

ENVELOPES AND COVERS BY MODULES OF FINITE FP-INJECTIVE AND FLAT DIMENSIONS

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Let R be a ring, n a fixed non-negative integer and $\mathcal{F}\mathcal{I}_n$ (\mathcal{F}_n) the class of all right (left) R -modules of FP-injective (flat) dimension at most n . We prove that $(\mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n^\perp)$ is a perfect cotorsion theory if R is a right coherent ring with $FP-id(R_R) \leq n$. This result was proven by Aldrich, Enochs, Jenda, and Oyonarte in Noetherian case. The modules in \mathcal{F}_n^\perp are also studied. Some applications are given.

Key Words: Cotorsion theory; n -Cotorsion module; (Pre)Cover; (Pre)Envelope.

2000 Mathematics Subject Classification: 16E10; 16D50; 16D40.

1. NOTATION

In this section, we shall recall some known notions and definitions which we need in the later sections.

Throughout this article, R is an associative ring with identity and all modules are unitary. M_R (${}_R M$) denotes a right (left) R -module. For an R -module M , $E(M)$ denotes the injective envelope of M , the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ . $wD(R)$ stands for the weak global dimension of a ring R . $fd(M)$ and $id(M)$ denote the flat and injective dimensions of M , respectively. Let M and N be R -modules. $\text{Hom}(M, N)$ ($\text{Ext}^n(M, N)$) means $\text{Hom}_R(M, N)$ ($\text{Ext}_R^n(M, N)$), and similarly $M \otimes N$ ($\text{Tor}_n(M, N)$) denotes $M \otimes_R N$ ($\text{Tor}_n^R(M, N)$) for an integer $n \geq 1$.

Let M be a right R -module. M is called *FP-injective* (Stenström, 1970) if $\text{Ext}^1(N, M) = 0$ for all finitely presented right R -modules N . Following Stenström (1970), the *FP-injective dimension* of M , denoted by $FP-id(M)$, is defined to be the smallest integer $n \geq 0$ such that $\text{Ext}^{n+1}(F, M) = 0$ for every finitely presented right R -module F (if no such n exists, set $FP-id(M) = \infty$), and r . $FP-dim(R)$ is defined as $\sup\{FP-id(M) : M \text{ is a right } R\text{-module}\}$.

Received October 14, 2005; Revised January 24, 2006. Communicated by R. Wisbauer.

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In what follows, the symbol $\mathcal{F}\mathcal{F}_n$ (\mathcal{F}_n) denotes the class of all right (left) R -modules with FP -injective (flat) dimension less than or equal to a fixed non-negative integer n .

Let \mathcal{C} be a class of right R -modules and M a right R -module. Following Enochs (1981), we say that a homomorphism $\phi : M \rightarrow C$ is a \mathcal{C} -preenvelope if $C \in \mathcal{C}$ and the Abelian group homomorphism $\text{Hom}_R(\phi, C') : \text{Hom}(C, C') \rightarrow \text{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A \mathcal{C} -preenvelope $\phi : M \rightarrow C$ is said to be a \mathcal{C} -envelope if every endomorphism $g : C \rightarrow C$ such that $g\phi = \phi$ is an isomorphism. A \mathcal{C} -envelope $\phi : M \rightarrow F$ is said to have the *unique mapping property* (Ding, 1996) if for any homomorphism $f : M \rightarrow F'$ with $F' \in \mathcal{C}$, there is a unique homomorphism $g : F \rightarrow F'$ such that $g\phi = f$. Dually, we have the definitions of a \mathcal{C} -precover and a \mathcal{C} -cover (with the unique mapping property). \mathcal{C} -envelopes (\mathcal{C} -covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let \mathcal{C} be a class of R -modules and M an R -module. A *right \mathcal{C} -resolution* of M (Enochs and Jenda, 2000) is a $\text{Hom}(-, \mathcal{C})$ exact complex $0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$ with each $C^i \in \mathcal{C}$.

If $0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$ is a right \mathcal{C} -resolution of M , let

$$L^0 = M, \quad L^1 = \text{coker}(M \rightarrow C^0), \quad L^i = \text{coker}(C^{i-2} \rightarrow C^{i-1}) \quad \text{for } i \geq 2.$$

The n th cokernel L^n ($n \geq 0$) is called the n th \mathcal{C} -cosyzygy of M .

If \mathcal{C} is the class of injective modules, then L^n is simply called the n th cosyzygy.

Let R be a right coherent ring. Then every finitely presented left R -module M has a right \mathcal{F}_0 -resolution $0 \rightarrow M \rightarrow P^0 \rightarrow P^1 \rightarrow \dots$ with each P^i finitely generated projective by Enochs and Jenda (2000, Example 8.3.10). So by the n th \mathcal{F}_0 -cosyzygy of a finitely presented left R -module, we will mean the n th cokernel in such a right \mathcal{F}_0 -resolution.

Given a class \mathcal{L} of right R -modules and a class \mathcal{L}' of left R -modules, we write

$$\mathcal{L}^\perp = \text{KerExt}^1(\mathcal{L}, -) = \{C : \text{Ext}^1(L, C) = 0 \text{ for all } L \in \mathcal{L}\},$$

$${}^\perp\mathcal{L} = \text{KerExt}^1(-, \mathcal{L}) = \{C : \text{Ext}^1(C, L) = 0 \text{ for all } L \in \mathcal{L}\},$$

$$\mathcal{L}^\top = \text{KerTor}_1(\mathcal{L}, -) = \{C : \text{Tor}_1(L, C) = 0 \text{ for all } L \in \mathcal{L}\},$$

$${}^\top\mathcal{L}' = \text{KerTor}_1(-, \mathcal{L}') = \{C : \text{Tor}_1(C, L) = 0 \text{ for all } L \in \mathcal{L}'\}.$$

A pair $(\mathcal{F}, \mathcal{C})$ of classes of R -modules is called a *cotorsion theory* (Enochs and Jenda, 2000) if $\mathcal{F}^\perp = \mathcal{C}$ and $\mathcal{F} = {}^\perp\mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *complete* (Trlifaj, 2000) if every R -module has a special \mathcal{C} -preenvelope (and a special \mathcal{F} -precover). A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *perfect* (Enochs et al., 1998; Garca Rozas, 1999) if every R -module has a \mathcal{C} -envelope and an \mathcal{F} -cover. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be *hereditary* (Enochs et al., 1998; Garca Rozas, 1999) if whenever $0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$ is exact with $L, L'' \in \mathcal{F}$ then L' is also in \mathcal{F} , or equivalently, if $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$ is exact with $C', C \in \mathcal{C}$ then C'' is also in \mathcal{C} .

General background materials can be found in Enochs and Jenda (2000), Rotman (1979), and Xu (1996).

2. INTRODUCTION

The problem of the existence of envelopes and covers by different classes of modules has become an active branch of algebra, especially after the appearance of these concepts in Enochs (1981) (with the terminology envelopes and covers) and in Auslander and Smalø (1980) (with the terminology minimal left and right approximations). So the problem has been studied by many authors (see, for example, Aldrich et al., 2001a,b; Angeleri Hügel et al., 2006; Asensio Mayor and Martinez Hernandez, 1988; Bican et al., 2001; Chen and Ding, 1996; Ding, 1996; Eklof and Trlifaj, 2001; Enochs, 1981, 1984; Enochs and Jenda, 2000; Enochs and Oyonarte, 2002; Enochs et al., 1998, 2004; García Rozas, 1999; García Rozas and Torrecillas, 1994; Guil Asensio and Herzog, 2005; Mao and Ding, 2005; Pinzon, 2005; Trlifaj, 2000; Xu, 1996).

