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FI-injective and FI-flat modules

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Abstract

Let R be a ring. A left R-module M (respectively right R-module N) is called FI-injective (respectively FI-flat) if $\mathrm{Ext}^1(G, M) = 0$ (respectively $\mathrm{Tor}_1(N, G) = 0$) for any FP-injective left R-module G. Suppose R is a left coherent ring. It is shown that a left R-module M is FI-injective if and only if M is a direct sum of an injective left R-module and a reduced FI-injective left R-module; a finitely presented right R-module M is FI-flat if and only if M is a cokernel of a flat preenvelope of a right R-module. These modules together with the left derived functors of Hom are used to study the FP-injective dimensions of modules and rings.

Keywords: FP-injective module; FP-injective dimension; FI-injective module; FI-flat module; (Pre)cover; (Pre)envelope

1. Introduction

We first recall some known notions and facts needed in the sequel.

Let R be a ring. A left R-module M is called *FP-injective* (or *absolutely pure*) [15,19] if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented left R-modules N. The *FP-injective dimension of M*, denoted by $FP\text{-}id(M)$, is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for every finitely presented left R-module F (if no such n exists, set $FP\text{-}id(M) = \infty$), and $l.FP\text{-}\dim(R)$ is defined as $\sup\{FP\text{-}id(M)\colon\ M \text{ is a left } R\text{-module}\}$.

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Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [7], we say that a homomorphism $\phi: M \to C$ is a *$\mathcal{C}$-preenvelope* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\phi, C'): \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi: M \to C$ is said to be a *$\mathcal{C}$-envelope* if every endomorphism $g: C \to C$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of a *$\mathcal{C}$-precover* and a *$\mathcal{C}$-cover*. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

In what follows, we write ${}_R\mathcal{M}$ and $\mathcal{FI}$ for the categories of all left R-modules and all *FP*-injective left R-modules, respectively. Recall that a ring R is called *left coherent* if every finitely generated left ideal is finitely presented. It has been recently proven that every left R-module has an *FP*-injective cover over a left coherent ring R (see [16]), so every left R-module M has a *left $\mathcal{FI}$-resolution*, that is, there is a $\mathrm{Hom}(\mathcal{FI}, -)$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ (not necessarily exact) with each F_i *FP*-injective. Write

$$K_0 = M, \qquad K_1 = \ker(F_0 \to M), \qquad K_i = \ker(F_{i-1} \to F_{i-2}) \quad \text{for } i \geqslant 2.$$

The nth kernel K_n $(n \geqslant 0)$ is called the *nth $\mathcal{FI}$-syzygy of M*.

On the other hand, every left R-module M has an *FP*-injective preenvelope over any ring R (see [11]). So M has a *right $\mathcal{FI}$-resolution*, that is, there is a $\mathrm{Hom}(-, \mathcal{FI})$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ with each F^i *FP*-injective. Obviously, the complex is exact. Let

$$L^0 = M, \qquad L^1 = \mathrm{coker}\big(M \to F^0\big), \qquad L^i = \mathrm{coker}\big(F^{i-2} \to F^{i-1}\big) \quad \text{for } i \geqslant 2.$$

The nth cokernel L^n $(n \geqslant 0)$ is called the *nth $\mathcal{FI}$-cosyzygy of M*.

