On PDE analysis of flows of quasi-incompressible fluids

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Abstract

We study mathematical properties of quasi-incompressible fluids. These are mixtures in which the density depends on the concentration of one of their components. Assuming that the mixture meets mass and volume additivity constraints, this density-concentration relationship is given explicitly. We show that such a constrained mixture can be written in the form similar to compressible Navier-Stokes equations with a singular relation between the pressure and the density. This feature automatically leads to the density bounded from below and above. After addressing the choice of thermodynamically compatible boundary conditions, we establish the large data existence of weak solution to the relevant initial and boundary value problem. We then investigate one possible limit from the quasi-compressible regime to the incompressible regime.

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1 Introduction

Flows of salty water, red blood cells in plasma, or in general a solute in a solvent are often described by the following system of equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}) = 0\,, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{v}) + \mathrm{div}_x(\varrho \mathbf{v} \otimes \mathbf{v}) = \mathrm{div}_x \mathbf{T}\,, \tag{1.2}$$

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{v}) = -\mathrm{div}_x\, \mathbf{j}\,, \tag{1.3}$$

where ϱ and $\mathbf{v}$ are the density and the velocity of the solvent and c is the concentration of the solute. The quantities $\mathbf{T}$ and $\mathbf{j}$ stand for the Cauchy stress and the diffusive flux that are specified via the constitutive equations such as

$$\mathbf{T} = -p(\varrho, c)\mathbf{I} + 2\mu(\varrho, c)\mathbf{D}(\mathbf{v}) + \lambda(\varrho, c)\mathrm{div}_x \mathbf{v}\mathbf{I} \tag{1.4}$$

and

$$\mathbf{j} = -\beta(\varrho, c)\nabla c\,, \tag{1.5}$$

where (1.4) characterizes the Navier-Stokes fluid and (1.5) is known as Fick's law.

1.1 Quasi-incompressibility

The closed system of partial differential equations (PDE) (1.1)-(1.5), sometimes called the Navier-Stokes-Fick system, is often 'simplified' by additional assumptions such as *incompressibility* stating that

$$\operatorname{div}_x \mathbf{v} = 0\,, \tag{1.6}$$

or *quasi-incompressibility* when

$$\varrho = \tilde{\varrho}(c)\,. \tag{1.7}$$

Note that (1.7) together with (1.1) and (1.3) implies that

$$\operatorname{div}_x \mathbf{v} = -\frac{\tilde{\varrho}'(c)}{\tilde{\varrho}^2(c)}\left(\varrho\partial_t c + \varrho\mathbf{v}\cdot\nabla c\right) = \frac{\tilde{\varrho}'(c)}{\tilde{\varrho}^2(c)}\operatorname{div}_x \mathbf{j}\,. \tag{1.8}$$

Comparing (1.6) and (1.8) gives a clear distinction between incompressibility and quasi-incompressibility; the latter makes volume changes due to solute diffusion (or diffusion effects due to volume changes) possible. Obviously, (1.8) reduces to (1.6) if $\operatorname{div}_x \mathbf{j} = 0$ or ϱ does not vary with concentration; (1.8) is less restrictive than (1.6).

1.2 Ill-posed problems

As quasi-incompressibility links (1.1) and (1.3), it can happen that one can assign inappropriate boundary conditions to such reduced problems which can finally result in ill-posed problems; see for example [8]. The aim of this paper is to study flows of quasi-incompressible fluids, i.e. the flows described by (1.1)-(1.5) and (1.7), that are built upon clear assumptions of the theory of interacting continua, are consistent with the second law of thermodynamics, and provide the boundary conditions that are compatible with the second law and PDE analysis. In doing so, we recall only those concepts of the theory of mixtures that suffice for the formulation of the problem we analyze below in Sections 3 and 4.

1.3 Structure of the paper

In Section 2, we first introduce basic volume and mass measures associated with interacting and coexisting continua. Restricting ourselves to mixtures fulfilling mass and volume additivity constraints and considering the material densities of each individual constituent constant, one obtains an *explicit* formula between the density ϱ and the concentration c of the form (1.7). Then, stemming from the mass balance equations for each constituent and the balance of momenta and energy for the mixture as a whole, we provide a derivation of the PDE as well as boundary conditions that are compatible with the second law of thermodynamics. In Section 3, we rewrite the problem setting in the form more amenable to its mathematical analysis. Motivated by a logarithmic (i.e. logarithmically singular) dependence of the chemical potential on the concentrations, we consider pressures that exhibit polynomial singularity. We then establish large data existence of weak solutions. In Section 4, we study one possible limit from quasi-incompressibility to 'incompressibility'. This limit as well as the constraint (1.7) thus represent two types of model reduction used in this study. We put the word incompressibility in quotation marks as we arrive at the condition (1.6) due to the fact that the mixture as a whole becomes homogeneous. Section 5 provides a brief summary.

2 Derivation of a class of mathematical models

The objective of this section is to provide a derivation of a thermodynamically compatible model including boundary conditions. Our approach is based on the theory of interacting continua. We follow a simplified procedure that leads directly to a system of the type (1.1) we are interested in. We refer the interested reader to [15] for a more detailed study that includes in addition mass conversion between the individual constituents and thermal effects. Special attention is paid to mixtures composed of two or three constituents.

2.1 Theory of mixtures - mass and volume measures

The theory of interacting continua (see e.g. [19, 18, 12]) starts with the assumption that the individual constituents coexist, in a homogenized sense, at each point x of the domain Ω occupied by a mixture at a current time t. This approach allows us to introduce, for each individual ith constituent ($i = 1, \dots, N$), its density $\varrho^i = \frac{dM^i}{dV}$, concentration $c^i = \frac{dM^i}{dM}$, volume fraction $\phi^i = \frac{dV^i}{dV}$, and the material densities $\varrho^i_m = \frac{dM^i}{dV^i}$, and for the mixture as a whole the density $\varrho = \frac{dM}{dV}$, where M and V stand for the mass and volume measures related to the mixture as a whole and M^i and V^i are the mass and volume measures of the ith constituent, $i = 1, \dots, N$. It follows from these definitions that

$$c^i \varrho = \varrho^i \quad \text{and} \quad \varrho^i = \varrho^i_m \phi^i \,. \tag{2.1}$$

2.2 Theory of mixtures - mass balance equation for each constituent

The assumption of the co-occupancy helps to introduce, analogously as in the case of a single continuum, the velocities $\mathbf{v}^i$ and other kinematic quantities, and one can write down the balance equations (for mass, linear and angular momenta and energy) for each constituent. In this study, we will consider a simplified framework and associate to the ith constituent, $i = 1, \dots, N$, solely the mass balance equation that takes the form

$$\partial_t \varrho^i + \operatorname{div}_x(\varrho^i \mathbf{v}^i) = 0 \,. \tag{2.2}$$

Here, we do not assume any conversion of mass of one constituent to the other. We shall consider the balance of linear and angular momentum and the balance of energy for the mixture as a whole. We shall introduce them later.

2.3 Theory of mixtures - constraints

We formulate four constraints, which lead to a natural introduction of further quantities.

The first constraint - *mass additivity* - says that $\sum_{i=1}^N M_i(P) = M(P)$ for each $P \subset \Omega$. In terms of the densities this leads to

$$\varrho = \sum_{i=1}^N \varrho^i \,. \tag{2.3}$$

With this relation, summing (2.2) over $i = 1, \ldots, N$ and introducing the (mixture or barycentric) velocity $\mathbf{v}$ as

$$\varrho\mathbf{v} := \sum_{i=1}^N \varrho^i \mathbf{v}^i \,, \tag{2.4}$$

we obtain the mass balance equation for the mixture as a whole in the standard form

$$\partial_t \varrho + \mathrm{div}_x(\varrho\mathbf{v}) = 0 \,. \tag{2.5}$$

The *volume additivity* constraint expresses the assumption that $V(P) = \sum_{i=1}^N V_i(P)$, which implies that

$$1 = \sum_{i=1}^N \phi^i \,. \tag{2.6}$$

We shall also consider the constraint stating that *the material densities are constant*, i.e.,

$$\varrho_m^i \text{ are constants.} \tag{2.7}$$

For the sake of completeness we will finally recall the constraint of *incompressibility* associated with the mixture as a whole. It takes the form

$$\mathrm{div}_x \mathbf{v} = 0 \,. \tag{2.8}$$

For the purpose of this study, (2.8) is too restrictive.

