1}^{2n}\sum_{k=0}^{2n-j} \frac{A^jB^k}{j!k!} - \sum_{k=n+1}^{2n}\sum_{j=0}^{2n-k} \frac{A^jB^k}{j!k!} \\
&= -\sum_{j=n+1}^{2n}\sum_{k=0}^{2n-j} \frac{A^jB^k}{j!k!} - \sum_{k=n+1}^{2n}\sum_{j=0}^{2n-k} \frac{A^jB^k}{j!k!}.
\end{aligned} \tag{1.68}$$

This implies that

$$\begin{aligned}
\|S_n(A)S_n(B) - S_{2n}(A+B)\| &\le \sum_{j=n+1}^{2n} \frac{\|A\|^j}{j!} \sum_{k=0}^{2n-j} \frac{\|B\|^k}{k!} + \sum_{k=n+1}^{2n} \frac{\|B\|^k}{k!} \sum_{j=0}^{2n-k} \frac{\|A\|^j}{j!} \\
&\le e^{\|B\|} \sum_{j=n+1}^{2n} \frac{\|A\|^j}{j!} + e^{\|A\|} \sum_{k=n+1}^{2n} \frac{\|B\|^k}{k!}.
\end{aligned} \tag{1.69}$$

Passing $n \to \infty$ in (1.68) and using Proposition 1.59 implies (1.65).

□

Proposition 1.64 (The geometric series)**.** *Let X be a Banach space over $\mathbb{F}$ with $X \neq \{0\}$. For each operator $A \in L(X,X)$ with $\|A\| < 1$, the infinite series*

$$B := \sum_{j=0}^{\infty} A^j$$

converges absolutely to an operator $B \in L(X,X)$. In addition,

$$B = (I - A)^{-1},$$

where I is the identity operator.

证明. The classical geomotric series

$$\sum_{j=0}^{\infty} z^j$$

converges absolutely for all $z \in \mathbb{C}$ with $|z| < 1$. Thus the series

$$\sum_{j=0}^{\infty} A^j$$

converges absolutely provided $\|A\| < 1$.

Clearly,

$$(I - A)B = B - AB = \sum_{j=0}^{\infty} A^j - \sum_{j=0}^{\infty} A^{j+1} = A^0 = I$$

and

$$B(I - A) = B - BA = \sum_{j=0}^{\infty} A^j - \sum_{j=0}^{\infty} A^{j+1} = A^0 = I$$

Hence $B = (I - A)^{-1}$.

□

Let X and Y be Banach spaces over $\mathbb{F}$ with $X \neq \{0\}$ and $Y \neq \{0\}$. Denote $L_{inv}(X, Y)$ the set of all operators $A \in L(X, Y)$ such that the inverse operator A^{-1} exists and $A^{-1} \in L(Y, X)$.

Proposition 1.65. *If $A \in L_{inv}(X, Y)$ and $B \in L(Y, X)$ with*

$$\|B\| < \|A^{-1}\|^{-1},$$

then $A - B \in L_{inv}(X, Y)$.

证明. Let $A \in L_{inv}(X, Y)$. It follows from $AA^{-1} = I$ that $A^{-1} \neq 0$. Hence $\|A^{-1}\| \neq 0$ and $\|A^{-1}\|^{-1}$ is well defined.

Since

$$\|A^{-1}B\| \leq \|A^{-1}\|\|B\| < 1,$$

we have that $(I - A^{-1}B) \in L(X, X)$ is invertible with

$$(I - A^{-1}B)^{-1} = \sum_{j=0}^{\infty} (A^{-1}B)^j.$$

Hence,

$$A(I - A^{-1}B) = A - B$$

is invertible.

□

Corollary 1.66. *The subset $L_{inv}(X, Y)$ is open in $L(X, Y)$.*

1.25 Applications to linear differential equations in Banach spaces

Definition 1.67. *Let X be a normed space over $\mathbb{F}$, let $U(t_0) \subset \mathbb{R}$ be a open neighborhood of the point $t_0 \in \mathbb{R}$, and let*

$$u : U(t_0) \subset \mathbb{R} \to X$$

be a function with values in X. We define the derivative

$$u'(t_0) := \lim_{h \to 0} \frac{u(t_0 + h) - u(t_0)}{h}$$

provided the limit exists in X.

Proposition 1.68. *If the derivative $u'(t_0)$ exists, then the function u is continuous at the point t_0.*

证明. The identity

$$u(t_0 + h) - u(t_0) = h \cdot \frac{u(t_0 + h) - u(t_0)}{h}$$

yields

$$u(t_0 + h) - u(t_0) \to 0 \text{ in } X, \text{ as } h \to 0.$$

□

Proposition 1.69. *Let X be a Banach space over $\mathbb{F}$, let $u : \mathbb{R} \to X$, and let $A \in L(X, X)$. Given initial datum $u_0 \in X$, the following initial-valued problem:*

$$\begin{aligned} &u'(t) = Au(t), \quad -\infty < t < +\infty, \\ &u(0) = u_0 \end{aligned} \tag{1.70}$$

admits a unique solution given by

$$u(t) = e^{tA} u_0, \quad \textit{for all } t \in \mathbb{R}. \tag{1.71}$$

Example 1.70. $X = \mathbb{R}^d$, $A = (a_{jk})_{1 \le j,k \le d}$ *is a real $d \times d$ matrix, $x(t) = (x_1, \cdots, x_d)(t) \in \mathbb{R}^d$. Then the equation*

$$x'(t) = Ax(t), \ x(0) = x_0 \in \mathbb{R}^d$$

has a unique solution $x(t) = e^{tA} x_0$.

证明. **Existence.** Let $h \in \mathbb{R}$. It follows from

$$e^{hA} = \sum_{j=0}^{\infty} \frac{1}{j!} (hA)^j = I + hA + \frac{h^2}{2!} A^2 + \cdots$$

that

$$\begin{aligned} \|h^{-1}(e^{hA} - I) - A\| = \|\sum_{j=2}^{\infty} \frac{h^{j-1}}{j!} A^j\| &\le |h| \sum_{j=2}^{\infty} \frac{|h|^{j-2}}{j!} \|A\|^j \\ &\le \|A\|^2 e^{|h|\|A\|} |h| \to 0, \quad \text{as } h \to 0. \end{aligned} \tag{1.72}$$

Since

$$e^{(t+h)A} = e^{tA}e^{hA} = e^{hA}e^{tA} \quad \text{for all } t, h \in \mathbb{R},$$

we get for $u(t) = e^{tA}u_0$ that

$$\begin{aligned} \|h^{-1}(u(t+h) - u(t)) - Au(t)\| &= \|h^{-1}(e^{(t+h)A}u_0 - e^{tA}u_0) - Ae^{tA}u_0\| \\ &= \|\big(h^{-1}(e^{hA} - I) - A\big)e^{tA}u_0\| \\ &\le \|h^{-1}(e^{hA} - I) - A\| \cdot \|e^{tA}\| \cdot \|u_0\| \to 0, \quad \text{as } h \to 0. \end{aligned} \tag{1.73}$$

This implies that $u'(t) = Au(t)$ for all $t \in \mathbb{R}$. In addition $u(0) = e^{0A}u_0 = u_0$.

Uniqueness. Let $u(t)$ and $v(t)$ be two solution to (1.70). Let $w(t) = u(t) - v(t)$. Then

$$w'(t) = Aw(t), \quad t \in \mathbb{R}; \quad w(0) = 0. \tag{1.74}$$

We shall show that $w(t) = 0$ for all $t \in \mathbb{R}$. Given $f \in X^*$. By (1.74), we have

$$\big\langle f, w'(t)\big\rangle = \langle f, Aw(t)\rangle \quad \text{for all } t \in \mathbb{R}.$$

Since $f : X \to \mathbb{R}$ is linear and continuous, we thus have

$$\begin{aligned} \frac{d}{dt}\langle f, w(t)\rangle &= \lim_{h\to 0} \frac{\langle f, w(t+h)\rangle - \langle f, w(t)\rangle}{h} \\ &= \lim_{h\to 0} \left\langle f, \frac{w(t+h) - w(t)}{h} \right\rangle \\ &= \left\langle f, \lim_{h\to 0} \frac{w(t+h) - w(t)}{h} \right\rangle \\ &= \big\langle f, w'(t)\big\rangle = \langle f, Aw(t)\rangle \quad \text{for all } t \in \mathbb{R}. \end{aligned} \tag{1.75}$$

Since $w(t)$ is differentiable at each $t \in \mathbb{R}$, so $w(t) : \mathbb{R} \to X$ is continuous at each $t \in \mathbb{R}$. Thus the function

$$t \to \langle f, Aw(t)\rangle$$

is continuous at all $t \in \mathbb{R}$, due to the continuity of f and A. Integrating (1.75) in t and observing $\langle f, Aw(0)\rangle = 0$, we obtain

$$\langle f, w(t)\rangle = \int_0^t \big\langle f, Aw(t')\big\rangle\, dt' \quad \text{for all } t \in \mathbb{R}. \tag{1.76}$$

Thus, for all t with $|t| \le h$ where $h > 0$, there holds

$$\begin{aligned} |\langle f, w(t)\rangle| &\le \int_0^t \big|\big\langle f, Aw(t')\big\rangle\big|\, dt' \\ &\le \int_0^t \|f\|\,\|A\|\,\|w(t')\| dt'. \end{aligned} \tag{1.77}$$

By Corollary 1.49, we have for all $t \in [-h, h]$ that

$$\|w(t)\| = \sup_{f\in X^*, \|f\|\le 1} |\langle f, w(t)\rangle| \le \int_0^t \|A\|\,\|w(t')\|dt' \le h\|A\| \max_{|t'|\le h} \|w(t')\|. \tag{1.78}$$

This implies that

$$\max_{|t|\le h} \|w(t)\| \le h\|A\| \max_{|t|\le h} \|w(t)\|. \tag{1.79}$$

If $A = 0$, clearly $w(t) = 0$ for all $t \in \mathbb{R}$.

If $A \neq 0$, we choose $h := \frac{1}{2\|A\|}$. It follows from (1.79) that

$$\max_{|t|\le h} \|w(t)\| \le h\|A\| \max_{|t|\le h} \|w(t)\| = \frac{1}{2} \max_{|t|\le h} \|w(t)\|,$$

and a consequence is that $\max_{|t|\le h} \|w(t)\| = 0$. We proved that $w(t) = 0$ for all $t \in [-h, h]$.

Now we apply the same result to the initial-value problems

$$w'(t) = Aw(t), \quad t \in \mathbb{R}; \quad w(\pm h) = 0, \tag{1.80}$$

to deduce that $w(t) = 0$ for all $t \in [-2h, 2h]$. Continuing this, we obtain $w(t) = 0$ for all $t \in \mathbb{R}$.

□

1.26 Applying to the spectrum

Definition 1.71. *Let $A \in L(X, X)$ with X a nontrivial Banach space over $\mathbb{C}$.*

- *A complex number λ is called an* eigenvalue *of the operator A provided there exists a nonzero vector $u \in X$ such that*

$$Au = \lambda u. \tag{1.81}$$

- *The* resolvent set *$\rho(A)$ of A is defined to be the collection of all complex numbers λ such that $(A - \lambda I)^{-1} : X \to X$ exists and $(A - \lambda I)^{-1} \in L(X, X)$. If $\lambda \in \rho(A)$, the operator $(A - \lambda I)^{-1}$ is called a resolvent of A.*

- *The* spectrum *$\sigma(A)$ is defined as $\sigma(A) = \rho(A)^c$.*

Proposition 1.72. *Let $A \in L(X, X)$ with X a nontrivial Banach space over $\mathbb{C}$. Then*

(i) The resolvent set $\rho(A)$ is open in $\mathbb{C}$.

(ii) The spectrum $\sigma(A)$ is compact in $\mathbb{C}$ and

$$|\lambda| \le \|A\|, \quad \text{for all } \lambda \in \sigma(A).$$

(iii) Each eigenvalue of A belongs to the spectrum of A.

证明. (i). Let $\lambda \in \rho(A)$, then $A - \lambda I \in L_{inv}(X, X)$. Let $\mu \in \mathbb{C}$, we have

$$\|(A - \lambda I) - (A - \mu I)\| = \|(\mu - \lambda)I\| = |\lambda - \mu|.$$

By (1.66), we know that $L_{inv}(X, X)$ is an open set in $L(X, X)$. Thus for all μ with $|\lambda - \mu|$ sufficient small, we have $(A - \mu I) \in L_{inv}(X, X)$, i.e. $\mu \in \rho(A)$. This means $\rho(A)$ is open.

(ii). If $\lambda > \|A\|$, then

$$\|\lambda^{-1} A\| = |\lambda^{-1}| \, \|A\| < 1.$$

It follows from (1.65) that

$$\lambda^{-1} A - I \in L_{inv}(X, X).$$

This implies that

$$A - \lambda I = \lambda(\lambda^{-1} A - I) \in L_{inv}(X, X).$$

This means $\lambda \in \rho(A)$, i.e. $\lambda \notin \sigma(A)$. Thus, for any $\lambda \in \sigma(A)$, there holds $|\lambda| \leq \|A\|$. Consequently, the spectrum $\sigma(A) = \rho(A)^c$ is closed and bounded in $\mathbb{C}$, i.e. $\sigma(A)$ is compact in $\mathbb{C}$.

(iii). If $\lambda \notin \sigma(A)$, i.e. $\lambda \in \rho(A)$, by definition we know that $(A - \lambda I) \in L_{inv}(X, X)$. If there exists $u \in X$ such that $Au = \lambda u$, there must hold

$$u = (A - \lambda I)^{-1}(A - \lambda I)u = (A - \lambda I)^{-1}(0) = 0.$$

This means λ is not an eigenvalue of A.

□

2 Hilbert spaces, orthogonality, and variational problems

2.1 Hilbert spaces

Definition 2.1.

- *Let X be a linear space over $\mathbb{K}$. An semi-inner product on X is a function, denoted by $(\cdot, \cdot)$ or $(\cdot|\cdot) : X \times X \to \mathbb{K}$, such that for all $u, v, w \in X$ and all $\alpha, \beta \in \mathbb{K}$:*

  (i) $(u, u) \geq 0$.

  (ii) $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w)$.

  (iii) $(u, v) = \overline{(v, u)}$, *where $\overline{\alpha}$ denotes the complex conjugate of α for each $\alpha \in \mathbb{K}$.*

- *The property (ii) implies that for all $u, v \in X$:*

$$(u, \mathbf{0}) = (u, 0 \cdot \mathbf{0}) = 0(u, \mathbf{0}) = 0, \quad (\mathbf{0}, v) = (0 \cdot \mathbf{0}, v) = 0(\mathbf{0}, v) = 0.$$

  In particular $(\mathbf{0}, \mathbf{0}) = 0$.

- *An inner product is a semi-inner product that also satisfies the following property:* $(u, u) = 0$ *iff* $u = \mathbf{0}$.
- *Let* $u, v \in X$. *We say* u *is orthogonal to* v *iff* $(u, v) = 0$.

We recall some basic properties about inner product. The proofs can be found in many textbooks on functional analysis, see for example [4].

Proposition 2.2. *Let* X *be a linear space with semi-inner product* $(\cdot, \cdot)$. *Then*

$$|(u, v)| \leq (u, u)^{\frac{1}{2}} (v, v)^{\frac{1}{2}}, \quad \textit{for all } u, v \in X.$$

Moreover, equality occurs iff there are $\alpha, \beta \in \mathbb{F}$ *both not* 0, *such that*

$$(\alpha u + \beta v, \alpha u + \beta v) = 0.$$

Proposition 2.3. *Let* X *be a linear space with inner product* $(\cdot, \cdot)$. *Then* X *is a normed space with respect to the norm*

$$\|u\| := (u, u)^{\frac{1}{2}}, \quad \textit{for all } u \in X.$$

Let X be a linear space with inner product $(\cdot, \cdot)$. We know from Proposition 2.3 that X is a normed space. In the sequel, we will give the natural topology to each linear space with inner product $(\cdot, \cdot)$ where the norm is defined as in Proposition 2.3.

Proposition 2.4. *Let* X *be a linear space with inner product* $(\cdot, \cdot)$. *Then*

- *The inner product is continuous in the sense that if*

  $$u_n \to u, \; v_n \to v, \quad \textit{as } n \to \infty,$$

  then

  $$(u_n, v_n) \to (u, v), \quad \textit{as } n \to \infty.$$

- *Let* M *be a dense subset of* X. *If*

  $$(u, v) = 0 \quad \textit{for all } v \in M,$$

  then $u = 0$.

Definition 2.5 (Hilbert space)**.** *Let* X *be a linear space with inner product* $(\cdot, \cdot)$. *If* X *is a Banach space with respect to the natural norm given in Proposition 2.3, we say* X *is a Hilbert space.*

Example 2.6. $\mathbb{R}^d$, $L^2(\Omega)$, $W^{1,2}(\Omega)$, $H^s(\mathbb{R}^d)$, *and so on.*

2.2 Friedrichs' mollifier and density of smooth functions in L^p spaces

The standard Friedrichs' mollifier is base on the C_c^∞ function

$$\phi(x) = \tilde{\phi}(|x|) := c\, e^{\frac{-1}{1-|x|^2}},\ |x| < 1; \quad \phi(x) := 0,\ |x| \geq 1, \tag{2.1}$$

where c is the renormalized constant defined as

$$c = \left(\int_{|x|<1} e^{\frac{-1}{-1-|x|^2}} \, \mathrm{d}x \right)^{-1}.$$

It can be shown that

- $\phi \geq 0, \quad \phi \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}\phi \subset \overline{B(0,1)}.$
- $\int_{\mathbb{R}^d} \phi(x) \, \mathrm{d}x = 1.$

We remark that any C_c^∞ function satisfying the above two properties can be used to define the standard Friedrichs' mollifier, not necessarily the precise form in (2.1).