Recently, Aldrich et al. (2001b) studied envelopes and covers by modules of finite injective and projective dimensions. In the present discussion, we shall consider envelopes and covers by modules of finite FP -injective and flat dimensions.

In Section 3, we prove that $(\mathcal{F}\mathcal{I}_n, \mathcal{F}\mathcal{I}_n^\perp)$ is a perfect cotorsion theory if R is a right coherent ring with $FP-id(R_R) \leq n$. This result was proven by Aldrich et al. (2001b, Theorem 2.8) for right Noetherian rings.

In Section 4, n -cotorsion modules are defined and studied. A left R -module M is called n -cotorsion if $M \in \mathcal{F}\mathcal{I}_n^\perp$, that is, $\text{Ext}^1(N, M) = 0$ for any $N \in \mathcal{F}\mathcal{I}_n$. For a right coherent ring R with $FP-id(R_R) \leq n$, we prove that a left R -module M is n -cotorsion if and only if M is a kernel of an $\mathcal{F}\mathcal{I}_n$ -precover $f: A \rightarrow B$ with A injective if and only if M is a direct sum of an injective left R -module and a reduced n -cotorsion left R -module.

Section 5 concerns cokernels of $\mathcal{F}\mathcal{I}_n$ -preenvelopes and $\mathcal{F}\mathcal{I}_n^\perp$ -preenvelopes. For a right coherent ring R with $FP-id(R_R) \leq n$, it is shown that a finitely presented right R -module M belongs to ${}^\top\mathcal{F}\mathcal{I}_n$ if and only if M is a cokernel of an $\mathcal{F}\mathcal{I}_n$ -preenvelope $K \rightarrow F$ of a right R -module K with F projective. We also get that over any right coherent ring R , a finitely presented left R -module M belongs to $\mathcal{F}\mathcal{I}_n^\top$ if and only if M is a cokernel of an $\mathcal{F}\mathcal{I}_n^\perp$ -preenvelope $K \rightarrow F$ of a left R -module K with F projective.

Section 6 is dedicated to applications. Some results obtained in the last three sections are used to characterize rings with finite weak global dimension in terms of, among others, n -cotorsion modules. It is proven that $wD(R) \leq n$ if and only if every n -cotorsion left R -module is injective if and only if every n -cotorsion left R -module belongs to $\mathcal{F}\mathcal{I}_n$ if and only if $id(M) \leq m$ for some m with $0 \leq m \leq n$ and any $(n - m)$ -cotorsion left R -module M . It is also shown that if every n -cotorsion left R -module has an $\mathcal{F}\mathcal{I}_n$ -envelope with the unique mapping property, then $wD(R) \leq n + 2$. We conclude the article by proving that $\mathcal{F}\mathcal{I}_n$ is closed under direct limits if every right R -module has an $\mathcal{F}\mathcal{I}_n$ -cover with the unique mapping property.

3. GENERAL RESULTS

We begin with some known facts needed frequently in the sequel.

Lemma 3.1 (Fieldhouse, 1972, Theorem 2.1). *Let R be any ring and M an R -module. Then $fd(M) = id(M^+) = FP-id(M^+)$.*

Lemma 3.2 (Fieldhouse, 1972, Theorem 2.2). *Let R be a right coherent ring and M a right R -module. Then $fd(M^+) = FP-id(M)$.*

Lemma 3.3 (Aldrich et al., 2001a, Corollary 2.13). *Let \mathcal{F} be a class of modules closed under direct sums, extensions, continuous well-ordered unions, and contain all projective modules. If $\mathcal{F}^\perp = S^\perp$ for a set $S \subseteq \mathcal{F}$, then $(\mathcal{F}, \mathcal{F}^\perp)$ is a cotorsion theory.*

For a fixed non-negative integer n , let $\mathcal{F}\mathcal{J}_n$ (\mathcal{F}_n) be the class of all right (left) R -modules of FP -injective (flat) dimension at most n . Now we have the following theorem.

Theorem 3.4. *Let n be a fixed non-negative integer. The following hold:*

- (1) *For a right coherent ring R with $FP-id(R_R) \leq n$, $(\mathcal{F}\mathcal{J}_n, \mathcal{F}\mathcal{J}_n^\perp)$ is a perfect cotorsion theory;*
- (2) *For any ring R , $(\mathcal{F}_n, \mathcal{F}_n^\perp)$ is a perfect hereditary cotorsion theory.*

Proof. (1) Let $\text{Card}(R) = \aleph_\beta$ and $F \in \mathcal{F}\mathcal{J}_n$. By Enochs and Jenda (2000, Lemma 5.3.12), for each $x \in F$, there is a pure submodule S of F with $x \in S$ such that $\text{Card}(S) \leq \aleph_\beta$ (simply let $N = Rx$ and $f = id_N$ in the lemma). So we can write F as a union of a continuous chain $(F_\alpha)_{\alpha < \lambda}$ of pure submodules of F such that $\text{Card}(F_0) \leq \aleph_\beta$ and $\text{Card}(F_{\alpha+1}/F_\alpha) \leq \aleph_\beta$ whenever $\alpha + 1 < \lambda$. If N is a right R -module such that $\text{Ext}^1(F_0, N) = 0$ and $\text{Ext}^1(F_{\alpha+1}/F_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\text{Ext}^1(F, N) = 0$ by Eklof and Trlifaj (2001, Lemma 1) or Enochs and Jenda (2000, Theorem 7.3.4). Since F_α is a pure submodule of F for any $\alpha < \lambda$, $F^+ \rightarrow F_\alpha^+ \rightarrow 0$ is split. Then $F_\alpha^+ \in \mathcal{F}_n$ since $F^+ \in \mathcal{F}_n$ by Lemma 3.2, and so $F_\alpha \in \mathcal{F}\mathcal{J}_n$ by Lemma 3.2 again. On the other hand, F_α is a pure submodule of $F_{\alpha+1}$ whenever $\alpha + 1 < \lambda$, so the exact sequence $0 \rightarrow F_\alpha \rightarrow F_{\alpha+1} \rightarrow F_{\alpha+1}/F_\alpha \rightarrow 0$ induces the split exact sequence $0 \rightarrow (F_{\alpha+1}/F_\alpha)^+ \rightarrow F_{\alpha+1}^+ \rightarrow F_\alpha^+ \rightarrow 0$. Thus $(F_{\alpha+1}/F_\alpha)^+ \in \mathcal{F}_n$ since $F_{\alpha+1}^+ \in \mathcal{F}_n$ by Lemma 3.2, and hence $F_{\alpha+1}/F_\alpha \in \mathcal{F}\mathcal{J}_n$. Let X be a set of representatives of all modules $G \in \mathcal{F}\mathcal{J}_n$ with $\text{Card}(G) \leq \aleph_\beta$. Then $\mathcal{F}\mathcal{J}_n^\perp = X^\perp$.