We shall however assume that the constraints (2.3), (2.6) and (2.7) hold. We also restrict ourselves to mixtures consisting of two constituents (solvent and solute), i.e. $N = 2$. Then the constraints considered simplify to

$$\varrho = \varrho^1 + \varrho^2 \,, \quad 1 = \phi^1 + \phi^2 \,, \quad \text{and} \quad \varrho_m^1, \varrho_m^2 \text{ are constants.}$$

Setting

$$\phi := \phi^1 \quad \text{and} \quad c := \frac{\varrho^1}{\varrho}\,,$$

we rewrite (2.3) as

$$\varrho = \phi\varrho^1_m + (1-\phi)\varrho^2_m\,. \tag{2.9}$$

Consequently

$$\frac{1}{c} = \frac{\phi\varrho^1_m + (1-\phi)\varrho^2_m}{\phi\varrho^1_m}\,,$$

which implies that

$$\phi = \frac{c\varrho^2_m}{(1-c)\varrho^1_m + c\varrho^2_m}\,. \tag{2.10}$$

Inserting (2.10) into (2.9) finally leads to

$$\varrho = \tilde{\varrho}(c) := \frac{\varrho^1_m\varrho^2_m}{(1-c)\varrho^1_m + c\varrho^2_m} \quad \Longleftrightarrow \quad c = \hat{c}(\varrho) := \frac{\varrho^1_m(\varrho^2_m - \varrho)}{\varrho(\varrho^2_m - \varrho^1_m)}\,. \tag{2.11}$$

Since

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c\mathbf{v}) = \partial_t\varrho^1 + \mathrm{div}_x(\varrho^1\mathbf{v}) = \partial_t\varrho^1 + \mathrm{div}_x(\varrho^1\mathbf{v}^1) - \mathrm{div}_x(\varrho^1(\mathbf{v}^1 - \mathbf{v}))\,,$$

we observe that equations (2.2) for ϱ^1 and ϱ^2 can be equivalently replaced by the equation (2.3) for ϱ (see also (1.1)) and the equation (1.3) for c with $\mathbf{j}$ given through

$$\mathbf{j} = \varrho^1(\mathbf{v}^1 - \mathbf{v})\,. \tag{2.12}$$

Consequently, the equation (1.8) holds, and, due to (2.11), it further simplifies to

$$\mathrm{div}_x\mathbf{v} = r_*\mathrm{div}_x\mathbf{j}, \quad \text{with} \quad r_* := \frac{\varrho^1_m - \varrho^2_m}{\varrho^1_m\varrho^2_m} \in \mathbb{R}. \tag{2.13}$$

If $\varrho^1_m = \varrho^2_m$, i.e. the density of the solvent and the solute equals, we obtain $\mathrm{div}_x\mathbf{v} = 0$ and the fluid becomes homogeneous. The limit $r_* \to 0$ is studied in Section 4 rigorously for the problem specified below.

2.4 Theory of mixtures - balance equations of the simplified setting

In the simplified setting considered, the equations for ϱ and c (i.e. (2.5) and (1.3)) are completed by the balance equations for linear momentum and energy for the mixture as a whole. Then the set of governing equations takes the form (compare with (1.1)-(1.3))

$$
\begin{aligned}
\dot{\varrho} &= -\varrho \operatorname{div}_x \mathbf{v}, \\
\varrho \dot{c} &= -\operatorname{div}_x \mathbf{j}, \\
\varrho \dot{\mathbf{v}} &= \operatorname{div}_x \mathbf{T} + \varrho \mathbf{b}, \\
\mathbf{T} &= \mathbf{T}^T, \\
\varrho (e + \frac{1}{2}|\mathbf{v}|^2)^{\cdot} &= \operatorname{div}_x(\mathbf{T}\mathbf{v} - \mathbf{q}_e) + \varrho \mathbf{b} \cdot \mathbf{v},
\end{aligned}
\tag{2.14}
$$

where the notation for material time derivative is used, i.e. $\dot{z} := \partial_t z + \mathbf{v} \cdot \nabla_x z$ for any scalar quantity z.

We thus have a system of equations for $(\varrho, c, \mathbf{v}, e)$, where the diffusive flux $\mathbf{j}$, the Cauchy stress $\mathbf{T}$, which is supposed to be symmetric, and the energy flux $\mathbf{q}_e$ are determined via a thermodynamically compatible constitutive theory that follows. Note that the velocity $\mathbf{v}^1$ is not specified by the system (2.14) and, consequently, $\mathbf{j}$ cannot be determined by (2.12).

Note that by subtracting the result of a scalar multiplication of $(2.14)_3$ by $\mathbf{v}$ from $(2.14)_5$ one obtains another form of the energy balance equation, namely,

$$
\varrho \dot{e} = \mathbf{T} \cdot \mathbf{D} - \operatorname{div}_x \mathbf{q}_E \,, \tag{2.15}
$$

where

$$
\mathbf{D} = \frac{1}{2} \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) .
$$

Introducing the notation

$$
m := \frac{1}{3} \operatorname{tr} \mathbf{T}
$$

and $\mathbf{A}^d := \mathbf{A} - \frac{1}{3}(\operatorname{tr} \mathbf{A})\mathbf{I}$ for any tensor $\mathbf{A}$, the equation (2.15) takes the form

$$
\varrho \dot{e} = \mathbf{T}^d \cdot \mathbf{D}^d + m \operatorname{div}_x \mathbf{v} - \operatorname{div}_x \mathbf{q}_E \,. \tag{2.16}
$$

2.5 Specification of constitutive equations

The approach we have described (that is well presented in [17]) is based on the specification of the constitutive equation for the entropy η, which is in our situation equivalent to the constitutive equation for the internal energy, and a subsequent derivation of the 'balance equation' for the entropy. This results in an identification of individual contributions to the rate of the entropy production ζ (or the rate of dissipation ξ for the isothermal processes). Since these individual contributions are in the form of products we fulfill the requirement that the rate of dissipation is non-negative (i.e. the second law of thermodynamics holds) by imposing the simplest linear relations between the factors of individual products (with non-negative coefficients).

More specifically, starting with the assumption that the specific entropy η depends on the internal energy e, the density ϱ and the concentration c, i.e. $\eta = \tilde{\eta}(e, \varrho, c)$, and also requiring that $\frac{\partial \tilde{\eta}}{\partial e} > 0$, we obtain

$$e = \tilde{e}(\eta, \varrho, c). \tag{2.17}$$

We further introduce the thermodynamic temperature θ, the thermodynamic pressure p and the chemical potential μ:

$$\theta := \frac{\partial \tilde{e}}{\partial \eta}, \quad p := \frac{\partial \tilde{e}}{\partial \varrho}, \quad \mu := \frac{\partial \tilde{e}}{\partial c}. \tag{2.18}$$

Applying the material derivative to (2.17), and using the balance equations (2.16), $(2.14)_{1-2}$ we arrive at

$$\varrho\theta\dot{\eta} = \mathbf{T}^d : \mathbf{D}^d + (m+p)\mathrm{div}_x\mathbf{v} - \mathrm{div}_x\mathbf{q}_e + \mu\,\mathrm{div}_x\mathbf{j}\,. \tag{2.19}$$

Next, inserting (2.13) into (2.19), we obtain

$$\begin{aligned} \varrho\theta\dot{\eta} &= \mathbf{T}^d : \mathbf{D}^d - \mathrm{div}_x\mathbf{q}_e + \big(\mu + r_*(m+p)\big)\mathrm{div}_x\mathbf{j} \\ &= \mathbf{T}^d : \mathbf{D}^d + \mathrm{div}_x\Big(\big(\mu + r_*(m+p)\big)\mathbf{j} - \mathbf{q}_e\Big) - \mathbf{j}\cdot\nabla(\mu + r_*(m+p)). \end{aligned} \tag{2.20}$$

If we set $\mathbf{q}_\eta := \mathbf{q}_e - \big(\mu + r_*(m+p)\big)\mathbf{j}$, then (2.20) gives

$$\varrho\theta\dot{\eta} + \mathrm{div}_x\mathbf{q}_\eta = \mathbf{T}^d : \mathbf{D}^d - \mathbf{j}\cdot\nabla(\mu + r_*(m+p)). \tag{2.21}$$

Furthermore, dividing by $\theta > 0$ the above equation can be written as

$$\varrho\dot{\eta} + \mathrm{div}_x\left(\frac{\mathbf{q}_\eta}{\theta}\right) = \frac{1}{\theta}\left(\mathbf{T}^d : \mathbf{D}^d - \mathbf{j}\cdot\nabla_x(\mu + r_*(m+p)) - \frac{\mathbf{q}_\eta\cdot\nabla_x\theta}{\theta}\right) =: \zeta\,. \tag{2.22}$$

For the isothermal processes (i.e. $\theta = const.$), ζ leads to the following form for the rate of dissipation ξ:

$$\xi = \mathbf{T}^d : \mathbf{D}^d - \mathbf{j}\cdot\nabla_x(\mu + r_*(m+p))\,. \tag{2.23}$$

If we impose the relations

$$\mathbf{T}^d = 2\nu_*\mathbf{D}^d, \qquad \nu_* \in (0,\infty)\,, \tag{2.24}$$

$$\mathbf{j} = -\beta_*\nabla_x(\mu + r_*(m+p)), \qquad \beta_* \in (0,\infty)\,, \tag{2.25}$$

then $\xi \geq 0$ and the system of governing equations for ϱ, $\mathbf{v}$ and m takes the form

$$\begin{aligned} \dot{\varrho} &= -\varrho\,\mathrm{div}_x\mathbf{v}, \\ \varrho\dot{\mathbf{v}} &= \nabla m + \nu_*\Delta\mathbf{v} + \frac{\nu_*}{3}\nabla\mathrm{div}_x\mathbf{v} + \varrho\mathbf{b}, \\ -\Delta(m + p + r_*^{-1}\mu) &= r_*^{-2}\beta_*\mathrm{div}_x\mathbf{v}\,. \end{aligned} \tag{2.26}$$

Note that above we use (2.13) and replace $\mathrm{div}_x \mathbf{v}$ in (2.19) by $r_* \mathrm{div}_x \mathbf{j}$. We could however proceed differently and replace $\mathrm{div}_x \mathbf{j}$ by $r_*^{-1} \mathrm{div}_x \mathbf{v}$. The same procedure as above would then lead to the compressible Navier-Stokes system with the Cauchy stress tensor of the form

$$\mathbf{T} = -p\mathbf{I} - r_*^{-1} \mu \mathbf{I} + 2\nu_* \mathbf{D} + \lambda_* (\mathrm{div}_x \mathbf{v}) \mathbf{I} \, .$$

As we are primarily interested in processes driven by diffusion (and not by compression) we proceed with the system (2.26) derived above.

