HOMOGENIZATION OF STOKES EQUATIONS IN PERFORATED DOMAINS: A UNIFIED APPROACH

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Abstract. We consider the homogenization of the Stokes equations in a domain perforated with a large number of small holes which are periodically distributed. In [1, 2], Allaire gave a systematic study on this problem. In this paper, we introduce a unified proof for different sizes of holes for the homogenization of the Stokes equations by employing a generalized cell problem inspired by Tartar [17].

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1. Introduction

Homogenization problems in the framework of fluid mechanics have gain a lot interest both in mathematical analysis and numerical analysis. Such problems represent the study of fluid flows in domains perforated with a large number of tiny holes (obstacles). The goal is to describe the asymptotic behavior of fluid flows (governed by Stokes equations, Navier-Stokes equations, etc.) as the number of holes goes to infinity and the size of holes goes to zero simultaneously. The limit equations that describe the limit behavior of fluid flows are called *homogenized equations* which are defined in homogeneous domains without holes.

The perforated domain under consideration is described as follows. Let $\Omega \subset \mathbb{R}^d,\ d \geq 2$ be a bounded domain of class C^1. The holes in Ω are denoted by $T_{\varepsilon,k}$ which are assumed to satisfy

$$B(\varepsilon x_k, \delta_1 a_\varepsilon) \subset\subset T_{\varepsilon,k} = \varepsilon x_k + a_\varepsilon T \subset\subset B(\varepsilon x_k, \delta_2 a_\varepsilon) \subset\subset B(\varepsilon x_k, \delta_3 \varepsilon) \subset\subset \varepsilon Q_k, \tag{1.1}$$

where the cube $Q_k := (-\frac{1}{2}, \frac{1}{2})^d + k$ and $x_k = x_0 + k$ with $x_0 \in (-\frac{1}{2}, \frac{1}{2})^d$, for each $k \in \mathbb{Z}^d$; T is a model hole which is assumed to be closed, bounded, and simply connected, with C^1 boundary; δ_i, $i = 1, 2, 3$ are fixed positive numbers. The perforation parameters ε and a_ε are used to measure the mutual distance of holes and the size of holes, respectively, and $\varepsilon x_k = \varepsilon x_0 + \varepsilon k$ are the locations of the holes. Without loss of generality, we assume that $x_0 = 0$ and $0 < a_\varepsilon < \varepsilon \leq 1$. Otherwise it is sufficient to consider the domain with a shift of εx_0 and consider different values of $\delta_i,\ i = 1, 2, 3$.

The perforated domain (fluid domain) Ω_ε is then defined as:

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{k \in K_\varepsilon} T_{\varepsilon,k}, \quad \text{where } K_\varepsilon := \{k \in \mathbb{Z}^d \ : \ \varepsilon \overline{Q}_k \subset \Omega\}. \tag{1.2}$$

Throughout the paper, we consider the following Dirichlet problem of Stokes equations in Ω_ε:

$$\begin{cases} -\Delta \mathbf{u}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}, & \text{in } \Omega_\varepsilon, \\ \operatorname{div} \mathbf{u}_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \mathbf{u}_\varepsilon = 0, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.3}$$

Here we take the external force $\mathbf{f} \in L^2(\Omega)$.

For each fixed $\varepsilon > 0$, the domain Ω_ε is bounded and is of C^1; the existence and uniqueness of the weak solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in W_0^{1,2}(\Omega_\varepsilon; \mathbb{R}^d) \times L_0^2(\Omega_\varepsilon)$ to (1.3) is known, see for instance [9]. Here $W_0^{1,2}$ denotes the Sobolev space with zero trace, and L_0^2 is the collection of all L^2 functions with zero average.

The behavior of the solution family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$ was studied by Tartar [17] for the case where the size of the holes is proportional to the mutual distance of the holes, i.e.

$$a_\varepsilon = a_* \varepsilon \quad \text{for some } a_* > 0. \tag{1.4}$$

Then Allaire [1, 2] considered general cases and showed that the homogenized equations are determined by the ratio σ_ε between the size and the mutual distance of the holes:

$$\sigma_\varepsilon := \left(\frac{\varepsilon^d}{a_\varepsilon^{d-2}}\right)^{\frac{1}{2}}, \quad d \geq 3; \qquad \sigma_\varepsilon := \varepsilon \left|\log \frac{a_\varepsilon}{\varepsilon}\right|^{\frac{1}{2}}, \quad d = 2. \tag{1.5}$$

Specifically, if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$ corresponding to the case of *large holes*, the homogenized system is the Darcy's law; if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \infty$ corresponding to the case of *small holes*, the motion of the fluid does not change much in the homogenization process and in the limit there arise the same Stokes equations in homogeneous domains; if $\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma_* \in (0, +\infty)$ corresponding to the case of *critical size of holes*, the homogenized equations are governed by the Brinkman's law — a combination of the Darcy's law and the original Stokes equations.

To obtain the limit system, a natural way is to pass $\varepsilon \to 0$ in the weak formulation of (1.3). In this process, one needs to pay special attention to the choice of test functions. Since the homogenized system is defined in Ω, so one needs to choose test functions in $C_c^\infty(\Omega)$. However, $C_c^\infty(\Omega)$ functions are *not* proper test functions for the original system (1.3) defined in Ω_ε where the test functions should be chosen in $C_c^\infty(\Omega_\varepsilon)$. Hence, a proper surgery on the test functions need to be done and this surgery plays a crucial role in the study of the homogenization problems in fluid mechanics. Tartar [17] and Allaire [1, 2] used different ideas to this issue. This will be explained with more details in the next section.

Later, the homogenization study is extended to more complicated models describing fluid flows: Mikelić [16] for the incompressible Navier-Stokes equations, Masmoudi [15] for the compressible Navier-Stokes equations, Feireisl, Novotný and Takahashi [8] for the complete Navier-Stokes-Fourier equations. In all these studies, only the case where the size of holes is proportional to the mutual distance of the holes (like (1.4)) is considered and the Darcy's law is recovered in the limit.

Recently, cases with different sizes of holes are also considered: Feireisl, Namlyeyeva and Nečasová [7] studied the case with critical size of holes for the incompressible Navier-Stokes equations and they derived Brinkman's law; in [6, 5, 14] the authors considered the case of small holes for the compressible Navier-Stokes equations and it is shown that the homogenized equations remain the same as the original ones. These results coincide with Allaire's study for the Stokes equations in [1, 2].

1.1. **A brief review of Tartar's idea and Allaire's idea.** As pointed out in the introduction, to obtain the limit system by passing $\varepsilon \to 0$ in the weak formulation of the Stokes equations, a proper surgery on $C_c^\infty(\Omega)$ test functions needs to be done such that the test functions vanish on the holes and then become *good* test functions for the original Stokes equations in Ω_ε. To this issue, Tartar [17] and Allaire [1, 2] used different methods.