For any $0 < \varepsilon < 1$, the standard Friedrichs' mollifier is defined as

$$\phi_\varepsilon(\cdot) = \frac{1}{\varepsilon^d} \phi(\frac{\cdot}{\varepsilon}). \tag{2.2}$$

Then

- $\phi_\varepsilon \geq 0, \quad \phi \in C_c^\infty(\mathbb{R}^d), \quad \mathrm{supp}\phi_\varepsilon \subset \overline{B(0,\varepsilon)}.$
- $\int_{\mathbb{R}^d} \phi_\varepsilon(x) \, \mathrm{d}x = 1.$

For any $u \in L^1_{loc}(\mathbb{R}^d)$, one can define its mollification

$$S_\varepsilon[u](x) := \int_{\mathbb{R}^d} \phi_\varepsilon(x-y) u(y) \, \mathrm{d}y, \tag{2.3}$$

and we have

Proposition 2.7. *Let $u \in L^1_{loc}(\mathbb{R}^d)$. Then $S_\varepsilon[u] \in C^\infty(\mathbb{R}^d)$.*

证明. Exercise. □

Moreover:

Proposition 2.8. *Let Ω be an open set in $\mathbb{R}^d$.*

(i) If $u \in C_c(\Omega)$, then

$$\|S_\varepsilon[u]\|_{L^\infty(\mathbb{R}^d)} \leq \|u\|_{L^\infty(\Omega)}, \quad S_\varepsilon[u] \to u \text{ in } L^\infty(\Omega).$$

(ii) If $u \in L^p(\Omega)$, $1 \le p < \infty$, then

$$\|S_\varepsilon[u]\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)}, \quad S_\varepsilon[u] \to u \text{ in } L^p(\Omega).$$

证明. We first prove (i). For any $x \in \Omega$, direct calculation gives

$$|S_\varepsilon[u](x)| \le \int_{\mathbb{R}^d} \phi_\varepsilon(x-y)|u(y)|\,\mathrm{d}y \le \|u\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d} \phi_\varepsilon(x-y)\,\mathrm{d}y = \|u\|_{L^\infty(\Omega)}, \tag{2.4}$$

and

$$\begin{aligned} S_\varepsilon[u](x) - u(x) &= \int_{\mathbb{R}^d} \phi_\varepsilon(x-y)u(y)\,\mathrm{d}y - u(x) = \int_{\mathbb{R}^d} \phi_\varepsilon(y)u(x-y)\,\mathrm{d}y - u(x) \\ &= \varepsilon^{-d} \int_{\mathbb{R}^d} \phi(y/\varepsilon)u(x-y)\,\mathrm{d}y - u(x) = \int_{\mathbb{R}^d} \phi(y)u(x-\varepsilon y)\,\mathrm{d}y - u(x) \\ &= \int_{\mathbb{R}^d} \phi(y)\big(u(x-\varepsilon y) - u(x)\big)\,\mathrm{d}y \le \int_{|y|\le 1} \phi(y)\,\mathrm{d}y \sup_{|y|\le 1} \big|(u(x-\varepsilon y) - u(x)\big|. \end{aligned} \tag{2.5}$$

Since $u \in C_c(\Omega)$ is continuous and of compact support, u is then uniform continuous on $\mathbb{R}^d$. Thus,

$$\sup_{x\in\Omega} \sup_{|y|\le\varepsilon} \big|(u(x-y) - u(x)\big| \to 0, \quad \text{as } \varepsilon \to 0.$$

Hence,

$$\sup_{x\in\Omega} \|S_\varepsilon[u](x) - u(x)\| \le \sup_{x\in\Omega} \sup_{|y|\le 1} \big|(u(x-\varepsilon y) - u(x)\big| \to 0. \tag{2.6}$$

We then prove (ii). Let $1 \le p < \infty$. By Minkowski's integral inequality, we have

$$\|S_\varepsilon[u](x)\|_{L^p(\Omega)} = \left\| \int_{\mathbb{R}^d} \phi_\varepsilon(y)u(x-y)\,\mathrm{d}y \right\|_{L^p(\Omega)} \le \int_{\mathbb{R}^d} \phi_\varepsilon(y)\|u(x-y)\|_{L^p(\Omega)}\,\mathrm{d}y = \|u\|_{L^p(\Omega)}. \tag{2.7}$$

Again by Minkowski's integral inequality, direct calculation gives

$$\begin{aligned} \|S_\varepsilon[u] - u\|_{L^p(\Omega)} &= \left\| \int_{|y|\le 1} \phi(y)\big(u(x-\varepsilon y) - u(x)\big)\,\mathrm{d}y \right\|_{L^p(\Omega)} \\ &\le \sup_{|y|\le 1} \|u(x-\varepsilon y) - u(x)\|_{L^p(\Omega)} \end{aligned} \tag{2.8}$$

Since $C_c(\Omega)$ is dense in $L^p(\Omega)$ for each $1 \le p < \infty$ (see Theorem 3.14 in [7]), for any $\delta > 0$, there exists $u^{(\delta)} \in C_c(\Omega)$ such that

$$\|u^{(\delta)} - u\|_{L^p(\Omega)} \le \delta.$$

Thus,

$$\begin{aligned} \|S_\varepsilon[u] - u\|_{L^p(\Omega)} &\le \left\|S_\varepsilon[u^{(\delta)}] - u^{(\delta)}\right\|_{L^p(\Omega)} + \left\|S_\varepsilon[u^{(\delta)} - u]\right\|_{L^p(\Omega)} + \left\|u^{(\delta)} - u\right\|_{L^p(\Omega)} \\ &\le \left\|S_\varepsilon[u^{(\delta)}] - u^{(\alpha)}\right\|_{L^p(\Omega)} + 2\delta \\ &\le \sup_{|y|\le 1} \left\|u^{(\delta)}(x-\varepsilon y) - u^{(\delta)}(x)\right\|_{L^p(\Omega)} + 2\delta. \end{aligned} \tag{2.9}$$

Passing $\varepsilon \to 0$ in above equation implies

$$\lim_{\varepsilon\to 0} \|S_\varepsilon[u] - u\|_{L^p(\Omega)} \le 2\delta, \tag{2.10}$$

where the limit

$$\lim_{\varepsilon\to 0} \sup_{|y|\le 1} \left\| u^{(\delta)}(x-\varepsilon y) - u^{(\delta)}(x) \right\|_{L^p(\Omega)} = 0$$

can be shown by the uniform continuity of u^δ on its compact support. Finally passing $\delta \to 0$ implies our desired convergence result.

□

Let Ω be an open set in $\mathbb{R}^d$. Let $C_0(\Omega)$ be the set of all continuous functions that vanish at infinity, i.e. a continuous function $u \in C_0(\Omega)$ iff for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \Omega$ such that $|u| \le \varepsilon$ in $\Omega \setminus K_\varepsilon$. Armed with the natural L^∞ norm for continuous functions, $C_0(\Omega)$ is a Banach space. We then have

Proposition 2.9. *The set $C_c^\infty(\Omega)$ is dense in $C_0(\Omega)$ and $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for all $1 \le p < \infty$.*

证明. We prove the first part. Given $u \in C_0(\Omega)$. We would like to show that for any given $\delta > 0$, there exists $v \in C_c^\infty(\Omega)$ such that

$$\sup_{x\in\Omega} |u(x) - v(x)| \le \delta. \tag{2.11}$$

Firstly, by the definition of $C_0(\Omega)$, there exists a compact set $K_\delta \subset \Omega$ such that

$$\sup_{x\in\Omega\setminus K_\delta} |u(x)| \le \delta/3. \tag{2.12}$$

Since K_δ is compact and Ω is open, and $K_\delta \subset \Omega$, there exists $\delta_0 > 0$ such that

$$K_{\delta,2\delta_0} := \{x \in \mathbb{R}^d : \operatorname{dist}(x, K_\delta) \le 2\delta_0\} \subset \Omega. \tag{2.13}$$

Clearly $K_{\delta,2\delta_0}$ is compact and $K \subset K_{\delta,2\delta_0}$.

We then introduce

$$u^{(\delta)}(x) = u(x) \text{ if } x \in K_{\delta,\delta_0}, \quad u^{(\delta)}(x) = 0 \text{ if } x \in \Omega \setminus K_{\delta,\delta_0}, \tag{2.14}$$

where

$$K_{\delta,\delta_0} := \{x \in \mathbb{R}^d : \operatorname{dist}(x, K_\delta) \le \delta_0\}.$$

Let $0 < \varepsilon < \delta_0$ and consider $S_\varepsilon[u^{(\delta)}]$. We introduce the following lemma, and the proof is left as an exercise:

Lemma 2.10. *For any $u \in L^1_{loc}(\Omega)$,*

$$\operatorname{supp} S_\varepsilon[u] \subset \overline{B(0,\varepsilon)} + \operatorname{supp} u = \{x + y : |x| \le \varepsilon,\ y \in \operatorname{supp} u\}. \tag{2.15}$$

By the above lemma,

$$\operatorname{supp} S_\varepsilon[u^{(\delta)}] \subset \overline{B(0,\varepsilon)} + K_{\delta,\delta_0} \subset K_{\delta,2\delta_0} \subset \Omega.$$

This means $S_\varepsilon[u^{(\delta)}] \in C_c^\infty(\Omega)$. Moreover, by the proof of Proposition 2.8, we have for all $x \in K_\delta$:

$$\begin{aligned} |S_\varepsilon[u^{(\delta)}](x) - u(x)| = |S_\varepsilon[u^{(\delta)}](x) - u^{(\delta)}(x)| &= \left| \int_{\mathbb{R}^d} \phi(y)\big(u(x-\varepsilon y) - u(x)\big)\,\mathrm{d}y \right| \\ &\leq \sup_{|y|\leq 1} \big|(u^{(\delta)}(x-\varepsilon y) - u^{(\delta)}(x)\big| \\ &\leq \sup_{|y|\leq 1} \big|(u(x-\varepsilon y) - u(x)\big| \end{aligned} \tag{2.16}$$

where we used the fact that $x - \varepsilon y \in K_{\delta,\delta_0}$ for all $|y| \leq 1,\ x \in K_\delta$. Since u is uniform continuous on compact set K_{δ,δ_0}, we thus have for ε small that

$$\sup_{x\in K_\delta} |S_\varepsilon[u^{(\delta)}](x) - u(x)| \leq \delta/3. \tag{2.17}$$

Thus for each $x \in \Omega \setminus K_\delta$,

$$\begin{aligned} |S_\varepsilon[u^{(\delta)}](x) - u(x)| &\leq |S_\varepsilon[u^{(\delta)}](x)| + |u(x)| \\ &\leq \int_{\mathbb{R}^d} \phi(y)|u^{(\delta)}(x-\varepsilon y)|\mathrm{d}y + \delta/3 \\ &\leq \sup_{|y|\leq\varepsilon} |u^{(\delta)}(x-y)| + \delta/3. \end{aligned} \tag{2.18}$$

Since $x \in \Omega \setminus K_\delta$ and $\Omega \setminus K_\delta$ is an open set, for ε sufficient small, we have $x - y \in \Omega \setminus K_\delta$ provided $|y| \leq \varepsilon$. Thus,

$$|S_\varepsilon[u^{(\delta)}](x) - u(x)| \leq \sup_{z\in\Omega\setminus K_\delta} |u^\delta(z)| + \delta/3 \leq \delta/3 + \delta/3 < \delta, \quad \text{for each } x \in \Omega \setminus K_\delta. \tag{2.19}$$

Then by (2.17) and (2.19), we obtain that

$$\sup_{x\in\Omega} |S_\varepsilon[u^{(\delta)}](x) - u(x)| \leq \delta, \tag{2.20}$$

where $S_\varepsilon[u^{(\delta)}] \in C_c^\infty(\Omega)$.

We now prove the second part. Given $u \in L^p(\Omega)$. We may employ the argument in the proof of the first part by observing the fact that for any $\delta > 0$ there exists a compact subset K_δ of Ω such that (why?)

$$\|u\|_{L^p(\Omega\setminus K_\delta)} \leq \delta.$$

We may also direct use the fact that $C_c(\Omega)$ is dense in $L^p(\Omega)$ to prove our result. Given $u \in L^p(\Omega)$ with $1 \leq p < \infty$. For any $\delta > 0$, there exists $u^{(\delta)} \in C_c(\Omega)$ such that

$$\|u - u^{(\delta)}\|_{L^p(\Omega)} \leq \delta/2.$$

By Proposition 2.9 and Lemma 2.10, we have for sufficient small ε that

$$S_\varepsilon[u^{(\delta)}] \in C_c^\infty(\Omega), \quad \|S_\varepsilon[u^{(\delta)}] - u^{(\delta)}\|_{L^p(\Omega)} \leq \delta/2.$$

Finally

$$\|S_\varepsilon[u^{(\delta)}] - u\|_{L^p(\Omega)} \leq \delta.$$

□

A direct corollary is the following:

Corollary 2.11. *Let Ω be a nonempty open set in $\mathbb{R}^d$ and let $u \in L^p(\Omega),\ 1 < p < \infty$. If*

$$\int_\Omega u\, v\, \mathrm{d}x = 0, \quad \text{for all } v \in C_c^\infty(\Omega),$$

then $u(x) = 0$ for almost all $x \in \Omega$.

Problem: What the case $p = 1,\ p = +\infty$?

证明. Exercise.

□

2.3 The space C_c^∞ and integration by parts

The classical integration by parts formula reads as follows:

$$\int_a^b u'\, v\, \mathrm{d}x = u\, v|_a^b - \int_a^b u\, v'\, \mathrm{d}x \tag{2.21}$$

with the *boundary integral*

$$u\, v|_a^b = u(b)\, v(b) - u(a)\, v(a). \tag{2.22}$$

In particular, if $v(a) = v(b) = 0$, there holds

$$\int_a^b u'\, v\, \mathrm{d}x = -\int_a^b u\, v'\, \mathrm{d}x \tag{2.23}$$

In higher dimensions, similar integration by parts formula holds:

Proposition 2.12.

- *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^1 boundary. Then for all $u, v \in C^1(\overline{\Omega})$, there holds*

$$\int_\Omega (\partial_j u)\, v\, \mathrm{d}x = \int_{\partial\Omega} u\, v\, n_j\, dS - \int_\Omega u\, (\partial_j v)\, \mathrm{d}x, \tag{2.24}$$

  where $\vec{n} = (n_1, \cdots, n_d)$ is the outer unit normal vector to the boundary $\partial\Omega$.

- *For all $u \in C^1(\Omega)$, $v \in C_c^1(\Omega)$ with $\Omega \subset \mathbb{R}^d$ an open set, there holds*

$$\int_\Omega (\partial_j u)\, v\, \mathrm{d}x = -\int_\Omega u\, (\partial_j v)\, \mathrm{d}x. \tag{2.25}$$

证明. Exercise.

□

2.4 Bilinear forms

Definition 2.13. *(i) Let X be a normed space over $\mathbb{F}$. By a bounded blinear form on X we mean a function*

$$a : X \times X \to \mathbb{F}$$

that has the following two properties:

- Bilinearity. *For all $u, v, w \in X$ and $\alpha, \beta \in \mathbb{F}$,*

$$a(\alpha u + \beta v, w) = \alpha a(u, w) + \beta a(v, w), \quad a(w, \alpha u + \beta v) = \alpha a(w, u) + \beta a(w, v).$$

- Boundedness. *There is a constant $C > 0$ such that*

$$|a(u, v)| \leq C\|u\|\|v\|, \quad \textit{for all } u, v \in X.$$

(ii) In addition, a bilinear form $a(\cdot, \cdot)$ is called symmetric *provided*

$$a(u, v) = a(v, u), \quad \textit{for all } u, v \in X.$$

(ii) Moreover, $a(\cdot, \cdot)$ is called positive *provided*

$$a(u, u) \geq 0, \quad \textit{for all } u \in X.$$

And $a(\cdot, \cdot)$ is called strictly positive *provided there is a constant $c > 0$ such that*

$$a(u, u) \geq c\|u\|^2, \quad \textit{for all } u \in X.$$

A bounded bilinear form is continuous:

Proposition 2.14. *Let $a : X \times X$ be a bounded bilinear form on normed space X. If $u_n \to u$, $v_n \to v$ in X as $n \to \infty$, then $a(u_n, v_n) \to a(u, v)$.*

2.5 Quadratic variational problems

In general, variational problems represent the problems of finding minimum or maximum values of functionals.

Theorem 2.15. *Let X be a real Hilbert space. Let $a : X \times X \to \mathbb{R}$ is a symmetric, bounded, strictly positive, blinear form, and let $b : X \to \mathbb{R}$ is a linear continuous functional. Define the functional $F : X \to \mathbb{R}$ as*

$$F(u) = \frac{1}{2} a(u, u) - b(u), \quad \textit{for all } u \in X.$$

Then the variational problem:

$$\textit{find } u \in X \textit{ such that } F(u) = \inf_{v \in X} F(v) \tag{2.26}$$

admits a unique solution $u \in X$; moreover, u is the unique solution to the so-called variational equation:

$$a(u,v) = b(v), \quad \textit{for all } v \in X. \tag{2.27}$$

证明. **Step 1. Existence.** Let $\alpha := \inf_{v\in X} F(v)$ and let $\{u_n\}$ be a sequence in X such that

$$F(u_n) \to \alpha, \quad \text{as } n \to \infty.$$

Since a is strictly positive and b is continuous, there holds

$$F(v) = \frac{1}{2}a(v,v) - b(v) \geq \frac{c\|v\|^2}{2} - \|b\|\|v\| \to +\infty, \quad \text{as } \|v\| \to \infty.$$

This implies that the sequence $\{u_n\}$ is bounded.

By the bilinearity and symmetry of a, one has

$$2a(u_n,u_n) + 2a(u_m,u_m) = a(u_n - u_m, u_n - u_m) + a(u_n + u_m, u_n + u_m).$$

Hence

$$\begin{aligned} 4\left[F(u_n) + F(u_m)\right] &= 2a(u_n,u_n) + 2a(u_m,u_m) - 4b(u_n) - 4b(u_m) \\ &= a(u_n - u_m, u_n - u_m) + a(u_n + u_m, u_n + u_m) - 4b(u_n + u_m) \\ &= a(u_n - u_m, u_n - u_m) + 8F\left(\frac{u_n + u_m}{2}, \frac{u_n + u_m}{2}\right) \\ &\geq c\|u_n - u_m\|^2 + 8\alpha, \end{aligned} \tag{2.28}$$

where $c > 0$ is the positive constant related to the strict positivity of a. This implies

$$c\|u_n - u_m\|^2 \leq 4\left[F(u_n) + F(u_m)\right] - 8\alpha \to 0, \tag{2.29}$$

as $m \to \infty, n \to \infty$. This means that $\{u_n\}$ is a Cauchy sequence. Since X is complete, we thus have

$$u_n \to u \text{ in } X, \quad \text{as } n \to \infty.$$

By the continuity of F, we thus have

$$F(u) = \lim_{n\to\infty} F(u_n) = \alpha = \inf_{v\in X} F(v) = \min_{v\in X} F(v).$$

This means u is a solution to the variational problem (2.26).

Step 2. Solution of the variational equation. Let u be a solution to the variational problem (2.26). Fix $v \in X$ and define

$$\varphi(t) := F(u + tv) = \frac{t^2}{2}a(v,v) + t[a(u,v) - b(v)] + \frac{1}{2}a(u,u) - b(u), \quad \forall t \in \mathbb{R}.$$

Since u is the point such that $F(v)$ achieve its minimum, the smooth function $\varphi(t)$ achieve its minimum at $t = 0$. Then necessarily $\varphi'(0) = 0$, which is exactly the variational equation

$$a(u, v) - b(v) = 0.$$

Step 3. Uniqueness. Let u_1 and u_2 be two solutions to the variational problem. Then

$$a(u_1, v) = b(v), \quad a(u_2, v) = b(v), \quad \forall v \in X.$$

Thus

$$a(u_1 - u_2, v) = 0 \quad \forall v \in X.$$

Taking $v = u_1 = u_2$ and using the strict positivity of bilinear form a implies that $u_1 = u_2$.

□

2.6 A variational problem: Dirichlet problem of Laplacian operator

Let Ω be a bounded open set in $\mathbb{R}^d$. Let $f \in L^2(\Omega)$ and let X be a suitable Banach space to be determined and we define the functional on X:

$$F(v) := \frac{1}{2}\int_\Omega |\nabla_x v|^2 \,\mathrm{d}x - \int_\Omega f\, v \,\mathrm{d}x, \quad \forall v \in X. \tag{2.30}$$

Let g be a suitable function. We then consider the following variational problem:

$$\text{find } u \in X_g := \{v \in X : v = g \text{ on } \partial\Omega\} \text{ such that } F(u) = \inf\nolimits_{v \in X_g} F(v). \tag{2.31}$$

Here X_g is called the set of admissible functions for the variational problem (2.31).

In the following subsections, we will study this variational problem step by step.

2.6.1 The Euler-Lagrange equation

Along with (2.30) and (2.31), we consider the following boundary-value problem of the Laplacian operator:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.32}$$

The boundary condition in (2.32) is about the value of the unknown on the boundary. Such a boundary condition is called the *Dirichlet boundary condition*, and the related boundary-value problem is called a *Dirichlet problem*. The connection between the variational problem (2.31) and the Dirichlet problem (2.32) is given in the following:

Proposition 2.16. *Let Ω be a bounded open set in $\mathbb{R}^d$ and let $f : \overline{\Omega} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be continuous functions. If $u \in C^2(\overline{\Omega})$ is a solution to the variational problem* (2.31) *in $X = C^2(\overline{\Omega})$, then u is a solution to the Dirichlet problem* (2.32)*.*

The equation (2.32) is called the Euler-Lagrange equation to the (2.31).