Note that $\mathrm{Hom}(-,-)$ is left balanced on ${}_R\mathcal{M} \times {}_R\mathcal{M}$ by $\mathcal{FI} \times \mathcal{FI}$ for a left coherent ring R (see [11, Definition 8.2.13]). Thus the *nth left derived functor* of $\mathrm{Hom}(-,-)$, which is denoted by $\mathrm{Ext}_n(-,-)$, can be computed using a right $\mathcal{FI}$-resolution of the first variable or a left $\mathcal{FI}$-resolution of the second variable. Following [11, Definition 8.4.1], the *left $\mathcal{FI}$-dimension of a left R-module M*, denoted by left $\mathcal{FI}$-dim M, is defined as $\inf\{n\colon$ there is a left $\mathcal{FI}$-resolution of the form $0 \to F_n \to \cdots \to F_0 \to M \to 0$ of $M\}$. If there is no such n, set left $\mathcal{FI}$-dim $M = \infty$. The *global left $\mathcal{FI}$-dimension of ${}_R\mathcal{M}$*, denoted by *gl* left $\mathcal{FI}$-$\dim_R \mathcal{M}$, is defined to be $\sup\{$left $\mathcal{FI}$-dim $M\colon M \in {}_R\mathcal{M}\}$ and is infinite otherwise. The right versions can be defined similarly.

Recall that a left R-module M is called *reduced* [11] if M has no nonzero injective submodules.

In Section 2 of this paper, we introduce the concepts of *FI*-injective and *FI*-flat modules. It is shown that a left R-module M is *FI*-injective if and only if M is a kernel of an *FP*-injective precover $A \to B$ with A injective. For a left coherent ring R, we prove that a left R-module M is *FI*-injective if and only if M is a direct sum of an injective left R-module and a reduced *FI*-injective left R-module; a finitely presented right R-module M is *FI*-flat if and only if M is a cokernel of a flat preenvelope of a right R-module.

In Section 3, we investigate the *FP*-injective dimensions of modules and rings in terms of *FI*-injective and *FI*-flat modules and the left derived functors $\mathrm{Ext}_n(-,-)$. Let R be a left coherent ring. We first give some characterizations of left semihereditary rings. It is proven that R is left semihereditary (i.e., $l.FP$-$\dim(R) \leqslant 1$) if and only if the canonical map $\sigma: \mathrm{Ext}_0(M, N) \to \mathrm{Hom}(M, N)$ is a monomorphism for all left R-modules M and N if and only if every *FI*-injective left R-module is injective if and only if every *FI*-flat right R-module is flat. Then it is shown that $l.FP$-$\dim(R) \leqslant n$ $(n \geqslant 2)$ if and only if $\mathrm{Ext}_{n+k}(M, N) = 0$ for all left R-modules M, N and all

$k \geqslant -1$. Moreover, we get that $l.FP\text{-dim}(R) \leqslant n$ $(n \geqslant 2)$ if and only if $\mathrm{Ext}_{n+k}(M, N) = 0$ for all pure-injective left R-modules M, N and all $k \geqslant -1$. Finally we prove that a ring R is left coherent and $l.FP\text{-dim}(R) \leqslant 2$ if and only if every left R-module has an FP-injective cover with the unique mapping property if and only if R is a left coherent ring and $\mathrm{Ext}_k(M, N) = 0$ for all left R-modules M, N and all $k \geqslant 1$ if and only if R is a left coherent ring and every finitely presented FI-flat right R-module has an epic flat (pre)envelope.

Throughout this paper, R is an associative ring with identity and all modules are unitary. M_R $({}_RM)$ denotes a right (left) R-module. For an R-module M, $E(M)$ stands for the injective envelope of M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, and $id(M)$ $(fd(M))$ is the injective (flat) dimension of M. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (respectively $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (respectively $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (respectively $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (respectively $\mathrm{Tor}^R_n(M, N)$) for an integer $n \geqslant 1$. For unexplained concepts and notations, we refer the reader to [11,18,20].

2. *FI*-injective modules and *FI*-flat modules

We begin with the following

Definition 2.1. A left R-module M is called *FI-injective* if $\mathrm{Ext}^1(G, M) = 0$ for any FP-injective left R-module G.

A right R-module N is said to be *FI-flat* if $\mathrm{Tor}_1(N, G) = 0$ for any FP-injective left R-module G.