In [17], Tartar considered the case where the size of the holes is proportional to the mutual distance of the holes as in (1.4). Without loss of generality, one may take $a_* = 1$. Then, near each single hole in εQ_k in the perforated domain Ω_ε, after a scaling of size ε^{-1}, there arises typically the following problem, named *cell problem*:

$$
\begin{cases}
-\Delta w^i + \nabla q^i = e^i, & \text{in } Q_0 \setminus T := \left(-\frac{1}{2}, \frac{1}{2}\right)^d \setminus T,\\
\operatorname{div} w^i = 0, & \text{in } Q_0 \setminus T,\\
w^i = 0, & \text{on } T,\\
(w^i, q^i) \text{ is } Q_0\text{-periodic.}
\end{cases}
\tag{1.6}
$$

Here $\{e^i\}_{i=1,\cdots,d}$ is the standard Euclidean coordinate of $\mathbb{R}^d$. The cell problem (1.6) admits a unique weak solution $(w^i, q^i) \in W^{1,2}(Q_0 \setminus T) \times L^2_0(Q_0 \setminus T)$. Then by scaling back to the original scale of the perforated domain, the *scaled cell solution* $(w^i_\varepsilon, q^i_\varepsilon)$ defined as

$$
w^i_\varepsilon(\cdot) := w^i\big(\frac{\cdot}{\varepsilon}\big), \quad q^i_\varepsilon(\cdot) := q^i\big(\frac{\cdot}{\varepsilon}\big)
\tag{1.7}
$$

solves

$$
\begin{cases}
-\varepsilon^2 \Delta w^i_\varepsilon + \varepsilon \nabla q^i_\varepsilon = e^i, & \text{in } \varepsilon Q_0 \setminus \varepsilon T,\\
\operatorname{div} w^i_\varepsilon = 0, & \text{in } \varepsilon Q_0 \setminus \varepsilon T,\\
w^i_\varepsilon = 0, & \text{on } \varepsilon T,\\
(w^i_\varepsilon, q^i_\varepsilon) \text{ is } \varepsilon Q_0\text{-periodic.}
\end{cases}
\tag{1.8}
$$

Clearly w^i_ε vanishes on the holes in Ω_ε. Given each scalar function $\phi \in C^\infty_c(\Omega)$, $w^i_\varepsilon \phi$ is a good test function to (1.3). Then choosing $w^i_\varepsilon \phi$ as a test function in the weak formulation of (1.3), and passing $\varepsilon \to 0$, together with the property of w^i_ε and the optimal uniform estimates for $\mathbf{u}_\varepsilon$ and p_ε, gives the limit model—Darcy's law with permeability tensor A defined as

$$
A_{i,j} = \int_{Q_0} \nabla w^i : \nabla w^j \,\mathrm{d}x = \int_{Q_0} (w^i)_j \,\mathrm{d}x,
\tag{1.9}
$$

where w^i is the solution to (1.6) and is extended by zero on Q_0. In this paper, we will generalize Tartar's idea so that we can cover also obstacles of smaller sizes which were considered by Allaire. Since the case that a_ε is proportional to ε is already considered by Tartar, so in this paper we mainly focus on the cases where a_ε is much smaller than ε such that $\eta := \frac{a_\varepsilon}{\varepsilon} \to 0$ as $\varepsilon \to 0$.

In [1, 2], Allaire used an abstract framework of hypotheses on the holes and verified the hypotheses in the case of a periodic distribution of the holes. This idea goes back to [4] for Laplacian operator instead of Stokes. For general cases where a_ε is much smaller than ε such that $\eta := \frac{a_\varepsilon}{\varepsilon} \to 0$ as $\varepsilon \to 0$, near each singular hole, after a scaling of size a_ε^{-1} such that the size of the holes becomes $O(1)$, one obtains a domain of the type $\eta^{-1} Q_0 \setminus T$ which converges to $\mathbb{R}^d \setminus T$ as $\varepsilon \to 0$. Allaire employed the following problem of Stokes equations in exterior domain $\mathbb{R}^d \setminus T$, namely *local problem*:

$$
\begin{cases}
-\Delta v^i + \nabla p^i = 0, & \text{in } \mathbb{R}^d \setminus T,\\
\operatorname{div} v^i = 0, & \text{in } \mathbb{R}^d \setminus T,\\
v^i = 0, & \text{on } T,\\
v^i = e^i, & \text{at infinity,}
\end{cases}
\tag{1.10}
$$

to construct a family of functions $(v^i_\varepsilon, p^i_\varepsilon) \in W^{1,2}(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$ which vanish on the holes in order to modify the $C^\infty_c(\Omega)$ test functions and derive the limit equations as $\varepsilon \to 0$. Allaire showed that the Dirichlet problem (1.10) is well-posed in $D^{1,2}(\mathbb{R}^d \setminus T; \mathbb{R}^d) \times L^2(\mathbb{R}^d \setminus T; \mathbb{R}^d)$ and showed decay estimates of the solutions at infinity. Here $D^{1,2}$ denotes the homogeneous Sobolev spaces. The corresponding $(v_\varepsilon^i, p_\varepsilon^i)$ is defined as follows: in cubes εQ_k that intersect with the boundary of Ω,

$$v_\varepsilon^i = e^i,\ p_\varepsilon^i = 0,\ \text{in } \varepsilon Q_k \cap \Omega, \quad \text{if } \varepsilon Q_k \cap \partial\Omega \neq \emptyset; \tag{1.11}$$

and in cubes εQ_k whose closures are contained in Ω,

$$\begin{aligned} & v_\varepsilon^i = e^i, \quad p_\varepsilon^i = 0, && \text{in } \varepsilon Q_k \setminus B(\varepsilon x_k, \delta_3 \varepsilon),\\ & -\Delta v_\varepsilon^i + \nabla p_\varepsilon^i = 0, \quad \operatorname{div} v_\varepsilon^i = 0, && \text{in } B(\varepsilon x_k, \delta_3\varepsilon) \setminus B(\varepsilon x_k, \delta_2 \varepsilon),\\ & v_\varepsilon^i(x) = v^i\big(\frac{x}{a_\varepsilon}\big), \quad p_\varepsilon^i(x) = \frac{1}{a_\varepsilon} p^i\big(\frac{x}{a_\varepsilon}\big), && \text{in } B(\varepsilon x_k, \delta_2\varepsilon) \setminus T_{\varepsilon,k},\\ & v_\varepsilon^i = 0, \quad p_\varepsilon^i = 0, && \text{in } T_{\varepsilon,k}. \end{aligned} \tag{1.12}$$

Such $(v_\varepsilon^i, p_\varepsilon^i) \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^d) \times L^2(\Omega_\varepsilon)$ fulfills the hypotheses in Allaire's abstract framework. In particular, $(v_\varepsilon^i, p_\varepsilon^i)$ vanishes on the holes. Thus, for each $\phi \in C_c^\infty(\Omega)$, the modified function $v_\varepsilon^i \phi$ becomes a good test function in the weak formulation of the original Stokes equations in Ω_ε. By careful analysis, passing $\varepsilon \to 0$ gives the desired homogenized systems.

1.2. **Main result.** Tartar employed the cell problem (1.6)–(1.8) to modify the test function, while he only covered the case $a_\varepsilon = a_* \varepsilon$ for some a_* independent of ε. Allaire employed the local problem (1.10)–(1.12) and covered general sizes of holes. We found that Tartar'a idea evolving the cell problem (1.6) could be more applicable when we impose soft restrictions on the distribution of the holes, which is the main topic in the forthcoming study [11]. Unfortunately, Tartar's method works only for a specific case. To cover the cases with general sizes of holes, a proper generalization needs to be done. Indeed, by introducing a generalized cell problem and establishing suitable estimates, we make it work for different sizes of holes. This gives a unified proof of Tartar's type for Allaire's homogenization results in [1, 2]. Along with others, such an idea of unified approach is also used recently in [10] for the study of Laplace equations in perforated domains.