We note that $\mathcal{F}\mathcal{J}_n$ is closed under direct sums, extensions, direct limits since R is right coherent, and contains all projective modules since $FP-id(R_R) \leq n$. Therefore $(\mathcal{F}\mathcal{J}_n, \mathcal{F}\mathcal{J}_n^\perp)$ is a cotorsion theory by Lemma 3.3.

Since $(\mathcal{F}\mathcal{J}_n, \mathcal{F}\mathcal{J}_n^\perp)$ is cogenerated by the set X , $(\mathcal{F}\mathcal{J}_n, \mathcal{F}\mathcal{J}_n^\perp)$ is a complete cotorsion theory by Eklof and Trlifaj (2001, Theorem 10). Moreover, $(\mathcal{F}\mathcal{J}_n, \mathcal{F}\mathcal{J}_n^\perp)$ is a perfect cotorsion theory by Enochs and Jenda (2000, Theorem 7.2.6) (for $\mathcal{F}\mathcal{J}_n$ is closed under direct limits).

(2) Note that \mathcal{F}_n is closed under direct sums, extensions, direct limits, pure submodules, cokernels of pure monomorphisms and \mathcal{F}_n contains all projective modules. An argument similar to that of (1) shows that $(\mathcal{F}_n, \mathcal{F}_n^\perp)$ is a perfect cotorsion theory. On the other hand, let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be exact with $B, C \in \mathcal{F}_n$, then $A \in \mathcal{F}_n$. So $(\mathcal{F}_n, \mathcal{F}_n^\perp)$ is hereditary. \square

Remark 3.5. (1) The perfect cotorsion theories of Theorem 3.4 could be used to define a model structure in the category of R -modules (see Hovey, 2002, for the interaction between cotorsion pairs and model category structures).

(2) We note that Theorem 3.4(1) extends the work of Aldrich et al. (2001b, Theorem 2.8) where the same result is obtained under the hypothesis that R is right Noetherian.

(3) Choose a field F , and set $F_i = F$ for $i = 1, 2, \dots$, $S = \prod_{i=1}^{\infty} F_i$. Then S is a commutative von Neumann regular ring. Let $R = S[X_1, X_2, \dots, X_n]$, the ring of polynomials in n indeterminates over S , then R is a coherent ring with $wD(R) = n$ (see Glaz, 1989). Clearly, the ring R satisfies the condition of Theorem 3.4(1), but it is not Noetherian.

(4) It is pointed out in Angeleri Hügel et al. (2006, p. 5) that $(\mathcal{F}_n, \mathcal{F}_n^{\perp})$ is a complete cotorsion theory.

The following proposition shows that over right coherent rings the existence of \mathcal{F}_n -preenvelopes and $\mathcal{F}\mathcal{F}_n$ -preenvelopes is always guaranteed.

Proposition 3.6. *The following hold for a right coherent ring R and a fixed integer $n \geq 0$:*

- (1) *Every left R -module has an \mathcal{F}_n -preenvelope;*
- (2) *$({}^{\perp}\mathcal{F}\mathcal{F}_n, \mathcal{F}\mathcal{F}_n)$ is a complete hereditary cotorsion theory.*

Proof. (1) Let M be any left R -module with $\text{Card}(M) = \aleph_{\beta}$. Then, by Enochs and Jenda (2000, Lemma 5.3.12), there is an infinite cardinal \aleph_{α} such that if $F \in \mathcal{F}_n$ and S is a submodule of F with $\text{Card}(S) \leq \aleph_{\beta}$, there is a pure submodule G of F with $S \subseteq G$ and $\text{Card}(G) \leq \aleph_{\alpha}$. Note that $G \in \mathcal{F}_n$, thus M has an \mathcal{F}_n -preenvelope by Enochs and Jenda (2000, Corollary 6.2.2) since the right coherence of R guarantees that \mathcal{F}_n is closed under direct products.

(2) $({}^{\perp}\mathcal{F}\mathcal{F}_n, \mathcal{F}\mathcal{F}_n)$ is a complete cotorsion theory by Mao and Ding (2005, Theorem 3.8) and hereditary by Stenström (1970, Lemma 3.1) and Enochs et al. (2004, Proposition 1.2). □

Proposition 3.7. *Let R be a right coherent ring and n a fixed non-negative integer. Then the following are equivalent:*

- (1) *$FP\text{-id}(R_R) \leq n$;*
- (2) *Every left R -module has a monic \mathcal{F}_n -preenvelope;*
- (3) *Every $(FP\text{-})$ injective left R -module belongs to \mathcal{F}_n ;*
- (4) *Every right R -module has an epic $\mathcal{F}\mathcal{F}_n$ -cover;*
- (5) *Every flat right R -module belongs to $\mathcal{F}\mathcal{F}_n$.*

Proof. (1) \Rightarrow (2) Let M be any left R -module. Then M has an \mathcal{F}_n -preenvelope $f: M \rightarrow F$ by Proposition 3.6(1). Since $(R_R)^+$ is a cogenerator in the category of left R -modules, there is an exact sequence $0 \rightarrow M \rightarrow \Pi(R_R)^+$. Note that $fd(R_R)^+ = FP\text{-id}(R_R) \leq n$ by Lemma 3.2, and so $fd(\Pi(R_R)^+) \leq n$. Thus f is monic, and hence (2) follows.

(2) \Rightarrow (3) Let M be an FP -injective left R -module. Then there exists a pure exact sequence $0 \rightarrow M \rightarrow F$ with $F \in \mathcal{F}_n$ by (2), and hence $F^+ \rightarrow M^+ \rightarrow 0$ splits. So $M \in \mathcal{F}_n$ by Lemma 3.1.

(3) \Rightarrow (1) Note that $(R_R)^+$ is injective, and so $fd(R_R)^+ \leq n$ by (3). Thus $FP-id(R_R) = fd(R_R)^+ \leq n$ by Lemma 3.2.

(1) \Rightarrow (4) follows from Theorem 3.4 (1).

(4) \Rightarrow (1) is clear since R_R has an epic $\mathcal{F}\mathcal{F}_n$ -cover.

(3) \Rightarrow (5) Let M be a flat right R -module. Then $FP-id(M) = fd(M^+) \leq n$ by (3) and Lemma 3.2.

(5) \Rightarrow (1) is obvious. \square

Corollary 3.8. *Let R be a commutative coherent ring. Then the following are equivalent:*

(1) $FP-id(R) \leq n$;

(2) $({}^\perp\mathcal{F}_n, \mathcal{F}_n)$ is a complete hereditary cotorsion theory.

Proof. (1) \Rightarrow (2) We note that $\mathcal{F}_n = \mathcal{F}\mathcal{F}_n$ by Proposition 3.7. Thus (2) follows from Proposition 3.6(2).

(2) \Rightarrow (1) Since every injective R -module belongs to \mathcal{F}_n by (2), (1) holds by Proposition 3.7. \square

4. n -COTORSION MODULES

Let R be a ring and n a fixed non-negative integer. In Section 3, it is shown that $(\mathcal{F}_n, \mathcal{F}_n^\perp)$ is a perfect hereditary cotorsion theory. Recall that a left R -module C is called *cotorsion* (Eochs, 1984) provided that $\text{Ext}^1(F, C) = 0$ for any flat left R -module F . Clearly, cotorsion modules are exactly the modules in the class \mathcal{F}_0^\perp . In this section, n -cotorsion modules are defined to be the modules in the class \mathcal{F}_n^\perp .