Remark 2.2. (1) We note that any FI-injective left R-module is copure injective in sense of [9] and any FI-flat right R-module is copure flat in sense of [10]. If R is a left noetherian ring, then FI-injective left R-modules and FI-flat right R-modules coincide with copure injective left R-modules and copure flat right R-modules, respectively.

(2) A right R-module M is FI-flat if and only if M^+ is FI-injective by the standard isomorphism: $\mathrm{Ext}^1(N, M^+) \cong \mathrm{Tor}_1(M, N)^+$ for any left R-module N.

Proposition 2.3. *The following hold for a left coherent ring R:*

(1) *A left R-module M is injective if and only if M is FI-injective and $FP\text{-}id(M) \leqslant 1$.*
(2) *A right R-module N is flat if and only if N is FI-flat and $fd(N) \leqslant 1$.*

Proof. (1) “Only if” part is trivial.

“If” part. Let M be an FI-injective left R-module and $FP\text{-}id(M) \leqslant 1$. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Note that L is FP-injective by [19, Lemma 3.1] since R is a left coherent ring. So the exact sequence is split, and hence M is injective.

(2) “Only if” part is trivial.

“If” part. For any FI-flat right R-module N with $fd(N) \leqslant 1$, we have N^+ is FI-injective by Remark 2.2(2). Thus N^+ is injective by (1) since $FP\text{-}id(N^+) \leqslant 1$. So N is flat. $\square$

Proposition 2.4. *The following are equivalent for a left R-module M:*

(1) *M is FI-injective.*

(2) *For every exact sequence* $0 \to M \to E \to L \to 0$*, where* E *is FP-injective,* $E \to L$ *is an FP-injective precover of* L.
(3) M *is a kernel of an FP-injective precover* $f : A \to B$ *with* A *injective.*
(4) M *is injective with respect to every exact sequence* $0 \to A \to B \to C \to 0$*, where* C *is FP-injective.*

Proof. (1) ⇒ (2) and (1) ⇒ (4) are clear by definitions.

(2) ⇒ (3) is obvious since there exists a short exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$.

(3) ⇒ (1). Let M be a kernel of an *FP*-injective precover $f : A \to B$ with A injective. Then we have an exact sequence $0 \to M \to A \to A/M \to 0$. So, for any *FP*-injective left R-module N, the sequence $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to \mathrm{Ext}^1(N, M) \to 0$ is exact. It is easy to verify that $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to 0$ is exact by (3). Thus $\mathrm{Ext}^1(N, M) = 0$, and so (1) follows.

(4) ⇒ (1). For each *FP*-injective left R-module N, there exists a short exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1(N, M) \to 0$. Note that $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(N, M) = 0$, as desired. □

Proposition 2.5. *Let* R *be a left coherent ring. Then the following are equivalent for a left* R*-module* M:

(1) M *is a reduced FI-injective left* R*-module.*
(2) M *is a kernel of an FP-injective cover* $f : A \to B$ *with* A *injective.*

Proof. (1) ⇒ (2). By Proposition 2.4, the natural map $\pi : E(M) \to E(M)/M$ is an *FP*-injective precover. Note that $E(M)/M$ has an *FP*-injective cover, and $E(M)$ has no nonzero direct summand K contained in M since M is reduced. It follows that $\pi : E(M) \to E(M)/M$ is an *FP*-injective cover by [20, Corollary 1.2.8], and hence (2) follows.

(2) ⇒ (1). Let M be a kernel of an *FP*-injective cover $\alpha : A \to B$ with A injective. By Proposition 2.4, M is *FI*-injective. Now let K be an injective submodule of M. Suppose $A = K \oplus L$, $p : A \to L$ is the projection and $i : L \to A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$ since $\alpha(K) = 0$. Therefore ip is an isomorphism since α is a cover. Thus i is epic, and hence $A = L$, $K = 0$. So M is reduced. □

Theorem 2.6. *Let* R *be a left coherent ring. Then a left* R*-module* M *is FI-injective if and only if* M *is a direct sum of an injective left* R*-module and a reduced FI-injective left* R*-module.*

Proof. “If” part is clear.