Before stating the theorem (see also in [1, 2]), we recall the extension of $(\mathbf{u}_\varepsilon, p_\varepsilon) \in W_0^{1,2}(\Omega_\varepsilon; \mathbb{R}^d) \times L_0^2(\Omega_\varepsilon)$ which is the unique solution to (1.3) in Ω_ε. For the velocity, since it has zero trace on the boundary, it is nature to use its zero extension defined as:

$$\tilde{\mathbf{u}}_\varepsilon = \mathbf{u}_\varepsilon \text{ in } \Omega_\varepsilon; \quad \tilde{\mathbf{u}}_\varepsilon = 0 \text{ on } \Omega \setminus \Omega_\varepsilon, \tag{1.13}$$

which satisfies

$$\tilde{\mathbf{u}}_\varepsilon \in W_0^{1,2}(\Omega; \mathbb{R}^d), \quad \|\tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)} = \|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \quad \|\nabla \tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)} = \|\nabla \mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)}. \tag{1.14}$$

The extension of the pressure is more delicate and is defined by employing the so-called *restriction operator* due to Allaire [1, 2] for general sizes of holes, and due to Tartar [17] for the case where the size of the holes is proportional to their mutual distance. For Ω_ε defined through (1.1) and (1.2), there exists a linear operator, named *restriction operator*, $R_\varepsilon : W_0^{1,2}(\Omega; \mathbb{R}^d) \to W_0^{1,2}(\Omega_\varepsilon; \mathbb{R}^d)$ such that:

$$\begin{aligned} & \mathbf{u} \in W_0^{1,2}(\Omega_\varepsilon; \mathbb{R}^d) \Longrightarrow R_\varepsilon(\tilde{\mathbf{u}}) = \mathbf{u} \text{ in } \Omega_\varepsilon, \text{ where } \tilde{\mathbf{u}} := \begin{cases} \mathbf{u} & \text{in } \Omega_\varepsilon,\\ 0 & \text{on } \Omega \setminus \Omega_\varepsilon, \end{cases}\\ & \mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^d),\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega \Longrightarrow \operatorname{div} R_\varepsilon(\mathbf{u}) = 0 \text{ in } \Omega_\varepsilon,\\ & \mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^d) \Longrightarrow \|\nabla R_\varepsilon(\mathbf{u})\|_{L^2(\Omega_\varepsilon)} \leq C\left(\|\nabla \mathbf{u}\|_{L^2(\Omega)} + (1 + \sigma_\varepsilon^{-1})\|\mathbf{u}\|_{L^2(\Omega)}\right). \end{aligned} \tag{1.15}$$

Then the extension $\tilde{p}_\varepsilon \in L_0^2(\Omega)$ is defined through the following dual formulation:

$$\langle \nabla \tilde{p}_\varepsilon, \varphi \rangle_\Omega = \langle \nabla p_\varepsilon, R_\varepsilon(\varphi) \rangle_{\Omega_\varepsilon}, \quad \forall\, \varphi \in W_0^{1,2}(\Omega; \mathbb{R}^d). \tag{1.16}$$

The above formulation (1.16) is well defined due to the three properties in (1.15); moreover $\nabla \tilde{p}_\varepsilon \in W^{-1,2}(\Omega; \mathbb{R}^d)$ and up to a constant, $\tilde{p}_\varepsilon \in L_0^2(\Omega)$; in particular, $\tilde{p}_\varepsilon = p_\varepsilon$ in Ω_ε. Indeed, by property

2 of (1.15), one has $\operatorname{div} R_\varepsilon(\varphi) = 0$ for each $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^d)$ with $\operatorname{div}\varphi = 0$, then one deduces naturally from (1.16) that $\langle \nabla \tilde{p}_\varepsilon, \varphi\rangle = \langle \nabla p_\varepsilon, R_\varepsilon(\varphi)\rangle = 0$. For each $f \in L^2(\Omega_\varepsilon)$, we employ the Bogovskii operator $\mathcal{B}_{\Omega_\varepsilon} : L^2(\Omega_\varepsilon) \to W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^d)$ and introduce

$$\varphi := \mathcal{B}_{\Omega_\varepsilon}(f - \langle f\rangle) \in W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^d) \text{ with } \langle f\rangle := \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} f \,\mathrm{d}x$$

such that $\operatorname{div}\varphi = f - \langle f\rangle$. Let $\tilde{\varphi}, \tilde{f}$ be the zero extension of φ, f. Since p_ε and $\tilde{p}_\varepsilon$ are both of mean zero, together with property 1 of (1.15), one has

$$\begin{aligned}\int_{\Omega_\varepsilon} \tilde{p}_\varepsilon f \,\mathrm{d}x &= \int_\Omega \tilde{p}_\varepsilon \tilde{f}\,\mathrm{d}x = \int_\Omega \tilde{p}_\varepsilon(\tilde{f} - \langle f\rangle)\,\mathrm{d}x = \int_\Omega \tilde{p}_\varepsilon \operatorname{div}\tilde{\varphi}\,\mathrm{d}x = \langle \nabla\tilde{p}_\varepsilon, \tilde{\varphi}\rangle_\Omega \\ &= \langle \nabla p_\varepsilon, R_\varepsilon(\tilde{\varphi})\rangle_{\Omega_\varepsilon} = \langle \nabla p_\varepsilon, \varphi\rangle_{\Omega_\varepsilon} = \langle p_\varepsilon, \operatorname{div}\varphi\rangle_{\Omega_\varepsilon} = \int_{\Omega_\varepsilon} p_\varepsilon(f - \langle f\rangle)\,\mathrm{d}x = \int_{\Omega_\varepsilon} p_\varepsilon f\,\mathrm{d}x.\end{aligned}$$

This holds for all $f \in L^2(\Omega_\varepsilon)$ and therefore $\tilde{p}_\varepsilon = p_\varepsilon$ in Ω_ε.

We now state the homogenization results of Allaire [1, 2]. The limits are taken up to possible extractions of subsequences.

Theorem 1.1. *For each $\varepsilon > 0$ small, let $(\mathbf{u}_\varepsilon, p_\varepsilon) \in W_0^{1,2}(\Omega_\varepsilon;\mathbb{R}^d) \times L_0^2(\Omega_\varepsilon)$ be the unique solution to the Dirichlet problem of Stokes equations* (1.3) *in Ω_ε. Let $(\tilde{\mathbf{u}}_\varepsilon, \tilde{p}_\varepsilon)$ be their extension in Ω defined through* (1.13)–(1.16)*. Then we have the following description of the limit system related to different sizes of holes:*

(i) *If $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \infty$ corresponding to the case of small holes, then*

$$(\tilde{\mathbf{u}}_\varepsilon, \tilde{p}_\varepsilon) \to (\mathbf{u}, p) \text{ strongly in } W_0^{1,2}(\Omega;\mathbb{R}^d) \times L_0^2(\Omega),$$

where $(\mathbf{u}, p)$ is the unique (weak) solution to the Stokes equations:

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

(ii) *If $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$ corresponding to the case of large holes, then*

$$\frac{\tilde{\mathbf{u}}_\varepsilon}{\sigma_\varepsilon^2} \to \mathbf{u} \text{ weakly in } L^2(\Omega;\mathbb{R}^d), \quad \tilde{p}_\varepsilon \to p \text{ strongly in } L_0^2(\Omega),$$

where $(\mathbf{u}, p)$ satisfies the Darcy's law:

$$\begin{cases} \mathbf{u} = A(\mathbf{f} - \nabla p), & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.18}$$

where $\mathbf{n}$ is the unit normal vector on the boundary of Ω.

(iii) *If $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \sigma_* \in (0, +\infty)$ corresponding to the case of critical size of holes, then*

$$(\tilde{\mathbf{u}}_\varepsilon, \tilde{p}_\varepsilon) \to (\mathbf{u}, p) \text{ weakly in } W_0^{1,2}(\Omega;\mathbb{R}^d) \times L_0^2(\Omega),$$

where $(\mathbf{u}, p)$ is the unique (weak) solution to the system of Brinkman's law:

$$\begin{cases} -\Delta\mathbf{u} + \nabla p + \sigma_*^{-2}A^{-1}\mathbf{u} = \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

Here in (1.18) *and* (1.19)*, the permeability tensor A is a constant positive definite matrix determined in the following way: if $\frac{a_\varepsilon}{\varepsilon} = 1$, A is given earlier in* (1.9)*; if $\lim_{\varepsilon\to 0}\frac{a_\varepsilon}{\varepsilon} = 0$, A is given later in* (2.22)*. In particular, A is solely determined by the model hole T.*

In Theorem 1.1, both the cases $a_\varepsilon = \varepsilon$ and $a_\varepsilon \ll \varepsilon$ are included. While, we need to point out that, even the limit models are both the Darcy's law for the case $a_\varepsilon = \varepsilon$ and the case $\lim_{\varepsilon\to 0} \frac{a_\varepsilon}{\varepsilon} = 0$ satisfying $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, the related permeability tensors A are different. However, the permeability tensors A are the same as long as $\lim_{\varepsilon\to 0} \frac{a_\varepsilon}{\varepsilon} = 0$ for the cases (ii) and (iii). Since the case $a_\varepsilon = \varepsilon$ is already considered by Tartar, in this paper we mainly focus on the cases where a_ε is much smaller than ε such that $\eta := \frac{a_\varepsilon}{\varepsilon} \to 0$ as $\varepsilon \to 0$.