We start with the following definition.

Definition 4.1. Let R be a ring and n a fixed non-negative integer. A left R -module M is called *n -cotorsion* if $M \in \mathcal{F}_n^\perp$, that is, $\text{Ext}^1(N, M) = 0$ for any $N \in \mathcal{F}_n$.

Remark 4.2. (1) 0-cotorsion modules are precisely cotorsion modules. If $m \geq n$, then m -cotorsion modules are n -cotorsion.

(2) Recall that a left R -module C is called *strongly cotorsion* (Xu, 1996) if $\text{Ext}^1(F, C) = 0$ for any left R -module F with $fd(F) < \infty$. Obviously, for any non-negative integer n , we have the following implications:

strongly cotorsion modules $\Rightarrow n$ -cotorsion modules \Rightarrow cotorsion modules.

(3) Let R be an *n -Gorenstein ring* (that is, R is a left and right Noetherian ring with $id({}_R R) \leq n$ and $id(R_R) \leq n$) and N an R -module. Then $fd(N) \leq n$ if and only if $fd(N) < \infty$ by Eochs and Jenda (2000, Theorem 9.1.10). Therefore, an R -module M is n -cotorsion if and only if M is strongly cotorsion if and only if M is Gorenstein injective by Eochs and Jenda (2000, Corollary 11.2.2).

Some general properties of n -cotorsion modules follow below.

Proposition 4.3. *Let R be a ring, m and n two non-negative integers.*

- (1) *If M is an n -cotorsion left R -module, then $\text{Ext}^{j+1}(N, M) = 0$ for any integer $j \geq m$ and any $N \in \mathcal{F}_{m+n}$.*
- (2) *The m th cosyzygy of any n -cotorsion left R -module is $(m+n)$ -cotorsion.*

Proof. (1) For any $N \in \mathcal{F}_{m+n}$, consider the exact sequence

$$0 \rightarrow K_m \rightarrow P_{m-1} \rightarrow P_{m-2} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow N \rightarrow 0,$$

where each P_i is projective. It is clear that $K_m \in \mathcal{F}_n$. Therefore, $\text{Ext}^{m+1}(N, M) \cong \text{Ext}^1(K_m, M) = 0$ since M is n -cotorsion, and the result follows by induction.

(2) Let N be any n -cotorsion left R -module and L^m the m th cosyzygy of N . Note that $\text{Ext}^1(F, L^m) \cong \text{Ext}^{m+1}(F, N) = 0$ for any $F \in \mathcal{F}_{m+n}$ by (1). Thus L^m is $(m+n)$ -cotorsion. □

Proposition 4.4. *Let R be a right coherent ring with $\text{FP-id}(R_R) \leq n$. Then the following are equivalent for a left R -module M :*

- (1) *M is n -cotorsion;*
- (2) *For every exact sequence $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ with E injective, $E \rightarrow L$ is an \mathcal{F}_n -precover of L ;*
- (3) *M is a kernel of an \mathcal{F}_n -precover $f: A \rightarrow B$ with A injective;*
- (4) *M is injective with respect to every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $C \in \mathcal{F}_n$.*

Proof. (1) \Rightarrow (2) is easy since $E \in \mathcal{F}_n$ by Proposition 3.7.

(2) \Rightarrow (3) follows from the short exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow L \rightarrow 0$.

(3) \Rightarrow (1) Let M be a kernel of an \mathcal{F}_n -precover $f: A \rightarrow B$ with A injective. Then we have an exact sequence $0 \rightarrow M \rightarrow A \rightarrow B \rightarrow 0$. So, for any $N \in \mathcal{F}_n$, the sequence $\text{Hom}(N, A) \rightarrow \text{Hom}(N, B) \rightarrow \text{Ext}^1(N, M) \rightarrow 0$ is exact. Thus $\text{Ext}^1(N, M) = 0$ since $\text{Hom}(N, A) \rightarrow \text{Hom}(N, B) \rightarrow 0$ is exact by (3), and so (1) follows.

(1) \Rightarrow (4) is clear by definition.

(4) \Rightarrow (1) For each $N \in \mathcal{F}_n$, there exists a short exact sequence $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$ with P projective, which induces an exact sequence $\text{Hom}(P, M) \rightarrow \text{Hom}(K, M) \rightarrow \text{Ext}^1(N, M) \rightarrow 0$. Note that $\text{Hom}(P, M) \rightarrow \text{Hom}(K, M) \rightarrow 0$ is exact by (4). Hence $\text{Ext}^1(N, M) = 0$, as desired. □

Recall that an R -module M is called *reduced* (Enochs and Jenda, 2000) if M has no nonzero injective submodules.

Proposition 4.5. *Let R be a right coherent ring with $\text{FP-id}(R_R) \leq n$. Then the following are equivalent for a left R -module M :*

- (1) *M is a reduced n -cotorsion left R -module;*
- (2) *M is a kernel of an \mathcal{F}_n -cover $f: A \rightarrow B$ with A injective.*

Proof. (1) \Rightarrow (2) By Proposition 4.4, the natural map $\pi : E(M) \rightarrow E(M)/M$ is an \mathcal{F}_n -precover. Thus $E(M)$ has no nonzero direct summand K contained in M since M is reduced. Note that $E(M)/M$ has an \mathcal{F}_n -cover by Theorem 3.4(2). It follows that $\pi : E(M) \rightarrow E(M)/M$ is an \mathcal{F}_n -cover by Xu (1996, Corollary 1.2.8), and hence (2) follows.

(2) \Rightarrow (1) Let M be a kernel of an \mathcal{F}_n -cover $\alpha : A \rightarrow B$ with A injective. By Proposition 4.4, M is n -cotorsion. Now let K be an injective submodule of M . Suppose $A = K \oplus L$, $p : A \rightarrow L$ is the projection and $i : L \rightarrow A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$ since $\alpha(K) = 0$. Therefore ip is an isomorphism, and hence i is epic. Thus $A = L$, $K = 0$, and so M is reduced. \square

Theorem 4.6. *Let R be a right coherent ring with $FP-id(R_R) \leq n$. Then a left R -module M is n -cotorsion if and only if M is a direct sum of an injective left R -module and a reduced n -cotorsion left R -module.*

Proof. “ \Leftarrow ” is clear.

