Gauss-Bonnet on certain open manifolds (Joint work with Tian)

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Introduction

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- The question is as follows:

  What is the Gauss-Bonnet-Chern formula on conformally compact four manifolds?

Introduction

- Let M be the interior of a compact manifold with boundary. According to Penrose, a complete metric g on M is *conformally compact* if there is a smooth defining function ρ on $\bar{M} = M \cup \partial M$, i.e. $\rho(\partial M) = 0$, $d\rho \neq 0$ on ∂M and $\rho > 0$ on M, such that the metric

$$\bar{g} = \rho^2 \cdot g, \tag{1}$$

extends to a smooth metric on $\bar{M}$.

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extends to a smooth metric on $\bar{M}$.

- ρ is called special if $|d\rho|^2_{\bar{g}} = 1$ on a neighborhood of the boundary.

Introduction

- Under mild conditions, the Gauss-Bonnet-Chern formula for a conformally compact manifolds has the following form:

$$\frac{1}{8\pi^2}\int_M (|W|^2 - \frac{1}{2}|z|^2 + \frac{1}{24}(s+12)^2)dV$$

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$$= \chi(M) - \frac{3}{4\pi^2}\hat{V},$$

- W: Weyl tensor, z: trace-free Ricci tensor, s: scalar curvature.

Gauss-Bonnet in 19th Century

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- Gauss, 1828: For a geodesic triangle ABC in a surface in R^3, one has

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- Gauss, 1828: For a geodesic triangle ABC in a surface in R^3, one has

$$\alpha + \beta + \gamma - \pi = \int_{ABC} k \mathrm{d}s.$$

- Bonnet, 1848: extended the formula to smooth curves on surfaces.

Gauss-Bonnet in 20th Century

- (Gauss-Bonnet)Let Σ be a smooth closed oriented surface in R^3, then

$$\int_{\Sigma} k\mathrm{d}s = 2\pi\chi(\Sigma).$$

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- Hopf, 1925: For a hypersurface M^n in R^{n+1}(n even), one has

$$\int_M k\mathrm{d}v = \frac{1}{2}\mathrm{vol}(S^n)\chi(M),$$

where k is the Gauss-Kronecker curvature.

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- Allendoerfer and Weil, 1943: For any abstract oriented riemannian manifolds, one has

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- Remarks: For odd n, $\Theta = 0$; They use the local isometric embedding theorem to obtain the global formula.

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- Results for open manifolds:
- Cohn-Vossen, 1935: For complete surface M, if $\dim H_1(M,R)$ is finite, then

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- Huber, 1957: Extended the above result to general 2-manifolds.

Gauss-Bonnet on open manifolds

- Walter, 1975: For complete 4-manifolds with non-negative sectional curvature,

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- Cheeger and Gromov, 1985: They considered complete manifolds with bounded curvature and finite volume.

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- Fang, 2005: Considered a class of complete locally conformally flat manifolds.

Conformally compact manifolds

- Let's now return back to conformally compact manifolds. When (M, g) is a complete conformally compact Einstein metric with $Ric_g = -(n-1)g$, then the sectional curvatures of g necessarily approach -1 uniformly at infinity at an exponential rate, i.e, the manifolds are asymptotically hyperbolic.

Conformally compact manifolds

- Let's now return back to conformally compact manifolds. When (M, g) is a complete conformally compact Einstein metric with $Ric_g = -(n-1)g$, then the sectional curvatures of g necessarily approach -1 uniformly at infinity at an exponential rate, i.e, the manifolds are asymptotically hyperbolic.
- The study of this kind of manifolds has become very active recently due to the so called AdS/CFT correspondence in string theory.

Renormalized volume

- Let ρ be a special defining function. Graham observed that, in even dimensions,

$$\int_{\rho>\varepsilon} dvol_g = C_0\varepsilon^{1-n} + C_2\varepsilon^{3-n} + \ldots \text{(odd powers)}$$

$$\ldots + C_{n-2}\varepsilon^{-1} + \widehat{V} + o(1),$$

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$$\ldots + C_{n-2}\varepsilon^{-1} + \widehat{V} + o(1),$$

- $\widehat{V}$ is known as the renormalized volume, it does not depend on the choice of special defining functions.

Gauss-Bonnet Renormalized

- Anderson (2001) showed that, for 4-dim conformally compact Einstein manifolds,

$$\frac{1}{8(2\pi)^2}\int_M |W|^2 + \frac{3}{(2\pi)^2}\hat{V} = \chi(M),$$

where W is the Weyl curvature tensor.

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  where W is the Weyl curvature tensor.

- This formula can be thought as a Renormalized Gauss-Bonnet formula. From it one can also see that the renormalized volume $\hat{V}$ is only depend on (M, g).

Gauss-Bonnet Renormalized

- Albin (2005) then proved a Renormalized Gauss-Bonnet formula for any even dimensional conformally compact Einstein manifolds:

$$ {}^{R}\!\!\int_{M} \Theta = \chi(M). $$

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- A particular case was also obtained by Epstein (2001) for convex cocompact hyperbolic manifold:

$$ \frac{(-1)^{m/2}}{2^{m/2}(2\pi)^{m/2}} \frac{m!}{(m/2)!} \hat{V} = \chi(M). $$

Gauss-Bonnet Renormalized

- Also, Chang, Qing, and Yang (2004) obtained the following general formula:

$$\int_M \widetilde{W}\, dvol_g + (-1)^{\frac{m}{2}} \frac{\Gamma\frac{m+1}{2}}{\pi^{\frac{m+1}{2}}} \hat{V} = \chi(M),$$

where $\widetilde{W}$ is a full contraction of the Weyl tensor and its covariant derivatives.