2.6 Boundary conditions and energy estimates

Here, we follow the goal to add to the governing equations boundary conditions that guarantee that the total energy integrated over Ω is conserved (if $\mathbf{b} = \mathbf{0}$), that do not contribute to the rate of entropy production (integrated over Ω), that do not generate any kind of surface energy on the boundary and that are simple enough for a PDE analysis. For this purpose, we require that

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad (0, T) \times \partial\Omega. \tag{2.27}$$

If $\mathbf{b} = \mathbf{0}$, we observed that the energy $\mathcal{E}(t)$ defined as

$$\mathcal{E}(t) := \int_\Omega \varrho \big(e + \frac{1}{2} |\mathbf{v}|^2 \big) dx \tag{2.28}$$

is certainly conserved if (see $(2.14)_4$)

$$\mathbf{q}_e \cdot \mathbf{n} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega. \tag{2.29}$$

Recalling the definition of $\mathbf{q}_\eta$ (see the line above (2.21)), applying the Gauss theorem to (2.22) and using (2.29) we conclude that

$$\mathbf{j} \cdot \mathbf{n} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega \, , \tag{2.30}$$

which is more than sufficient to make sure that that there is no contribution to the rate of the entropy production due to the boundary integral $\int_{\partial\Omega} \mathbf{q}_\eta \cdot \, dS$. The condition (2.30) together with (2.25) leads to

$$\frac{\partial(\mu + r_*(m + p))}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial\Omega \, . \tag{2.31}$$

The boundary conditions (2.27) and (2.31) will be associated with the PDE system (2.26) in what follows.

3 PDE analysis of a selected problem

In this and the following sections, we study the mathematical properties of the PDE system (2.26) completed with the boundary conditions (2.27) and (2.31). We note that $(2.26)_3$ together with the boundary condition (2.31) is well posed up to a constant. To ensure uniqueness of the solution to $(2.26)_3$-(2.31), we demand that the integral mean value of $m+p+r_*^{-1}\mu$ over the domain Ω is equal to 0. We then use it to calculate m and insert it into the second equation of (2.26). Thus, denoting by $(-\Delta_N)^{-1}$ the solution operator of the homogeneous Neumann problem associated to the Laplace operator and writing the standard Helmholtz decomposition in the form

$$\mathbf{z} = \mathbf{H}[\mathbf{z}] + \mathbf{H}^{\perp}[\mathbf{z}]\,, \quad \mathbf{H}^{\perp}[\mathbf{z}] := -\nabla_x(-\Delta_N)^{-1}\mathrm{div}_x\mathbf{z},$$

we can rewrite the considered problem in the following compact form:

$$\begin{aligned}
&\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{v}) = 0 \quad \text{in } (0,T)\times\Omega\,,\\
&\partial_t(\varrho\mathbf{v}) + \mathrm{div}_x(\varrho\mathbf{v}\otimes\mathbf{v}) = -\nabla(p + r_*^{-1}\mu) + \Delta\mathbf{v} + \frac{1}{3}\nabla\mathrm{div}_x\mathbf{v} - r_*^{-2}\mathbf{H}^{\perp}[\mathbf{v}]\\
&\qquad\qquad + \varrho\mathbf{b} \quad \text{in } (0,T)\times\Omega\,,\\
&\mathbf{v} = \mathbf{0} \quad \text{in } (0,T)\times\partial\Omega\,,
\end{aligned} \tag{3.1}$$

where we set for simplicity $\nu_* = \beta_* = 1$.

Since

$$c = \frac{\varrho^1}{\varrho_1+\varrho_2}, \quad \varrho(c) = \frac{\varrho_m^1\varrho_m^2}{(1-c)\varrho_m^1 + c\varrho_m^2}, \quad r_* = \frac{\varrho_m^1 - \varrho_m^2}{\varrho_m^1\varrho_m^2}, \tag{3.2}$$

we observe that $0 \le c \le 1$. Following the ansatz corresponding to the mixture of ideal gases (see [16]) it is reasonable to choose $\mu(c)$ of the form

$$\mu(c) = \frac{1}{2}\big(\ln c - \ln(1-c)\big). \tag{3.3}$$

Note that 0 and 1 are singular points. Without loss of generality, we suppose $\varrho_m^1 \ge \varrho_m^2$ and we let $\varrho_m^1 = 1$, $\varrho_m := \varrho_m^2$. Then

$$\begin{aligned}
r_* &= \frac{1-\varrho_m}{\varrho_m} \in (0,+\infty),\\
\varrho(c) &= \frac{\varrho_m}{1-(1-\varrho_m)c} = \frac{1}{(1+r_*) - r_*c},\\
c(\varrho) &= \frac{\varrho-\varrho_m}{\varrho(1-\varrho_m)} = \frac{\varrho r_* + (\varrho - 1)}{\varrho r_*},
\end{aligned} \tag{3.4}$$

and we have

$$\mu(c(\varrho)) = \ln\big((1+r_*)\varrho - 1\big) - \ln(1-\varrho). \tag{3.5}$$

3.1 A Navier-Stokes setting with singular pressure

We rewrite the system (3.1) with the external force chosen to be zero $\mathbf{b} = \mathbf{0}$ as:

$$\begin{aligned} &\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}) = 0, \\ &\partial_t(\varrho \mathbf{v}) + \mathrm{div}_x(\varrho \mathbf{v} \otimes \mathbf{v}) + \frac{1}{r_*}\nabla_x q(\varrho) = 2\mathrm{div}_x \mathbf{D}^d(\mathbf{v}) - \frac{1}{r_*^2}\mathbf{H}^\perp[\mathbf{v}], \end{aligned} \tag{3.6}$$

where

$$q(\varrho) := r_* p(\varrho) + \mu(c(\varrho)). \tag{3.7}$$

Inspired by (3.5), we suppose that the 'pressure' term $q(\varrho)$ is a C^3 and strictly increasing function on $(\frac{1}{1+r_*}, 1)$, satisfying

$$\lim_{\varrho \to \frac{1}{1+r_*}+} q(\varrho) = -\infty, \quad \lim_{\varrho \to 1-} q(\varrho) = +\infty. \tag{3.8}$$

The initial data are assumed to satisfy

$$[\varrho, \mathbf{v}](0, \cdot) = [\varrho_0, \mathbf{v}_0], \quad \frac{1}{1+r_*} < \varrho_0 < 1, \quad \mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^3), \quad \int_\Omega Q(\varrho_0) dx < \infty, \tag{3.9}$$

where the spatial domain $\Omega \subset \mathbb{R}^3$ is open and bounded with smooth boundary, and we assume the Dirichlet boundary condition:

$$\mathbf{v}|_{\partial\Omega} = 0. \tag{3.10}$$

The functional Q in (3.9) is called *pressure potential* and is defined as

$$Q(\varrho) := \varrho \int_{\varrho_*}^{\varrho} \frac{q(z)}{z^2} dz, \tag{3.11}$$

where ϱ_* is the only zero point of $q(\cdot)$, implying that $q(\varrho) < 0$ for $(1 + r_*) < \varrho < \varrho_*$, $q(\varrho_*) = 0$, and $q(\varrho) > 0$ for $\varrho_* < \varrho < 1$. It can be checked that $Q(\varrho)$ is always nonnegative and

$$Q''(\varrho) = \frac{q'(\varrho)}{\varrho}. \tag{3.12}$$

3.1.1 Growth of the pressure and global existence of weak solution

We first give the definition of global weak solution:

Definition 3.1. *[Finite energy weak solution] We call* $[\varrho, \mathbf{v}]$ *a global finite energy weak solution for problem* (3.6), (3.9) *and* (3.10)*, if for any* $T > 0$*:*

- *There holds*

$$\frac{1}{1+r_*} \le \varrho \le 1, \quad \textit{a.e. in } (0,T)\times\Omega, \quad \mathbf{v}|_{(0,T)\times\partial\Omega} = 0$$

and

$$q(\varrho) \in L^1((0,T)\times\Omega), \quad \mathbf{v} \in C_w(0,T; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)).$$

- *For any $0 \le \tau \le T$ and any test functions $\varphi \in C_c^\infty([0,T]\times\Omega)$, $\psi \in C_c^\infty([0,T]\times\Omega;\mathbb{R}^3)$, there holds*

$$\int_0^\tau \int_\Omega \varrho\partial_t\varphi + \varrho\mathbf{v}\cdot\nabla_x\varphi \,dx\,dt + \int_\Omega \varrho_0\varphi(0,\cdot)\,dx = \int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)\,dx \tag{3.13}$$

and

$$\begin{aligned}&\int_0^\tau \int_\Omega \varrho\mathbf{v}\cdot\partial_t\psi + \varrho\mathbf{v}\otimes\mathbf{v} : \nabla_x\psi + \frac{1}{r_*}q(\varrho)\mathrm{div}_x\psi \,dx\,dt + \int_\Omega \varrho_0\mathbf{v}_0\cdot\psi(0,\cdot)\,dx \\ &= \int_0^\tau \int_\Omega 2\mathbf{D}^d(\mathbf{v}) : \nabla_x\psi + \frac{1}{r_*^2}\mathbf{H}^\perp[\mathbf{v}]\cdot\psi\,dx\,dt + \int_\Omega \varrho\mathbf{v}(\tau,\cdot)\cdot\psi(\tau,\cdot)\,dx.\end{aligned} \tag{3.14}$$

- *For a.e. $\tau \in (0,T)$, there holds the energy inequality for any constant $\bar\varrho$:*

$$\begin{aligned}&\int_\Omega \left(\frac12\varrho|\mathbf{v}|^2 + \frac{1}{r_*}(Q(\varrho) - Q(\bar\varrho) - Q'(\bar\varrho)(\varrho-\bar\varrho))\right)(\tau,\cdot)\,dx \\ &\quad + \int_0^\tau\int_\Omega 2|\mathbf{D}^d(\mathbf{v})|^2 + \frac{1}{r_*^2}|\mathbf{H}^\perp[\mathbf{v}]|^2\,dx\,dt \\ &\le \int_\Omega \left(\frac12\varrho_0|\mathbf{v}_0|^2 + \frac{1}{r_*}(Q(\varrho_0) - Q(\bar\varrho) - Q'(\bar\varrho)(\varrho_0-\bar\varrho))\right)\,dx.\end{aligned} \tag{3.15}$$

Remark 3.2. *By the continuity equation $(3.6)_1$ and the complete boundary condition (3.10), there holds*

$$\int_\Omega \varrho(\tau,\cdot)\,dx = \int_\Omega \varrho_0\,dx \quad \textit{a.e. } \tau \in [0,T].$$

Then (3.15) is equivalent to

$$\begin{aligned}&\int_\Omega \left(\frac12\varrho|\mathbf{v}|^2 + \frac{1}{r_*}Q(\varrho)\right)(\tau,\cdot)\,dx + \int_0^\tau\int_\Omega 2|\mathbf{D}^d(\mathbf{v})|^2 + \frac{1}{r_*^2}|\mathbf{H}^\perp[\mathbf{v}]|^2\,dx\,dt \\ &\le \int_\Omega \left(\frac12\varrho_0|\mathbf{v}_0|^2 + \frac{1}{r_*}Q(\varrho_0)\right)\,dx.\end{aligned} \tag{3.16}$$

However, it is more convenient to use (3.15) in studying the asymptotic behavior of the solution $\mathbf{v}$ as $r_* \to 0$ (see later in Section 4).