2. Proof of Theorem 1.1

In this section, we will introduce a generalized cell problem of Tartar [17] and prove Theorem 1.1 based on this generalized cell problem. Throughout the paper, we use C to denote a positive constant independent of ε.

2.1. **Uniform estimates for** $(\tilde{\mathbf{u}}_\varepsilon, \tilde{p}_\varepsilon)$**.** We recall the estimates for $(\tilde{\mathbf{u}}_\varepsilon, \tilde{p}_\varepsilon)$ that have been shown in Allaire [1, 2]. Direct energy estimate and the properties of the restriction operator gives

$$\|\tilde{\mathbf{u}}_\varepsilon\|_{W_0^{1,2}(\Omega)} \leq C, \quad \|\tilde{p}_\varepsilon\|_{L_0^2(\Omega)} \leq C. \tag{2.1}$$

Then, up to a subsequence, as $\varepsilon \to 0$:

$$\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ weakly in } W_0^{1,2}(\Omega); \quad \tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ strongly in } L^2(\Omega); \quad \tilde{p}_\varepsilon \to p \text{ weakly in } L^2(\Omega). \tag{2.2}$$

The divergence free condition $\operatorname{div} \mathbf{u} = 0$ follows from $\operatorname{div} \mathbf{u}_\varepsilon = 0$.

In perforated domains, one can benefit from the zero boundary condition on the holes and obtain the following perforation version of Poincaré inequality (see Lemma 3.4.1 in [2]):

$$\|u\|_{L^2(\Omega_\varepsilon)} \leq C \min\{1, \sigma_\varepsilon\} \|\nabla u\|_{L^2(\Omega_\varepsilon)}, \quad \text{for each } u \in W_0^{1,2}(\Omega_\varepsilon). \tag{2.3}$$

Then for the case of large holes with $\lim_{\varepsilon\to 0} \sigma_\varepsilon = 0$, the above estimate constant in (2.3) becomes σ_ε. By (2.3), direct energy estimate and the properties of the restriction operator gives

$$\|\nabla \tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)} \leq C\sigma_\varepsilon, \quad \|\tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)} \leq C\sigma_\varepsilon^2, \tag{2.4}$$

$$\tilde{p}_\varepsilon = \tilde{p}_\varepsilon^{(1)} + \sigma_\varepsilon \tilde{p}_\varepsilon^{(2)} \text{ with } \|\tilde{p}_\varepsilon^{(1)}\|_{W^{1,2}(\Omega)} + \|\tilde{p}_\varepsilon^{(2)}\|_{L^2(\Omega)} \leq C. \tag{2.5}$$

Then, up to a subsequence, as $\varepsilon \to 0$:

$$\frac{\tilde{\mathbf{u}}_\varepsilon}{\sigma_\varepsilon^2} \to \mathbf{u} \text{ weakly in } L^2(\Omega), \quad \tilde{p}_\varepsilon \to p \text{ strongly in } L^2(\Omega). \tag{2.6}$$

Since $\tilde{\mathbf{u}}_\varepsilon \in W_0^{1,2}(\Omega)$ and $\operatorname{div} \tilde{\mathbf{u}}_\varepsilon = 0$, there holds $\operatorname{div} \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

2.2. **The generalized cell problem.** Near each single hole, after a scaling of size ε^{-1} such that the controlling cube becomes of size $O(1)$, one obtains a domain of the form $Q_0 \setminus (\eta T)$ with $\eta := \frac{a_\varepsilon}{\varepsilon}$. Without loss of generality we may assume $0 < \eta < 1$. We then consider the following *modified cell problem*:

$$\begin{cases} -\Delta w_\eta^i + \nabla q_\eta^i = c_\eta^2 e^i, & \text{in } Q_\eta := Q_0 \setminus (\eta T), \\ \operatorname{div} w_\eta^i = 0, & \text{in } Q_\eta, \\ w_\eta^i = 0, & \text{on } \eta T, \\ (w_\eta^i, q_\eta^i) \text{ is } Q_0\text{-periodic}. \end{cases} \tag{2.7}$$

Again $\{e^i\}_{i=1,\cdots,d}$ is the standard Euclidean coordinate of $\mathbb{R}^d$; c_η is defined as

$$c_\eta := |\log \eta|^{-\frac{1}{2}}, \quad \text{if } d = 2; \quad c_\eta := \eta^{\frac{d-2}{2}}, \quad \text{if } d \geq 3. \tag{2.8}$$

Clearly $c_\eta \to 0$ when $\eta \to 0$. When a_ε is proportional to ε, η becomes a positive constant independent of ε and Q_η becomes a fixed domain of type $Q_0 \setminus T$; this goes back to the case

(1.6) considered by Tartar. So we shall only focus on the general case $\eta = \frac{a_\varepsilon}{\varepsilon} \to 0$ as $\varepsilon \to 0$. The cell problem (2.7) becomes singular: the domain admits a shrinking hole and becomes non-uniformly Lipschitz. This may cause the solutions to be unbounded, see [12, 13] for the cases with zero boundary conditions.

To solve (2.7), we introduce the periodic Sobolev spaces:

$$W^{1,2}_p(Q_0) := \{u \in W^{1,2}(Q_0), u \text{ is } Q_0\text{-periodic}\}, \quad W^{1,2}_{0,p}(Q_\eta) := \{u \in W^{1,2}_p(Q_0), u = 0 \text{ on } \eta T\}.$$

Let $L^2_{0,p}(Q_\eta)$ be the collection of $L^2(Q_\eta)$ functions that are of zero average and Q_0-periodic.

For each fixed $\eta > 0$, by classical theory (energy estimates and compactness), we can show there exists a unique weak solution $(w^i_\eta, q^i_\eta) \in W^{1,2}_{0,p}(Q_\eta; \mathbb{R}^d) \times L^2_{0,p}(Q_\eta)$ to (2.7) in the weak sense:

$$\begin{aligned} &\int_{Q_\eta} w^i_\eta \cdot \nabla \phi \, \mathrm{d}x = 0, \quad \forall\, \phi \in W^{1,2}_{0,p}(Q_\eta) \\ &\int_{Q_\eta} \nabla w^i_\eta : \nabla \varphi \, \mathrm{d}x = c_\eta^2 \int_{Q_\eta} \varphi \cdot e^i, \quad \forall\, \varphi \in W^{1,2}_{0,p}(Q_\eta; \mathbb{R}^d),\ \operatorname{div} \varphi = 0. \end{aligned} \tag{2.9}$$

We shall deduce the explicit dependency of the norms $\|w^i_\eta\|_{W^{1,2}(Q_0)}$ and $\|q^i_\eta\|_{L^2(Q_0)}$ on η when $\eta \to 0$.

2.3. A Poincaré type inequality in Q_η.

We introduce the following lemma which gives a Poincaré type inequality in singular domain Q_η:

Lemma 2.1. *There exists a constant $C > 0$ such that for all $u \in W^{1,2}_{0,p}(Q_\eta)$ there holds*

$$\|u\|_{L^2(Q_0)} \le C c_\eta^{-1} \|\nabla u\|_{L^2(Q_0)}, \tag{2.10}$$

where c_η is given in (2.8).

Proof. Let $u \in W^{1,2}_{0,p}(Q_\eta)$. We assume in addition $u \in C^1(\overline{Q}_\eta)$. For general $u \in W^{1,2}_{0,p}(Q_\eta)$, the result follows from the classical density argument.