证明. **Step 1. Admissible functions**. Let u be a solution to the variational problem (2.31). Then, for each $v \in C_c^\infty(\Omega)$ and each $t \in \mathbb{R}$, the function

$$w := u + tv$$

is admissible for the variational problem (2.31) in $X = C^2(\overline{\Omega})$, i.e.

$$w \in C^2(\overline{\Omega}), \quad w = g \quad \text{on } \partial\Omega.$$

Step 2. Reduction to a minimum problem for real functions. For each fixed $v \in C_c^\infty(\Omega)$, we set

$$\phi(t) := F(u + tv) = \frac{1}{2}\int_\Omega |\nabla_x(u + tv)|^2 \,\mathrm{d}x - \int_\Omega f\,(u + tv)\,\mathrm{d}x, \quad \forall\, t \in \mathbb{R}.$$

Then ϕ is a differentiable function on $\mathbb{R}$. Moreover, since u is a solution to the variational problem, the function $\phi : \mathbb{R} \to \mathbb{R}$ admits a minimum at $t = 0$. Hence

$$\phi'(0) = \int_\Omega \nabla_x u \cdot \nabla_x v \,\mathrm{d}x - \int_\Omega f\, v \,\mathrm{d}x = 0. \tag{2.33}$$

This holds true for each $v \in C_c^\infty(\Omega)$.

Step 2. The Euler-Lagrange equation. Applying integration by parts to (2.33) implies

$$-\int_\Omega (\Delta_x u + f) v \,\mathrm{d}x = 0, \quad \forall\, v \in C_c^\infty(\Omega). \tag{2.34}$$

This implies $\Delta_x u + f = 0$ in G. (why?)

□

Remark 2.17 (Lack of classical solutions)**.** *By Proposition 2.16, each sufficient smooth solution to the variational problem* (2.31) *is also a solution to the Dirichlet problem* (2.32)*. However, the are reasonable situations where the variational problem* (2.31) *lacks smooth solutions.*

One may ask, why not to use general Theorem thm-variational-1 to solve the variational problem (2.31) in some reasonable space X? Then the functional $F(v)$ can be written as

$$F(v) = \frac{1}{2}a(v, v) - b(v)$$

where

$$a(u, v) := \int_\Omega \nabla_x u \cdot \nabla_x v \,\mathrm{d}x, \quad b(v) := \int_\Omega f\, v \,\mathrm{d}x.$$

To apply Theorem 2.15, we need a to be a symmetric, bounded, and strictly positive blinear form in Hilbert space X, and we need b to be bounded in X. Thus, the reasonable choice the norm for X is

$$\|u\|_X := \left(\int_\Omega (|\nabla_x u|^2 + |u|^2)\, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.35}$$

However, to define this norm we need $u \in C^1(\Omega)$. However, this norm $\|\cdot\|_X$ in (2.35) does not coincide with the norm of $C^1(\Omega)$. We need more general definition for derivatives. This is going to be done in the next subsection.

2.7 Generalized derivatives

The point of departure for the definition of generalized derivatives is the classical integration-by-parts formula: let $u \in C^1(\Omega)$ with $\Omega \subset \mathbb{R}^d$ an open set, then

$$\int_\Omega u(\partial_j v)\, \mathrm{d}x = -\int_\Omega (\partial_j u) v\, \mathrm{d}x, \quad \text{for all } v \in C_c^\infty(\Omega). \tag{2.36}$$

Setting $w = \partial_j u$ gives the formula

$$\int_\Omega u(\partial_j v)\, \mathrm{d}x = -\int_\Omega wv\, \mathrm{d}x, \quad \text{for all } v \in C_c^\infty(\Omega). \tag{2.37}$$

The point is that this formula remains valid for certain *nonsmooth* functions u and w.

Definition 2.18. *Let Ω be a nonempty open set in $\mathbb{R}^d$. Let $u \in L^1_{loc}(\Omega)$. If there exists $w \in L^1_{loc}(\Omega)$ such that* (2.37) *holds, we call w a generalized derivative of the function u in Ω. As in the classical case, we write $w = \partial_j u$.*

Proposition 2.19. *The generalized derivative is uniquely determined up to a set of measure zero.*

To prove this result, we need the show that Corollary 2.11 still holds for the case $p = 1$ and $p = \infty$. That is:

Corollary 2.20. *Let Ω be a nonempty open set in $\mathbb{R}^d$ and let $u \in L^p(\Omega),\ 1 \le p \le \infty$. If*

$$\int_\Omega u\, v\, \mathrm{d}x = 0, \quad \textit{for all } v \in C_c^\infty(\Omega), \tag{2.38}$$

then $u(x) = 0$ for almost all $x \in \Omega$.

Proof of Corollary 2.20. The cases $1 < p < \infty$ is already proved. We still need to handle the case $p = 1$ and $p = \infty$.

Case $p = \infty$. This case is easier to prove. Given $u \in L^\infty(\Omega)$ satisfying (2.11). Define

$$\Omega_n := \Omega \cap B(0, n), \quad n \in Z_+, \tag{2.39}$$

be a sequence of bounded open subsets of Ω such that

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n. \tag{2.40}$$

Thus, for any $n \in \mathbb{Z}_+$, one has $u \in L^p(\Omega_n)$ for all $1 \le p \le \infty$. Moreover,

$$\int_{\Omega_n} u\, v \,\mathrm{d}x = \int_{\Omega} u\, v \,\mathrm{d}x = 0, \quad \text{for all } v \in C_c^\infty(\Omega_n). \tag{2.41}$$

Applying the result of the case $1 < p < \infty$, we have $u(x) = 0$ for almost all $x \in \Omega_n$. Since a countable union of measure zero sets are still measure zero, we have $u(x) = 0$ for almost all $x \in \Omega = \bigcup_{n=1}^{\infty} \Omega_n$..

Case $p = 1$. Let $u \in L^1(\Omega)$ satisfy (2.38). For any compact set $K \subset \Omega$, we define v as

$$v = \begin{cases} \mathrm{sgn} u, & \text{on } K, \\ 0, & \text{on } \mathbb{R}^d \setminus K. \end{cases} \tag{2.42}$$

Then, for ε sufficient small, we have $S_\varepsilon[v] \in C_c^\infty(\Omega)$ where S_ε is the standard Friedrichs' mollification.

By Minkowski's inequality, there holds

$$\|S_\varepsilon[v]\|_{L^\infty} \le \|v\|_{L^\infty} \le 1.$$

Moreover, since $v \in L^p(\Omega)$ for any $p \in [1, \infty]$, there holds

$$S_\varepsilon[v] \to v \text{ in } L^p(\Omega).$$

This implies, up to a subsequence, that

$$S_\varepsilon[v] \to v \text{ a.e. in } \Omega.$$

By Lebesgue's dominated convergence theorem,

$$\int_K u\, v \,\mathrm{d}x = \int_\Omega u\, v \,\mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega u\, S_\varepsilon[v] \,\mathrm{d}x = 0.$$

By the definition of v, we finally obtain

$$\int_K |u| \,\mathrm{d}x = 0,$$

which implies

$$u = 0 \text{ a.e. in } K.$$

Since K is arbitrarily chosen in Ω, we derive that

$$u = 0 \text{ a.e. in } \Omega.$$

In this case, we may also employ Lusin's theorem which is recalled below:

Lemma 2.21. *Let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable function. Given $\varepsilon > 0$, for every measurable set A of finite measure there is a compact set E with $|A \setminus E| < \varepsilon$ such that f restricted to E is continuous. Moreover, we can find a continuous function $f_\varepsilon : \mathbb{R}^d \to \mathbb{C}$ with compact support that coincides with f on E and such that $\sup_{x\in\mathbb{R}^d} |f_\varepsilon(x)| \le \sup_{x\in\mathbb{R}^d} |f(x)|$.*

Again let Ω_n be defined as in (2.39). Clearly Ω_n is a bounded subset of Ω and is certainly of finite measure. Then, by Lusin's theorem, for each $\varepsilon > 0$, there exists a compact set $E_n \subset \Omega_n$ and a continuous function $u_\varepsilon : \Omega_n \to \mathbb{C}$ with compact support such that

$$u_\varepsilon(x) = u(x),\ \forall x \in E_{n,\varepsilon}, \quad \sup_{x\in\Omega_n} |u_\varepsilon(x)| \le \sup_{x\in\Omega_n} |u(x)|.$$

Since $E_{n,\varepsilon} \subset \Omega_n$, then for each $v \in C_c^\infty(E_{n,\varepsilon}) \subset C_c^\infty(\Omega_n) \subset C_c^\infty(\Omega)$, there holds

$$\int_{E_{n,\varepsilon}} f_\varepsilon\, v \,\mathrm{d}x = \int_{E_{n,\varepsilon}} f\, v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x = 0. \tag{2.43}$$

This implies

$$f(x) = f_\varepsilon(x) = 0,\ \forall\, x \in E_{n,\varepsilon}.$$

(why?). Since this holds for all $\varepsilon > 0$, we obtain

$$f(x) = 0,\ \forall\, x \in E := \bigcup_{k=1}^{\infty} E_{n,\varepsilon_k}$$

for each sequence ε_k satisfying $\varepsilon_k \to 0$ as $k \to \infty$. Moreover

$$|\Omega_n \setminus E| = |\bigcap_{k=1}^{\infty} \Omega_n \setminus E_{n,\varepsilon_k}| = 0$$

due to the fact

$$|\Omega_n \setminus E_{n,\varepsilon_k}| \le \varepsilon_k \to 0.$$

Hence,

$$f(x) = 0,\ a.a\ x \in \Omega_n$$

and furthermore

$$f(x) = 0,\ a.a\ x \in \Omega.$$

□

Proof of Proposition 2.19. Let Ω be a nonempty open set in $\mathbb{R}^d$ and define Ω_n as in (2.39). Let $u \in L^1_{loc}(\Omega)$. Suppose $w_1, w_2 \in L^1_{loc}(\Omega)$ both satisfies (2.37). Then

$$\int_\Omega (w_1 - w_2)\, v \,\mathrm{d}x = 0, \quad \text{for all } v \in C_c^\infty(\Omega).$$

This implies

$$\int_{\Omega_n} (w_1 - w_2)\, v \, \mathrm{d}x = 0, \quad \text{for all } v \in C_c^\infty(\Omega_n).$$

Since $w_1 - w_2 \in L^1(\Omega_n)$, applying Corollary 2.20 implies

$$w_1(x) = w_2(x) \quad a.a. \ x \in \Omega_n$$

and furthermore

$$w_1(x) = w_2(x) \quad a.a. \ x \in \Omega.$$

□

Example 2.22. *$u(x) = |x|, \ x \in (-1, 1)$. The generalized derivative $u' = w$ with*

$$w(x) = \begin{cases} 1, & 0 < x < 1, \\ -1, & -1 < x < 0, \end{cases}$$

Exercise: Prove that w does not admit a generalized derivative in the sense of Definition 2.18.

More generally, any continuous and piecewise C^1 function admits a generalized derivative that is in L^∞_{loc}.

Remark 2.23. *For every $u \in L^1_{loc}(\Omega)$, the* theory of distribution *gives a meaning to $\partial_{x_j} u$ as an element of the much larger space of distributions $\mathcal{D}'(\Omega)$, which is the space of bounded linear functionals on $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$. We will not go into the details of the theory of distribution. The students can learn from the book [3].*

2.8 Sobolev spaces

Now we give the definition of Sobolev spaces

Definition 2.24. *Let $\Omega \subset \mathbb{R}^d$ be a nonempty open set. For each $1 \le p \le \infty$, the Sobolev space $W^{1,p}(\Omega)$ consists precisely of all the functions*

$$u \in L^p(\Omega)$$

that have generalized derivatives

$$\partial_{x_j} u \in L^p(\Omega), \ j = 1, \cdots, d.$$

We also use the classical notation for the generalized gradient $\nabla u = (\partial_1 u, \cdots, \partial_d u)$.

Remark 2.25. *There are other ways to define the Sobolev spaces. These definitions are equivalent.*

One can also use the language of distributions. By the theory of distributions by L. Schwartz, each function $u \in L^1_{loc}(\Omega)$ admits a derivative in the sense of distributions:

$$\langle \partial_j u, v \rangle = - \langle u, \partial_j v \rangle = - \int_\Omega u \, (\partial_j v) \, \mathrm{d}x, \quad \textit{for all } v \in C_c^\infty(\Omega).$$

This derivative is an element of the much larger space of distributions $\mathcal{D}'(\Omega)$ which is the collection of all linear bounded functionals of the space $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$. We say that $u \in W^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and its distributional derivative happens to lie in $L^p(\Omega)$, which is a subspace of $\mathcal{D}'(\Omega)$.

When $\Omega = \mathbb{R}^d$ and $p = 2$, Sobolev spaces can also be defined by using the Fourier transform: $W^{1,2}(\mathbb{R}^2)$ consists of all functions

$$u \in L^2(\mathbb{R}^2)$$

for which the Fourier transform $\hat{u}$ satisfy

$$(1 + |\xi|^2)\hat{u}(\xi) \in L^2(\mathbb{R}^d).$$

To define the Fourier transform of some $u \in L^2(\mathbb{R}^d)$, one needs to use the Plancherel theorem and a density argument of Schwartz functions. (How?)

Proposition 2.26. *Let Ω be an open set of $\mathbb{R}^d$ and let $1 \le p \le \infty$. Equipped with the norm*

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega)} &:= \left(\|u\|_{L^p(\Omega)}^p + \|\nabla u\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \\ \|u\|_{W^{1,\infty}(\Omega)} &:= \|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)}, \quad p = \infty \end{aligned} \tag{2.44}$$

for all $u \in W^{1,p}(\Omega)$, the Sobolev space $W^{1,p}(\Omega)$ is a Banach space.

In particular if $p = 2$, the Sobolev space $W^{1,2}(\Omega)$ is a Hilbert space equipped with the inner product:

$$(u,v)_{W^{1,2}} = (u,v)_{L^2} + (\nabla u, \nabla v)_{L^2} = \int_\Omega u\,\bar{v}\,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla \bar{v}\,\mathrm{d}x \quad \textit{for all } u,v \in W^{1,2}(\Omega). \tag{2.45}$$

Moreover, $W^{1,p}(\Omega)$ is reflexive provided $1 < p < \infty$ and is separable provided $1 \le p < \infty$.

证明. **Step 1. Norm.**

Step 2. Completeness.

Step 3. Hilbert space

Step4. Reflexive and separable. We consider the linear mapping

$$F : W^{1,p}(\Omega) \to (L^p(\mathbb{R}^d))^{d+1}, \quad F(u) = (u, \nabla u). \tag{2.46}$$

Clearly F is an isometry from $W^{1,p}(\Omega)$ into $(L^p(\mathbb{R}^d))^{d+1}$. Indeed:

$$\|F(u)\|_{(L^p(\mathbb{R}^d))^{d+1}} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} = \|u\|_{W^{1,p}(\Omega)}. \tag{2.47}$$

Since $W^{1,p}(\Omega)$ is a Banach space, $F(W^{1,p}(\Omega))$ is a closed subspace of $(L^p(\mathbb{R}^d))^{d+1}$. If $1 < p < \infty$, $(L^p(\mathbb{R}^d))^{d+1}$ is reflexive. Thus, by the fact that any closed subspace of a reflexive Banach space is reflexive, we know that $F(W^{1,p}(\Omega))$ is reflexive. Thus $W^{1,p}(\Omega)$ is also reflexive.

If $1 \le p < \infty$, $(L^p(\mathbb{R}^d))^{d+1}$ is reflexive. By the fact that any subset of a separable metric space is also separable, we know that $F(W^{1,p}(\Omega))$ is separable. Thus $W^{1,p}(\Omega)$ is also separable.

□

We now give some properties of Sobolev spaces.

Proposition 2.27. *(i) Let Ω be an open set of $\mathbb{R}^d$. Let $\{u_n\}$ be a sequence in $W^{1,p}(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$ and ∇u_n converges to some limit in $L^p(\Omega)^d$. Then*

$$u \in W^{1,p}(\Omega), \quad u_n \to u \quad \textit{in } W^{1,p}(\Omega).$$

(ii) Let $1 < p \leq \infty$, let $u_n \to u$ in $L^p(\Omega)$ and $\{\nabla u_n\}$ be bounded in $L^p(\Omega)^d$. Then $u \in W^{1,p}(\Omega)$.

证明. (i). Suppose that $\nabla u_n \to v$ in $L^p(\Omega)^d$. Thus, for any test function $\phi \in C_c^\infty(\Omega)^d$, there holds

$$\begin{aligned} &\int_\Omega (\nabla u_n) \cdot \phi \,\mathrm{d}x \to \int_\Omega v \cdot \phi \,\mathrm{d}x, \\ &\int_\Omega (\nabla u_n) \cdot \phi \,\mathrm{d}x = -\int_\Omega u_n \mathrm{div}\, \phi \,\mathrm{d}x \to -\int_\Omega u \mathrm{div}\, \phi \,\mathrm{d}x. \end{aligned} \tag{2.48}$$

This implies that

$$\int_\Omega u \mathrm{div}\, \phi \,\mathrm{d}x = -\int_\Omega v \cdot \phi \,\mathrm{d}x. \tag{2.49}$$

This means $\nabla u = v$ in the generalized sense. Hence

$$u \in W^{1,p}(\Omega), \quad u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

(ii). When $1 < p \leq \infty$, we know that each bounded sequence in $L^p(\Omega)$ admits a weakly-star convergent subsequence. So there exists $v \in L^p(\Omega)^d$ such that

$$\int_\Omega \nabla u_n \cdot \phi \,\mathrm{d}x = \int_\Omega v \cdot \phi \,\mathrm{d}x, \quad \text{for all } \phi \in C_c^\infty(\Omega)^d.$$

Similarly as above, we then have that $\nabla u = v$ in the generalized sense. Hence $\nabla u \in L^p(\Omega)$ and $u \in W^{1,p}(\Omega)$.

□

We remark that in case (ii), we know that $u \in W^{1,p}(\Omega)$, but we do not know whether $u_n \to u$ in $u \in W^{1,p}(\Omega)$.

We introduce the zero extension of a function in Ω. Given a function f defined in Ω, we denote by $\tilde{f}$ its zero extension in $\mathbb{R}^d$, that is,

$$\tilde{f}(x) = f(x) \text{ for all } x \in \Omega, \quad \tilde{f}(x) = 0 \text{ for all } x \in \mathbb{R}^d \setminus \Omega.$$

Proposition 2.28. *Let Ω be an open set of $\mathbb{R}^d$, let $u \in W^{1,p}(\Omega)$ and let $v \in C_c^\infty(\Omega)$. Then*

$$\widetilde{uv} \in W^{1,p}(\mathbb{R}^d), \quad \partial_{x_j}(\widetilde{uv}) = \widetilde{v\partial_{x_j} u} + \widetilde{u\partial_{x_j} v}.$$

证明. It suffices to prove that

$$\partial_{x_j}(\widetilde{uv}) = \widetilde{v\partial_{x_j}u} + \widetilde{u\partial_{x_j}v}$$

in the generalized sense in $\mathbb{R}^d$. Given a test function $\phi \in C_c^\infty(\mathbb{R}^d)$, there holds $v\phi \in C_c^\infty(\Omega)$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{uv}(\partial_j\phi)\,\mathrm{d}x &= \int_\Omega uv(\partial_j\phi)\,\mathrm{d}x = \int_\Omega u(\partial_j(v\phi))\,\mathrm{d}x - \int_\Omega u(\partial_j v)\phi\,\mathrm{d}x \\ &= -\int_\Omega (\partial_j u)(v\phi)\,\mathrm{d}x - \int_\Omega u(\partial_j v)\phi\,\mathrm{d}x \\ &= -\int_\Omega \big((\partial_j u)v + u(\partial_j v)\big)\phi\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \big(\widetilde{v\partial_{x_j}u} + \widetilde{u\partial_{x_j}v})\big)\phi\,\mathrm{d}x.\end{aligned} \tag{2.50}$$

This means

$$\partial_{x_j}(\widetilde{uv}) = \widetilde{v\partial_{x_j}u} + \widetilde{u\partial_{x_j}v}$$

in the generalized sense in $\mathbb{R}^d$. Since $u \in W^{1,p}(\Omega)$, we have

$$\widetilde{uv} \in L^p(\mathbb{R}^d), \quad \nabla(\widetilde{uv}) = \widetilde{v\nabla u} + \widetilde{u\nabla v} \in L^p(\mathbb{R}^d).$$

This means $\widetilde{uv} \in W^{1,p}(\mathbb{R}^d)$.

□

Remark 2.29. *Given $u \in W^{1,p}(\Omega)$, in general $\tilde{u} \notin W^{1,p}(\mathbb{R}^d)$. (why?)*

We know investigate the density of smooth functions in Sobolev spaces.