We show that for any fixed $r_* \in (0,+\infty)$, if additionally $q(\varrho)$ enjoys some additional growth near the two critical points $(1+r_*)^{-1}$ and 1 such that

$$\liminf_{\varrho\to\frac{1}{1+r_*}+} |q(\varrho)(\varrho-\frac{1}{1+r_*})^{\beta_0}|>0, \quad \liminf_{\varrho\to 1-}|q(\varrho)(1-\varrho)^{\beta_0}|>0 \quad \text{for some } \beta_0>5/2, \tag{3.17}$$

the initial-boundary value problem (3.6), (3.9), (3.10) admits a global weak solution in the sense stated in Definition 3.1:

Theorem 3.3. *Given condition* (3.17)*, the initial-boundary value problem* (3.6)*,* (3.9)*,* (3.10) *admits global finite energy weak solution.*

3.2 Proof of Theorem 3.3

We prove Theorem 3.3 in this section by employing the idea of Feireisl and Zhang in [10]. The main difference is that, in [10] the authors assume the growth rate $\beta_0 \geq 3$ in (3.17), while here we improve this number to $\beta_0 > 5/2$. The reason that condition (3.17) is needed and how we improve the growth power from $\beta_0 \geq 3$ to $\beta_0 > 5/2$ is explained later on in Section 3.2.3.

3.2.1 Approximate solutions

We consider the following regularized pressure for $\alpha > 0$ small and $\gamma > 3/2$ large:

$$q_\alpha(\varrho) := \begin{cases} q(\frac{1}{1+r_*}+\alpha), & \varrho \leq \frac{1}{1+r_*}+\alpha, \\ q(\varrho), & \frac{1}{1+r_*}+\alpha \leq \varrho \leq 1-\alpha, \\ q(1-\alpha)+(\varrho-2)_+^\gamma, & \varrho \geq 1-\alpha. \end{cases} \tag{3.18}$$

By replacing the pressure term q by q_α in (3.6), we obtain the approximate system:

$$\begin{aligned} &\partial_t\varrho_\alpha + \mathrm{div}_x(\varrho_\alpha \mathbf{v}_\alpha) = 0, \\ &\partial_t(\varrho_\alpha\mathbf{v}_\alpha) + \mathrm{div}_x(\varrho_\alpha\mathbf{v}_\alpha\otimes\mathbf{v}_\alpha) + \frac{1}{r_*}\nabla_x q_\alpha(\varrho_\alpha) = 2\mathrm{div}_x\mathbf{D}^d(\mathbf{v}_\alpha) - \frac{1}{r_*^2}\mathbf{H}^\perp[\mathbf{v}_\alpha]. \end{aligned} \tag{3.19}$$

Armed with initial and boundary condition (3.9)-(3.10), the global existence of weak solution $[\varrho_\alpha, \mathbf{v}_\alpha]$ to this approximate system is known (see [14] and the improvement to the case $\gamma > 3/2$ in [9]). Moreover the weak solution constructed in [14] and [9] enjoys the following energy inequality:

$$\begin{aligned} &\int_\Omega \left(\frac{1}{2}\varrho_\alpha|\mathbf{v}_\alpha|^2 + \frac{1}{r_*}Q_\alpha(\varrho_\alpha)\right)(\tau,\cdot)\,dx + \int_0^\tau\int_\Omega 2|\mathbf{D}^d(\mathbf{v}_\alpha)|^2 + \frac{1}{r_*^2}|\mathbf{H}^\perp[\mathbf{v}_\alpha]|^2\,dx\,dt \\ &\leq \int_\Omega\left(\frac{1}{2}\varrho_0|\mathbf{v}_0|^2 + \frac{1}{r_*}Q_\alpha(\varrho_0)\right)\,dx, \end{aligned} \tag{3.20}$$

where

$$Q_\alpha(\varrho) := \varrho \int_{\varrho_*}^{\varrho} \frac{q_\alpha(z)}{z^2} dz \geq 0. \tag{3.21}$$

We will show that the approximate solution family $\{[\varrho_\alpha, \mathbf{v}_\alpha]\}_{\alpha>0}$ converges to a global finite energy weak solution to the initial boundary problem (3.6), (3.9) and (3.10) as $\alpha \to 0$.

By the definition of the regularized pressure q_α in (3.18) and the assumption on the initial data in (3.9), we have $Q_\alpha(\varrho_0) \nearrow Q(\varrho_0)$ in $L^1(\Omega)$, and moreover the right-hand side of the energy inequality (3.20) is uniformly bounded in α. This gives the following uniform bounds for any $T > 0$ and some $\alpha_0 > 0$ small:

$$\begin{aligned}
&\{\varrho_\alpha |\mathbf{v}_\alpha|^2\}_{0<\alpha<\alpha_0} \text{ bounded in } L^\infty(0,T;L^1(\Omega;\mathbb{R}^3)),\\
&\{Q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \text{ bounded in } L^\infty(0,T;L^1(\Omega;\mathbb{R}^3)),\\
&\{\mathbf{v}_\alpha\}_{0<\alpha<\alpha_0} \text{ bounded in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)).
\end{aligned} \tag{3.22}$$

This implies that, up to a subtraction of a subfamily,

$$\begin{aligned}
&\varrho_\alpha \to \varrho \text{ weakly(*) in } L^\infty(0,T;L^\gamma(\Omega;\mathbb{R}^3)),\\
&\mathbf{v}_\alpha \to \mathbf{v} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)).
\end{aligned} \tag{3.23}$$

Moreover, by the assumption on the growth of q in (3.17), the uniform bound in $Q_\alpha(\varrho_\alpha)$ in (3.22) gives

$$\frac{1}{1+r_*} \leq \varrho \leq 1 \quad \text{a.e. in } [0,T]\times\Omega. \tag{3.24}$$

Following the classical result in [14, see Theorem 5.1], one has the following convergence results (at least in the sense of distribution):

$$\varrho_\alpha \mathbf{v}_\alpha \to \varrho \mathbf{v}, \quad \varrho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha \to \varrho \mathbf{v} \otimes \mathbf{v}, \tag{3.25}$$

and one has the following energy inequality by letting $\alpha \to 0$ in (3.20):

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{v}|^2 + \frac{1}{r_*}Q(\varrho)\right)(\tau,\cdot)\,dx + \int_0^\tau \int_\Omega 2|\mathbf{D}^d(\mathbf{v})|^2 + \frac{1}{r_*^2}|\mathbf{H}^\perp[\mathbf{v}]|^2\,dx\,dt\\
&\leq \int_\Omega \left(\frac{1}{2}\varrho_0|\mathbf{v}_0|^2 + \frac{1}{r_*}Q(\varrho_0)\right)dx.
\end{aligned} \tag{3.26}$$

By (3.23) and (3.25), to show the limit $[\varrho, \mathbf{v}]$ satisfies the weak formulation of (3.6), which are (3.13) and (3.14), it suffices to show the weak convergence of the pressure family:

$$q_\alpha(\varrho_\alpha) \to q(\varrho) \text{ weakly in } L^1((0,T)\times\Omega)). \tag{3.27}$$

This is done in the following two subsections.

3.2.2 Uniform bounds for the pressure

As observed in [10], uniform bound $(3.22)_2$ does not generally imply any uniform bounds for $\{q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0}$, not even in $L^1((0,T)\times\Omega)$. Indeed, for singular $q(\varrho)$ such that

$$
\begin{aligned}
q(\varrho) &= O\Big(\frac{1}{(\varrho-\frac{1}{1+r_*})^{\beta_1}}\Big) \quad \text{near} \quad \frac{1}{1+r_*}+, \\
q(\varrho) &= O\Big(\frac{1}{(1-\varrho)^{\beta_2}}\Big) \quad \text{near} \quad 1-,
\end{aligned} \tag{3.28}
$$

where β_1 and β_2 are numbers larger than $5/2$, the pressure potential functional Q defined by (3.11) is less singular:

$$
\begin{aligned}
Q(\varrho) &= O\Big(\frac{1}{(\varrho-\frac{1}{1+r_*})^{\beta_1-1}}\Big) \quad \text{near} \quad \frac{1}{1+r_*}+, \\
Q(\varrho) &= O\Big(\frac{1}{(1-\varrho)^{\beta_2-1}}\Big) \quad \text{near} \quad 1-\,.
\end{aligned} \tag{3.29}
$$

To show the integrability of $q_\alpha(\varrho_\alpha)$, we introduce the *test function*

$$
\varphi = \psi(t)\mathcal{B}(\varrho_\alpha - \langle\varrho_\alpha\rangle), \quad \langle\varrho_\alpha\rangle := \frac{1}{|\Omega|}\int_\Omega \varrho_\alpha \, dx,
$$

with $\psi \in C_c^\infty(0,T)$ and $\mathcal{B}$ a bounded linear operator from $\{g \in L^p(\Omega), \quad \langle g\rangle = 0\}$ to $W_0^{1,p}(\Omega;\mathbb{R}^3)$ for $1<p<\infty$ such that

$$
\mathrm{div}_x \mathcal{B}(g) = g, \quad \mathcal{B}(g)|_{\partial\Omega} = 0.
$$

The existence of such an operator $\mathcal{B}$ is proven in Galdi [4] (see Chapter 3).