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- Question 1. Both formulas are the generalizations of the Gauss-Bonnet-Chern formula. What's the relation between them?

Conformally compact 4-manifolds

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Conformally compact 4-manifolds

- Question 2. What happens if the manifolds are not Einstein?
- To our knowledge, the answer to question 1 is unclear up to now. We consider question 2 for the case of dimension 4.
- Let M be a 4-dimensional open manifold with a complete metric g. Suppose ρ is a positive function on M such that $\rho^2 \cdot g$ can be extended to a metric $\bar{g}$ on $\bar{M} = M \cup \partial M$. So $\rho|_{\partial M} = 0$.

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- $$\bar{K_{ij}} = \rho^{-2}(K_{ij} + |\bar{\nabla}\rho|^2)$$
$$-\rho^{-1}[\bar{D}^2\rho(\bar{e_i}, \bar{e_i}) + \bar{D}^2\rho(\bar{e_j}, \bar{e_j})]$$

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- Assume that

$i).|\bar{\nabla}\rho| = 1$ near ∂M, $ii).\bar{D}^2\rho = O(\rho)$.

Conformally compact 4-manifolds

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  i.e, (M, g) is asymtotically hyperbolic. Also

- $$Ric + 3 = \rho^2 \cdot \bar{Ric} + 2\rho \cdot \bar{\nabla}^2 \rho + \rho \cdot \bar{\Delta}\rho,$$

$$s + 12 = \rho^2 \cdot \bar{s} + 6\rho \cdot \bar{\Delta}\rho.$$

Conformally compact 4-manifolds

- Let $\rho = e^{-r}$, and $\lambda_i, \bar{\lambda}_i$ be the eigenvalues of $D^2 r$ and $\bar{D}^2 \rho$ respectively. We have

$$\lambda_i = 1 - \rho \cdot \bar{\lambda}_i$$

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- Since $|\bar{\nabla}\rho| = 1$ near ∂M, the integral curves of $\bar{\nabla}\rho$ are geodesics. So along these geodesics, we have the Ricatti equation:

$$\bar{H}' + |\bar{A}|^2 + \bar{Ric}(\bar{\nabla}\rho, \bar{\nabla}\rho) = 0.$$

Where $\bar{H}$ is the mean curvature of ∂M.

Conformally compact 4-manifolds

- In particular, Since

$$Ric(4,4) + 3 = \rho^2 \cdot \bar{Ric}(4,4) + \rho \cdot \bar{H},$$

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we have the following estimate

- $$Ric(4,4) + 3 = -\frac{1}{3}\rho^3 \cdot \bar{H}''(0) + O(\rho^4)$$

which means $Ricci$ along normal direction decays at rate of order at least 3.

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- The idea of proof of the renormalized Gauss-bonnet-Chern formula is to apply the above computations to manifolds with boundary.

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$$\frac{1}{8\pi^2}\int_D (|R|^2 - 4|z|^2) = \chi(D) -$$
$$\frac{1}{2\pi^2}\int_{\partial D}\prod_{i=1}^{3}\lambda_i - \frac{1}{8\pi^2}\int_{\partial D}\sum_{\sigma\in S_3} K_{\sigma_1\sigma_2}\cdot\lambda_{\sigma_3}$$

Conformally compact 4-manifolds

- Take $D = B(r) = \{\log \rho^{-1} \leq r\} \subset M$, $\partial D = S(r)$. It follows that

$$\frac{1}{8\pi^2}\int_{B(r)} [|W|^2 - \frac{1}{2}|z|^2 + \frac{1}{24}(s+12)^2]$$

$$= \chi(B(r)) - \frac{3}{4\pi^2}[I + II + III] + O(\rho),$$

$$I = vol B(r) - \frac{1}{3} vol S(r) = \frac{1}{3}\rho^{-1} \cdot \int_{\bar{S}(0)} \bar{H}'$$

$$-\frac{1}{6}\log\rho \cdot \int_{\bar{S}(0)} \bar{H}'' + C_1 + o(1)$$

Conformally compact 4-manifolds

- $$II = \frac{1}{6}\int_{B(r)} (s+12) - \frac{1}{6}\int_{S(r)} (s+12)$$
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- $$III = \frac{1}{3}\int_{S(r)}(\rho^2 \bar{H}' - 2\rho\bar{H})$$
$$= -\frac{1}{3}\rho^{-1}\int_{\bar{S}(0)} \bar{H}'(0) + O(\rho)$$

Conformally compact 4-manifolds

- Thus we have

$$\frac{1}{8\pi^2}\int_{B(r)}[|W|^2 - \frac{1}{2}|z|^2 + \frac{1}{24}(s+12)^2]$$

$$= \chi(B(r)) + C_3 \cdot \log\rho + C_4 + o(1)$$

This implies that the constants C_3 is 0.

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This implies that the constants C_3 is 0.

- The final formula:

$$\frac{1}{8\pi^2}\int_{M}[|W|^2-\frac{1}{2}|z|^2+\frac{1}{24}(s+12)^2]$$

$$=\chi(M)-\frac{3}{4\pi^2}\hat{V}$$

Conformally compact 4-manifolds

- where $\hat{V}$ is the following limit:

$$\hat{V} = \lim_{r \to +\infty} [vol B(r) - \frac{1}{3} vol S(r) + \frac{1}{6} \int_{B(r)} (s + 12)$$

$$- \frac{1}{6} \int_{S(r)} (s + 12) + \frac{1}{3} \int_{S(r)} (\rho^2 \bar{H}' - 2\rho \bar{H})]$$

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- $\hat{V}$ is called the renormalized volume.

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- Acknowledgement: THANKS FOR YOUR PATIENCE!