“ \Rightarrow ” Let M be an n -cotorsion left R -module. Consider the short exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$. Note that $E(M) \rightarrow E(M)/M$ is an \mathcal{F}_n -precover of $E(M)/M$ by Proposition 4.4. But $E(M)/M$ has an \mathcal{F}_n -cover $L \rightarrow E(M)/M$ by Theorem 3.4(2), so we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M \longrightarrow 0 \\
 & & \phi \downarrow & & \gamma \downarrow & & \parallel \\
 0 & \longrightarrow & M & \xrightarrow{\alpha} & E(M) & \longrightarrow & E(M)/M \longrightarrow 0 \\
 & & \sigma \downarrow & & \beta \downarrow & & \parallel \\
 0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M \longrightarrow 0.
 \end{array}$$

Since $\beta\gamma$ is an isomorphism, $E(M) = \ker(\beta) \oplus \text{im}(\gamma)$. So L and $\ker(\beta)$ are injective (for $\text{im}(\gamma) \cong L$). Therefore K is a reduced n -cotorsion module by Proposition 4.5. Note that $\sigma\phi$ is an isomorphism by Five Lemma. Thus $M = \ker(\sigma) \oplus \text{im}(\phi)$, where $\text{im}(\phi) \cong K$. On the other hand, we get the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \ker(\sigma) & \longrightarrow & \ker(\beta) & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \xrightarrow{\alpha} & E(M) & \longrightarrow & E(M)/M \longrightarrow 0 \\
 & & \sigma \downarrow & & \beta \downarrow & & \parallel \\
 0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

Hence $\ker(\sigma) \cong \ker(\beta)$ by 3×3 Lemma (Rotman, 1979, Exercise 6.16, p. 175). This completes the proof. \square

The following result generalizes Xu (1996, Proposition 3.3.1).

Theorem 4.7. *Let R be a ring. Then the following are equivalent for a fixed non-negative integer n :*

- (1) Every left R -module is n -cotorsion;
- (2) Every left R -module in \mathcal{F}_n is projective;
- (3) Every flat left R -module is n -cotorsion;
- (4) Every projective left R -module is n -cotorsion;
- (5) ${}_R R$ is n -cotorsion and every left R -module has an \mathcal{F}_n^\perp -precover.

Proof. (1) \Leftrightarrow (2) holds by Theorem 3.4(2).

(1) \Rightarrow (3) \Rightarrow (4) and (1) \Rightarrow (5) are trivial.

(4) \Rightarrow (1) Let M be a left R -module. Then there exists an exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with $F \in \mathcal{F}_n$ and $K \in \mathcal{F}_n^\perp$ by Theorem 3.4(2). Since every projective left R -module is cotorsion by (4), R is left perfect by Guil Asensio and Herzog (2005, Corollary 10). Thus F is n -cotorsion by (4) and Enochs et al. (2004, Proposition 1.2) since $(\mathcal{F}_n, \mathcal{F}_n^\perp)$ is hereditary. So M is n -cotorsion.

(5) \Rightarrow (4) By García Rozas and Torrecillas (1994, Proposition 1), \mathcal{F}_n^\perp is closed under direct sums. Hence every free (projective) left R -module is n -cotorsion since ${}_R R$ is n -cotorsion. \square

5. COKERNELS OF $\mathcal{F}\mathcal{F}_n$ -PREENVELOPES AND \mathcal{F}_n -PREENVELOPES

In this section, we shall investigate some properties of the cokernels of $\mathcal{F}\mathcal{F}_n$ -preenvelopes and \mathcal{F}_n -preenvelopes.

Proposition 5.1. *The following are true:*

- (1) If M is a cokernel of an $\mathcal{F}\mathcal{F}_n$ -preenvelope $K \rightarrow F$ of a right R -module K with F flat, then $M \in {}^\top \mathcal{F}_n$;
- (2) If R is a right coherent ring, M is a cokernel of an \mathcal{F}_n -preenvelope $L \rightarrow F$ of a left R -module L with F flat, then $M \in \mathcal{F}\mathcal{F}_n^\top$.

Proof. (1) Assume M is a cokernel of an $\mathcal{F}\mathcal{F}_n$ -preenvelope $K \rightarrow F$ of a right R -module K with F flat. Then $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ is exact. Note that $E^+ \in \mathcal{F}\mathcal{F}_n$ for any $E \in \mathcal{F}_n$ by Lemma 3.1. Thus we obtain an exact sequence $\text{Hom}(F, E^+) \rightarrow \text{Hom}(K, E^+) \rightarrow 0$, which gives rise to the exactness of $(F \otimes E)^+ \rightarrow (K \otimes E)^+ \rightarrow 0$. So the sequence $0 \rightarrow K \otimes E \rightarrow F \otimes E$ is exact. But the flatness of F implies the exactness of $0 \rightarrow \text{Tor}_1(M, E) \rightarrow K \otimes E \rightarrow F \otimes E$, and hence $\text{Tor}_1(M, E) = 0$.

(2) Suppose M is a cokernel of an \mathcal{F}_n -preenvelope $L \rightarrow F$ of a left R -module L with F flat. Let $K = \text{im}(L \rightarrow F)$, then $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ is exact and $K \rightarrow F$ is an \mathcal{F}_n -preenvelope of K . We can prove that $M \in \mathcal{F}\mathcal{F}_n^\top$ in a way similar to that of (1) using Lemma 3.2 in place of Lemma 3.1. \square

Corollary 5.2. *The following are true for a right coherent ring R :*

- (1) *Every $(n + 1)$ th \mathcal{F}_0 -cosyzygy of any finitely presented left R -module belongs to $\mathcal{F}\mathcal{F}_n^\top$;*
- (2) *${}^\perp\mathcal{F}_n \subseteq \mathcal{F}\mathcal{F}_n^\top$.*

Proof. (1) Let M be a finitely presented left R -module and $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$ be any right \mathcal{F}_0 -resolution of M with each F^i finitely generated projective. By Enochs and Jenda (2000, Remark 8.4.35) or Chen and Ding (1994, Lemma 2.1), $L^n \rightarrow F^n$ is an \mathcal{F}_n -preenvelope, where L^n is the n th \mathcal{F}_0 -cosyzygy of M . Thus the $(n + 1)$ th \mathcal{F}_0 -cosyzygy L^{n+1} belongs to $\mathcal{F}\mathcal{F}_n^\top$ by Proposition 5.1(2).

(2) Let $M \in {}^\perp\mathcal{F}_n$. Consider the exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ with P projective. It is easy to see that $K \rightarrow P$ is an \mathcal{F}_n -preenvelope of K . Thus $M \in \mathcal{F}\mathcal{F}_n^\top$ by Proposition 5.1(2). □

Theorem 5.3. *Let R be a right coherent ring.*

- (1) *If M is a finitely presented right R -module and $FP-id(R_R) \leq n$, then $M \in {}^\top\mathcal{F}_n$ if and only if M is a cokernel of an $\mathcal{F}\mathcal{F}_n$ -preenvelope $K \rightarrow P$ of a right R -module K with P projective.*
- (2) *If M is a finitely presented left R -module, then $M \in \mathcal{F}\mathcal{F}_n^\top$ if and only if M is a cokernel of an \mathcal{F}_n -preenvelope $K \rightarrow F$ of a left R -module K with F projective if and only if $M \in {}^\perp\mathcal{F}_n$.*

Proof. (1) “ \Leftarrow ” follows from Proposition 5.1(1).