By taking φ as a test function in the weak formulation of $(3.19)_2$, by the same argument as in Section 3.4 of [10], we obtain that

$$
\{q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \text{ bounded in } L^1((0,T)\times\Omega). \tag{3.30}
$$

3.2.3 Equi-integrability of the pressure

A bounded family in L^1 is weakly pre-compact in $\mathcal{M}$, which is the space of bounded measures, but is not necessarily weakly pre-compact in L^1 itself. By Dunford-Pettis' Theorem, to show weak compactness of $\{q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0}$ in $L^1((0,T)\times\Omega)$, one still needs to show its equi-integrability, which means that

$$
\forall\varepsilon>0, \quad \exists M>0, \quad \text{s.t.} \quad \sup_{0<\alpha<\alpha_0}\int_{|q_\alpha(\varrho_\alpha)|\geq M} |q_\alpha(\varrho_\alpha)| \, dx\, dt \leq \varepsilon.
$$

This is where we need condition (3.17).

We introduce a new test function

$$\varphi = \psi(t)\mathcal{B}(\eta_\alpha(\varrho_\alpha) - \langle\eta_\alpha(\varrho_\alpha)\rangle),$$

where $\psi \in C_c^\infty(0,T)$ and

$$\eta_\alpha(s) := \begin{cases} \log\big(s - \frac{1}{1+r*}\big) - \log(1-s), & \frac{1}{1+r_*} + \alpha \le s \le 1-\alpha, \\ \log\big(1-\alpha - \frac{1}{1+r*}\big) - \log(\alpha), & s \ge 1-\alpha, \\ \log(\alpha) - \log\big(1-\alpha-\frac{1}{1+r*}\big), & s \le \frac{1}{1+r_*} + \alpha. \end{cases}$$

The idea is to test $(3.19)_2$ by φ and to show a uniform bound of the following quantity:

$$\{q_\alpha(\varrho_\alpha)\eta_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^1((0,T)\times\Omega).$$

This implies the equi-integrability of $\{q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0}$.

As observed in Section 3.5 of [10], to show the equi-integrability of $\{q_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0}$, the most difficult part is to show that the following term is uniformly bounded:

$$I := \int_0^T \int_\Omega \psi\varrho_\alpha \mathbf{v}_\alpha \mathcal{B}\left(\eta'_\alpha(\varrho_\alpha)\varrho_\alpha \mathrm{div}_x\mathbf{v}_\alpha - \langle\eta'_\alpha(\varrho_\alpha)\varrho_\alpha\mathrm{div}_x\mathbf{v}_\alpha\rangle\right)\,dx\,dt. \tag{3.31}$$

In this paper, we treat this term in a different way compared to [10], and this allows us to improve the growth rate β_0 in (3.17) from $\beta_0 \ge 3$ to $\beta_0 > 5/2$. In [10], the authors use the integrability of $Q_\alpha(\varrho_\alpha)$ to gain some integrability of $\eta'(\varrho_\alpha)$. Indeed, by (3.17) and the observation from (3.28) to (3.29), one can get that

$$Q_\alpha(\varrho_\alpha) \ge \frac{c_1}{|\varrho_\alpha - \frac{1}{1+r_*}|^{\beta_0-1}} + \frac{c_1}{|1-\varrho_\alpha|^{\beta_0-1}} - c_2, \quad \text{for all} \quad \frac{1}{1+r_*} + \alpha \le \varrho_\alpha \le 1-\alpha$$

for some positive constants c_1 and c_2. This implies

$$Q_\alpha(\varrho_\alpha) \ge c_1|\eta'_\alpha(\varrho_\alpha)|^{\beta_0-1} - c_2, \quad \text{for all} \quad \varrho_\alpha \ge 0.$$

By the uniform bound $(3.22)_2$, one has

$$\{\eta'_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^\infty(0,T;L^{\beta_0-1}(\Omega)). \tag{3.32}$$

Here instead of using the uniform bound of $Q_\alpha(\varrho_\alpha)$, we use the uniform bound of $q_\alpha(\varrho_\alpha)$ obtained in (3.30) to handle $\eta'_\alpha(\varrho_\alpha)$. By (3.17), it is straightforward to obtain

$$|q_\alpha(\varrho_\alpha)| \ge \frac{c_1}{|\varrho_\alpha - \frac{1}{1+r_*}|^{\beta_0}} + \frac{c_1}{|1-\varrho_\alpha|^{\beta_0}} - c_2 \quad \text{for all} \quad \frac{1}{1+r_*} + \alpha \le \varrho_\alpha \le 1-\alpha$$

and

$$|q_\alpha(\varrho_\alpha)| \geq c_1 |\eta'_\alpha(\varrho_\alpha)|^{\beta_0} - c_2 \quad \text{for all} \quad \varrho_\alpha \geq 0.$$

Then

$$\{\eta'_\alpha(\varrho_\alpha)\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^{\beta_0}(0,T;L^{\beta_0}(\Omega)). \tag{3.33}$$

Compared to (3.32), in (3.33) we gain more integrability with respect to the spatial variable, from L^{β_0-1} to L^{β_0}. The drawback is also obvious: we lose integrability respect to the time variable. However, we observe that γ in (3.18) can be taken arbitrarily large such that ϱ_α is in $L^\infty(0,T;L^{\infty-}(\Omega))$ where $\infty-$ denotes any positive number, and this allows us show the uniform boundedness of I in (3.31).

To make the index in using the interpolation property and in using Hölder's inequality clear, we let $r+$ denote any number in the interval $(r, r+\varepsilon_1)$ and $r-$ denote any number in the interval $(r-\varepsilon_1, r)$ for some small $\varepsilon_1 > 0$. For simplicity, we use $L^p(L^q)$ to denote function space $L^p(0,T;,L^q(\Omega;\mathbb{R}^3))$ or $L^p(0,T;L^q(\Omega))$, and so on.

By $(3.22)_3$, (3.33) with $\beta_0 = \frac{5}{2}+$ and $\varrho_\alpha \in L^\infty(0,T;L^{\infty-}(\Omega))$, we have

$$\{\eta'_\alpha(\varrho_\alpha)\varrho_\alpha \mathrm{div}_x \mathbf{v}_\alpha\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^{\frac{10}{9}+}(L^{\frac{10}{9}+}).$$

Then

$$\{\mathcal{B}\left(\eta'_\alpha(\varrho_\alpha)\varrho_\alpha \mathrm{div}_x \mathbf{v}_\alpha - \langle \eta'_\alpha(\varrho_\alpha)\varrho_\alpha \mathrm{div}_x \mathbf{v}_\alpha\rangle\right)\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^{\frac{10}{9}+}(W_0^{1,\frac{10}{9}+}).$$

Sobolev embedding gives

$$\{\mathcal{B}\left(\eta'_\alpha(\varrho_\alpha)\varrho_\alpha \mathrm{div}_x \mathbf{v}_\alpha - \langle \eta'_\alpha(\varrho_\alpha)\varrho_\alpha \mathrm{div}_x \mathbf{v}_\alpha\rangle\right)\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^{\frac{10}{9}+}(L^{\frac{30}{17}+}). \tag{3.34}$$

One the other hand, by $(3.22)_1$, $(3.22)_2$ and $\varrho_\alpha \in L^\infty(0,T;L^{\infty-}(\Omega))$, together with Sobolev embedding, we have

$$\{\varrho_\alpha \mathbf{v}_\alpha\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^\infty(L^{2-}) \cap L^2(L^{6-}).$$

By interpolation, we obtain

$$\{\varrho_\alpha \mathbf{v}_\alpha\}_{0<\alpha<\alpha_0} \quad \text{bounded in} \quad L^{10}(L^{\frac{30}{13}-}). \tag{3.35}$$

Uniform bounds in (3.34) and (3.35) imply that I is uniformly bounded. Then we can employ the argument in [10] to prove that

$$q_\alpha(\varrho_\alpha) \to \overline{q(\varrho)} \text{ weakly in } L^1((0,T)\times\Omega)). \tag{3.36}$$

3.2.4 Strong convergence of density and conclusion of the proof

From (3.36) to (3.27), it is left to show that $\overline{q(\varrho)} = q(\varrho)$. The proof of this point is exactly as in Section 3.6 in [10], where it was shown that $\overline{\varrho\log\varrho} = \varrho\log\varrho$ and furthermore that $\varrho_\alpha \to \varrho$ a.a in $(0,T)\times\Omega$. This completes the proof of Theorem 3.3.

4 From quasi-incompressibility to incompressibility

We study the singular limit $r_* = \varepsilon \to 0$. We keep the subscript ε accordingly as we denote by $[\varrho_\varepsilon, \mathbf{v}_\varepsilon]$ the solution obtained in Theorem 3.3 and by $[\varrho_{0,\varepsilon}, \mathbf{v}_{0,\varepsilon}]$ the associated initial datum.