By (1.1), there holds

$$B(0, \delta_1 \eta) \subset \eta T \subset B(0, \delta_2 \eta) \subset B(0, \delta_3) \subset Q_0 \subset B(0, 1). \tag{2.11}$$

By Q_0 periodicity of u, we have

$$\|\nabla u\|^2_{L^2(Q_0)} \le \|\nabla u\|^2_{L^2(B(0,1))} \le \|\nabla u\|^2_{L^2((-1,1)^d)} = 2^d \|\nabla u\|^2_{L^2(Q_0)}. \tag{2.12}$$

For each $x \in B(0,1) \setminus (\eta T) \subset B(0,1) \setminus B(0, \delta_1 \eta)$, we denote $r_x := |x|$ and $\omega_x := \frac{x}{|x|}$. By the fact $u = 0$ on ηT, we have

$$u(x) = u(r_x \omega_x) = u(r_x \omega_x) - u(\delta_1 \eta \omega_x) = \int_{\delta_1 \eta}^{r_x} \frac{\mathrm{d}}{\mathrm{d}s} u(s \omega_x) \, \mathrm{d}s = \int_{\delta_1 \eta}^{r_x} (\nabla u)(s \omega_x) \cdot \omega_x \, \mathrm{d}s.$$

By Hölder's inequality, direct calculation gives

$$
\begin{aligned}
\|u\|_{L^2(Q_0)}^2 &\le \int_{B(0,1)\setminus B(0,\delta_1\eta)} |u(x)|^2\,\mathrm{d}x = \int_{\delta_1\eta}^1 \int_{\mathbb{S}^{d-1}} |u(r_x\omega_x)|^2 r_x^{d-1}\,\mathrm{d}\omega_x\,\mathrm{d}r_x \\
&= \int_{\delta_1\eta}^1 \int_{\mathbb{S}^{d-1}} \left| \int_{\delta_1\eta}^{r_x} (\nabla u)(s\omega_x)\cdot\omega_x\,\mathrm{d}s \right|^2 r_x^{d-1}\,\mathrm{d}\omega_x\,\mathrm{d}r_x \\
&\le \int_{\mathbb{S}^{d-1}} \int_{\delta_1\eta}^1 r_x^{d-1} \left(\int_{\delta_1\eta}^{r_x} s^{-d+1}\,\mathrm{d}s \right) \left(\int_{\delta_1\eta}^{r_x} s^{d-1} |\nabla u(s\omega_x)|^2\,\mathrm{d}s \right) \mathrm{d}r_x\,\mathrm{d}\omega_x \\
&\le \left(\int_{\delta_1\eta}^1 r_x^{d-1} \left(\int_{\delta_1\eta}^{r_x} s^{-d+1}\,\mathrm{d}s \right) \mathrm{d}r_x \right) \left(\int_{\mathbb{S}^{d-1}} \int_{\delta_1\eta}^1 s^{d-1} |\nabla u(s\omega_x)|^2\,\mathrm{d}s\,\mathrm{d}\omega_x \right) \\
&\le C \int_{\delta_1\eta}^1 s^{-d+1}\,\mathrm{d}s \int_{B(0,1)} |\nabla u(x)|^2\,\mathrm{d}x.
\end{aligned}
\tag{2.13}
$$

We then deduce from (2.13) that

$$
\begin{aligned}
&\|u\|_{L^2(Q_0)}^2 \le C|\log\eta| \|\nabla u\|_{L^2(B(0,1))}^2, \quad \text{if } d=2, \\
&\|u\|_{L^2(Q_0)}^2 \le C\eta^{-d+2} \|\nabla u\|_{L^2(B(0,1))}^2, \quad \text{if } d\ge 3.
\end{aligned}
\tag{2.14}
$$

Combining (2.12) and (2.14) implies our desired estimate (2.10).

□

2.4. **A Bogovskii type operator in Q_η.** We then introduce a Bogovskii type operator in Q_η:

Lemma 2.2. *There exists a linear mapping $\mathcal{B}_{Q_\eta} : L^2_{0,p}(Q_\eta) \to W^{1,2}_{0,p}(Q_\eta;\mathbb{R}^d)$ such that for each $f \in L^2_{0,p}(Q_\eta)$, there holds*

$$
\operatorname{div} \mathcal{B}_{Q_\eta}(f) = f \ \text{ in } Q_\eta, \ \|\mathcal{B}_{Q_\eta}(f)\|_{W^{1,2}_{0,p}(Q_\eta)} \le C\|f\|_{L^2(Q_\eta)}.
$$

Proof. Given $f \in L^2_{0,p}(Q_\eta)$. Let $\tilde f \in L^2_{0,p}(Q_0)$ be the zero extension of f in Q_0. Since $\tilde f$ is Q_0-periodic and is of zero average, we have the following expression of Fourier series:

$$
\tilde f(x) = \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} f_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot x}, \ x \in Q_0.
$$

Here $f_{\mathbf{k}}$, $\mathbf{k}\in\mathbb{Z}^d$ are the Fourier coefficients of $\tilde f$. Let

$$
\tilde u := \nabla\Delta^{-1} f := \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{-i\mathbf{k}}{2\pi|\mathbf{k}|^2} f_{\mathbf{k}} e^{2\pi i\mathbf{k}\cdot x}.
$$

Then $\tilde u \in W^{1,2}_p(Q_0;\mathbb{R}^d)$ satisfying

$$
\operatorname{div}\tilde u = \tilde f \text{ in } Q_0, \quad \|\tilde u\|_{W^{1,2}(Q_0)} \le C\|\tilde f\|_{L^2(Q_0)}.
$$

Recall (2.11) and consider the following problem in v near the hole:

$$
\begin{cases}
\operatorname{div} v = \operatorname{div}\tilde u = f, & \text{in } B(0,\delta_2\eta)\setminus(\eta T), \\
v = \tilde u, & \text{on } \partial(B(0,\delta_2\eta)), \\
v = 0, & \text{on } \partial(\eta T).
\end{cases}
\tag{2.15}
$$

By employing the proof of Lemma 2.1.4 in Allaire [1], there exists a solution v to (2.15) satisfying

$$
\|v\|_{W^{1,2}(B(0,\delta_2\eta)\setminus(\eta T))} \le C\|\tilde u\|_{W^{1,2}(Q_0)} \le C\|\tilde f\|_{L^2(Q_0)} = C\|f\|_{L^2(Q_\eta)}.
$$

Finally, the following linear operator

$$\mathcal{B}_{Q_\eta}(f) := \begin{cases} \tilde{u}, & \text{in } Q_0 \setminus B(0,\delta_2\eta), \\ v, & \text{in } B(0,\delta_2\eta) \setminus (\eta T) \end{cases}$$

is well defined and fulfills our desired properties stated in Lemma 2.2.

□

2.5. **Estimates for** (w_η^i, q_η^i)**.** Taking w_η^i as a test function for (2.7) in the weak formulation $(2.9)_2$ and using Lemma 2.1 gives

$$\|\nabla w_\eta^i\|_{L^2(Q_\eta)}^2 \le c_\eta^2 \|w_\eta^i\|_{L^2(Q_\eta)} \le C c_\eta \|\nabla w_\eta^i\|_{L^2(Q_\eta)}. \tag{2.16}$$

This implies, again using Lemma 2.1, that

$$\|\nabla w_\eta^i\|_{L^2(Q_\eta)} \le C c_\eta, \quad \|w_\eta^i\|_{L^2(Q_\eta)} \le C. \tag{2.17}$$

Taking $\mathcal{B}_{Q_\eta}(q_\eta^i)$ as a test function for (2.7) and using Lemmas 2.1 and 2.2 implies

$$\|q_\eta^i\|_{L^2(Q_\eta)}^2 \le c_\eta^2 \|\mathcal{B}_{Q_\eta}(q_\eta^i)\|_{L^2(Q_\eta)} + \|\nabla w_\eta^i\|_{L^2(Q_\eta)} \|\nabla \mathcal{B}_{Q_\eta}(q_\eta^i)\|_{L^2(Q_\eta)}. \tag{2.18}$$

By Lemma 2.2, (2.17) and (2.18), we get

$$\|q_\eta^i\|_{L^2(Q_\eta)} \le C c_\eta. \tag{2.19}$$

By (2.17) and compact Sobolev embedding, we have, up to a subsequence, that

$$w_\eta^i \to w^i \text{ weakly in } W^{1,2}(Q_0), \quad w_\eta^i \to w^i \text{ strongly in } L^2(Q_0). \tag{2.20}$$

In particular, when $\eta \to 0$ as $\varepsilon \to 0$ such that $c_\eta \to 0$, by (2.17), there holds $\nabla w^i = 0$ meaning that the limit w^i is a constant vector.