“ \Rightarrow ” Since M is a finitely presented right R -module, there is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ with P finitely generated projective and K finitely generated. It is clear that $P \in \mathcal{F}\mathcal{F}_n$ since $FP-id(R_R) \leq n$. We claim that $K \rightarrow P$ is an $\mathcal{F}\mathcal{F}_n$ -preenvelope. In fact, for any $F \in \mathcal{F}\mathcal{F}_n$, we have $F^+ \in \mathcal{F}_n$ by Lemma 3.2 since R is right coherent. Thus $\text{Tor}_1(M, F^+) = 0$, and so we get the exact commutative diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & K \otimes F^+ & \xrightarrow{\alpha} & P \otimes F^+ \\
 & & \sigma_K \downarrow & & \sigma_P \downarrow \\
 & & \text{Hom}(K, F)^+ & \xrightarrow{\theta} & \text{Hom}(P, F)^+.
 \end{array}$$

On the other hand, there exists an exact sequence $Q \rightarrow K \rightarrow 0$ with Q finitely generated projective since K is finitely generated. So we have the exact commutative diagram

$$\begin{array}{ccccc}
 Q \otimes F^+ & \longrightarrow & K \otimes F^+ & \longrightarrow & 0 \\
 \sigma_Q \downarrow & & \sigma_K \downarrow & & \\
 \text{Hom}(Q, F)^+ & \longrightarrow & \text{Hom}(K, F)^+ & \longrightarrow & 0.
 \end{array}$$

Note that σ_Q is an isomorphism by Rotman (1979, Lemma 3.59), so σ_K is epic. Thus θ is a monomorphism since σ_P is an isomorphism, and hence the sequence $\text{Hom}(P, F) \rightarrow \text{Hom}(K, F) \rightarrow 0$ is exact, as desired.

(2) The proof of the first equivalence is similar to that of (1) by Lemma 3.1 and Proposition 5.1(2). The second equivalence is easy by Corollary 5.2(2). \square

Recall that an R -module M is said to be *coreduced* (Chen, 1996) if it has no nonzero projective quotient modules. A ring R is called *semiregular* (Nicholson, 1976) if idempotents lift modulo the Jacobson radical $J(R)$ and $R/J(R)$ is von Neumann regular.

Proposition 5.4. *Let R be a semiregular 2-sided coherent ring and M a finitely presented left R -module. Then the following are equivalent for an integer $n \geq 0$:*

- (1) $M \in {}^\perp \mathcal{F}_n$ and M is coreduced;
- (2) M is a cokernel of an \mathcal{F}_n -envelope $K \rightarrow P$ of a left R -module K with P projective.

Proof. (2) \Rightarrow (1) $M \in {}^\perp \mathcal{F}_n$ holds by Theorem 5.3(2), and M is coreduced by Ding (1996, Lemma 3.7).

(1) \Rightarrow (2) Consider the exact sequence $0 \rightarrow K \xrightarrow{\varphi} P \rightarrow M \rightarrow 0$ with P finitely generated projective and K finitely generated. Then $\varphi : K \rightarrow P$ is an \mathcal{F}_n -preenvelope (for $M \in {}^\perp \mathcal{F}_n$). Since R is left coherent, K is finitely presented. So K has an \mathcal{F}_0 -envelope $\alpha : K \rightarrow Q$ by Asensio Mayor and Martinez Hernandez (1988, Corollary 3) since R is semiregular and right coherent. Thus there exist $f : Q \rightarrow P$ and $g : P \rightarrow Q$ such that $f\alpha = \varphi$ and $g\varphi = \alpha$, and hence $(gf)\alpha = \alpha$. It follows that gf is an isomorphism, $P = \text{im}(f) \oplus \ker(g)$, and $\alpha : K \rightarrow Q$ is an \mathcal{F}_n -envelope. Note that $\text{im}(\varphi) \subseteq \text{im}(f)$, and so $P/\text{im}(\varphi) \rightarrow P/\text{im}(f) \rightarrow 0$ is exact. But $P/\text{im}(\varphi)$ is coreduced, and hence $P/\text{im}(f) = 0$, that is, $P = \text{im}(f)$. So f is an isomorphism, and then $\varphi : K \rightarrow P$ is an \mathcal{F}_n -envelope. \square

Corollary 5.5. *Let R be a semiregular 2-sided coherent ring, and M a finitely presented left R -module. Then $M \in {}^\perp \mathcal{F}_n$ if and only if $M = P \oplus N$, where P is a projective left R -module, $N \in {}^\perp \mathcal{F}_n$ and N is coreduced.*

Proof. “ \Leftarrow ” is clear.

“ \Rightarrow ” can be proven in a way dual to that of Theorem 4.6 using Proposition 5.4 and its proof. \square

6. APPLICATIONS

In this section, some results obtained in the last three sections are used to characterize rings with finite weak global dimension in terms of, among others, n -cotorsion modules.

To this aim, we need the following lemmas.

Lemma 6.1. *The following are equivalent for a left R -module M and an integer $n \geq 0$:*

- (1) $M \in \mathcal{F}\mathcal{F}_n^+$;
- (2) $M^+ \in \mathcal{F}\mathcal{F}_n^+$;

- (3) $M \in {}^\perp \mathcal{C}$, where $\mathcal{C} = \{B^+ : B \in \mathcal{F}\mathcal{J}_n\}$;
- (4) For every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $C \in \mathcal{F}\mathcal{J}_n$, the functor $-\otimes M$ preserves the exactness.

Proof. By Cartan and Eilenberg (1956, VI. 5.1) or Rotman (1979, p. 360), for any right R -module N , there are the following standard isomorphisms:

$$\text{Ext}^1(M, N^+) \cong \text{Tor}_1(N, M)^+ \cong \text{Ext}^1(N, M^+).$$

Thus (1) \Leftrightarrow (2) \Leftrightarrow (3) follows. (1) \Leftrightarrow (4) is easy. □

Corollary 6.2. *The following hold for a right coherent ring R and an integer $n \geq 0$:*

- (1) A right R -module M is injective if and only if $M \in \mathcal{F}\mathcal{J}_n^\perp$ and $M \in \mathcal{F}\mathcal{J}_{n+1}$;
- (2) A left R -module N is flat if and only if $N \in \mathcal{F}\mathcal{J}_n^\top$ and $N \in \mathcal{F}_{n+1}$.

Proof. (1) “ \Rightarrow ” is trivial.

“ \Leftarrow ” Let $M \in \mathcal{F}\mathcal{J}_n^\perp$ and $M \in \mathcal{F}\mathcal{J}_{n+1}$. Consider the exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$. Note that $FP\text{-id}(E(M)/M) \leq n$ since $FP\text{-id}(M) \leq n + 1$. So $\text{Ext}^1(E(M)/M, M) = 0$, and hence the above sequence is split. Thus M is injective.

(2) “ \Rightarrow ” is trivial.

“ \Leftarrow ” Let $N \in \mathcal{F}\mathcal{J}_n^\top$ and $N \in \mathcal{F}_{n+1}$. Then $N^+ \in \mathcal{F}\mathcal{J}_n^\perp$ by Lemma 6.1. Thus N^+ is injective by (1) since $FP\text{-id}(N^+) = fd(N) \leq n + 1$. Hence N is flat. □

Lemma 6.3. *Let R be a right coherent ring with $FP\text{-id}(R_R) \leq n$ and $n \geq 1$. If $M \in \mathcal{F}\mathcal{J}_{n-1}^\top$, then there is an exact sequence $0 \rightarrow M \rightarrow F \rightarrow L \rightarrow 0$ such that F is flat and $L \in \mathcal{F}\mathcal{J}_n^\top$.*

Proof. Consider the following pushout diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M^+ \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & E(P) & \longrightarrow & Q \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & C & \xlongequal{\quad} & C \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

where P is projective and $P \rightarrow E(P)$ is an injective envelope. Note that $FP\text{-id}(C) \leq n - 1$ by Stenström (1970, Lemma 3.1) since $FP\text{-id}(P) \leq n$. So $\text{Ext}^1(C, M^+) = 0$ (for

$M^+ \in \mathcal{FF}_{n-1}^\perp$ by Lemma 6.1), then the sequence $0 \rightarrow M^+ \rightarrow Q \rightarrow C \rightarrow 0$ is split. Therefore, there exists an exact sequence $E(P) \rightarrow M^+ \rightarrow 0$, which in turn yields the exactness of $0 \rightarrow M^{++} \rightarrow E(P)^+$. So M embeds in a flat left R -module (for $E(P)^+$ is flat).