Theorem 2.30 (Friedrichs)**.** *Let $u \in W^{1,p}(\Omega)$ with $1 \le p < \infty$. Then there exists a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^d)$ such that*

$$u_n \to u \quad \text{in } L^p(\Omega\ )$$

and

$$\nabla u_n \to \nabla u \quad \text{in } L^p(\omega) \text{ for all } \omega \subset\subset \Omega.$$

For the case $\Omega = \mathbb{R}^d$, there exists a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \to u \quad \text{in } W^{1,p}(\mathbb{R}^d).$$

To prove this theorem, we need the following lemma:

Lemma 2.31. *Let $u \in W^{1,p}(\mathbb{R}^d)$ with $1 \le p \le \infty$ and $v \in L^1(\mathbb{R}^d)$. Then the convolution $v * u \in W^{1,p}(\mathbb{R}^d)$ and*

$$\partial_{x_j}(v * u) = v * (\partial_{x_j}u),\ j = 1, \cdots, d.$$

Proof of Lemma 2.31. By Young's inequality, we have that $v * u \in L^p(\mathbb{R}^d)$ and

$$\|v * u\|_{L^p(\mathbb{R}^d)} \leq \|v\|_{L^1(\mathbb{R}^d)}\|u\|_{L^p(\mathbb{R}^d)}.$$

Thus, given each test function $\phi \in C_c^\infty(\mathbb{R}^d)$, by Hölder's inequality and Young's inequality,

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} |v(x-y)||u(y)||\phi(x)|\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^d}|\phi(x)|\int_{\mathbb{R}^d}|v(x-y)||u(y)|\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \|\phi(x)\|_{L^{p'}}\||v|*|u|\|_{L^p} \\ &\leq C\|v\|_{L^1(\mathbb{R}^d)}\|u\|_{L^p(\mathbb{R}^d)} \\ &< \infty.\end{aligned} \tag{2.51}$$

This implies that

$$F(x,y) = v(x-y)u(y)\phi(x) \in L^1(\mathbb{R}^d\times\mathbb{R}^d), \quad \forall \phi \in C_c^\infty.$$

Then, by Fubini's theorem, by setting $\check{v}(\cdot) = v(\cdot)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} v*u(x)(\partial_j\phi)(x)\,\mathrm{d}x &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} v(x-y)u(y)\,\mathrm{d}y(\partial_j\phi)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} u(y)\,\mathrm{d}y\int_{\mathbb{R}^d} v(x-y)(\partial_j\phi)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} u(y)\big(\check{v}*(\partial_j\phi)\big)(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} u(y)\partial_j\big(\check{v}*\phi\big)(y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d}(\partial_j u)(y)\big(\check{v}*\phi)\big)(y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d}(\partial_j u)(y)\int_{\mathbb{R}^d}\check{v}(y-x)\phi(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^d}(v*\partial_j u)(x)\phi(x)\,\mathrm{d}x,\end{aligned} \tag{2.52}$$

where we used the fact that (why?)

$$\check{v}*(\partial_j\phi) = \partial_j\big(\check{v}*\phi\big).$$

We thus derive that

$$\partial_{x_j}(v*u) = v*(\partial_{x_j}u) \in L^p(\mathbb{R}^d),\ j = 1,\cdots,d.$$

Hence, the convolution $v*u \in W^{1,p}(\mathbb{R}^d)$ and we complete the proof.

□

Proof of Theorem 2.30. Given $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. We consider the zero extension $\tilde{u}$ defined as

$$\tilde{u}(x) = u(x) \text{ for all } x \in \Omega, \quad \tilde{u}(x) = 0 \text{ for all } x \in \mathbb{R}^d\setminus\Omega.$$

Let ϕ_n be the standard Friedrichs' mollifier with $\varepsilon = 1/n$, $n \in \mathbb{Z}_+$, and consider the mollification sequence

$$u_n := S_n[\tilde{u}] := \phi_n * \tilde{u} \in C_c^\infty(\Omega).$$

By Proposition 2.8, we know that

$$\|u_n\|_{L^p(\mathbb{R}^d)} \le \|\tilde{u}\|_{L^p(\mathbb{R}^d)} = \|u\|_{L^p(\Omega)}, \quad u_n \to \tilde{u} \text{ in } L^p(\mathbb{R}^d).$$

In particular,

$$u_n \to \tilde{u} = u \;\text{ in } L^p(\Omega).$$

We next show that

$$\nabla u_n \to \nabla u \quad \text{in } L^p(\omega) \text{ for all } \omega \subset\subset \Omega.$$

Given $\omega \subset\subset \Omega$, we fix a function $\chi \in C_c^\infty(\Omega)$ such that

$$0 \le \chi \le 1, \quad \chi = 1 \text{ on a neighborhood of } \omega.$$

By the property on the support of the convolution, we have for n large enough that

$$\begin{aligned}\operatorname{supp}(\phi_n * (\widetilde{\chi u}) - \phi_n * \tilde{u}) &= \operatorname{supp}(\phi_n * ((\tilde{\alpha} - 1)\tilde{u}) \\ &\subset \operatorname{supp}\phi_n + \operatorname{supp}((\tilde{\alpha} - 1)\tilde{u}) \\ &\subset \overline{B(0, 1/n)} + \operatorname{supp}(\tilde{\alpha} - 1) \\ &\subset \omega^c.\end{aligned} \tag{2.53}$$

Thus, if n is large enough, there holds

$$\phi_n * (\widetilde{\chi u}) = \phi_n * \tilde{u} \text{ on } \omega. \tag{2.54}$$

By Proposition 2.28 and Lemma 2.31, we know that

$$\partial_{x_j}(\phi_n * (\widetilde{\chi u})) = \phi_n * \partial_{x_j}(\widetilde{\chi u}) = \phi_n * \left(\widetilde{\chi \partial_{x_j} u} + \widetilde{\partial_{x_j} \chi u}\right) \tag{2.55}$$

in $\mathbb{R}^d$ in the generalized sense. Since

$$\left(\widetilde{\chi \partial_{x_j} u} + \widetilde{\partial_{x_j} \chi u}\right) \in L^p(\mathbb{R}^d),$$

then

$$\partial_{x_j}(\phi_n * (\widetilde{\chi u})) \to \left(\widetilde{\chi \partial_{x_j} u} + \widetilde{\partial_{x_j} \chi u}\right) \text{ in } L^p(\mathbb{R}^d). \tag{2.56}$$

In particualar,

$$\partial_{x_j}(\phi_n * (\widetilde{\chi u})) \to \left(\widetilde{\chi \partial_{x_j} u} + \widetilde{\partial_{x_j} \chi u}\right) = \partial_{x_j} u \text{ in } L^p(\omega). \tag{2.57}$$

Together with (2.54), we obtain

$$\partial_{x_j}(\phi_n * (\tilde{u})) \to \partial_{x_j} u \text{ in } L^p(\omega). \tag{2.58}$$

We see that u_n satisfies our desired convergence properties, except that $u_n \in C^\infty(\mathbb{R}^d)$ is not necessarily compactly supported. While, this can be remided be introducing a classical cut off functions in the following way. Let $\chi_0 \in C_c^\infty(B(0,1))$ be a cut-off function satisfying

$$0 \leq \chi_0 \leq 1, \quad \chi_0 = 1 \text{ on } B(0,1/2) \tag{2.59}$$

and set

$$\chi_n(\cdot) = \chi_0(\cdot/n) \text{ satisfying } \chi_n \in C_c^\infty(B(0,n)),\ 0 \leq \chi_n \leq 1,\ \chi_n = 1 \text{ on } B(0,n/2). \tag{2.60}$$

Then, it can be shown that the sequence $\chi_n u_n \in C_c^\infty$ fulfills our desired request (why?).

In the case $\Omega = \mathbb{R}^d$, it can be shown that the sequence

$$u_n = \chi_n(\phi_n * u)$$

has the desired properties.

□

Remark 2.32.

- *It can be shown (Meyers-Serrin's theorem) that if $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^d$ an open set, then there exists a sequence $\{u_n\}$ in $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. The proof of this result is fairly delicate (see, e.g., R. Adams [1]).*
- *In general, if Ω is an arbitrary open set and if $u \in W^{1,p}(\Omega)$, there need not exist a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^d) \cap W^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$, even when $p < \infty$. However, this is true if Ω is regular, of class C^1.*

Here is a simple characterization of $W^{1,p}$ functions:

Proposition 2.33. *Let $u \in L^p(\Omega)$ with $1 < p \leq \infty$. The following properties are equivalent:*

(i) $u \in W^{1,p}(\Omega)$.

(ii) There exists a constant C such that

$$\left| \int_\Omega u \partial_{x_j} \phi \,\mathrm{d}x \right| \leq C \|\phi\|_{L^{p'}(\Omega)}, \quad \forall \phi \in C_c^\infty(\Omega),\ \forall j = 1, 2, \cdots, d.$$

(iii) There exists a constant C such that for all $\omega \subset\subset \Omega$ and all $h \in \mathbb{R}^d$ with $|h| < \operatorname{dist}(\omega, \partial\Omega)$ there holds

$$\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|,$$

where $\tau_h u(\cdot) := u(\cdot + h)$. (Note that $\tau_h u(x) = u(x+h)$ makes sense for $x \in \omega$ and $|h| < \operatorname{dist}(\omega, \partial\Omega)$.)

Moreover, in (ii) and (iii) one can take $C = \|\nabla u\|_{L^p(\Omega)}$.

If $\Omega = \mathbb{R}^d$, *there holds*

$$\|\tau_h u - u\|_{L^p(\mathbb{R}^d)} \le |h| \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

证明. (i) $\Rightarrow$ (ii). This is rather obvious.

(ii) $\Rightarrow$ (i). Since $1 < p \le \infty$, we have that $1 \le p' < \infty$. Thus $C_c^\infty(\Omega)$ is dense in $L^{p'}(\Omega)$. By density argument and property (ii),

$$\Phi(v) := -\int_\Omega u \partial_{x_j} v \, \mathrm{d}x, \ \forall v \in L^{p'}(\Omega) \tag{2.61}$$

defines a linear bounded functional in $L^{p'}(\Omega)$ for each $j = 1, 2, \cdots, d$. Since the dual space of $L^{p'}(\Omega)$ is $L^p(\Omega)$ when $1 \le p' < \infty$, there exists $w \in L^p(\Omega)$ such that

$$\Phi(v) = \int_\Omega w \, v \, \mathrm{d}x, \ \forall v \in L^{p'}(\Omega). \tag{2.62}$$

By (2.61) and (2.62), we know that

$$\partial_{x_j} u = w \in L^p(\Omega). \tag{2.63}$$

This is true for each $j = 1, 2, \cdots, d$. Hence $u \in W^{1,p}(\Omega)$.

(i) $\Rightarrow$ (iii). Let $\omega \subset\subset \Omega$ and $h \in \mathbb{R}^d$ with $|h| < \mathrm{dist}\,(\omega, \partial\Omega)$. Assume first that $u \in C_c^\infty(\mathbb{R}^d)$. Then

$$u(x+h) - u(x) = \int_0^1 \frac{d}{dt}(u(x+th)) \, \mathrm{d}t = \int_0^1 h \cdot \nabla u(x+th) \, \mathrm{d}t, \quad \forall x \in \omega. \tag{2.64}$$

This implies for $1 \le p < \infty$ that

$$|u(x+h) - u(x)|^p \le |h|^p \int_0^1 |\nabla u(x+th)|^p \, \mathrm{d}t, \quad \forall x \in \omega, \tag{2.65}$$

where we used Hölder's inequality. Then

$$\begin{aligned}
\|\tau_h u - u\|_{L^p(\omega)}^p &= \int_\omega |u(x+h) - u(x)|^p \, \mathrm{d}x \\
&\le |h|^p \int_\omega \int_0^1 |\nabla u(x+th)|^p \, \mathrm{d}t \, \mathrm{d}x \\
&\le |h|^p \int_0^1 \int_{\omega'} |\nabla u(y)|^p \, \mathrm{d}y \, \mathrm{d}t \\
&\le |h|^p \int_{\omega'} |\nabla u(y)|^p \, \mathrm{d}y,
\end{aligned} \tag{2.66}$$

where $\omega' := \{y : \mathrm{dist}\,(y, \omega) < |h|\} \subset\subset \Omega$. This gives that

$$\|\tau_h u - u\|_{L^p(\omega)} \le |h| \|\nabla u\|_{L^p(\omega')}, \quad \forall u \in C_c^\infty(\mathbb{R}^d). \tag{2.67}$$

Assume now that u $W^{1,p}(\Omega\ )$ with $1 \leq p < \infty$. By Theorem 2.30, there exists a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \to u \quad \text{in } L^p(\Omega\ )$$

and

$$\nabla u_n \to \nabla u \quad \text{in } L^p(\omega) \text{ for all } \omega \subset\subset \Omega.$$

Applying (2.67) to $\{u_n\}$ and passing $n \to \infty$, we obtain (iii) for every $u \in W^{1,p}(\Omega\ ),\ \ 1 \leq p < \infty$.

Applying the above result for $p < \infty$ and passing $p \to \infty$ in (2.67) gives our desired result for $p = \infty$.

(iii) $\Rightarrow$ (ii). Given $\phi \in C_c^\infty(\Omega\ )$. Let ω be a neighborhood of $\operatorname{supp}\phi$ such that $\operatorname{supp}\phi \subset \omega \subset\subset \Omega$. By (iii), for all $h \in \mathbb{R}^d$ with $|h| < \operatorname{dist}(\omega, \partial\Omega)$ there holds

$$\|\tau_h u - u\|_{L^p(\omega)} \leq C|h|,$$

Thus

$$\left|\int_\Omega (\tau_h u - u)\phi \,\mathrm{d}x\right| = \left|\int_\omega (\tau_h u - u)\phi \,\mathrm{d}x\right| \leq C|h|\|\phi\|_{L^{p'}(\Omega)}. \tag{2.68}$$

On the other hand, since

$$\int_\Omega (\tau_h u - u)\phi \,\mathrm{d}x = \int_\omega (u(x+h) - u(x))\phi(x) \,\mathrm{d}x = \int_\omega u(y)(\phi(y-h) - \phi(y)) \,\mathrm{d}y, \tag{2.69}$$

it follows that

$$\left|\int_\omega u(y)\frac{(\phi(y-h) - \phi(y))}{|h|} \,\mathrm{d}y\right| \leq C|h|\|\phi\|_{L^{p'}(\Omega)}. \tag{2.70}$$

Choosing $h = te_j,\ t \in \mathbb{R}$, and passing $t \to 0$ implies (ii).

Now we prove that for each $1 \leq p \leq \infty$, there holds

$$\|\tau_h u - u\|_{L^p(\mathbb{R}^d)} \leq |h|\|\nabla u\|_{L^p(\mathbb{R}^d)}, \quad \forall u \in W^{1,p}(\mathbb{R}^d).$$

When $1 \leq p < \infty$, we can apply Theorem 2.30 and using density argument to prove it, as in the case for general domain Ω.

For $p = \infty$ and $u \in W^{1,\infty}(\mathbb{R}^d)$, let $h \in \mathbb{R}^d$. We consider $B(0, n),\ n \in \mathbb{Z}_+$. Applying the result for $\Omega := B(0, n + |h| + 1)$ gives

$$\|\tau_h u - u\|_{L^\infty(B(0,n))} \leq |h|\|\nabla u\|_{L^\infty(B(0,n+|h|+1))} \leq |h|\|\nabla u\|_{L^\infty(\mathbb{R}^d)}.$$

Passing $n \to \infty$ gives our desired result. □

Remark 2.34. *From the proof of Proposition 2.33, we see that the constrain $p > 1$ is used only when proving (i) by using (ii). When $p = 1$, the following implications remain true:*

$$(i) \Rightarrow (ii) \Leftrightarrow (iii).$$

The functions that satisfy (ii) (or (iii)) with $p = 1$ are called functions of bounded variation. In the language of distributions, a function of bounded variation is an L^1 function such that all its first derivatives, in the sense of distributions, are bounded measures. This space plays an important role in many applications.

Remark 2.35. *From Proposition 2.33, we see that each function $u \in W^{1,\infty}(\Omega)$ has a continuous representative on Ω, that is there exists a continuous function $\tilde{u} \in C(\Omega)$ such that $u = \tilde{u}$ a.e. in Ω, and we will no longer distinguish them two. Moreover, if Ω is connected then*

$$|u(x) - u(y)| \leq \|\nabla u\|_{L^\infty(\Omega)} \mathrm{dist}_\Omega(x, y), \quad \forall x, y \in \Omega. \tag{2.71}$$

where $\mathrm{dist}_\Omega(x, y)$ denotes the geodesic distance from x to y in Ω ; in particular, if Ω is convex then $\mathrm{dist}_\Omega(x, y) = |x - y|$.

From here one can also deduce that if u $W^{1,p}(\Omega)$ for some $1 \leq p \leq \infty$ and some open set Ω, and if $\nabla u = 0$ a.e. in Ω, then u is constant on each connected component of Ω.

Proposition 2.36 (differentiation of a product)**.** *Let Ω be an open set in $\mathbb{R}^d$ and let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \leq p \leq \infty$. Then the product $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and*

$$\partial_j(uv) = (\partial_j u)v + u(\partial_j v), \quad j = 1, \cdots, d \tag{2.72}$$

in the generalized sense in Ω.

证明. It is sufficient to prove (2.72) (why?). That is for each $\phi \in C_c^\infty(\Omega)$, there holds

$$\int_\Omega uv(\partial_j \phi)\,\mathrm{d}x = -\int_\Omega \partial_j(uv)\phi\,\mathrm{d}x = -\int_\Omega [(\partial_j u)v + u(\partial_j v)]\phi\,\mathrm{d}x. \tag{2.73}$$

We start with the case $1 \leq p < \infty$. By Theorem 2.30, there exists two sequences $\{u_n\}$ and $\{v_n\}$ in $C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \to u, \ v_n \to v, \quad \text{in } L^p(\Omega)$$

and

$$\nabla u_n \to \nabla u, \ \nabla v_n \to \nabla v \quad \text{in } L^p(\omega) \text{ for all } \omega \subset\subset \Omega.$$

Moreover, from the proof of Theorem 2.30, we know that

$$\|u_n\|_{L^\infty(\mathbb{R}^d)} \leq \|u\|_{L^\infty(\Omega)}, \quad \|v_n\|_{L^\infty(\mathbb{R}^d)} \leq \|v\|_{L^\infty(\Omega)}.$$

For each $\phi \in C_c^\infty(\Omega)$, integration by parts gives

$$\int_\Omega u_n v_n(\partial_j \phi)\,\mathrm{d}x = -\int_\Omega \partial_j(u_n v_n)\phi\,\mathrm{d}x = -\int_\Omega [(\partial_j u_n)v_n + u_n(\partial_j v_n)]\phi\,\mathrm{d}x. \tag{2.74}$$

Passing $n \to \infty$ in (2.74) implies (2.73). (why?)

We then consider the case $p=\infty$. Given $\phi \in C_c^\infty(\Omega)$, there exists a bounded open set $\omega \subset\subset \Omega$ such that $\operatorname{supp}\phi \subset \omega$. To prove (2.73), we can work in ω instead of Ω. Since $u,v \in W^{1,\infty}(\omega)\cap L^\infty(\omega)$ then $u,v \in W^{1,p}(\omega)\cap L^\infty(\omega)$ for all $1 \le p \le \infty$. Then employing the proof of the case $1 \le p < \infty$ gives us (2.73).