We deduce from (2.19), up to a subsequence, that

$$c_\eta^{-1} q_\eta^i \to q^i \text{ weakly in } L^2(Q_0). \tag{2.21}$$

Define $A(\eta) \in M^{d\times d}$ as

$$A(\eta)_{i,j} := c_\eta^{-2} \int_{Q_\eta} \nabla w_\eta^i : \nabla w_\eta^j \,\mathrm{d}x.$$

Clearly $A(\eta)$ is semi-positive definite. Taking w_η^j as a test function in (2.7) gives

$$A(\eta)_{i,j} = \int_{Q_\eta} w_\eta^j \cdot e^i \,\mathrm{d}x = \int_{Q_\eta} (w_\eta^j)_i \,\mathrm{d}x.$$

By (2.20) where we have shown the weak convergence of w_η^j in L^2 as $\eta \to 0$ up to a subsequence, we then define A as the limit of $A(\eta)$:

$$A_{i,j} = \lim_{\eta\to 0} A(\eta)_{i,j} = \lim_{\eta\to 0} c_\eta^{-2} \int_{Q_\eta} \nabla w_\eta^i : \nabla w_\eta^j \,\mathrm{d}x = \lim_{\eta\to 0} \int_{Q_\eta} (w_\eta^j)_i \,\mathrm{d}x =: (\bar{w}^j)_i. \tag{2.22}$$

We see that the matrix $A = (\bar{w}_i^j)_{1\le i,j\le d}$ is symmetric. Moreover, the main Theorem in [3, Section 0] says that

$$\lim_{\eta\to 0} A_\eta = A = M^{-1}, \tag{2.23}$$

where M is the permeability tensor introduced by Allaire, which is positive definite. Actually, the permeability tensor M is defined by (see [1, 2] or [3])

$$M := \pi\mathbb{I}, \text{ if } d = 2; \quad M := \Big(\frac{1}{2^d} \int_{\mathbb{R}^d\setminus T} \nabla v^i : \nabla v^j \,\mathrm{d}x\Big)_{1\le i,j\le d}, \text{ if } d \ge 3,$$

where v^i is the solution to the local problem (1.10).

2.6. **The scaled cell solutions.** Starting from the solution (w^i_η, q^i_η) to the cell problem (2.7), we define

$$w^i_{\eta,\varepsilon}(\cdot) := w^i_\eta\big(\frac{\cdot}{\varepsilon}\big),\ q^i_{\eta,\varepsilon}(\cdot) := q^i_\eta\big(\frac{\cdot}{\varepsilon}\big) \tag{2.24}$$

solving

$$\begin{cases} -\varepsilon^2\Delta w^i_{\eta,\varepsilon} + \varepsilon\nabla q^i_{\eta,\varepsilon} = c^2_\eta e^i, & \text{in } \varepsilon Q_0 \setminus (a_\varepsilon T),\\ \operatorname{div} w^i_{\eta,\varepsilon} = 0, & \text{in } \varepsilon Q_0 \setminus (a_\varepsilon T),\\ w^i_{\eta,\varepsilon} = 0, & \text{on } a_\varepsilon T,\\ (w^i_{\eta,\varepsilon}, q^i_{\eta,\varepsilon}) \text{ is } \varepsilon Q_0\text{-periodic.} \end{cases} \tag{2.25}$$

By (2.16)–(2.19), (2.24), direct calculation gives

$$\begin{aligned} &\|w^i_{\eta,\varepsilon}\|_{L^2(\Omega)} \le C\|w^i_\eta\|_{L^2(Q)} \le C,\\ &\|q^i_{\eta,\varepsilon}\|_{L^2(\Omega)} \le C\|q^i_\eta\|_{L^2(Q)} \le Cc_\eta,\\ &\|\nabla w^i_{\eta,\varepsilon}\|_{L^2(\Omega)} \le C\varepsilon^{-1}\|\nabla w^i_\eta\|_{L^2(Q)} \le C\varepsilon^{-1}c_\eta \le C\sigma_\varepsilon^{-1}, \end{aligned} \tag{2.26}$$

where we observed that $\varepsilon^{-1}c_\eta = \sigma_\varepsilon^{-1}$ from (1.5) and (2.8). Thus, by the convergence we have shown in (2.20) and (2.21), using the periodicity of $(w^i_{\eta,\varepsilon}, q^i_{\eta,\varepsilon})$, we can obtain

$$w^i_{\eta,\varepsilon} \to \bar w^i \text{ weakly in } L^2(\Omega), \quad c_\eta^{-1} q^i_{\eta,\varepsilon} \to \bar q^i := \int_{Q_0} q^i \,\mathrm{d}x \text{ weakly in } L^2(\Omega), \tag{2.27}$$

as $\varepsilon \to 0$, up to a subsequence.

2.7. **Homogenization process.** Clearly $w^i_{\eta,\varepsilon}$ vanishes on the holes in Ω_ε. Given any scalar function $\phi \in C^\infty_c(\Omega)$, taking $w^i_{\eta,\varepsilon}\phi$ as a test function to (1.3) gives

$$\int_{\Omega_\varepsilon} \nabla\mathbf{u}_\varepsilon : \nabla(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x - \int_{\Omega_\varepsilon} p_\varepsilon \operatorname{div}(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x = \int_{\Omega_\varepsilon} \mathbf{f}\cdot(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x. \tag{2.28}$$

By the fact that $w^i_{\eta,\varepsilon}$ vanishes on the holes and that $(\tilde{\mathbf{u}}_\varepsilon, \tilde p_\varepsilon)$ coincides with $(\mathbf{u}_\varepsilon, p_\varepsilon)$ in Ω_ε, the integral equality (2.28) is equivalent to

$$\int_{\Omega} \nabla\tilde{\mathbf{u}}_\varepsilon : \nabla(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x - \int_{\Omega} \tilde p_\varepsilon \operatorname{div}(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x = \int_{\Omega} \mathbf{f}\cdot(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x \tag{2.29}$$

We will pass $\varepsilon \to 0$ case by case in the following subsections. The limits are taken up to a subsequence and we will not repeat this point.

2.7.1. *The case with small holes.* We start with the case of small holes such that $\lim_{\varepsilon\to 0}\sigma_\varepsilon \to +\infty$.

By (2.26) and (2.27), we have $\|\nabla w^i_{\eta,\varepsilon}\|_{L^2(\Omega)} \le C\sigma_\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$; moreover $w^i_{\eta,\varepsilon} \to \bar w^i$ srtongly in $L^2(\Omega)$ by Rellich-Kondrachov compact embedding theorem. Thus, as $\varepsilon \to 0$,

$$\begin{aligned} \int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : \nabla(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x &= \int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : w^i_{\eta,\varepsilon}\otimes\nabla\phi\,\mathrm{d}x + \int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : \nabla w^i_{\eta,\varepsilon}\phi\,\mathrm{d}x\\ &\to \int_\Omega \nabla\mathbf{u} : \bar w^i\otimes\nabla\phi\,\mathrm{d}x = \int_\Omega \nabla\mathbf{u} : \nabla(\bar w^i\phi)\,\mathrm{d}x, \end{aligned} \tag{2.30}$$

$$\int_\Omega \tilde p_\varepsilon \operatorname{div}(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x = \int_\Omega \tilde p_\varepsilon\, w^i_{\eta,\varepsilon}\cdot\nabla\phi\,\mathrm{d}x \to \int_\Omega p\,\bar w^i\cdot\nabla\phi\,\mathrm{d}x = \int_\Omega p \operatorname{div}(\bar w^i\phi)\,\mathrm{d}x, \tag{2.31}$$

and

$$\int_\Omega \mathbf{f}\cdot(w^i_{\eta,\varepsilon}\phi)\,\mathrm{d}x \to \int_\Omega \mathbf{f}\cdot\bar w^i\phi\,\mathrm{d}x. \tag{2.32}$$