Now let $\beta : M \rightarrow F$ be a flat preenvelope of M , then β is monic. So we have the exact sequence $0 \rightarrow M \rightarrow F \rightarrow L \rightarrow 0$. Note that $L \in \mathcal{FF}_0^\top$ by Proposition 5.1(2). We claim that $L \in \mathcal{FF}_n^\top$. In fact, let $X \in \mathcal{FF}_n$. Consider the exact sequence $0 \rightarrow X \rightarrow E(X) \rightarrow D \rightarrow 0$. Then $D \in \mathcal{FF}_{n-1}$. Thus we get the induced exact sequence

$$0 = \text{Tor}_2(D, F) \rightarrow \text{Tor}_2(D, L) \rightarrow \text{Tor}_1(D, M) = 0.$$

Therefore $\text{Tor}_2(D, L) = 0$. On the other hand, the short exact sequence $0 \rightarrow X \rightarrow E(X) \rightarrow D \rightarrow 0$ induces the exactness of the sequence

$$0 = \text{Tor}_2(D, L) \rightarrow \text{Tor}_1(X, L) \rightarrow \text{Tor}_1(E(X), L) = 0.$$

Therefore, $\text{Tor}_1(X, L) = 0$, as desired. □

We are now in a position to prove the following theorem which improves Xu (1996, Theorem 3.3.2).

Theorem 6.4. *The following are equivalent for a ring R and a fixed non-negative integer n :*

- (1) $wD(R) \leq n$;
- (2) Every n -cotorsion left R -module is injective;
- (3) $id(M) \leq n$ for any 0-cotorsion left R -module M ;
- (4) Every n -cotorsion left R -module belongs to \mathcal{FF}_n ;
- (5) $id(M) \leq m$ for any m with $0 \leq m \leq n$ and any $(n - m)$ -cotorsion left R -module M ;
- (6) $id(M) \leq m$ for some m with $0 \leq m \leq n$ and any $(n - m)$ -cotorsion left R -module M .

If R is a right coherent ring and $n \geq 1$, then the above conditions are also equivalent to:

- (7) $r.FP\text{-dim}(R) \leq n$;
- (8) Every $((n - 1)$ -cotorsion) left R -module has an epic \mathcal{FF}_{n-1} -envelope;
- (9) Every right R -module M (with $M \in {}^\perp\mathcal{FF}_{n-1}$) has a monic \mathcal{FF}_{n-1} -cover;
- (10) $FP\text{-id}(R_R) \leq n$ and every right R -module in \mathcal{FF}_n^\perp is injective;
- (11) Every right R -module in ${}^\perp\mathcal{FF}_n$ is projective;
- (12) $FP\text{-id}(R_R) \leq n$ and every right R -module in \mathcal{FF}_n^\perp belongs to \mathcal{FF}_n ;
- (13) Every right R -module in ${}^\perp\mathcal{FF}_n$ belongs to \mathcal{FF}_n ;
- (14) $FP\text{-id}(R_R) \leq n$ and M is flat for any $M \in \mathcal{FF}_n^\top$;
- (15) $FP\text{-id}(R_R) \leq n$ and M is flat for any $M \in \mathcal{FF}_{n-1}^\top$.

Proof. (1) \Leftrightarrow (2) follows from Theorem 3.4 (2). (1) \Rightarrow (4), (5) \Rightarrow (6) are trivial.

(4) \Rightarrow (1) Let M be any left R -module. By Theorem 3.4 (2), there is a short exact sequence $0 \rightarrow M \rightarrow C \rightarrow L \rightarrow 0$ with $C \in \mathcal{F}_n^\perp$ and $L \in \mathcal{F}_n$. Then $C \in \mathcal{F}_n$ by (4), and hence $M \in \mathcal{F}_n$. Thus $wD(R) \leq n$.

(1) \Rightarrow (5) Let M be any $(n - m)$ -cotorsion left R -module and N any left R -module. Since $fd(N) \leq n$, $\text{Ext}^{m+1}(N, M) = 0$ by Proposition 4.3(1). So $id(M) \leq m$.

(6) \Rightarrow (3) Let M be any 0-cotorsion left R -module. Then the $(n - m)$ th cosyzygy L^{n-m} of M is $(n - m)$ -cotorsion by Proposition 4.3(2). Thus $id(L^{n-m}) \leq m$ by (6), and so $id(M) \leq n$.

(3) \Rightarrow (1) follows since $fd(N) = id(N^+) \leq n$ for any right R -module N by Lemma 3.1.

(1) \Leftrightarrow (7) holds by Stenström (1970, Theorem 3.3).

(7) \Leftrightarrow (10) follows from Theorem 3.4(1). (1) \Leftrightarrow (9) \Leftrightarrow (13) follow from the equivalence of (1), (6) and (9) in Mao and Ding (2005, Theorem 4.1).

(7) \Leftrightarrow (11) follows from Proposition 3.6(2).

(7) \Rightarrow (12) and (15) \Rightarrow (14) are clear.

(1) \Rightarrow (8) Let M be a left R -module. Then M has an \mathcal{F}_{n-1} -preenvelope $\alpha : M \rightarrow N$ by Proposition 3.6 (1). It is easy to check that $\text{im}(\alpha) \in \mathcal{F}_{n-1}$ since $N/\text{im}(\alpha) \in \mathcal{F}_n$. Thus $M \rightarrow \text{im}(\alpha)$ is an epic \mathcal{F}_{n-1} -envelope.

(8) \Rightarrow (1) Let M be a left R -module. Then by Theorem 3.4 (2), there exists the exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with $F \in \mathcal{F}_{n-1}$ and $K \in \mathcal{F}_{n-1}^\perp$. Since K has an epic \mathcal{F}_{n-1} -envelope by (8), $K \in \mathcal{F}_{n-1}$. Thus $M \in \mathcal{F}_n$, and so $wD(R) \leq n$.

(12) \Rightarrow (7) Let M be any right R -module. By Theorem 3.4 (1), there is a short exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with $K \in \mathcal{F}\mathcal{F}_n^\perp$ and $F \in \mathcal{F}\mathcal{F}_n$. Then $K \in \mathcal{F}\mathcal{F}_n$ by (12), and hence $M \in \mathcal{F}\mathcal{F}_n$.

(10) \Rightarrow (14) follows from Lemma 6.1.

(14) \Rightarrow (15) holds by Lemma 6.3.