□

Proposition 2.37 (differentiation of a composition). *Let Ω be an open set in $\mathbb{R}^d$ and let $u \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $1 \le p \le \infty$. Let $G \in C^1(\mathbb{R})$ be such that $G(0)=0$ and $|G'(s)| \le M$, $\forall s \in \mathbb{R}$ for some constant M. Then the composition*

$$G\circ u \in W^{1,p}(\Omega), \quad \partial_j(G\circ u) = (G'\circ u)(\partial_j u),\ j=1,\cdots,d. \tag{2.75}$$

证明. We have that $|G(s)| \le M|s|$ for all $s\in\mathbb{R}$. Thus $|G\circ u| \le M|u|$ and $G\circ u \in L^p(\Omega)$. Similarly, $|(G'\circ u)(\partial_j u)| \le M|\partial_j u| \in L^p(\Omega)$. It is left to verify for each $\phi \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega (G\circ u)(\partial_j\phi)\,\mathrm{d}x = -\int_\Omega (G'\circ u)(\partial_j u)\phi\,\mathrm{d}x, \qquad j=1,\cdots,d. \tag{2.76}$$

When $1 \le p < \infty$, by Theorem 2.30, there exists a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \to u, \quad \text{in } L^p(\Omega\ )$$

and

$$\nabla u_n \to \nabla u, \quad \text{in } L^p(\omega) \text{ for all } \omega \subset\subset \Omega.$$

Moreover, from the proof of Theorem 2.30, we know that

$$\|u_n\|_{L^\infty(\mathbb{R}^d)} \le \|u\|_{L^\infty(\Omega)}.$$

On the other hand, integration by parts implies

$$\int_\Omega (G\circ u_n)(\partial_j\phi)\,\mathrm{d}x = -\int_\Omega (G'\circ u_n)(\partial_j u)\phi\,\mathrm{d}x, \qquad j=1,\cdots,d. \tag{2.77}$$

Then passing $n\to\infty$ in (2.77) implies (2.76). (why?)

When $p=\infty$, we consider a bounded open set $\omega\subset\subset\Omega$ such that $\operatorname{supp}\phi\subset\omega$. Then $u \in W^{1,p}(\omega)$ for all $1\le p\le\infty$. Then applying the argument for the case $p<\infty$ implies (2.76).

□

Proposition 2.38 (change of variables formula). *Let Ω and Ω' be two open sets in $\mathbb{R}^d$ and let $H:\Omega'\to\Omega$ be a bijective map such that $H\in C^1(\Omega')$, $H^{-1}\in C^1(\Omega)$, and the Jacobian matrices* $\operatorname{Joc} H = (\partial_j H_i)_{1\le i,j\le d} \in L^\infty(\Omega')$, $\operatorname{Joc} H^{-1} = (\partial_j H_i^{-1})_{1\le i,j\le d} \in L^\infty(\Omega)$. *Let $u \in W^{1,p}(\Omega)$ with $1\le p\le\infty$. Then $u\circ H \in W^{1,p}(\Omega')$ and*

$$\partial_{y_j} u(H(y)) = \sum_{i=1}^d (\partial_{x_i}u)(H(y))\partial_{y_j}H_i(y),\ j=1,\cdots,d. \tag{2.78}$$

证明. Clearly, by the property of H, we have

$$u \circ H \in L^p(\Omega'), \quad (\partial_{x_i} u)(H(y))\partial_{y_j} H_i(y) \in L^p(\Omega')$$

for all $1 \leq i, j \leq d$. It remains to show (2.78). That is for all $\psi \in C_c^\infty(\Omega')$ and all $j = 1, \cdots, d$, there holds

$$\int_{\Omega'} u(H(y))\partial_{y_j}\psi(y)\,\mathrm{d}y = -\int_{\Omega'} \sum_{i=1}^{d} (\partial_{x_i} u)(H(y))\partial_{y_j} H_i(y)\,\mathrm{d}y, \tag{2.79}$$

When $1 \leq p < \infty$, by Theorem 2.30, there exists a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \to u, \quad \text{in } L^p(\Omega\ )$$

and

$$\nabla u_n \to \nabla u, \quad \text{in } L^p(\omega) \text{ for all } \omega \subset\subset \Omega.$$

Thus, by the property of H, we have (why?)

$$u_n \circ H \to u \circ H, \quad \text{in } L^p(\Omega\ )$$

and

$$(\partial_{x_i} u_n)(H(y))\partial_{y_j} H_i(y) \to (\partial_{x_i} u)(H(y))\partial_{y_j} H_i(y), \quad \text{in } L^p(\omega') \text{ for all } \omega' \subset\subset \Omega'.$$

On the other hand, integration by parts implies

$$\int_{\Omega'} u_n(H(y))\partial_{y_j}\psi(y)\,\mathrm{d}y = -\int_{\Omega'} \sum_{i=1}^{d} (\partial_{x_i} u_n)(H(y))\partial_{y_j} H_i(y)\,\mathrm{d}y, \tag{2.80}$$

for all $\psi \in C_c^\infty(\Omega')$ and all $j = 1, \cdots, d$. Then passing $n \to \infty$ in (2.80) implies (2.79).

When $p = \infty$, we consider a bounded open set $\omega \subset\subset \Omega$ such that $\operatorname{supp}\phi \subset \omega$. Then $u \in W^{1,p}(\omega)$ for all $1 \leq p \leq \infty$. Then applying the argument for the case $p < \infty$ implies (2.79).

□

2.9 The spaces $W^{m,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ be an open set, let $m \geq 2$ be an integer and let $1 \leq p \leq \infty$. We define

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : \partial_x^\alpha u \in L^p(\Omega) \text{ for all multi-index } \alpha \text{ such that } |\alpha| \leq m\} \tag{2.81}$$

where the derivative $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_d}^{\alpha_d}$ is defined in the generalized sense in Ω.

The space $W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}(\Omega)} := \sum_{0\leq|\alpha|\leq m} \|\partial_x^\alpha u\|_{L^p(\Omega)} \tag{2.82}$$

is a Banach space.

The space $H^m(\Omega) := W^{m,2}(\Omega)$ equipped with the inner product

$$(u,v)_{H^m(\Omega)} := \sum_{0\leq|\alpha|\leq m} (\partial^a u, \partial^\alpha v)L^2(\Omega) \tag{2.83}$$

is a Hilbert space.

Remark 2.39. *One can show that if Ω is 'smooth enough' with $\partial\Omega$ bounded, then the norm on $W^{m,p}(\Omega)$ is equivalent to the norm*

$$\|u\|_{L^p(\Omega)} + \sum_{|\alpha|=m} \|\partial_x^\alpha u\|_{L^p(\Omega)}. \tag{2.84}$$

More precisely, it can be proved that for every multi-index β with $0 < |\beta| < m$ and for every $\varepsilon > 0$, there exists a constant C depending on Ω, ε, α, such that

$$\|\partial^\beta u\|_{L^p(\Omega)} \leq \varepsilon \sum_{|\alpha|=m} \|\partial_x^\alpha u\|_{L^p(\Omega)} + C\|u\|_{L^p(\Omega)}, \quad \forall u \in W^{m,p}(\Omega). \tag{2.85}$$

We refer to Adams [1] for the details and the proofs.

2.10 Extension operators

It is often convenient to establish properties of functions in $W^{1,p}(\Omega)$ by beginning with the case $\Omega = \mathbb{R}^d$, for example the Sobolev Inequalities. It is therefore useful to be able to extend a function $u \in W^{1,p}(\Omega)$ to a function $u \in W^{1,p}(\mathbb{R}^d)$. This is not always possible for a general domain Ω. However, if Ω is 'smooth', such an extension can be constructed. Let us begin by making precise the notion of a smooth open set.

We first introduce some notations. Given $x \in \mathbb{R}^d$, we write

$$x = (x', x_d) \text{ with } x' \in \mathbb{R}^{d-1},\ x' = (x_1, x_2, \cdots, x_{d-1}),$$

and set

$$|x'| = \left(\sum_{i=1}^{d-1} x_i^2\right)^{\frac{1}{2}}.$$

We define

$$\begin{aligned}
\mathbb{R}^d_+ &:= \{x = (x\ , x_d) \in \mathbb{R}^d : x_d > 0\},\\
\mathbb{R}^d_- &:= \{x = (x\ , x_d) \in \mathbb{R}^d : x_d < 0\},\\
Q &:= \{x = (x\ , x_d) \in \mathbb{R}^d : |x'| < 1,\ |x_d| < 1\},\\
Q_+ &:= Q \cap \mathbb{R}^d_+,\\
Q_- &:= Q \cap \mathbb{R}^d_-,\\
Q_0 &= \{x = (x', 0) : |x\ | < 1\}.
\end{aligned} \tag{2.86}$$

Definition 2.40. *Let $\Omega \in \mathbb{R}^d$ be an open set with boundary $\Gamma := \partial\Omega$. We say Ω is of class C^1 if:*

(i) There exists an open cover $\{\Omega_i\}_{i\geq 0}$ of Ω such that

$$\Omega_0 \subset \Omega, \ \operatorname{dist}(\Omega_0, \partial\Omega) > 0,$$

and

$$\text{for each } i \geq 1,\ \Omega_i \text{ is bounded and } \Omega_i \cap \partial\Omega \neq \emptyset,\ \partial\Omega \subset \bigcup_{i\geq 1} \Omega_i,$$

and either the family $\{\Omega_i\}_{i\geq 0}$ is finite or

$$\text{there exists } k_0 \in \mathbb{Z}_+ \text{ such that } |i-j| \geq k_0 \Longrightarrow \Omega_i \cap \Omega_j = \emptyset.$$

(ii) For each $i \geq 1$, there exists a bijective map $H_i : Q \to \Omega_i$ such that

$$H_i \in C^1(\overline{Q}),\ H_i^{-1} \in C^1(\overline{\Omega}_i),\ H_i(Q_+) = \Omega_i \cap Q,\ H_i(Q_0) = \Omega_i \cap (\partial\Omega)\ . \tag{2.87}$$

The map H_i is called a local chart (or local coordinates) of the boundary $\partial\Omega$.

(iii) There exist a C^∞ partition of unity $\{\varphi\}_{i\geq 0}$ subordinate to the cover $\{\Omega_i\}_{i\geq 0}$ and constants C_1 and C_2 such that

$$\sup_{i\geq 0} \|\varphi_i\|_{W^{1,\infty}(\mathbb{R}^d)} \leq C_1, \quad \sup_{i\geq 1} \|(H_i, H_i^{-1})\|_{W^{1,\infty}(Q)\times W^{1,\infty}(\Omega_i)} \leq C_2. \tag{2.88}$$

The C^1 character (or C^1 norm) of Ω is defined by $C_1 + C_2$.

- *We say Ω is of class Lipschitz if in the above definition, the local coordinates H_i, $i \geq 1$ are merely uniform Lipschitz functions. The rest of the definition remains unchanged.*
- *Similarly, one can define a C^m, $m \geq 2$ domain or a $C^{m,\alpha}$, $m \geq 0$ domain.*

In the above definition, we used a C^∞ partition of unity. Here we recall its definition:

Definition 2.41. *A C^∞ partition of unity subordinate to an open cover $\{\Omega_i\}_{i\in\mathbb{N}}$ of the open set Ω is a sequence of smooth functions $\{\varphi_i\}_{i\in\mathbb{N}}$ with the following properties:*

(i) For every i, $\varphi_i \in C_c^\infty(\Omega_i)$, $0 \leq \varphi_i \leq 1$.

(ii) For each compact subset $K \subset \Omega$, only a finite number of the functions φ_i are not zero on K.

(iii) For all $x \in \Omega$, $\sum_{i\geq 0} \varphi_i(x) = 1$.

A classical result on the C^∞ partition of unity is the following:

Lemma 2.42. *Let $\Omega \subset \mathbb{R}^d$ be an open set with bounded boundary $\Gamma := \partial\Omega$. Then there exists a finite open cover $\{\Omega_i\}_{i=0}^N$ of Ω such that*

$$\Omega_0 \subset \Omega,\ \operatorname{dist}(\Omega_0, \partial\Omega) > 0,$$

and

$$\text{for each } i \geq 1,\ \Omega_i \text{ is bounded and } \Omega_i \cap \partial\Omega \neq \emptyset,\ \partial\Omega \subset \bigcup_{i\geq 1} \Omega_i.$$

Moreover, there exists a C^∞ partition of unity $\{\varphi\}_{i\geq 0}$ subordinate to the cover $\{\Omega_i\}_{i\geq 0}$.

证明. Exercise.

□

The main result is the following:

Theorem 2.43. *Suppose that Ω is of class C^1 in the sense of Definition 2.40 (or $\Omega = \mathbb{R}^d_+$). Then there exists a linear extension operator*

$$E : W^{1,p}(\Omega\ ) \to W^{1,p}(\mathbb{R}^d),\ 1 \leq p \leq \infty,$$

such that for all $u \in W^{1,p}(\Omega\ )$, there holds

$$Eu|_\Omega\ = u, \quad \|Eu\|_{L^p(\mathbb{R}^d)} \leq C\|Eu\|_{L^p(\Omega)}, \quad \|u\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\Omega)}, \tag{2.89}$$

where C depends only on Ω through C_1, C_2 in Definition 2.40.

Remark 2.44. *We remark that the same result holds if Ω is of class Lipschitz, slightly weaker than C^1. This can be seen from the proof that the estimate constant C in* (2.89) *depends only on the Lipschitz norm, that is $W^{1,\infty}$ norm, of the local chart and the C^∞ partition of unity. The proof is left to the students.*

We shall begin by proving a simple but fundamental lemma concerning the extension by reflection.

Lemma 2.45. *Given $u \in W^{1,p}(Q_+)$ with $1 \leq p \leq \infty$, one defines the function u^* on Q to be the extension by reflection, that is,*

$$u^*(x', x_d) = \begin{cases} u(x', x_d), & \text{if } x_d > 0, \\ u(x', -x_d), & \text{if } x_d < 0. \end{cases} \tag{2.90}$$

Then $u^* \in W^{1,p}(Q)$ and

$$\|u^*\|_{L^p(Q)} \leq 2\|u\|_{L^p(Q_+)}, \quad \|u^*\|_{W^{1,p}(Q)} \leq 2\|u\|_{W^{1,p}(Q_+)}. \tag{2.91}$$

证明. First of all, it is clear that $\|u^*\|_{L^p(Q)} \le 2\|u\|_{L^p(Q_+)}$. We shall furthermore prove that

$$\frac{\partial u^*}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^* \text{ for } 1 \le i \le d-1, \quad \frac{\partial u^*}{\partial x_d} = \left(\frac{\partial u}{\partial x_d}\right)^{\#}, \tag{2.92}$$

where $\left(\frac{\partial u}{\partial x_i}\right)^*$ denotes the extension by reflection of $\frac{\partial u}{\partial x_i}$ as in (2.90), and $\left(\frac{\partial u}{\partial x_d}\right)^{\#}$ denotes the extension by minus-reflection in Q defined as

$$f^{\#}(x', x_d) = \begin{cases} f(x', x_d), & \text{if } x_d > 0, \\ -f(x', -x_d), & \text{if } x_d < 0 \end{cases} \tag{2.93}$$

for a function f defined in Q_+.

Choose $\eta \in C^\infty(\mathbb{R})$ satisfying

$$\eta(t) = 0,\ t \le 1/2; \quad \eta(t) = 1,\ t \ge 1.$$

Then define

$$\eta_k(\cdot) = \eta(k\cdot),\ k \in \mathbb{Z}_+.$$

Given $\varphi \in C_c^\infty(Q)$. For $1 \le i \le d-1$, we have

$$\begin{aligned} \int_Q u^* \frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x &= \int_{Q_+} u(x', x_d) \frac{\partial \varphi}{\partial x_i}(x)\,\mathrm{d}x + \int_{Q_-} u(x', -x_d) \frac{\partial \varphi}{\partial x_i}(x)\,\mathrm{d}x \\ &= \int_{Q_+} u(x', x_d) \frac{\partial \varphi}{\partial x_i}(x', x_d)\,\mathrm{d}x + \int_{Q_+} u(x', x_d) \frac{\partial \varphi}{\partial x_i}(x', -x_d)\,\mathrm{d}x. \end{aligned} \tag{2.94}$$

We would like to switch the partial derivative in (2.94) to u by the definition of generalized derivatives, but this cannot be done directly because a $C_c^\infty(Q)$ function is not in general a proper test function for u in Q_+. While, by using the property of η, we know that $\operatorname{supp}(\eta_k(x_d)\varphi(x', \pm x_d)) \subset Q_+$ for all $k \in \mathbb{Z}_+$. So $\eta_k(x_d)\varphi(x', \pm x_d)$ is a good test function in Q_+ and there holds

$$\begin{aligned} \int_{Q_+} u(x', x_d) \frac{\partial\big(\eta_k(x_d)\varphi(x', x_d)\big)}{\partial x_i}\,\mathrm{d}x &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i}\big(\eta_k(x_d)\varphi(x', x_d)\big)\,\mathrm{d}x, \\ \int_{Q_+} u(x', x_d) \frac{\partial\big(\eta_k(x_d)\varphi(x', -x_d)\big)}{\partial x_i}\,\mathrm{d}x &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i}\big(\eta_k(x_d)\varphi(x', -x_d)\big)\,\mathrm{d}x. \end{aligned} \tag{2.95}$$

Since for $1 \le i \le d-1$,

$$\frac{\partial}{\partial x_i}(\eta_k(x_d)\varphi(x', \pm x_d)) = \eta_k(x_d) \frac{\partial \varphi(x', \pm x_d)}{\partial x_i},$$

we then have

$$\begin{aligned} \int_{Q_+} u(x', x_d)\eta_k(x_d) \frac{\partial \varphi(x', x_d)}{\partial x_i}\,\mathrm{d}x &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i}\big(\eta_k(x_d)\varphi(x', x_d)\big)\,\mathrm{d}x, \\ \int_{Q_+} u(x', x_d)\eta_k(x_d) \frac{\partial \varphi(x', -x_d)}{\partial x_i}\,\mathrm{d}x &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i}\big(\eta_k(x_d)\varphi(x', -x_d)\big)\,\mathrm{d}x. \end{aligned} \tag{2.96}$$

By Lebesgue's dominated convergence theorem, passing $k \to \infty$ in (2.96) implies

$$\begin{aligned} \int_{Q_+} u(x', x_d) \frac{\partial \varphi(x', x_d)}{\partial x_i} \,\mathrm{d}x &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i} \varphi(x', x_d) \,\mathrm{d}x, \\ \int_{Q_+} u(x', x_d) \frac{\partial \varphi(x', -x_d)}{\partial x_i} \,\mathrm{d}x &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i} \varphi(x', -x_d) \,\mathrm{d}x. \end{aligned} \tag{2.97}$$

Combining (2.94) and (2.97) , we are led to

$$\begin{aligned} \int_Q u^* \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i} \varphi(x', x_d) \,\mathrm{d}x - \int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i} \varphi(x', -x_d) \,\mathrm{d}x \\ &= -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_i} \varphi(x', x_d) \,\mathrm{d}x - \int_{Q_-} \frac{\partial u(x', -x_d)}{\partial x_i} \varphi(x', x_d) \,\mathrm{d}x \\ &= -\int_Q \left(\frac{\partial u}{\partial x_i} \right)^* (x) \varphi(x) \,\mathrm{d}x, \ 1 \le i \le d-1. \end{aligned} \tag{2.98}$$

This means

$$\frac{\partial u^*}{\partial x_i} = \left(\frac{\partial u}{\partial x_i} \right)^* \text{ for } 1 \le i \le d-1.$$

Now we turn to consider the derivative on x_d. Given $\varphi \in C_c^\infty(Q)$, direct calculation gives

$$\begin{aligned} \int_Q u^* \frac{\partial \varphi}{\partial x_d} \,\mathrm{d}x &= \int_{Q_+} u(x', x_d) \frac{\partial \varphi}{\partial x_d}(x) \,\mathrm{d}x + \int_{Q_-} u(x', -x_d) \frac{\partial \varphi}{\partial x_d}(x) \,\mathrm{d}x \\ &= \int_{Q_+} u(x', x_d) \frac{\partial \varphi}{\partial x_d}(x', x_d) \,\mathrm{d}x + \int_{Q_+} u(x', x_d) \frac{\partial \varphi}{\partial x_d}(x', -x_d) \,\mathrm{d}x \\ &= \int_{Q_+} u(x', x_d) \frac{\partial (\varphi(x', x_d))}{\partial x_d} \,\mathrm{d}x - \int_{Q_+} u(x', x_d) \frac{\partial (\varphi(x', -x_d))}{\partial x_d} \,\mathrm{d}x \\ &= \int_{Q_+} u \frac{\partial \psi}{\partial x_d} \,\mathrm{d}x, \end{aligned} \tag{2.99}$$

where

$$\psi(x', x_d) := \varphi(x', x_d) - \varphi(x', -x_d).$$

Note that $\psi(x', 0) = 0$, then there exists a constant M such that $|\psi(x', x_d)| \le M|x_d|$ on Q.