Let $\mathbf{H}$ denote the standard Helmholtz projection onto the space of solenoidal functions. Under these circumstances, the energy inequality reads

$$
\begin{aligned}
&\int_\Omega \left[\frac{1}{2}\varrho_\varepsilon |\mathbf{v}_\varepsilon|^2 + \frac{1}{\varepsilon}\big(Q(\varrho_\varepsilon) - Q'(\overline{\varrho}_\varepsilon)(\varrho_\varepsilon - \overline{\varrho}_\varepsilon) - Q(\overline{\varrho}_\varepsilon)\big)\right](\tau,\cdot)\ dx \\
&\qquad + \int_0^\tau \int_\Omega \left(2|\mathbf{D}^d(\mathbf{v}_\varepsilon)|^2 + \frac{1}{\varepsilon^2}|\mathbf{H}^\perp[\mathbf{v}_\varepsilon]|^2 \right)\ dx\ dt \\
&\leq \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{v}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon}\big(Q(\varrho_{0,\varepsilon}) - Q'(\overline{\varrho}_\varepsilon)(\varrho_{0,\varepsilon} - \overline{\varrho}_{0,\varepsilon}) - Q(\overline{\varrho}_\varepsilon)\big)\right]\ dx,
\end{aligned} \tag{4.1}
$$

where we have set

$$
\overline{\varrho}_\varepsilon = \frac{1}{|\Omega|}\int_\Omega \varrho_{0,\varepsilon}\ dx. \tag{4.2}
$$

The first observation is that, unconditionally,

$$
\sup_{t\in(0,T)} \|\varrho_\varepsilon(\tau,\cdot) - 1\|_{L^\infty(\Omega)} \leq \varepsilon. \tag{4.3}
$$

In order to control the velocity fields, we have to keep the right-hand side of (4.1) bounded uniformly as $\varepsilon \to 0$. To this end, we suppose that

$$
\|\mathbf{v}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \tag{4.4}
$$

and

$$
\frac{1}{|\Omega|}\int_\Omega Q(\varrho_{0,\varepsilon})\ dx \leq Q(\overline{\varrho}_\varepsilon) + \varepsilon c. \tag{4.5}
$$

It follows from the energy inequality (4.1) that, at least for a suitable subsequence,

(4.6)

$$
\mathbf{v}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)) \text{ and weakly-* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)),
$$

and, moreover,

$$
\mathbf{H}^\perp[\mathbf{v}_\varepsilon] \to 0 \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{4.7}
$$

Finally, we take a solenoidal test function in the momentum balance to deduce that

$$
t \mapsto \int_\Omega \mathbf{v}_\varepsilon \cdot \varphi\ dx \text{ is precompact in } C([0,T]) \text{ for any } \varphi \in C_c^1(\Omega;\mathbb{R}^3),\ \mathrm{div}\,\varphi = 0; \tag{4.8}
$$

whence, by means of the standard Lions-Aubin argument,

$$\mathbf{H}[\mathbf{v}_\varepsilon] \to \mathbf{U} \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{4.9}$$

and, consequently,

$$\mathbf{v}_\varepsilon \to \mathbf{U} \text{ in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{4.10}$$

where $\mathbf{U}$ is a *weak* solution to the incompressible Navier-Stokes system

$$\mathrm{div}\mathbf{U} = 0, \tag{4.11}$$

$$\partial_t \mathbf{U} + \nabla\mathbf{U}\cdot\mathbf{U} + \nabla P = \Delta\mathbf{U},\ \mathbf{U}|_{\partial\Omega} = 0, \tag{4.12}$$

supplemented with the initial condition

$$\mathbf{U}(0,\cdot) = \mathbf{H}[\mathbf{v}_0], \tag{4.13}$$

where $\mathbf{v}_0$ is a weak limit of $\mathbf{v}_{0,\varepsilon}$ in $L^2(\Omega;\mathbb{R}^3)$.

Thus, under very mild and almost necessary restrictions (4.4), (4.5) imposed on the initial data, the solutions approach, in the singular regime $r_* \to 0$, a weak solution of the incompressible Navier-Stokes system.

In the next section, for pressures $q(\varrho)$ which take the explicit form

$$q(\varrho) = \frac{-a_1}{(\varrho - \frac{1}{1+\varepsilon})^{\beta_1}} + \frac{a_2}{(1-\varrho)^{\beta_2}}, \quad a_j > 0,\ \beta_j > 5/2,\ j = 1,2, \tag{4.14}$$

we exhibit a necessary and sufficient condition on the initial density (see (4.24)) such that the mild assumption (4.5) is satisfied; we give in addition the rate of convergence $\mathbf{v}_\varepsilon \to \mathbf{U}$ in the case of well-prepared initial data, for which the initial velocity fields converge strongly to the initial state of the limit system, the latter possessing a (unique) smooth solution $\mathbf{U}$ defined on a maximal time interval. We should mention that the argument applies to many other choices of $q(\varrho)$, as long as the explicit growth rates of $q(\varrho)$ in the two critical points 1 and $1/(1+\varepsilon)$ is known.

4.1 The singular limit $\varepsilon \to 0$ and well-prepared initial data

Under the assumption (4.14), we will exhibit a necessary and sufficient condition on the initial density such that (4.5) is satisfied, and furthermore the right-hand side of (4.1) is uniformly bounded. Moreover, we give quantitative convergence results from quasi-incompressibility to incompressibility.

Initial density is assumed to be in $(\frac{1}{1+\varepsilon}, 1)$ (see (3.9)), and this implies for ε small:

$$\varrho_{0,\varepsilon}(x) = 1 - \varepsilon\varrho_{0,\varepsilon}^{(1)}(x), \quad \varrho_{0,\varepsilon}^{(1)}(x) \in (0,1) \text{ for all } x \in \Omega. \tag{4.15}$$

Here we focus on the case where there holds

$$\liminf_{\varepsilon\to 0}\inf_{x\in\Omega}\varrho_{0,\varepsilon}^{(1)}(x)\in(0,1),\quad \limsup_{\varepsilon\to 0}\sup_{x\in\Omega}\varrho_{0,\varepsilon}^{(1)}(x)\in(0,1). \tag{4.16}$$

Remark that the above conditions in (4.16) are equivalent to

$$\inf_{0<\varepsilon<\varepsilon_0}\inf_{x\in\Omega}\varrho_{0,\varepsilon}^{(1)}(x)\in(0,1),\quad \sup_{0<\varepsilon<\varepsilon_0}\sup_{x\in\Omega}\varrho_{0,\varepsilon}^{(1)}(x)\in(0,1),\ \text{for some } \varepsilon_0>0. \tag{4.17}$$

Then

$$\bar\varrho_\varepsilon=\frac{1}{|\Omega|}\int_\Omega \varrho_{0,\varepsilon}\,dx=1-\varepsilon\overline{\varrho_\varepsilon^{(1)}},\quad \overline{\varrho_\varepsilon^{(1)}}:=\frac{1}{|\Omega|}\int_\Omega\varrho_{0,\varepsilon}^{(1)}\,dx.$$

We calculate by using Taylor's formula,

$$\begin{aligned}
&\frac{1}{\varepsilon}\big(Q(\varrho_{0,\varepsilon})-Q'(\bar\varrho_\varepsilon)(\varrho_{0,\varepsilon}-\bar\varrho_\varepsilon)-Q(\bar\varrho_\varepsilon)\big)\\
&\quad=\frac{1}{\varepsilon}Q''\left(\bar\varrho_\varepsilon+\varepsilon\theta\big(\overline{\varrho_\varepsilon^{(1)}}-\varrho_\varepsilon^{(1)}\big)\right)(\varrho_{0,\varepsilon}-\bar\varrho_\varepsilon)^2\\
&\quad=\varepsilon Q''\left(1-\varepsilon\big[(1-\theta)\overline{\varrho_\varepsilon^{(1)}}+\theta\varrho_\varepsilon^{(1)}\big]\right)\left(\overline{\varrho_\varepsilon^{(1)}}-\varrho_{0,\varepsilon}^{(1)}\right)^2,
\end{aligned} \tag{4.18}$$

where $\theta\in(0,1)$. Let $\varrho_{\varepsilon,\theta}:=(1-\theta)\overline{\varrho_\varepsilon^{(1)}}+\theta\varrho_\varepsilon^{(1)}\in(0,1)$. Then by (3.12) and (4.14), we have

(4.19)

$$\begin{aligned}
Q''(1-\varepsilon\varrho_{\varepsilon,\theta})=\frac{q'(1-\varepsilon\varrho_{\varepsilon,\theta})}{1-\varepsilon\varrho_{\varepsilon,\theta}}&=\frac{1}{1-\varepsilon\varrho_{\varepsilon,\theta}}\left(\frac{a_1\beta_1}{(1-\varepsilon\varrho_{\varepsilon,\theta}-\frac{1}{1+\varepsilon})^{\beta_1+1}}+\frac{a_2\beta_2}{(\varepsilon\varrho_{\varepsilon,\theta})^{\beta_2+1}}\right)\\
&=\frac{1}{1-\varepsilon\varrho_{\varepsilon,\theta}}\left(\frac{1}{\varepsilon^{(\beta_1+1)}}\frac{a_1\beta_1(1+\varepsilon)^{\beta_1+1}}{(1-\varrho_{\varepsilon,\theta}-\varepsilon\varrho_{\varepsilon,\theta})^{\beta_1+1}}+\frac{1}{\varepsilon^{(\beta_2+1)}}\frac{a_2\beta_2}{(\varrho_{\varepsilon,\theta})^{\beta_2+1}}\right).
\end{aligned}$$