(14) \Rightarrow (1) By Corollary 5.2, the $(n + 1)$ th \mathcal{F}_0 -cosyzygy L^{n+1} of any finitely presented left R -module M belongs to $\mathcal{F}\mathcal{F}_n^\top$. Therefore L^{n+1} is flat by (14), and hence projective. So $wD(R) \leq n + 3 < \infty$ by Enochs and Jenda (2000, Corollary 8.4.28). Thus $wD(R) = FP-id(R_R) \leq n$ by Stenström (1970, Proposition 3.5).

□

Corollary 6.5. *The following are equivalent for a ring R and a fixed integer $n \geq 0$:*

- (1) R is a semisimple Artinian ring;
- (2) Every n -cotorsion left R -module is projective.

Proof. (1) \Rightarrow (2) is trivial.

(2) \Rightarrow (1). Note that R is a QF ring since every injective left R -module is projective. It follows that $wD(R) \leq n$ by Theorem 6.4. Thus $wD(R) = 0$, and hence R is semisimple Artinian. □

Theorem 6.6. *Assume a ring R satisfies one of the following conditions:*

- (1) *Every n -cotorsion left R -module has an \mathcal{F}_n -envelope with the unique mapping property;*
- (2) *Every finitely presented left R -module has an \mathcal{F}_n -envelope with the unique mapping property;*
- (3) *R is right coherent, and every right R -module M with $M \in {}^\perp\mathcal{F}_n$ has an \mathcal{F}_n -cover with the unique mapping property.*

Then $wD(R) \leq n + 2$.

Proof. Assume (1). Let M be any left R -module. Then we have the exact sequences

$$0 \longrightarrow C \xrightarrow{i} F_0 \xrightarrow{\alpha} M \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow F_2 \xrightarrow{\psi} F_1 \xrightarrow{\beta} C \longrightarrow 0$$

by Theorem 3.4 (2), where $\alpha : F_0 \rightarrow M$ and $\beta : F_1 \rightarrow C$ are \mathcal{F}_n -covers, C and F_2 are n -cotorsion. Thus we get an exact sequence

$$0 \longrightarrow F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi=i\beta} F_0 \xrightarrow{\alpha} M \longrightarrow 0.$$

Let $\theta : F_2 \rightarrow H$ be an \mathcal{F}_n -envelope with the unique mapping property. Then there exists $\delta : H \rightarrow F_1$ such that $\psi = \delta\theta$. Thus $\varphi\delta\theta = \varphi\psi = 0$, and hence $\varphi\delta = 0$, which implies that $\text{im}(\delta) \subseteq \ker(\varphi) = \text{im}(\psi)$. So there exists $\gamma : H \rightarrow F_2$ such that $\psi\gamma = \delta$, and hence we get the following exact commutative diagram:

$$\begin{array}{ccccccc}
 & & H & & & & \\
 & & \uparrow \theta & \searrow \delta & & & \\
 & \gamma \downarrow & & & & & \\
 0 & \longrightarrow & F_2 & \xrightarrow{\psi} & F_1 & \xrightarrow{\varphi} & F_0 \xrightarrow{\alpha} M \longrightarrow 0.
 \end{array}$$

Note that $\psi\gamma\theta = \psi$, and so $\gamma\theta = 1_{F_2}$ since ψ is monic. Thus F_2 is isomorphic to a direct summand of H , and hence $F_2 \in \mathcal{F}_n$. Therefore, $fd(M) \leq n + 2$, and so $wD(R) \leq n + 2$.

Assume (2). By Ding (1996, Lemma 3.2), every left R -module has an \mathcal{F}_n -envelope with the unique mapping property since \mathcal{F}_n is closed under direct limits. So the result follows since condition (1) is satisfied.

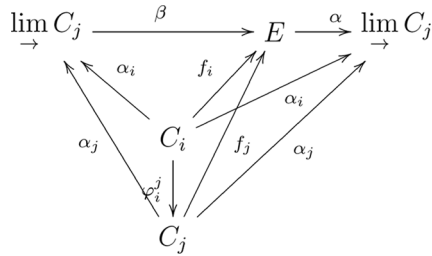
Assume (3). We can prove that $wD(R) \leq n + 2$ in a way dual to that of (1) using Proposition 3.6 (2) and the fact that $wD(R) = FP\text{-}id(R_R)$ for a right coherent ring R . □

Finally, we prove the following result which may be of independent interest.

Proposition 6.7. *Let n be a fixed non-negative integer. If every right R -module has an \mathcal{F}_n -cover with the unique mapping property, then \mathcal{F}_n is closed under direct limits.*

Proof. Let $\{C_j, \varphi_j^i\}$ be any direct system with $C_j \in \mathcal{F}_n$. By hypothesis, $\lim C_j$ has an \mathcal{F}_n -cover $\alpha : E \rightarrow \lim C_j$ with the unique mapping property. Let $\alpha_j : C_j \rightarrow \lim C_j$ with $\alpha_j = \alpha_j \varphi_j^i$ whenever $i \leq j$. Then there exists $f_i : C_i \rightarrow E$ such that $\alpha_i = \alpha f_i$ for any

$i \leq j$. It follows that $\alpha f_i = \alpha f_j \varphi_j^i$, and so $f_i = f_j \varphi_j^i$. Therefore, by the definition of direct limits, there exists $\beta : \lim_{\rightarrow} C_j \rightarrow E$ such that the following diagram is commutative:



Thus $f_i = \beta \alpha_i$, and so $(\alpha \beta) \alpha_i = \alpha(\beta \alpha_i) = \alpha f_i = \alpha_i$ for any $i \leq j$. Therefore, $\alpha \beta = 1_{\lim_{\rightarrow} C_j}$ by the definition of direct limits, and hence $\lim_{\rightarrow} C_j$ is a direct summand of E . So $\lim_{\rightarrow} C_j \in \mathcal{F}\mathcal{I}_n$. □

Remark 6.8. By Proposition 6.7, if every right R -module M has an $\mathcal{F}\mathcal{I}_0$ -cover with the unique mapping property, then $\mathcal{F}\mathcal{I}_0$ is closed under direct limits, and so R is right coherent by Stenström (1970, Theorem 3.2). By Theorem 3.4 (1), if R is a right coherent right self- FP -injective ring, then every right R -module M has an epic $\mathcal{F}\mathcal{I}_0$ -cover. By Chen and Ding (1996, Corollary 8), R is a right coherent ring with $wD(R) \leq 1$ if and only if every right R -module M has a monic $\mathcal{F}\mathcal{I}_0$ -cover. So it seems reasonable to conjecture that a ring R is right coherent if and only if every right R -module has an $\mathcal{F}\mathcal{I}_0$ -cover. In fact, Pinzon (see Pinzon, 2005) has proven recently that if R is right coherent, then every right R -module has an $\mathcal{F}\mathcal{I}_0$ -cover. We also wonder whether every right R -module over any ring R has an $\mathcal{F}\mathcal{I}_0$ -precover.

ACKNOWLEDGMENTS

This research was partially supported by SRFDP (No. 20050284015), NSFC (No. 10331030), China Postdoctoral Science Foundation (No. 20060390926), Collegial Natural Science Research Program of Education Department of Jiangsu Province (No. 06KJB110033), and Jiangsu Planned Projects for Postdoctoral Research Funds (No. 0601021B).

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