Since $\eta_k(x_d)\psi(x', x_d) \in C_c^\infty(Q_+)$, then

$$\int_{Q_+} u(x', x_d) \frac{\partial \big(\eta_k(x_d)\psi(x', x_d)\big)}{\partial x_d} \,\mathrm{d}x = -\int_{Q_+} \frac{\partial u(x', x_d)}{\partial x_d} \big(\eta_k(x_d)\psi(x', x_d)\big) \,\mathrm{d}x. \tag{2.100}$$

Here

$$\frac{\partial \big(\eta_k(x_d)\psi(x', x_d)\big)}{\partial x_d} = \eta_k(x_d) \frac{\partial \psi(x', x_d)}{\partial x_d} + k\eta'(kx_d)\psi(x', x_d) \tag{2.101}$$

We claim that

$$\int_{Q_+} u(x', x_d) k\eta'(kx_d)\psi(x', x_d) \,\mathrm{d}x \to 0, \text{ as } k \to \infty. \tag{2.102}$$

Indeed, we have

$$\begin{aligned}\left|\int_{Q_+} u(x',x_d)k\eta'(kx_d)\psi(x',x_d)\,\mathrm{d}x\right| &\le k\|\eta'\|_{L^\infty}M\int_{x\in Q_+,\frac{1}{2k}\le x_d\le\frac{1}{k}}|u(x',x_d)||x_d|\,\mathrm{d}x\\ &\le \|\eta'\|_{L^\infty}M\int_{x\in Q_+,\frac{1}{2k}\le x_d\le\frac{1}{k}}|u(x',x_d)|\,\mathrm{d}x\end{aligned} \tag{2.103}$$

which goes to zero as $k\to\infty$ because u is integrable in Q_+.

By (2.100), (2.101) and (2.102) and passing $k\to\infty$, we have

$$\int_{Q_+} u\frac{\partial\psi}{\partial x_d}\,\mathrm{d}x = -\int_{Q_+}\frac{\partial u}{\partial x_d}\psi\,\mathrm{d}x. \tag{2.104}$$

Together with (2.99), we finally obtain

$$\int_Q u^*\frac{\partial\varphi}{\partial x_d}\,\mathrm{d}x = \int_{Q_+} u\frac{\partial\psi}{\partial x_d}\,\mathrm{d}x = -\int_{Q_+}\frac{\partial u}{\partial x_d}\psi\,\mathrm{d}x = -\int_Q\left(\frac{\partial u}{\partial x_d}\right)^{\#}\varphi\,\mathrm{d}x. \tag{2.105}$$

This means

$$\frac{\partial u^*}{\partial x_d} = \left(\frac{\partial u}{\partial x_d}\right)^{\#}.$$

□

Remark 2.46. *The conclusion of Lemma 2.45 remains valid if Q_+ is replaced by $\mathbb{R}^d_+$ (the proof is unchanged). This establishes Theorem 2.43 for $\Omega = \mathbb{R}^d_+$.*

Now we are ready to prove Theorem 2.43:

Proof of Theorem 2.43. The case $\Omega = \mathbb{R}_+$ can be proved by using the extension by reflection, as in Lemma 2.45. Suppose that Ω is of class Lipschitz with $\partial\Omega$ bounded. By Definition 2.40, there exists an open cover $\{\Omega\}_{i\ge0}$ and a local chart $\{H_i\}_{i\ge0}$ satisfy the stated property in Definition 2.40. Let $\{H_i\}_{i\ge0}$ be the related local chart. By Lemma 2.42, there exists a C^∞ partition of unity $\{\varphi\}_{i\ge0}$ related to the open cover $\{\Omega_i\}_{i\ge0}$.

Let $u\in W^{1,p}(\Omega)$. By the definition of a C^∞ partition of unity, we have the decomposion

$$u = \sum_{i\ge0}\varphi_i u = \sum_{i\ge0} u_i \text{ in } \Omega, \quad u_i := \varphi_i u. \tag{2.106}$$

Now we extend each of the functions u_i to $\mathbb{R}^d$, distinguishing u_0 and $u_i,\ i\ge1$.

Extension of u_0. Since $\varphi_0\in C_c^\infty(\Omega_0)$ with $\Omega_0\subset\Omega$, by Proposition 2.28, the zero extension of $u_0=\varphi_0 u$, denoted by $\tilde u_0$, is in $W^{1,p}(\mathbb{R}^d)$, and there holds

$$\partial_{x_j}(\widetilde{u_0}) = \partial_{x_j}(\widetilde{u\varphi_0}) = \widetilde{\varphi_0\partial_{x_j}u} + \widetilde{u\partial_{x_j}\varphi_0}.$$

Thus

$$\|\tilde u_0\|_{L^p(\mathbb{R}^d)}\le C\|u\|_{L^p(\Omega)}, \quad \|\tilde u_0\|_{W^{1,p}(\mathbb{R}^d)}\le C\|u\|_{W^{1,p}(\Omega)}.$$

Extension of $u_i,\ i \geq 1$.

Consider the restriction of u to $\Omega_i \cap \Omega$ and "transfer" this function to Q_+ with the help of H_i . More precisely, set

$$v_i(y) = u(H_i(y)), \quad y \in Q_+.$$

We know from Proposition 2.38 that $v_i \in W^{1,p}(Q_+)$. Then define the extension of v_i on Q by reflection of v_i as in Lemma 2.45, and we denote this extension by v_i^*. We know that $v_i^* \in W^{1,p}(Q)$. Then we "retransfer" v_i^* to Ω_i using H_i^{-1}:

$$w_i = v_i^*(H_i^{-1}(x)), \quad x \in \Omega_i.$$

Then $w_i \in W^{1,p}(\Omega_i)$, $w_i = u$ in $\Omega_i \cap \Omega$, and

$$\|w_i\|_{L^p(\Omega_i)} \leq C\|u\|_{L^p(\Omega_i\cap\Omega)}, \quad \|w_i\|_{W^{1,p}(\Omega_i)} \leq C\|u\|_{W^{1,p}(\Omega_i\cap\Omega)}. \tag{2.107}$$

Finally define $\hat{u}_i$ a function on $\mathbb{R}^d$ as

$$\hat{u}_i := \widetilde{\varphi_i(x)w_i}(x) := \begin{cases} \varphi_i(x)w_i(x), & x \in \Omega_i, \\ 0, & x \in \mathbb{R}^d \setminus \Omega_i. \end{cases}$$

Then $\hat{u}_i \in W^{1,p}(\mathbb{R}^d)$, $\hat{u}_i = u_i$ in Ω, and

$$\|\hat{u}_i\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|w_i\|_{W^{1,p}(\Omega_i)} \leq C\|u\|_{W^{1,p}(\Omega\cap\Omega_i)}. \tag{2.108}$$

Extension opeartor. Our desired extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ as

$$Eu := \tilde{u}_0 + \sum_{i\geq 1} \hat{u}_i = \varphi_0 u + \sum_{i\geq 1} \varphi_i(x)w_i(x), \tag{2.109}$$

where naturally we omit the zero extension notation ˜ outside of the support of suppφ_i. Clearly

$$\operatorname{supp} \tilde{u}_0 \subset \Omega_0, \quad \operatorname{supp} \hat{u}_i \subset \Omega_i,\ i \geq 1. \tag{2.110}$$

Now we show that $Eu \in W^{1,p}(\mathbb{R}^d)$. By the property (i) in Definition 2.40, for each $x \in \mathbb{R}^d$, the definition of $Eu(x)$ in (2.109) is a finite sum; moreover, for each $x \in \mathbb{R}^d$, wewrite

$$Eu(x) = \tilde{u}_0(x) + \sum_{j=1}^{k_0} \sum_{m\geq 0} \hat{u}_{mk_0+j}(x) =: \tilde{u}_0(x) + \sum_{j=1}^{k_0} v_j(x), \tag{2.111}$$

with

$$v_j := \sum_{m\geq 0} \hat{u}_{mk_0+j},\ 1 \leq j \leq k_0.$$

Then there holds for each $1 \leq j \leq k_0$ that

$$\operatorname{supp} \hat{u}_{mk_0+j} \subset \Omega_{mk_0+j}, \quad \Omega_{m_1k_0+j} \cap \Omega_{m_2k_0+j} = \emptyset, \quad \forall\ m_1 \neq m_2. \tag{2.112}$$

When $1 \le p < \infty$, for for each $1 \le j \le k_0$,

$$\begin{aligned}\int_{\mathbb{R}^d} |v_j(x)|^p \,\mathrm{d}x &= \int_{\cup_{m\ge0}\Omega_{mk_0+j}} |\sum_{m\ge0} \hat{u}_{mk_0+j}(x)|^p \,\mathrm{d}x \\ &= \int_{\cup_{m\ge0}\Omega_{mk_0+j}} |\sum_{m\ge0} \hat{u}_{mk_0+j}(x)|^p \,\mathrm{d}x \\ &= \sum_{m\ge0} \int_{\Omega_{mk_0+j}} |\sum_{m\ge0} \hat{u}_{mk_0+j}(x)|^p \,\mathrm{d}x \\ &= \sum_{m\ge0} \int_{\Omega_{mk_0+j}} |\hat{u}_{mk_0+j}(x)|^p \,\mathrm{d}x.\end{aligned} \tag{2.113}$$

Together with (2.119) and (2.108), we deduce

$$\begin{aligned}\int_{\mathbb{R}^d} |v_j(x)|^p \,\mathrm{d}x &= \sum_{m\ge0} \int_{\Omega_{mk_0+j}} |\hat{u}_{mk_0+j}(x)|^p \,\mathrm{d}x \\ &\le C^p \sum_{m\ge0} \int_{\Omega_{mk_0+j}} |w_{mk_0+j}(x)|^p \,\mathrm{d}x \\ &\le C^p \sum_{m\ge0} \int_{\Omega\cap\Omega_{mk_0+j}} |u(x)|^p \,\mathrm{d}x \\ &= C^p \int_{\cup_{m\ge0}\Omega\cap\Omega_{mk_0+j}} |u(x)|^p \,\mathrm{d}x \\ &\le C^p \int_{\Omega} |u(x)|^p \,\mathrm{d}x.\end{aligned} \tag{2.114}$$

That is

$$\|v_j\|_{L^p(\mathbb{R}^d)} \le C\|u\|_{L^p(\Omega)}. \tag{2.115}$$

Now we consider $\|\nabla v_j\|_{L^p(\mathbb{R}^d)}$. For each $\varphi \in C_c^\infty(\mathbb{R}^d)$, there are only finite number of φ_i that are nonzero on $\operatorname{supp}\varphi$. Thus

$$\begin{aligned}\int_{\mathbb{R}^d} v_j \partial\varphi \,\mathrm{d}x &= \int_{\mathbb{R}^d} \big(\sum_{m\ge0} \hat{u}_{mk_0+j}\big)\partial\varphi \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \big(\sum_{m\ge0} \varphi_{mk_0+j} w_{mk_0+j}\big)\partial\varphi \,\mathrm{d}x \\ &= \sum_{m\ge0} \int_{\mathbb{R}^d} \big(\varphi_{mk_0+j} w_{mk_0+j}\big)\partial\varphi \,\mathrm{d}x \\ &= -\sum_{m\ge0} \int_{\mathbb{R}^d} \big(\partial\varphi_{mk_0+j} w_{mk_0+j} + \varphi_{mk_0+j}\partial w_{mk_0+j}\big)\varphi \,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \sum_{m\ge0} \big(\partial\varphi_{mk_0+j} w_{mk_0+j} + \varphi_{mk_0+j}\partial w_{mk_0+j}\big)\varphi \,\mathrm{d}x.\end{aligned} \tag{2.116}$$

This implies that

$$\nabla v_j = \sum_{m\ge0} \big(\nabla\varphi_{mk_0+j} w_{mk_0+j} + \varphi_{mk_0+j}\nabla w_{mk_0+j}\big). \tag{2.117}$$

Again by the property (2.112), similarly as the derivation of (2.114) and (2.115), we can deduce

$$\|\nabla v_j\|_{L^p(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\Omega)}. \tag{2.118}$$

Hence $Eu \in W^{1,p}(\mathbb{R}^d)$ and

$$\begin{aligned} \|Eu\|_{L^p(\mathbb{R}^d)} &\le \|\tilde{u}_0\|_{L^p(\mathbb{R}^d)} + \sum_{1\le j\le k_0} \|v_j\|_{L^p(\mathbb{R}^d)} \le C\|u\|_{L^p(\Omega)}, \\ \|\nabla Eu\|_{L^p(\mathbb{R}^d)} &\le \|\nabla\tilde{u}_0\|_{L^p(\mathbb{R}^d)} + \sum_{1\le j\le k_0} \|\nabla v_j\|_{L^p(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\Omega)}. \end{aligned} \tag{2.119}$$

We complete the proof.

□

Remark 2.47. *If $\partial\Omega$ is bounded and then is compact, there exists a finite subcover $\{\Omega_i\}_{i=0}^N$ of the cover $\{\Omega\}_{i\ge1}$. By Lemma 2.42, there exists a C^∞ partition of unity $\{\varphi\}_{i=0}^N$ related to the finite subcover $\{\Omega_i\}_{i=0}^N$. Let $u \in W^{1,p}(\Omega)$. By the definition of a C^∞ partition of unity, we have the decomposition of finite sum*

$$u = \sum_{i=0}^N \varphi_i u = \sum_{i=0}^N u_i \text{ in } \Omega, \quad u_i := \varphi_i u. \tag{2.120}$$

Corollary 2.48. . *Assume that Ω is of class C^1, and let $u \in W^{1,p}(\Omega)$ with $1 \le p < \infty$. Then there exists a sequence $\{u_n\}$ in $C_c^\infty(\mathbb{R}^d)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. In other words, $C_c^\infty(\mathbb{R}^d)$ functions form a dense subspace of $W^{1,p}(\Omega)$.*

证明. By Theorem 2.43, there exists an extension $Eu \in W^{1,p}(\mathbb{R}^d)$ such that $Eu = u$ in Ω. From the proof of Theorem 2.30, we know the sequence $\chi_n S_{1/n}[Eu]$ converges to Eu in $W^{1,p}(\mathbb{R}^d)$ and thus it answers the problem.

□

2.11 Sobolev inequalities

2.11.1 The case $\Omega = \mathbb{R}^d$

Theorem 2.49. *Let $p \ge 2$ and $1 \le p < d$. Then*

$$W^{1,p}(\mathbb{R}^d) \subset L^{p^*}(\mathbb{R}^d), \quad \text{with } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}, \tag{2.121}$$

and there exists a constant $C = C(p,d)$ such that

$$\|u\|_{L^{p^*}(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W^{1,p}(\mathbb{R}^d). \tag{2.122}$$

Remark 2.50. *The constant $C = C(p,d)$ can be chosen as*

$$C(p,d) = \frac{p(d-1)}{d-p}.$$

But this constant is not optimal. For the optimal constants, we refer to the paper Best constant in Sobolev in equality *by Giorgio Talenti.*

Remark 2.51. *The value p^* can be obtained by a simple scaling argument (scaling arguments, dear to the physicists, sometimes give useful information with a minimum of effort). Indeed, assume that there exist constants C and q with $1 \le q \le \infty$ such that*

$$\|u\|_{L^q(\mathbb{R}^d)} \le C\|\nabla u\|_{L^p(\mathbb{R}^d)}, \quad \forall u \in C_c^\infty(\mathbb{R}^d),$$

then necessarily $q = p^*$. To see this, fix any function $u \in C_c^\infty(\mathbb{R}^d)$, and plug into the above equality with $u_\lambda(x) = u(\lambda x)$. We obtain

$$\|u\|_{L^q(\mathbb{R}^d)} \le C\lambda^{1+d\left(\frac{1}{q}-\frac{1}{p}\right)}\|\nabla u\|_{L^p(\mathbb{R}^d)}, \quad \forall \lambda > 0.$$

This implies $1 + d\left(\frac{1}{q} - \frac{1}{p}\right) = 0$, i.e., $q = p^*$.

To prove Theorem 2.49, we need the following lemma:

Lemma 2.52. *Let $d \ge 2$ and let $f_1, f_2, \cdots, f_d \in L^{d-1}(\mathbb{R}^{d-1})$. For $x = (x_1, x_2, \cdots, x_d) \in R_d$ we set*

$$\tilde{x}_i = (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_d) \in R^{d-1}, \quad 1 \le i \le d,$$

i.e., x_i is omitted from $x = (x_1, x_1, \cdots, x_d)$. Then the function

$$f(x) = f_1(\tilde{x}_1) f_2(\tilde{x}_2) \cdots f_d(\tilde{x}_d), \quad x \in \mathbb{R}_d,$$

belongs to $L^1(\mathbb{R}^d)$ and

$$\|f\|_{L^1(\mathbb{R}^d)} \le \prod_{i=1}^d \|f_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

证明. When $d = 2$, we have

$$f(x_1, x_2) = f_1(x_2) f_2(x_1)$$

with $f_1,\ f_2 \in L^1(\mathbb{R}^1)$. Direct calculation gives

$$\|f\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |f(x_1,x_2)|\,\mathrm{d}x_1\mathrm{d}x_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} |f_1(x_2)|\,|f_2(x_1)|\,\mathrm{d}x_1\mathrm{d}x_2 = \|f_1\|_{L^1(\mathbb{R}^1)}\|f_2\|_{L^1(\mathbb{R}^1)}. \tag{2.123}$$

We then complete the proof for the case $d = 2$.

By induction, we assume the result holds true for d and we want to prove the result for $d+1$. We first fix x_{d+1}. By Hölder's inequality, there holds

$$\int_{\mathbb{R}^d} |f(x)|\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_d \le \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \left[\int_{\mathbb{R}^d} |f_1\cdots f_d|^{d'}\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_d\right]^{\frac{1}{d'}}, \tag{2.124}$$

with $d' = d/(d-1)$. Applying the induction assumption to the functions $|f_1|^{d'}, \cdots, |f_d|^{d'}$, we deduce

$$\int_{\mathbb{R}^d} |f_1|^{d'} \cdots |f_d|^{d'} \, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_d \leq \prod_{i=1}^{d} \| |f_i|^{d'} \|_{L^{d-1}(\mathbb{R}^d)} = \prod_{i=1}^{d} \|f_i\|_{L^d(\mathbb{R}^{d-1})}^{d'}. \tag{2.125}$$

By (2.124) and (2.125), we obtain

$$\int_{\mathbb{R}^d} |f(x)| \, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_d \leq \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^{d} \|f_i\|_{L^d(\mathbb{R}^{d-1})}, \tag{2.126}$$

where the integral is taken without x_{d+1} variable (fixing x_{d+1}).

Now we vary x_{d+1}. Since $f_i(\tilde{x}_i) \in L^d(\mathbb{R}^d),\ 1 \leq i \leq d$, then the function $x_{d+1} \to \|f_i\|_{L^d(\mathbb{R}^{d-1})}$ belongs to $L^d(\mathbb{R}),\ 1 \leq i \leq d$. As a consequence, their product function

$$x_{d+1} \to \prod_{i=1}^{d} \|f_i\|_{L^d(\mathbb{R}^{d-1})}$$

belongs to $L^1(\mathbb{R})$ and (why?)

$$\int_{\mathbb{R}^d} |f(x)| \, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_d \mathrm{d}x_{d+1} \leq \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^{d} \|f_i\|_{L^d(\mathbb{R}^d)} = \prod_{i=1}^{d+1} \|f_i\|_{L^d(\mathbb{R}^d)}. \tag{2.127}$$

□

Now we are ready to prove Theorem 2.49.