Then

$$\frac{1}{\varepsilon}\big(Q(\varrho_{0,\varepsilon})-Q'(\bar\varrho_\varepsilon)(\varrho_{0,\varepsilon}-\bar\varrho_\varepsilon)-Q(\bar\varrho_\varepsilon)\big)=R_\varepsilon\left(\overline{\varrho_\varepsilon^{(1)}}-\varrho_{0,\varepsilon}^{(1)}\right)^2, \tag{4.20}$$

where

$$R_\varepsilon:=\frac{1}{1-\varepsilon\varrho_{\varepsilon,\theta}}\left(\frac{1}{\varepsilon^{\beta_1}}\frac{a_1\beta_1(1+\varepsilon)^{\beta_1+1}}{(1-\varrho_{\varepsilon,\theta}-\varepsilon\varrho_{\varepsilon,\theta})^{\beta_1+1}}+\frac{1}{\varepsilon^{\beta_2}}\frac{a_2\beta_2}{(\varrho_{\varepsilon,\theta})^{\beta_2+1}}\right). \tag{4.21}$$

Condition (4.16) implies for some constant $C>0$ independent of ε that

$$C^{-1}\varepsilon^{-\beta_0}\le R_\varepsilon\le C\varepsilon^{-\beta_0}\quad \beta_0:=\max\{\beta_1,\beta_2\}.$$

To keep the right-hand side of (4.1) bounded uniformly for $\varepsilon\to 0$, the initial density in (4.15) must satisfy the following condition:

$$\overline{\varrho_\varepsilon^{(1)}}-\varrho_{0,\varepsilon}^{(1)}=\varepsilon^{\beta_0/2}\varrho_{0,\varepsilon}^{(2)}, \tag{4.22}$$

where

$$\limsup_{\varepsilon\to 0}\|\varrho_{0,\varepsilon}^{(2)}\|_{L^2} < +\infty \Longleftrightarrow \sup_{0<\varepsilon<\varepsilon_0}\|\varrho_{0,\varepsilon}^{(2)}\|_{L^2} < +\infty \text{ for some } \varepsilon_0 > 0. \tag{4.23}$$

Without loss of generality, here ε_0 is taken to be the same as in (4.17).

Then under the condition (4.22) and (4.23), the right-hand side of (4.1) is uniformly bounded, which is equivalent to assumption (4.5). We deduced the *sufficient* and *necessary* condition on $\varrho_{0,\varepsilon}$ such that (4.5) is satisfied by combining (4.15), (4.22) and (4.23) as we write:

$$\varrho_{0,\varepsilon} = 1 - \varepsilon\,\overline{\varrho_\varepsilon^{(1)}} + \varepsilon^{\beta_0/2+1}\varrho_{0,\varepsilon}^{(2)}, \quad \sup_{0<\varepsilon<\varepsilon_0}\|\varrho_{0,\varepsilon}^{(2)}\|_{L^2} < +\infty \text{ for some } \varepsilon_0 > 0. \tag{4.24}$$

Accordingly, we write ϱ_ε as

$$\varrho_\varepsilon = 1 - \varepsilon\,\overline{\varrho_\varepsilon^{(1)}} + \varepsilon^{\beta_0/2+1}\varrho_\varepsilon^{(2)}, \quad \varrho_\varepsilon^{(2)} := \varepsilon^{-\beta_0/2-1}\big(\varrho_\varepsilon - (1-\varepsilon\,\overline{\varrho_\varepsilon^{(1)}})\big). \tag{4.25}$$

We suppose that (4.4) is also satisfied and we define

$$d := \sup_{0<\varepsilon<\varepsilon_0}(\|\mathbf{v}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\varrho_{0,\varepsilon}^{(2)}\|_{L^2(\Omega)}) < \infty.$$

Then from (4.1), we obtain the uniform bounds for any $T>0$ and $0<\varepsilon<\varepsilon_0$:

$$\begin{aligned} &\operatorname*{ess\,sup}_{\tau\in(0,T)}\|\mathbf{v}_\varepsilon(\tau,\cdot)\|_{L^2(\Omega;\mathbb{R}^3)} + \operatorname*{ess\,sup}_{\tau\in(0,T)}\left\|\varrho_\varepsilon^{(2)}(\tau,\cdot)\right\|_{L^2(\Omega)} \le C(d,T),\\ &\int_0^\tau\int_\Omega 2|\mathbf{D}^d(\mathbf{v}_\varepsilon)|^2 + \frac{1}{\varepsilon^2}|\mathbf{H}^\perp[\mathbf{v}_\varepsilon]|^2\,dx\,dt \le C(d,T). \end{aligned} \tag{4.26}$$

By Korn's inequality, we have as $\varepsilon\to 0$,

$$\begin{aligned} &\varrho_\varepsilon^{(2)} \to \varrho^{(2)} \quad \text{weakly-(*) in} \quad L^\infty(0,T;L^2(\Omega),\\ &\mathbf{v}_\varepsilon \to \mathbf{U} \quad \text{weakly-(*) in} \quad L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),\\ &\|\mathbf{H}^\perp[\mathbf{v}_\varepsilon]\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le C(d,T)\varepsilon, \text{ for } 0<\varepsilon<\varepsilon_0. \end{aligned} \tag{4.27}$$

As regards the limit density $\varrho^{(2)}$, by multiplying $\varepsilon^{-\beta_0/2-1}$ in (3.14) and letting $\varepsilon\to 0$, we obtain

$$\nabla_x\varrho^{(2)} = 0, \quad \varrho^{(2)} = \bar{\varrho}_2 \text{ a constant.} \tag{4.28}$$

Now we summarize the results obtained above and give a quantitative convergence result for (4.27) provided the limit solution $\mathbf{U}$ is regular enough.

Theorem 4.1. *Let $[\varrho_\varepsilon, \mathbf{v}_\varepsilon]$ be a finite energy weak solution to* (3.6)*,* (3.9)*,* (3.10) *and* (4.14) *for which* (4.4) *and* (4.5) *are satisfied. Then under condition* (4.16)*, the initial density satisfies* (4.24)*, and there holds* (4.27) *with $\varrho^{(2)} = \bar{\varrho}_2$ a constant and $\mathbf{U}$ a solution to the incompressible Navier-Stokes equation* (4.11)*,* (4.12) *and* (4.13)*. If moreover*

$$\nabla_x \mathbf{U} \in L^1(0,T;L^\infty(\Omega)), \quad \nabla_x P \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.29}$$

then there holds for $0 < \varepsilon < \varepsilon_0$:

$$\begin{aligned} \|\mathbf{v}_\varepsilon - \mathbf{U}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)} + \|\nabla_x(\mathbf{v}_\varepsilon - \mathbf{U})\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} + \left\|\varrho_\varepsilon^{(2)} - \bar{\varrho}_2\right\|^2_{L^\infty(0,T;L^2(\Omega)} \\ \leq C(D,T)\left(\|\mathbf{v}_{0,\varepsilon} - \mathbf{H}[\mathbf{v}_0]\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\varrho^{(2)}_{0,\varepsilon} - \bar{\varrho}_2\|^2_{L^2(\Omega)} + \varepsilon\right). \end{aligned} \tag{4.30}$$

The constant $C(D,T)$ is only dependent on T and D defined as

$$\begin{aligned} D := \sup_{0<\varepsilon<\varepsilon_0} &(\|\varrho^{(2)}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{v}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)}) \\ &+ \|\nabla_x \mathbf{U}\|_{L^1(0,T;L^\infty(\Omega;\mathbb{R}^{3\times 3}))} + \|\nabla_x P\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}. \end{aligned}$$

Remark 4.2. *Condition* (4.29) *can be satisfied provided the initial data $\mathbf{U}(0,\cdot)$ is regular, for example is in $W^{2,2}_0(\Omega;\mathbb{R}^3)$. In this case, there is a $T > 0$ such that* (4.11)*-*(4.12) *admit a unique solution in the space $L^\infty(0,T;W^{2,2}_0(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W^{3,2}_0(\Omega;\mathbb{R}^3))$. One can then deduce that $\nabla_x P \in L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and $\nabla_x \mathbf{U} \in L^2(0,T;W^{2,2}_0(\Omega;\mathbb{R}^{3\times 3})) \subset L^1(0,T;L^\infty(\Omega;\mathbb{R}^{3\times 3}))$.*

The next section is devoted to proving this theorem.

4.2 Proof of Theorem 4.1

To prove Theorem 4.1, it is left to show (4.30). This is done by using a relative entropy inequality.