Then using (2.30)–(2.32) and passing $\varepsilon \to 0$ in (2.29) implies

$$\int_\Omega \nabla \mathbf{u} : \nabla(\bar{w}^i \phi)\,\mathrm{d}x - \int_\Omega p \,\mathrm{div}\,(\bar{w}^i\phi)\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot \bar{w}^i \phi\,\mathrm{d}x.$$

This gives

$$\int_\Omega \nabla \mathbf{u} : \nabla(A\varphi) - p\,\mathrm{div}\,(A\varphi) - \mathbf{f}\cdot(A\varphi)\,\mathrm{d}x = 0,\ \forall\,\varphi \in C_c^\infty(\Omega;\mathbb{R}^d),$$

which means

$$A(-\Delta \mathbf{u} + \nabla p - \mathbf{f}) = 0$$

in the weak sense. Here $A = (w_j^i)_{1\le i,j\le d}$ is the permeability matrix defined in (2.22) and satisfies (2.23). Since A is positive definite, together with the results in (2.1), we deduce the Stokes equations in non perforated domain Ω:

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f},\ \mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega; \quad \mathbf{u} = 0 \text{ on } \partial\Omega. \tag{2.33}$$

We show the strong convergence of $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ in $W_0^{1,2}(\Omega;\mathbb{R}^d)$. Taking $\mathbf{u}_\varepsilon$ as a test function in the weak formulation of (1.3), using the property that $\tilde{\mathbf{u}}_\varepsilon = \mathbf{u}_\varepsilon$ in Ω_ε and the weak convergence of $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ in $W_0^{1,2}(\Omega;\mathbb{R}^d)$, passing $\varepsilon \to 0$ implies

$$\lim_{\varepsilon\to 0} \|\nabla \tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)}^2 = \int_\Omega \mathbf{u}\cdot \mathbf{f}\,\mathrm{d}x. \tag{2.34}$$

Taking $\mathbf{u}$ as a test function to (2.33) gives

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 = \int_\Omega \mathbf{u}\cdot\mathbf{f}\,\mathrm{d}x. \tag{2.35}$$

Thus $\lim_{\varepsilon\to 0}\|\nabla\tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)} = \|\nabla \mathbf{u}\|_{L^2(\Omega)}$ resulting in $\nabla\tilde{\mathbf{u}}_\varepsilon \to \nabla\mathbf{u}$ strong in $L^2(\Omega)$ and finally $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ in $W_0^{1,2}(\Omega;\mathbb{R}^d)$. The strong convergence $\tilde{p}_\varepsilon \to p$ in $L^2(\Omega)$ follows from the strong convergence $\nabla\tilde{p}_\varepsilon \to \nabla p$ in $W^{-1,2}(\Omega)$ and employing the Bogovskii operator on Ω.

2.7.2. *The case with large holes.* We then consider the case with large holes: $\lim_{\varepsilon\to 0}\sigma_\varepsilon \to 0$. By (2.25), direct calculation gives

$$\begin{aligned}
\int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : \nabla(w_{\eta,\varepsilon}^i\phi)\,\mathrm{d}x &= \int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : w_{\eta,\varepsilon}^i \otimes \nabla\phi\,\mathrm{d}x + \int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : \nabla w_{\eta,\varepsilon}^i \phi\,\mathrm{d}x\\
&= \int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : w_{\eta,\varepsilon}^i \otimes \nabla\phi\,\mathrm{d}x + \int_\Omega \nabla(\phi\tilde{\mathbf{u}}_\varepsilon) : \nabla w_{\eta,\varepsilon}^i\,\mathrm{d}x - \int_\Omega \nabla\phi\otimes\tilde{\mathbf{u}}_\varepsilon : \nabla w_{\eta,\varepsilon}^i\,\mathrm{d}x\\
&= \int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : w_{\eta,\varepsilon}^i \otimes \nabla\phi\,\mathrm{d}x - \int_\Omega \nabla\phi\otimes\tilde{\mathbf{u}}_\varepsilon : \nabla w_{\eta,\varepsilon}^i\,\mathrm{d}x\\
&\quad + \varepsilon^{-1}\int_\Omega \mathrm{div}\,(\phi\tilde{\mathbf{u}}_\varepsilon)\,q_{\eta,\varepsilon}^i\,\mathrm{d}x + \varepsilon^{-2}c_\eta^2\int_\Omega (\phi\tilde{\mathbf{u}}_\varepsilon)\cdot e^i\,\mathrm{d}x.
\end{aligned} \tag{2.36}$$

By (2.26), (2.27), (2.4), we have

$$\left|\int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : w_{\eta,\varepsilon}^i\otimes\nabla\phi\,\mathrm{d}x\right| \le C\|\nabla\tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)}\|w_{\eta,\varepsilon}^i\|_{L^2(\Omega)} \le C\sigma_\varepsilon \to 0,$$

$$\left|\int_\Omega \nabla\phi\otimes\tilde{\mathbf{u}}_\varepsilon : \nabla w_{\eta,\varepsilon}^i\,\mathrm{d}x\right| \le C\|\tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)}\|\nabla w_{\eta,\varepsilon}^i\|_{L^2(\Omega)} \le C\sigma_\varepsilon^2\sigma_\varepsilon^{-1} = C\sigma_\varepsilon \to 0.$$

Moreover, using the divergence free condition $\mathrm{div}\,\tilde{\mathbf{u}}_\varepsilon = 0$ and observing $\varepsilon^{-1}c_\eta = \sigma_\varepsilon^{-1}$ implies

$$\left|\varepsilon^{-1}\int_\Omega \mathrm{div}\,(\phi\tilde{\mathbf{u}}_\varepsilon)\,q_{\eta,\varepsilon}^i\,\mathrm{d}x\right| \le C\varepsilon^{-1}\|\tilde{\mathbf{u}}_\varepsilon\|_{L^2(\Omega)}\|q_{\eta,\varepsilon}^i\|_{L^2(\Omega)} \le C\varepsilon^{-1}\sigma_\varepsilon^2 c_\eta = C\sigma_\varepsilon \to 0.$$

By (2.6) and observing $\varepsilon^{-2}c_\eta^2 = \sigma_\varepsilon^{-2}$, we have

$$\varepsilon^{-2}c_\eta^2 \int_\Omega (\phi\tilde{\mathbf{u}}_\varepsilon)\cdot e^i \,\mathrm{d}x = \int_\Omega \phi \frac{\tilde{\mathbf{u}}_\varepsilon}{\sigma_\varepsilon^2}\cdot e^i \,\mathrm{d}x \to \int_\Omega \phi\mathbf{u}\cdot e^i \,\mathrm{d}x.$$

For the term related to the pressure, by (2.5) and (2.6),

$$\int_\Omega \tilde{p}_\varepsilon \,\mathrm{div}\,(w_{\eta,\varepsilon}^i\phi)\,\mathrm{d}x = \int_\Omega \tilde{p}_\varepsilon\, w_{\eta,\varepsilon}^i\cdot\nabla\phi\,\mathrm{d}x \to \int_\Omega p\,\bar{w}^i\cdot\nabla\phi\,\mathrm{d}x = \int_\Omega p\,\mathrm{div}\,(\bar{w}^i\phi)\,\mathrm{d}x.$$

Then passing $\varepsilon\to 0$ in (2.29) implies

$$\int_\Omega \phi\mathbf{u}\cdot e^i\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\bar{w}^i\phi\,\mathrm{d}x + \int_\Omega p\,\mathrm{div}\,(\bar{w}^i\phi)\,\mathrm{d}x. \tag{2.37}$$

Together with the results in Section 2.1, from (2.37) we deduce the Darcy's law in Ω:

$$\mathbf{u} = A(\mathbf{f}-\nabla p),\ \mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega; \quad \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \tag{2.38}$$