Proof of Theorem 2.49. We begin with the case $p = 1$ and $u \in C_c^1(\mathbb{R}^d)$. We have

$$\begin{aligned} |u(x_1, \cdots, x_d)| &= \left| \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(t, x_2, \cdots, x_d) \, \mathrm{d}t \right| \\ &\leq \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_1}(x_1, x_2, \cdots, x_d) \right| \mathrm{d}x_1, \end{aligned} \tag{2.128}$$

and similarly,

$$|u(x_1, \cdots, x_d)| \leq \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_i}(x_1, x_2, \cdots, x_d) \right| \mathrm{d}x_i =: f_i(\tilde{x}_i), \quad \text{for each } 1 \leq i \leq d. \tag{2.129}$$

Thus,

$$|u(x)|^d \leq \prod_{i=1}^{d} f_i(\tilde{x}_i), \tag{2.130}$$

and

$$|u(x)|^{d/(d-1)} \leq \prod_{i=1}^{d} |f_i(\tilde{x}_i)|^{1/(d-1)}. \tag{2.131}$$

We deduce from Lemma 2.52 that

$$\int_{\mathbb{R}^d} |u(x)|^{d/(d-1)} \, \mathrm{d}x \leq \prod_{i=1}^{d} \|f_i\|_{L^1(\mathbb{R}^{d-1})}^{1/(d-1)} = \prod_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{1/(d-1)}. \tag{2.132}$$

As a consequence, we have

$$\|u\|_{L^{d/(d-1)}} \leq \prod_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{1/d}. \tag{2.133}$$

This completes the proof of the SGN inequality (2.122) with $p = 1$ and $u \in C_c^1(\mathbb{R}^d)$.

We now turn to the case $1 < p < d$, still with $u \in C_c^1(\mathbb{R}^d)$. Let $m \geq 1$. Applying (2.133) to $|u|^{m-1}u$ implies

$$\|u\|_{L^{md/(d-1)}}^m \leq m \prod_{i=1}^{d} \left\| |u|^{m-1} \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{1/d} \leq m \|u\|_{L^{(m-1)p'}}^{m-1} \prod_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}^{1/d}. \tag{2.134}$$

Then choose m such that

$$md/(d-1) = (m-1)p', \quad \text{i.e. } m = (d-1)p^*/d.$$

Clearly $m \geq 1$ when $1 < p < d$. Hence

$$\|u\|_{L^{p^*}} \leq m \prod_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}^{1/d}. \tag{2.135}$$

We thus complete the proof for $1 \leq p < d$ with $u \in C_c^1(\mathbb{R}^d)$.

Finally, we use density argument to finish the proof for general $u \in W^{1,p}(\mathbb{R}^d)$. By Theorem 2.30, there exists a sequence $\{u_n\}$ in C_c^∞ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^d)$. Moreover, one can also suppose, by extracting a subsequence if necessary, that $u_n \to u$ a.e. in $\mathbb{R}^d$. We have shown

$$\|u_n\|_{L^{p^*}} \leq C \|\nabla u_n\|_{L^p}.$$

It follows from Fatou's lemma that

$$u \in L^{p^*}, \quad \|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}.$$

□

Corollary 2.53. *Let $1 \leq p < d$. Then*

$$W^{1,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d), \quad \forall q \in [p, p^*]$$

with continuous injection.

证明. By interpolation. Exercise. □

Theorem 2.54. *Let $d \geq 2$. We have*

$$W^{1,d}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d), \quad \forall q \in [d, +\infty). \tag{2.136}$$

If $d = 1$, we have

$$W^{1,1}(\mathbb{R}) \subset L^\infty(\mathbb{R}). \tag{2.137}$$

证明. First consider the case $d = 1$. Assume $u \in C_c^1(\mathbb{R}^d)$. Taking $p = d$ in (2.134) implies

$$\begin{aligned}\|u\|_{L^{md/(d-1)}}^m &\le m \prod_{i=1}^d \left\| |u|^{m-1} \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{1/d} \\ &\le m\|u\|_{L^{(m-1)d/(d-1)}}^{m-1} \prod_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{L^d}^{1/d} \\ &\le \|u\|_{L^{(m-1)d/(d-1)}}^{m-1} \|\nabla u\|_{L^d}, \ \forall m \ge 1.\end{aligned} \tag{2.138}$$

By Young's inequality, we have

$$\|u\|_{L^{md/(d-1)}} \le C \left(\|u\|_{L^{(m-1)d/(d-1)}} + \|\nabla u\|_{L^d} \right), \ \forall m \ge 1. \tag{2.139}$$

Taking $m = d$ in (2.139) gives

$$\|u\|_{L^{d^2/(d-1)}} \le C(\|u\|_{L^d} + \|\nabla u\|_{L^d}) = C\|u\|_{W^{1,d}}. \tag{2.140}$$

Then by interpolation, we have

$$\|u\|_{L^q} \le C\|u\|_{W^{1,d}}, \quad \forall d \le q \le d^2(d-1). \tag{2.141}$$

Then we can reiterating this argument with $m = d + 1$, $m = d + 2$, etc., we arrive at

$$\|u\|_{L^q} \le C\|u\|_{W^{1,d}}, \quad \forall q \in [N, +\infty), \tag{2.142}$$

with a constant C depending on q and d. The proof will then be completed by density argument. (why?)

For $d = 1$. We consider $u \in C_1^\infty(\mathbb{R})$. Then

$$|u(x)| \le \left| \int_{-\infty}^x u'(t)\,\mathrm{d}t \right| \le \int_{-\infty}^x |u'(t)|\,\mathrm{d}t \le \|u'\|_{L^1}.$$

Now we can apply density argument to finish the proof. (How?)

□

Remark 2.55. *From the proof, we see that the constant C in Theorem 2.54 will go to infinity if q goes to infinity.*

Theorem 2.56 (Morrey). *Let $d \ge 1$ and $p > d$. We have*

$$W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d) \tag{2.143}$$

with continuous injection. Furthermore, for all $u \in W^{1,p}(\mathbb{R}^d)$, we have

$$|u(x) - u(y)| \le C|x - y|^\alpha \|\nabla u\|_{L^p}, \quad a.e.\ x, y \in \mathbb{R}^d, \tag{2.144}$$

where $\alpha = 1 - \frac{d}{p}$ and C is constant depending only on p and d.

Remark 2.57. *Inequality* (2.144) *implies the existence of a Hölder continuous function* $\tilde{u} \in C^\alpha(\mathbb{R}^d)$ *such that* $u = \tilde{u}$ *a.e. on* $\mathbb{R}^d$. *Indeed,let* $A \subset \mathbb{R}^d$ *be a set of measure zero such that* (2.144) *holds for* $x, y \in \mathbb{R}^d \setminus A$; *since* $\mathbb{R}^d \setminus A$ *is dense in* $\mathbb{R}^d$, *the function* $u|_{\mathbb{R}^d\setminus A}$ *admits a (unique) continuous extension to* $\mathbb{R}^d$. *In other words, every function* $u \in W^{1,p}(\mathbb{R}^d)$ *with* $p > d$ *admits a continuous representative. When it is useful, we replace* u *by its continuous representative, and we also denote by* u *this continuous representative.*

证明. Exercise. □

Remark 2.58. *From Theorem 2.56, we can deduce that* $W^{1,p}(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$ *for all* $d < p < \infty$. *Indeed, for each* $u \in W^{1,p}(\mathbb{R}^d)$ *with* $d < p < \infty$, *there exists a sequence* $\{u_n\}$ *in* $C_c^\infty(\mathbb{R}^d)$ *that converges to* u *in* $W^{1,p}(\mathbb{R}^d)$. *Applying Theorem 2.56, we have that* $u_n \to u$ *in* $L^\infty(\mathbb{R}^d)$, *which means that* $u \in C_0(\mathbb{R}^d)$.

Corollary 2.59. *Let* $m \geq 1$ *be an integer and* $p \in [1, +\infty]$. *We have*

$$\begin{aligned}
&W^{m,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d), \ \text{with } \frac{1}{q} = \frac{1}{p} - \frac{m}{d}, && \text{if } \frac{1}{p} - \frac{m}{d} > 0,\\
&W^{m,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d), \ \forall q \in [1, +\infty), && \text{if } \frac{1}{p} - \frac{m}{d} = 0,\\
&W^{m,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d), && \text{if } \frac{1}{p} - \frac{m}{d} < 0,
\end{aligned} \tag{2.145}$$

where all these injections are continuous.

Moreover, if $m - (d/p) > 0$ *is not an integer, we have for all* $u \in W^{m,p}(\mathbb{R}^d)$ *that*

$$\|\partial^\alpha u\|_{L^\infty(\mathbb{R}^d)} \leq C\|u\|_{W^{m,p}(\mathbb{R}^d)}, \quad \forall\ \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leq k, \tag{2.146}$$

and for all $x, y \in \mathbb{R}^d$,

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C\|u\|_{W^{m,p}(\mathbb{R}^d)}|x-y|^\theta, \quad \forall\ \alpha \in \mathbb{N}^d \text{ with } |\alpha| = k, \tag{2.147}$$

where

$$k = [m - (d/p)], \quad \theta = m - (d/p) - [m - (d/p)] \in (0, 1).$$

证明. Exercise. □

Remark 2.60. *The case* $p = 1$ *and* $m = d$ *is special. We have* $W^{d,1}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$. *(But it is not true, in general, that* $W^{m,p} \subset (\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ *for* $p > 1$ *and* $m = d/p$.*) Indeed, for* $u \in C_c^\infty(\mathbb{R}^d)$, *we have*

$$u(x_1, x_2, \cdots, x_d) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_d} \frac{\partial^d u}{\partial_{x_1}\partial_{x_2}\cdots\partial_{x_d}}(t_1, t_2, \cdots, t_d)\,\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_d,$$

and thus

$$\|u\|_{L^\infty} \leq \|\partial^d u\|_{L^1}, \quad \forall u \in C_c^\infty(\mathbb{R}^d).$$

The case of a general function $u \in W^{d,1}$ *follows by density.*

2.11.2 The case $\Omega \subset \mathbb{R}^d$

We first have the following Sobolev embedding theorem:

Theorem 2.61. *Let Ω be a domain of class Lipschitz and let $1 \le p \le \infty$. Then we have*

$$\begin{aligned} &W^{1,p}(\Omega) \subset L^{p^*}(\Omega), && \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}, && \text{if } p < d, \\ &W^{1,p}(\Omega) \subset L^q(\Omega), && \forall q \in [p, +\infty), && \text{if } p = d, \\ &W^{1,p}(\Omega) \subset L^\infty(\Omega), && && \text{if } p > d, \end{aligned} \tag{2.148}$$

and all these injections are continuous. Furthermore, for all $u \in W^{1,p}(\Omega)$, we have

$$|u(x) - u(y)| \le C|x-y|^\alpha \|\nabla u\|_{L^p}, \quad a.e.\ x, y \in \Omega, \tag{2.149}$$

where $\alpha := 1 - \frac{d}{p}$ and C is constant depending only on p and d and the Lipschitz character of Ω. This means, up to a choice of the continuous representative, $W^{1,p}(\Omega) \subset C^\alpha(\overline{\Omega})$.

证明. Using the extension operator in Theorem 2.43. Exercise. □

Remark 2.62. *An analogy as Corollary 2.59 for $W^{m,p}(\Omega)$ can be obtained simply by replacing $\mathbb{R}^d$ by Ω. We remark that one can prove such a result by using induction argument if Ω is Lipschitz. One can also prove such a result by employing an extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$, but this would require an extra hypothesis: Ω would have to be of class C^m to construct this extension operator.*

If moreover Ω is bounded, we then have the following compact embedding theorem:

Theorem 2.63 (Rellich-Kondrachov)**.** *Let Ω be a bounded domain of class Lipschitz and let $1 \le p \le \infty$. Then we have the following compact injections:*

$$\begin{aligned} &W^{1,p}(\Omega) \subset_c L^q(\Omega), && \forall q \in [1, p^*) \text{ with } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}, && \text{if } p < d, \\ &W^{1,p}(\Omega) \subset_c L^q(\Omega), && \forall q \in [p, +\infty), && \text{if } p = d, \\ &W^{1,p}(\Omega) \subset_c C^\infty(\overline{\Omega}), && && \text{if } p > d. \end{aligned} \tag{2.150}$$

To prove this compact embedding theorem, we shall need the following two results. The first one is the following classical Ascoli-Arzelá theorem:

Theorem 2.64 (Ascoli–Arzelá)**.** *Let K be a compact metric space and let H be a bounded subset of $C(K)$. Assume that H is uniformly equicontinuous, that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\operatorname{dist}(x_1, x_2) \le \delta \Longrightarrow |f(x_1) - f(x_2)| \le \varepsilon, \ \forall f \in H. \tag{2.151}$$

Then the closure of H in $C(K)$ is compact.

The proof of the Ascoli-Arzelá theorem can be found in many analysis books, see for example [7].

The second one is the following theorem which is an "L^p-versions" of the Ascoli–Arzelá theorem.

Theorem 2.65 (Kolmogorov–M. Riesz–Fréchet)**.** *Let A be a bounded set in $L^p(\mathbb{R}^d)$ with $1 \le p < \infty$ Assume that*

$$\lim_{h\to 0} \|\tau_h f - f\|_{L^p(\mathbb{R}^d)} = 0 \quad \textit{uniformly in } f \in A, \tag{2.152}$$

i.e., for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^d)} \le \varepsilon, \ \forall f \in A, \ \forall h \in \mathbb{R}^d, \ |h| \le \delta. \tag{2.153}$$

Then for any measurable set $\Omega \subset \mathbb{R}^d$ with finite measure, the closure of

$$A|_\Omega := \{f|_\Omega : f \in A\}$$

in $L^p(\Omega)$ is compact.

Proof of Theorem 2.65. **Step 1.** Given $\varepsilon > 0$. Let $\delta(\varepsilon) > 0$ be such that (2.153) holds. We claim that

$$\|S_{\delta'}[f] - f\|_{L^p(\mathbb{R}^d)} \le \varepsilon, \quad \forall f \in A, \ \forall 0 < \delta' \le \delta(\varepsilon), \tag{2.154}$$

where S_δ is the standard Fredrichs' mollifier with kernel $\phi_\delta = \frac{1}{\delta^d}\phi(\frac{\cdot}{\delta})$.

Indeed, we have

$$\begin{aligned} |S_{\delta'}[f](x) - f(x)| &\le \int |f(x-y) - f(x)| \phi_{\delta'}(y)\,\mathrm{d}y \\ &\le \left[\int |f(x-y) - f(x)|^p \phi_{1/n}(y)\right]^{1/p}. \end{aligned} \tag{2.155}$$

Thus for all $0 < \delta' \le \delta$, by using (2.153), we deduce

$$\begin{aligned} \int |S_{\delta'}[f](x) - f(x)|^p \,\mathrm{d}x &\le \int\int |f(x-y) - f(x)|^p \phi_{\delta'}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{B(0,\delta')} \phi_{\delta'}(y)\,\mathrm{d}y \int |f(x-y) - f(x)|^p \,\mathrm{d}x \\ &\le \varepsilon^p. \end{aligned} \tag{2.156}$$

Step 2. We then claim that, for each $\delta > 0$, there exists a constant $C(\delta)$ depending only on δ such that

$$\begin{aligned} &\|S_\delta[f]\|_{L^\infty} \le C_\delta \|f\|_{L^p}, \quad \forall f \in A, \\ &|S_\delta[f](x_1) - S_\delta[f](x_2)| \le C_\delta \|f\|_{L^p} |x_1 - x_2|, \quad \forall f \in A, \ \forall x_1, x_2 \in \mathbb{R}^d. \end{aligned} \tag{2.157}$$

Indeed, by Hölder's inequality, we have

$$\begin{aligned} |S_\delta[f](x)| &\le \|f\|_{L^p}\|\phi_\delta\|_{L^{p'}} \le \delta^{-d/p}\|\phi\|_{L^{p'}}\|f\|_{L^p}, \\ |\nabla S_\delta[f](x)| &\le \|f\|_{L^p}\|\nabla\phi_\delta\|_{L^{p'}} \le \delta^{-1-d/p}\|\nabla\phi\|_{L^{p'}}\|f\|_{L^p}. \end{aligned} \tag{2.158}$$

Step 3. Given $\varepsilon > 0$ and $\Omega \subset R^d$ of finite measure, there is a bounded measurable subset ω of such that

$$\|f\|_{L^p(\Omega\setminus\omega)} \le \varepsilon, \quad \forall f \in A. \tag{2.159}$$

Indeed, we write

$$\|f\|_{L^p(\Omega\setminus\omega)} \le \|f - S_\delta[f]\|_{L^p(\Omega\setminus\omega)} + \|S_\delta[f]\|_{L^p(\Omega\setminus\omega)} \le \|f - S_\delta[f]\|_{L^p(\mathbb{R}^d)} + \|S_\delta[f]\|_{L^\infty}|\Omega\setminus\omega|.$$

Then by the claims in Step 1 and Step 2, we have for $0 < \delta' \le \delta(\varepsilon/2)$ that

$$\|f\|_{L^p(\Omega\setminus\omega)} \le \varepsilon/2 + (\delta')^{-d/p}\|\phi\|_{L^{p'}}\|f\|_{L^p}|\Omega\setminus\omega|.$$

The claim can be proved by taking ω such that $|\Omega\setminus\omega|$ sufficient small.

Step 4. Since $L^p(\Omega)$ is a Banach space, by Proposition 1.18, it suffices to show that $A|_\Omega$ is totally bounded, i.e., given each $\varepsilon > 0$, there is a finite covering of $A|_\Omega$ by balls of radius ε. Given $\varepsilon > 0$, we fix a bounded measurable set ω such that (2.159) holds. Also we fix $\delta > 0$ such that (2.153) and (2.154) hold. The family

$$A' = \{S_\delta[f]|_{\bar\omega} : f \in A\}$$

satisfies all the assumptions of the Ascoli–Arzelá theorem by using Step 2. Therefore A' has compact closure in $C(\bar\omega)$; consequently A' also has compact closure in $L^p(\omega)$ due to the boundedness of ω. Hence we may cover A' by a finite number of balls of radius ε in $L^p(\omega)$, that is

$$A' \subset \bigcup_{j=1}^{J} B(g_j, \varepsilon), \text{ for some } g_j \in L^p(\omega).$$

We now consider the zero extensions $\tilde g_j : \Omega \to \mathbb{R}$ defined by

$$\tilde g_j = g_j \text{ on } \omega, \quad \tilde g_j = 0 \text{ on } \Omega\setminus\omega.$$

Clearly $g_j \in L^p(\Omega)$. We claim that the balls $B(g_j, 3\varepsilon),\ 1 \le j \le J$ covers $A|_\Omega$. Indeed, given $f \in A$, there is some g_j such that

$$\|S_\delta[f] - g_j\|_{L^p(\omega)} \le \varepsilon.$$

Since

$$\|f - \tilde g_j\|_p^p = \int_\omega |f - g_j|^p\,\mathrm{d}x + \int_{\Omega\setminus\omega} |f|^p\,\mathrm{d}x,$$

we have by using Step 1 and Step 3 that

$$\|f - \tilde{g}_j\|_p \leq \|f - S_\delta[f]\|_{L^p(\omega)} + \|S_\delta[f] - g_j\|_{L^p(\omega)} + \varepsilon \, \mathrm{d}x \leq 3\varepsilon.$$

We have shown that $A|_\Omega$ has a finite 3ε-net in $L^p(\Omega)$ for each $\varepsilon > 0$. This means $A|_\Omega$ is totally bounded and then relatively compact in $L^p(\Omega)$.

□

Remark 2.66. *When trying to establish that a family A in $L^p(\Omega)$ has compact closure in $L^p(\Omega)$, with Ω bounded, it is usually convenient to extend the functions to $\mathbb{R}^d$, then apply Theorem 2.65 and consider the restrictions to Ω .*

Remark 2.67. *Under the assumptions of Theorem 2.65, we cannot conclude in general that A itself has compact closure in $L^p(\mathbb{R}^d)$ (why?). An additional assumption is required; we describe it next:*

Corollary 2.68. *Let A be a bounded set in $L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$. Assume* (2.152) *and also that*

$$\begin{aligned}&\text{for all } \varepsilon > 0, \text{ there exists a bounded measurable set } \Omega \subset \mathbb{R}^d \text{ such that}\\ &\|f\|_{L^p(\mathbb{R}^d/\setminus\Omega)} \leq \varepsilon, \ \forall f \in A.\end{aligned} \tag{2.160}$$

Then A has compact closure in $L^p(\mathbb{R}^d)$.