4.2.1 Relative entropy inequality

It is convenient to use the relative entropy inequality in studying the asymptotic behavior of finite energy weak solutions as $\varepsilon \to 0$. For any finite energy weak solution $(\varrho_\varepsilon, \mathbf{v}_\varepsilon)$, the *relative entropy* is defined as

$$\mathcal{E}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon | r, \mathbf{v}\right) := \int_\Omega \left(\frac{1}{2}\varrho_\varepsilon|\mathbf{v}_\varepsilon - \mathbf{v}|^2 + \frac{1}{\varepsilon}\big(Q(\varrho_\varepsilon) - Q'(r)(\varrho_\varepsilon - r) - Q(r)\big)\right) dx, \tag{4.31}$$

where the *test function couple* $(r, \mathbf{v})$ is sufficiently smooth and is in the class

$$\mathbf{v}|_{\partial\Omega} = 0, \quad r > 0, \quad \mathbf{v} \text{ and } (r - \bar{\varrho}) \text{ are compactly supported in } \overline{\Omega}. \tag{4.32}$$

It can be shown, as in [6], that for any finite energy weak solution $(\varrho_\varepsilon, \mathbf{v}_\varepsilon)$ and any smooth test function $(r, \mathbf{v})$ satisfying (4.32), the following *relative entropy inequality* holds

$$
\begin{aligned}
&\mathcal{E}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon | r, \mathbf{v}\right)(\tau) + 2\int_0^\tau \int_\Omega \Big(\mathbf{D}^d(\mathbf{v}_\varepsilon) - \mathbf{D}^d(\mathbf{v})\Big) : \left(\nabla_x \mathbf{v}_\varepsilon - \nabla_x \mathbf{v}\right)\, dx\, dt \\
&\qquad + \frac{1}{\varepsilon^2}\int_\Omega |\mathbf{H}^\perp[\mathbf{v}_\varepsilon - \mathbf{v}]|^2\, dx. \\
&\leq \mathcal{E}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon | r, \mathbf{v}\right)(0) + \int_0^\tau \mathcal{R}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, r, \mathbf{v})\, dt,
\end{aligned} \tag{4.33}
$$

where

$$
\begin{aligned}
&\mathcal{R}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, r, \mathbf{v}) := 2\int_\Omega \mathbf{D}^d(\mathbf{v}) : \left(\nabla_x \mathbf{v} - \nabla_x \mathbf{v}_\varepsilon\right)\, dx \\
&+ \int_\Omega \varrho_\varepsilon\left(\partial_t \mathbf{v} + \mathbf{v}_\varepsilon \cdot \nabla_x \mathbf{v}\right) \cdot (\mathbf{v} - \mathbf{v}_\varepsilon)\, dx + \frac{1}{\varepsilon^2}\int_\Omega \mathbf{H}^\perp[\mathbf{v}] \cdot \left(\mathbf{v} - \mathbf{v}_\varepsilon\right)\, dx \\
&+ \frac{1}{\varepsilon}\int_\Omega \frac{r - \varrho_\varepsilon}{r} q'(r)\partial_t r + \frac{r\mathbf{v} - \varrho_\varepsilon \mathbf{v}_\varepsilon}{r} \cdot q'(r)\nabla_x r\, dx \\
&+ \frac{1}{\varepsilon}\int_\Omega \operatorname{div}_x \mathbf{v}\big(q(r) - q(\varrho_\varepsilon)\big)\, dx.
\end{aligned} \tag{4.34}
$$

It can be shown that (4.33) becomes (3.15) by choosing $r = \bar{\varrho}, \mathbf{v} = 0$. We remark that the regularity assumption on test function $(r, \mathbf{v})$ can be generalized to the class for which (4.31) and (4.33) make sense. In our argument, the test function will be chosen as the solution of the limit equation.

4.2.2 Test function as limit solution

To prove Theorem 4.1, we choose the test function in the relative entropy inequality (4.33) as the limit solution

$$
r = 1 - \varepsilon\, \overline{\varrho_\varepsilon^{(1)}} + \varepsilon^{\beta_0/2+1}\bar{\varrho}_2, \qquad \mathbf{v} = \mathbf{U}. \tag{4.35}
$$

Then

$$
\begin{aligned}
\int_0^\tau \mathcal{R}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, r, \mathbf{U})\, dt &= 2\int_0^\tau \int_\Omega \mathbf{D}^d(\mathbf{U}) : \left(\nabla_x \mathbf{U} - \nabla_x \mathbf{v}_\varepsilon\right)\, dx\, dt \\
&\quad + \int_0^\tau \int_\Omega \varrho_\varepsilon\left(\partial_t \mathbf{U} + \mathbf{v}_\varepsilon \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{v}_\varepsilon)\, dx\, dt.
\end{aligned} \tag{4.36}
$$

It can be checked that all the terms on the right-hand side of (4.36) are well defined. Moreover, we will show that the terms on the right-hand side of (4.36) are either small of order ε, or can be absorbed into the left hand side of entropy inequality (4.33) or the integral of the entropy functional with respect to the variable t.

4.2.3 Initial data

By (4.13), (4.24) and (4.20), we have

$$\mathcal{E}\left(\varrho_\varepsilon, \mathbf{v}_\varepsilon | r, \mathbf{U}\right)(0) \leq C\big(\|\mathbf{v}_{0,\varepsilon} - \mathbf{H}[\mathbf{v}_0]\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\varrho^{(2)}_{0,\varepsilon} - \bar{\varrho}_2\|^2_{L^2(\Omega)}\big). \tag{4.37}$$

4.2.4 The remainder

In this section, we estimate the remainder in (4.36).

Since $\frac{1}{1+\varepsilon} \leq \varrho_\varepsilon \leq 1$, then

$$\begin{aligned}
&\int_0^\tau \int_\Omega \varrho_\varepsilon \left(\partial_t \mathbf{U} + \mathbf{v}_\varepsilon \cdot \nabla_x \mathbf{U}\right) \cdot \left(\mathbf{U} - \mathbf{v}_\varepsilon\right) dx\, dt \\
&= \int_0^\tau \int_\Omega \left(\partial_t \mathbf{U} + \mathbf{v}_\varepsilon \cdot \nabla_x \mathbf{U}\right) \cdot \left(\mathbf{U} - \mathbf{v}_\varepsilon\right) dx\, dt + C(D,T)\varepsilon \\
&= \int_0^\tau \int_\Omega \left(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}\right) \cdot \left(\mathbf{U} - \mathbf{v}_\varepsilon\right) dx\, dt \\
&\quad + \int_0^\tau \int_\Omega \left((\mathbf{v}_\varepsilon - \mathbf{U}) \cdot \nabla_x \mathbf{U}\right) \cdot \left(\mathbf{U} - \mathbf{v}_\varepsilon\right) dx\, dt + C(D,T)\varepsilon.
\end{aligned} \tag{4.38}$$

We write

$$\int_0^\tau \mathcal{R}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, r, \mathbf{U})\, dt = I_1 + I_2 + C(D,T)\varepsilon,$$

where

$$\begin{aligned}
I_1 &:= \int_0^\tau \int_\Omega 2\mathbf{D}^d(\mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{v}_\varepsilon) + \left(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}\right) \cdot \left(\mathbf{U} - \mathbf{v}_\varepsilon\right) dx\, dt, \\
I_2 &:= \int_0^\tau \int_\Omega \left((\mathbf{v}_\varepsilon - \mathbf{U}) \cdot \nabla_x \mathbf{U}\right) \cdot \left(\mathbf{U} - \mathbf{v}_\varepsilon\right) dx\, dt.
\end{aligned} \tag{4.39}$$

For I_1, we consider the Helmholtz decomposition:

$$\mathbf{v}_\varepsilon = \mathbf{H}[\mathbf{v}_\varepsilon] + \mathbf{H}^\perp[\mathbf{v}_\varepsilon],$$

where

$$\mathbf{H}[\mathbf{v}_\varepsilon] \in L^2(0,T; W^{1,2}_0(\Omega;\mathbb{R}^3)), \quad \mathrm{div}_x \mathbf{H}[\mathbf{v}_\varepsilon] = 0.$$

We then have

$$\begin{aligned}
I_1 &= \int_0^\tau \int_\Omega \left(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} - \Delta_x \mathbf{U}\right) \cdot \Big[(\mathbf{U} - \mathbf{H}[\mathbf{v}_\varepsilon]) - \mathbf{H}^\perp[\mathbf{v}_\varepsilon]\Big]\, dx\, dt \\
&= \int_0^\tau \int_\Omega \nabla_x P \cdot \mathbf{H}^\perp[\mathbf{v}_\varepsilon]\, dx\, dt,
\end{aligned}$$

where we have used equation (4.12). Then by $(4.27)_3$ and (4.29), we obtain

$$I_1 \leq \|\nabla_x P\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \cdot \|\mathbf{H}^\perp[\mathbf{v}_\varepsilon]\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C(T,D)\varepsilon.$$

For I_2, it is direct to get

$$I_2 \le \int_0^\tau \|\nabla_x \mathbf{U}(t,\cdot)\|_{L^\infty} \int_\Omega |\mathbf{v}_\varepsilon - \mathbf{U}|^2(t,\cdot) dx\, dt.$$

By Gronwall's inequality, we obtain estimate (4.30) and complete the proof of Theorem 4.1.

5 Conclusions

Within the theory of interacting continua, we considered a mixture of two constituents (solute and solvent) characterized by the concentration of one constituent (solute) and by the density, the velocity and the internal energy of the mixture as a whole. Assuming further that the mass and volume additivity constraints hold, and the material densities are constants, we observed that the density depends on the concentration (i.e., the considered fluid is quasi-incompressible) and this dependence is explicit; see (2.11). Using a thermodynamic framework, we focused on the derivation of the (linear) constitutive equations involving the Cauchy stress, the diffusive flux and the energy flux. We also provided a simplified approach (a more complete approach is given in [15]) which helps to identify the boundary conditions that are thermodynamically compatible. The model achieved in this way is similar to the classical compressible Navier-Stokes-Fourier system but with one important difference. The pressure consists of two parts: a thermodynamical pressure and the chemical potential. The latter, inspired by the (logarithmic) form of the chemical potential for a mixture of ideal gases, makes the whole pressure singular: the pressure tends to plus infinity and minus infinity as the density tends its upper bound and its lower bound, respectively, where both the upper bound and the lower bound are positive constants representing the material densities of the solvent and the solute (see (3.8)).

We then restricted ourselves to isothermal flows and under a suitable assumption concerning the polynomial growth rate of the pressure term near the singular values (see (3.17)), we proved the existence of global-in-time finite energy weak solutions to a relevant initial-boundary-value problem for this model. We also studied the limit from quasi-compressibility to the state in which the density of the mixture is uniform, the fluid is thus homogeneous and consequently incompressible.

We have left open the possibility to reduce the growth rate of the pressure near the singular values to a lower order polynomial growth rate or to the logarithmic rate given by the chemical potential (3.5). In our opinion, the extension of the analysis presented in this study to thermal flows can be performed following the results proved in [7], or [1, 2, 11].

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