2.7.3. *The case with critical size of holes.* We finally consider the case $\lim_{\varepsilon\to 0}\sigma_\varepsilon = \sigma_* \in (0,+\infty)$. By (2.26) and (2.27), we have $\|w_{\eta,\varepsilon}^i\|_{W^{1,2}(\Omega)} \leq C$. Thus $w_{\eta,\varepsilon}^i \to \bar{w}^i$ weakly in $W^{1,2}(\Omega)$ and $w_{\eta,\varepsilon}^i \to \bar{w}^i$ srtongly in $L^2(\Omega)$. Together with (2.1), (2.2) and the strong convergence $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ and $w_{\eta,\varepsilon}^i \to \bar{w}^i$ in $L^2(\Omega)$, we have for the right-hand side of (2.36):

$$\begin{aligned}
&\int_\Omega \nabla\tilde{\mathbf{u}}_\varepsilon : w_{\eta,\varepsilon}^i \otimes\nabla\phi\,\mathrm{d}x \to \int_\Omega \nabla\mathbf{u} : \bar{w}^i\otimes\nabla\phi\,\mathrm{d}x = \int_\Omega \nabla\mathbf{u} : \nabla(\bar{w}^i\phi)\,\mathrm{d}x,\\
&\int_\Omega \nabla\phi\otimes\mathbf{u}_\varepsilon : \nabla w_{\eta,\varepsilon}^i\,\mathrm{d}x \to \int_\Omega \nabla\phi\otimes\mathbf{u} : \nabla\bar{w}^i\,\mathrm{d}x = 0,\\
&\varepsilon^{-1}\int_\Omega \mathrm{div}\,(\phi\tilde{\mathbf{u}}_\varepsilon)\,q_{\eta,\varepsilon}^i\,\mathrm{d}x = \varepsilon^{-1}c_\eta\int_\Omega \mathrm{div}\,(\phi\tilde{\mathbf{u}}_\varepsilon)\,(c_\eta^{-1}q_{\eta,\varepsilon}^i)\,\mathrm{d}x = \sigma_\varepsilon^{-1}\int_\Omega \nabla\phi\cdot\tilde{\mathbf{u}}_\varepsilon\,(c_\eta^{-1}q_{\eta,\varepsilon}^i)\,\mathrm{d}x\\
&\qquad\qquad \to \sigma_*^{-1}\int_\Omega \nabla\phi\cdot\mathbf{u}\,\bar{q}^i\,\mathrm{d}x = \sigma_*^{-1}\int_\Omega \mathrm{div}\,(\phi\mathbf{u})\,\bar{q}^i\,\mathrm{d}x = 0,
\end{aligned}$$

where we used the fact that $\bar{w}^i$ and $\bar{q}^i$ are constant.

Again by the strong convergence $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ and $w_{\eta,\varepsilon}^i \to \bar{w}^i$ in $L^2(\Omega)$, we obtain

$$\begin{aligned}
&\varepsilon^{-2}c_\eta^2\int_\Omega (\phi\tilde{\mathbf{u}}_\varepsilon)\cdot e^i\,\mathrm{d}x = \sigma_\varepsilon^{-2}\int_\Omega \phi\tilde{\mathbf{u}}_\varepsilon\cdot e^i\,\mathrm{d}x \to \sigma_*^{-2}\int_\Omega \phi\mathbf{u}\cdot e^i\,\mathrm{d}x,\\
&\int_\Omega p_\varepsilon\,\mathrm{div}\,(w_{\eta,\varepsilon}^i\phi)\,\mathrm{d}x = \int_\Omega \tilde{p}_\varepsilon\,w_{\eta,\varepsilon}^i\cdot\nabla\phi\,\mathrm{d}x \to \int_\Omega p\,\bar{w}^i\cdot\nabla\phi\,\mathrm{d}x = \int_\Omega p\,\mathrm{div}\,(\bar{w}^i\phi)\,\mathrm{d}x.
\end{aligned}$$

Finally, passing $\varepsilon\to 0$ in (2.29) implies

$$\int_\Omega \nabla\mathbf{u} : \nabla(\bar{w}^i\phi)\,\mathrm{d}x + \sigma_*^{-2}\int_\Omega \phi\mathbf{u}\cdot e^i\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\bar{w}^i\phi\,\mathrm{d}x + \int_\Omega p\,\mathrm{div}\,(\bar{w}^i\phi)\,\mathrm{d}x.$$

This is the Brinkmann's law in non perforated domain Ω:

$$\sigma_*^{-2}\mathbf{u} = A(\mathbf{f}-\nabla p+\Delta\mathbf{u}) \iff -\Delta\mathbf{u}+\nabla p+\sigma_*^{-2}A^{-1}\mathbf{u} = \mathbf{f}. \tag{2.39}$$

Moreover, by the results in Section 2.1, we have

$$\mathbf{u}\in W_0^{1,2}(\Omega;\mathbb{R}^d),\ p\in L_0^2(\Omega),\ \mathrm{div}\,\mathbf{u} = 0. \tag{2.40}$$

We thus complete the proof of Theorem 1.1.

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References

[1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. *Arch. Ration. Mech. Anal.*, **113**(3) (1990), 209-259.

[2] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. *Arch. Ration. Mech. Anal.*, **113**(3) (1990), 261-298.

[3] G. Allaire. Continuity of the Darcy's law in the low-volume fraction limit. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **4** (1991), 475-499.

[4] D. Cioranescu, F. Murat. Un terme étrange venu d'ailleurs, Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vols. 2 & 3, ed. by H. Brezis & J. L. Lions, Research Notes in Mathematics 60, pp. 98-138, and 70, pp. 154-178, Pitman, London (1982).

[5] L. Diening, E. Feireisl, Y. Lu. The inverse of the divergence operator on perforated domains with applications to homogenization problems for the compressible Navier-Stokes system. *ESAIM: Control Optim. Calc. Var.*, **23** (2017), 851-868.

[6] E. Feireisl, Y. Lu. Homogenization of stationary Navier-Stokes equations in domains with tiny holes. *J. Math. Fluid Mech.*, **17** (2015), 381-392.

[7] E. Feireisl, Y. Namlyeyeva, Š. Nečasová. Homogenization of the evolutionary Navier–Stokes system. *Manusc. Math.*, **149** (2016), 251-274.

[8] E. Feireisl, A. Novotný, T. Takahashi. Homogenization and singular limits for the complete Navier-Stokes-Fourier system. *J. Math. Pures Appl.*, **94**(1) (2010), 33-57.

[9] G.P. Galdi. An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems. Springer Science and Business Media, 2011.

[10] W. Jing. A unified homogenization approach for the Dirichlet problem in perforated domains. *Preprint, arXiv:1901.08251.*

[11] W. Jing, Y. Lu, C. Prange. Homogenization of Stokes equations in perforated domains with soft restriction on the distribution of holes. *In preparetion.*

[12] Y. Lu. On uniform estimates for Laplace equation in balls with small holes. *Calc. Var. Partial Differential Equations,* **55**(5) (2016), 55:110.

[13] Y. Lu. Uniform estimates for Stokes equations in a domain with a small hole and applications in homogenization problems. *Preprint, arXiv:1510.01678* .

[14] Y. Lu, S. Schwarzacher. Homogenization of the compressible Navier-Stokes equations in domains with very tiny holes. J. Differential Equations, **265**(4) (2018), 1371-1406.

[15] N. Masmoudi. Homogenization of the compressible Navier-Stokes equations in a porous medium. *ESAIM: Control Optim. Calc. Var.*, **8** (2002), 885-906.

[16] A. Mikelić. Homogenization of nonstationary Navier-Stokes equations in a domain with a grained boundary. *Ann. Mat. Pura Appl.*, **158** (1991), 167-179.

[17] L. Tartar. Incompressible fluid flow in a porous medium: convergence of the homogenization process, in *Nonhomogeneous media and vibration theory*, edited by E. Sánchez-Palencia, 1980, 368-377.

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