证明. Given $\varepsilon > 0$, we fix $\Omega \subset \mathbb{R}^d$ bounded measurable such that (2.160) holds. By Theorem 2.65, we know that $A|_\Omega$ has compact closure in $L^p(\Omega)$. Hence $A|_\Omega$ is totally bounded, that is $F|_\Omega$ admits a cover of a finite number of balls of radius ε in $Lp(\Omega\ )$:

$$A|_\Omega \subset \bigcup_{j=1}^{J} B(g_j, \varepsilon), \quad \text{for some } g_j \in L^p(\Omega).$$

By similar argument as Step 4 in the proof of Theorem 2.65, we can deduce that the balls $B(\tilde{g}_j, 2\varepsilon)$, $1 \leq j \leq J$ covers A in $L^p(\mathbb{R}^d)$. This implies that A is totally bounded and then is relatively compact.

□

Remark 2.69. *The converse of Corollary 2.68 is also true (why?). Therefore we have a complete characterization of compact sets in $L^p(\mathbb{R}^d)$.*

Proof of Theorem 2.63. The case $p > d$ follows from Theorem 2.61 and the Ascoli–Arzelá theorem. The case $p = d$ reduces to the case $p < d$. Therefore, we are left with the case $p < d$.

Let B be the unit ball in $W^{1,p}(\Omega)$ with $1 \leq p < d$. Let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ be the extension operator of Theorem 2.43. Set $A = E(B)$, so that $B = A|_\Omega$. We will invoke Theorem

2.65 to show that $B = A|_\Omega$ is relatively compact in $L^p(\Omega)$. Clearly A is a bounded set in $W^p(\mathbb{R}^d)$ by the boundedness of the extension operator. Moreover, Sobolev inequality implies that A is a bounded set in $L^p \cap L^{p^*}(\mathbb{R}^d)$. By Proposition 2.33, we know that

$$\|\tau_h u - u\|_{L^p(\mathbb{R}^d)} \leq |h| \, \|\nabla u\|_{L^p(\mathbb{R}^d)}, \quad \forall u \in A. \tag{2.161}$$

This implies that the assumptions in Theorem 2.65 are fulfilled. Since Ω is bounded, thus, by Theorem 2.65, $B = A|_\Omega$ is relatively compact in $L^p(\Omega)$. By interpolation, we know that B is relatively compact in $L^q(\Omega)$ for any $q \in [1, p^*)$.

□

Remark 2.70. *Theorem 2.63 is "almost optimal" in the following sense:*

(i) If Ω is not bounded, the injection $W^{1,p}(\Omega) \subset L^p(\Omega)$ is, in general, not compact.

(ii) The injection $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is never compact even if Ω is bounded and smooth.

Remark 2.71. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class Lipschitz. Then the norm*

$$|||u||| := \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^q(\Omega)}$$

is equivalent to the $W^{1,p}(\Omega)$ norm so long as

$$\begin{aligned} &1 \leq q \leq p^* && \text{if } 1 \leq p < d, \\ &1 \leq q < \infty && \text{if } p = d, \\ &1 \leq q \leq \infty && \text{if } p > d. \end{aligned}$$

(why?)

2.12 The space $W_0^{1,p}(\Omega)$

Definition 2.72. *Let Ω be an open set in $\mathbb{R}^d$ and let $1 \leq p < \infty$; $W_0^{1,p}(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The space $W_0^{1,p}$, equipped with the $W^{1,p}$ norm, is a separable Banach space, and is reflexive if $1 < p < \infty$. The space $W_0^{1,2}$, equipped with the $W^{1,2}(\Omega)$ inner product, is a Hilbert space.*

Remark 2.73.

- *Since $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\mathbb{R}^d)$, then $W_0^{1,p}(\mathbb{R}^d) = W^{1,p}(\mathbb{R}^d)$, for all $1 \leq p < \infty$.*
- *If $\Omega \neq \mathbb{R}^d$, in general $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$. However, it is possible that if $\mathbb{R}^d \setminus \Omega$ is "small enough", one could have $W_0^{1,p}(\Omega) = W^{1,p}(\Omega)$. For example, if $\Omega = \mathbb{R}^d \setminus \{0\}$ with $d \geq 2$, then $W_0^{1,2}(\Omega) = W^{1,2}(\Omega)$.*

The functions in $W_0^{1,2}(\Omega)$ are "roughly"those of $W^{1,2}(\Omega)$ that "vanish on the boundary $\partial\Omega$."It is delicate to make this precise, since a function $u \in W^{1,2}(\Omega)$ is defined only a.e. in Ω and the measure of $\partial\Omega$ is zero, and u need not have a continuous representative. The following characterizations suggest that we "really"have Sobolev functions that are "zero on $\partial\Omega$."We begin with a simple fact:

Lemma 2.74. *Let $u \in W^{1,p}(\Omega)$ with Ω an open set in $\mathbb{R}^d$ and $1 \le p < \infty$. Assume that* $\operatorname{supp} u$ *is a compact subset of Ω. Then $u \in W_0^{1,p}(\Omega)$.*

证明. Since $\operatorname{supp} u$ is a compact subset of the open set Ω, there exists a bounded open set Ω' such that $\operatorname{supp} u \subset \Omega' \subset\subset \Omega$. Let $\chi \in C_c^\infty(\Omega')$ such that $\chi = 1$ on $\operatorname{supp} u$. Then $u = \chi u$. We know that there exists a sequence $\{u_n\} \subset C_c^\infty(\mathbb{R}^d)$ such that

$$u_n \to u \text{ in } L^p(\Omega), \quad \nabla u_n \to \nabla u \text{ in } L^p(\Omega').$$

It follows that $\chi u_n \to \chi u = u$ in $W^{1,p}(\Omega)$. Thus $u \in W_0^{1,p}(\Omega)$.

□

Theorem 2.75. *Suppose that Ω is an open set of class C^1. Let $1 \le p < \infty$ and*

$$u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

The the following two properties are equivalent:

(i) $u = 0$ *on* $\partial\Omega$.

(ii) $u \in W_0^{1,p}(\Omega)$.

证明. **(i) $\Longrightarrow$ (ii).** Suppose first that $\operatorname{supp} u$ is bounded. Fix a function $G \in C^1(\mathbb{R})$ such that

$$|G(t)| \le |t|, \ 0 \le G'(t) \le 4, \ \forall\, t \in \mathbb{R}, \quad G(t) = \begin{cases} 0, & \text{if } |t| \le 1, \\ t & \text{if } |t| \ge 2. \end{cases} \tag{2.162}$$

We consider the sequence

$$u_n := \frac{1}{n} G(nu), \quad n \in \mathbb{Z}_+$$

satisfying

$$|u_n| \le |u|, \quad u_n = \begin{cases} 0, & \text{if } |u| \le \dfrac{1}{n}, \\ u & \text{if } |u| \ge \dfrac{2}{n}. \end{cases}$$

By Proposition 2.37 on the composition of Sobolev functions, we know that u_n belongs to $W^{1,p}(\Omega)$. We claim that $u_n \to u$ in $W^{1,p}(\Omega)$. Indeed, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} \int_\Omega |u_n - u|^p \,\mathrm{d}x &= \int_{|u|\geq \frac{2}{n}} |u_n - u|^p \,\mathrm{d}x + \int_{|u|<\frac{2}{n}} |u_n - u|^p \,\mathrm{d}x \\ &= \int_{|u|<\frac{2}{n}} |u_n - u|^p \,\mathrm{d}x \\ &\leq \int_{|u|<\frac{2}{n}} |u_n|^p \,\mathrm{d}x + \int_{|u|<\frac{2}{n}} |u|^p \,\mathrm{d}x \\ &\leq 2\int_{|u|<\frac{2}{n}} |u|^p \,\mathrm{d}x \\ &\leq 2\int_\Omega 1_{|u|<\frac{2}{n}} |u|^p \,\mathrm{d}x \to 0, \quad \text{as } n\to\infty \end{aligned} \tag{2.163}$$

and

$$\begin{aligned} \int_\Omega |\nabla u_n - \nabla u|^p \,\mathrm{d}x &= \int_\Omega |G'(nu)\nabla u - \nabla u|^p \,\mathrm{d}x \\ &= \int_{|u|>\frac{2}{n}} |G'(nu)\nabla u - \nabla u|^p \,\mathrm{d}x + \int_{|u|\leq\frac{2}{n}} |G'(nu)\nabla u - \nabla u|^p \,\mathrm{d}x \\ &= \int_{|u|\leq\frac{2}{n}} |G'(nu)\nabla u - \nabla u|^p \,\mathrm{d}x \\ &\leq \int_\Omega 1_{|u|\leq\frac{2}{n}} |G'(nu) - 1|^p |\nabla u|^p \,\mathrm{d}x \\ &\leq 3^p \int_\Omega 1_{|u|\leq\frac{2}{n}} |\nabla u|^p \,\mathrm{d}x \to 3^p \int_\Omega 1_{u=0} |\nabla u|^p \,\mathrm{d}x, \quad \text{as } n\to\infty. \end{aligned} \tag{2.164}$$

Now we show that a.e. on $\{u=0\} := \{x\in\Omega : u(x)=0\}$, there holds $\nabla u = 0$. By the definition of generalized derivatives, there holds

$$\int_\Omega \partial_j u \phi \,\mathrm{d}x = -\int_\Omega u \partial_j \phi \,\mathrm{d}x = 0, \qquad \text{for all } \phi \in C_c^\infty(\{u=0\}).$$

This implies that $\partial_j u = 0$ a.e. on $\{u=0\}$. Hence,

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^p \,\mathrm{d}x \leq 3^p \int_{u=0} |\nabla u|^p \,\mathrm{d}x = 0. \tag{2.165}$$

On the other hand, since $u=0$ on $\partial\Omega$, we have

$$\operatorname{supp} u_n \subset \{x\in\Omega : |u(x)| \geq 1/n\} \subset \Omega.$$

Thus $\operatorname{supp} u_n$ is a compact set contained in open set Ω. By Lemma 2.74, $u_n \in W_0^{1,p}(\Omega)$ and it follows $u \in W_0^{1,p}(\Omega)$.

In the general case in which $\operatorname{supp} u$ is not bounded, consider the sequence $chi_n u$ with $\chi_n(\cdot) = \chi(\frac{\cdot}{n})$ where $\chi \in C_c^\infty(B(0,1))$ satisfying $0\leq\chi\leq 1$, $\chi=1$ on $B(0,1/2)$. From the above, $\chi_n u \in W_0^{1,p}(\Omega)$ and since $\chi_n u \to u$ in $W_0^{1,p}(\Omega)$, we conclude that $u\in W_0^{1,p}(\Omega)$.

(ii) $\Longrightarrow$ (i). Using local charts it is reduced to the following problem (why?): let $u \in W_0^{1,p}(Q_+) \cap C(\overline{Q}_+)$, prove that $u = 0$ on Q_0.

Let u_n be a sequence in $C_c^\infty(Q_+)$ that converges to u in $W^{1,p}(\Omega)$. We have for $(x', x_d) \in Q_+$,

$$|u_n(x', x_d)| \le \int_0^{x_d} \left| \frac{\partial u_n}{\partial x_d}(x', t) \right| \mathrm{d}t.$$

Thus, for $0 < \varepsilon < 1$,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{|x'|<1}\int_0^\varepsilon |u_n(x', x_d)| \,\mathrm{d}x' \,\mathrm{d}x_d &\le \frac{1}{\varepsilon}\int_{|x'|<1}\int_0^\varepsilon \int_0^{x_d} \left| \frac{\partial u_n}{\partial x_d}(x', t) \right| \mathrm{d}t \,\mathrm{d}x' \,\mathrm{d}x_d \\ &\le \int_{|x'|<1}\int_0^\varepsilon \left| \frac{\partial u_n}{\partial x_d}(x', x_d) \right| \mathrm{d}x' \,\mathrm{d}x_d.\end{aligned}$$

Passing $n \to \infty$ implies

$$\frac{1}{\varepsilon}\int_{|x'|<1}\int_0^\varepsilon |u(x', x_d)| \,\mathrm{d}x' \,\mathrm{d}x_d \le \int_{|x'|<1}\int_0^\varepsilon \left| \frac{\partial u}{\partial x_d}(x', x_d) \right| \mathrm{d}x' \,\mathrm{d}x_d.$$

Since $u \in C(\overline{Q}_+)$ and $\partial_{x_d} u \in L^1(Q_+)$, passing $\varepsilon \to 0$ gives

$$\int_{|x'|<1} |u(x', 0)| \,\mathrm{d}x' = 0.$$

Thus $u = 0$ on Q_0.

□

Remark 2.76. *In the proof of (i) $\Longrightarrow$ (ii), we have not used the smoothness of Ω . However, the converse (ii) $\Longrightarrow$ (i) requires a smoothness hypothesis on Ω. (consider for example $\Omega = \mathbb{R}^d \setminus \{0\}$ with $d \ge 2$, $p \le d$).*

Here is another characterization of $W_0^{1,p}$.

Proposition 2.77. *Suppose Ω is of class C^1. Let $u \in L^p(\Omega)$ with $1 < p < \infty$. The following are equivalent:*

(i) $u \in W_0^{1,p}(\Omega)$.

(ii) There exists a constant C such that

$$\left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x \right| \le C\|\varphi\|_{L^{p'}(\Omega)} \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d),\ \forall i = 1, 2, \cdots, d.$$

(i) The function

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^d \setminus \Omega, \end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^d)$, and in this case

$$\frac{\partial \tilde{u}}{\partial x_i} = \widetilde{\frac{\partial u}{\partial x_i}}, \quad \forall i = 1, 2, \cdots, d.$$

证明. **(i) $\Longrightarrow$ (ii).** Let $\{u_n\}$ be a sequence in $C_c^\infty(\Omega)$ such that $u_n \to u$ in $W^{1,p}$. For each $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\left|\int_\Omega u_n \frac{\partial \varphi}{\partial x_i}\, dx\right| = \left|\int_\Omega \frac{\partial u_n}{\partial x_i} \varphi\, dx\right| \leq \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega)}. \tag{2.166}$$

Passing $n \to \infty$ in (2.166) implies (ii) with $C = \|\nabla u\|_{L^p(\Omega)}$.

□

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3 分析 II 开卷考试题

注意：1，可以查阅资料，但请勿抄袭。2，论证细节需要提供。

Exercise 1. [15 points] Let $a < b$ be two real numbers. Let $C[a, b]$ be the Banach space of all continuous functions on $[a, b]$ with norm $\|u\| := \max_{a\leq x\leq b} |u(x)|$. Let $f \in C[a, b]$ be a given function. Show that the nonlinear integral equation

$$u(x) = \int_a^b \sin u(x)\,\mathrm{d}x + f(x)$$

has a solution $u \in C[a, b]$

Exercise 2. [15 points]

Let $a < b$ be two real numbers. Let $K : [a, b] \times [a, b] \to \mathbb{R}$ be continuous with $0 \leq K(x, y) \leq k$ for all $x, y \in [a, b]$. Suppose $2(b - 1)k \leq 1/2$. Let $u_0 \equiv 0,\ v_0 \equiv 2$. Prove that both of the iteration sequences

$$u_{n+1}(x) := \int_a^b K(x, y)u_n(y)\,\mathrm{d}y + 1,$$
$$v_{n+1}(x) := \int_a^b K(x, y)v_n(y)\,\mathrm{d}y + 1$$

converge uniformly on $[a, b]$ to the unique solution $u \in C[a, b]$ of the integral equation

$$u(x) = \int_a^b K(x, y)u(y)\,\mathrm{d}y + 1.$$

Exercise 3.[15 points] Let B be a Banach algebra. A family $\{e_i\}$ in B is called an approximate identity for B if $\sup_i \|e_i\| < \infty$ and for each $x \in B$, there holds $e_i x \to x$ and $xe_i \to x$. Show that B has an approximate identity if and only if there is a bounded subset E of B such that for every $\varepsilon > 0$ and for every $x \in B$ there is an $e \in E$ such that $\|xe - x\| + \|ex - x\| < \varepsilon$.

Exercise 4. [15 points] Let Ω be a bounded measurable set in $\mathbb{R}^d$. Let $1 < p < \infty$. Let $K : \Omega \times \Omega \to \mathbb{C}$ be a measurable function such that

$$\sup_{x\in\Omega} \int_\Omega |K(x, y)|^{p'}\,\mathrm{d}y < \infty.$$

Prove that

$$Tu(x) := \int_\Omega K(x, y)u(y)\,\mathrm{d}y$$

defines a compact operator on $L^p(\Omega)$

Exercise 5.[20 points]

Let $\phi \in C_c^\infty(\mathbb{R})$ be a nonzero function. Define the sequence $\{u_n\}_{n=1}^\infty$ by $u_n(x) = \phi(x + n)$ for all $x \in \mathbb{R}$. Let $1 \leq p \leq \infty$.

- Show that $\{u_n\}_{n=1}^{\infty}$ is bounded in $W^{1,p}(\mathbb{R})$.
- Prove that there exists no subsequence $\{u_{n_k}\}$ converging strongly in $L^q(R)$, for any $1 \le q \le \infty$.

  (*This means the compact Sobolev embedding could be wrong for unbounded domains.*)
- Show that $u_n \to 0$ weakly in $W^{1,p}(\mathbb{R})$, for each $p \in (1,\infty)$.

Exercise 6. [15 points]

Let Ω be a smooth bounded domain in $\mathbb{R}^d$ and let $1 \le p \le \infty$. Prove that for each $\varepsilon > 0$, there exists $C = C(\varepsilon, \Omega)$ such that

$$\sum_{|\alpha|<m} \|\partial^\alpha u\|_{L^p(\Omega)} \le \varepsilon \sum_{|\alpha|=m} \|\partial^\alpha u\|_{L^p(\Omega)} + C(\varepsilon,\Omega)\|u\|_{L^p(\Omega)}, \quad \forall\, u \in W^{m,p}(\Omega).$$

Exercise 7. [20 points] (Stability of weak solutions)

Let Ω be a bounded C^1 domain in $\mathbb{R}^3$. We consider the stationary incompressible Navier-Stokes equations in Ω

$$\mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0, \tag{3.1}$$

subjected to the boundary condition

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Here $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is the vector valued unknown, the convective term $\mathbf{u}\cdot\nabla\mathbf{u}$ is defined as

$$\mathbf{u}\cdot\nabla\mathbf{u} := \sum_{i=1}^{3} \mathbf{u}_i \partial_i \mathbf{u}.$$

The source term $\mathbf{f}$ is supposed to be in $L^2(\Omega;\mathbb{R}^3)$.

We say $\mathbf{u} \in W_0^{1,2}(\Omega;\mathbb{R}^3)$ satisfying $\operatorname{div}\mathbf{u} = 0$ is a turbulent weak solution of (3.1)-(3.2) provided

$$\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\varphi\,\mathrm{d}x + \int_\Omega \nabla\mathbf{u} : \nabla\varphi\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\varphi\,\mathrm{d}x, \quad \forall\,\varphi \in C_c^\infty(\Omega;\mathbb{R}^3),\ \operatorname{div}\varphi = 0, \tag{3.3}$$

and

$$\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x \le \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x. \tag{3.4}$$

Let $\{\mathbf{u}_n\} \subset W_0^{1,2}(\Omega;\mathbb{R}^3)$ satisfying $\operatorname{div}\mathbf{u}_n = 0$ be a sequence of turbulent weak solutions to (3.1)-(3.2). Prove the following two statements.

- The solution sequence is bounded:

$$\sup_n \|\mathbf{u}_n\|_{W_0^{1,2}(\Omega)} < \infty.$$

- Thus, up to a subsequence, the sequence $\{\mathbf{u}_n\}$ admits a weak limit $\mathbf{u}$ in $W_0^{1,2}(\Omega)$. Moreover, the weak limit $\mathbf{u}$ is also a turbulent weak solution to (3.1)-(3.2).