“Only if” part. Let M be an *FI*-injective left R-module. Consider the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Note that $E(M) \to E(M)/M$ is an *FP*-injective precover of $E(M)/M$ by Proposition 2.4. But $E(M)/M$ has an *FP*-injective cover $L \to E(M)/M$, so we have the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\phi"'] & L \arrow[r] \arrow[d, "\gamma"'] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"'] \arrow[d, "\sigma"'] & E(M) \arrow[r] \arrow[d, "\beta"'] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "f"] & L \arrow[r] & E(M)/M \arrow[r] & 0.
\end{tikzcd}$$

Note that $\beta\gamma$ is an isomorphism, and so $E(M) = \ker(\beta) \oplus \mathrm{im}(\gamma)$. Thus L and $\ker(\beta)$ are injective (for $\mathrm{im}(\gamma) \cong L$). Therefore K is a reduced *FI*-injective module by Proposition 2.5. Since $\sigma\phi$ is an isomorphism by the Five Lemma, we have $M = \ker(\sigma) \oplus \mathrm{im}(\phi)$, where $\mathrm{im}(\phi) \cong K$. In addition, we get the commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\sigma) \arrow[r] \arrow[d] & \ker(\beta) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"'] \arrow[d, "\sigma"'] & E(M) \arrow[r] \arrow[d, "\beta"'] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "f"] \arrow[d] & L \arrow[r] \arrow[d] & E(M)/M \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence $\ker(\sigma) \cong \ker(\beta)$ by the 3×3 Lemma [18, Exercise 6.16, p.175]. This completes the proof. □

It is well known that R is a left coherent ring if and only if every right R-module has a flat preenvelope (see [7]). Here we have

Proposition 2.7. *Let R be a left coherent ring.*

(1) *If L is a cokernel of a flat preenvelope $f: K \to F$ of a right R-module K, then L is FI-flat.*
(2) *If M is a finitely presented FI-flat right R-module, then M is a cokernel of a flat preenvelope.*

Proof. (1) There is an exact sequence $0 \to \mathrm{im}(f) \xrightarrow{i} F \to L \to 0$. It is clear that $i : \mathrm{im}(f) \to F$ is a flat preenvelope. For any *FP*-injective left R-module N, N^+ is flat by [12, Theorem 2.2]. Thus we obtain an exact sequence $\mathrm{Hom}(F, N^+) \to \mathrm{Hom}(\mathrm{im}(f), N^+) \to 0$, which yields the exactness of $(F \otimes N)^+ \to (\mathrm{im}(f) \otimes N)^+ \to 0$. So the sequence $0 \to \mathrm{im}(f) \otimes N \to F \otimes N$ is exact. But the flatness of F implies the exactness of $0 \to \mathrm{Tor}_1(L, N) \to \mathrm{im}(f) \otimes N \to F \otimes N$, and hence $\mathrm{Tor}_1(L, N) = 0$.

(2) Let M be a finitely presented FI-flat right R-module. There is an exact sequence $0 \to K \to P \to M \to 0$ with P projective and both P and K finitely generated. We claim that $K \to P$ is a flat preenvelope. In fact, for any flat right R-module F, we have $\mathrm{Tor}_1(M, F^+) = 0$, and so we get the following commutative diagram with the first row exact:

$$\begin{tikzcd} 0 \arrow[r] & K \otimes F^+ \arrow[r, "\alpha"] \arrow[d, "\tau_{K,F}"'] & P \otimes F^+ \arrow[d, "\tau_{P,F}"'] \\ & \mathrm{Hom}(K, F)^+ \arrow[r, "\theta"] & \mathrm{Hom}(P, F)^+. \end{tikzcd}$$

Note that $\tau_{K,F}$ is an epimorphism and $\tau_{P,F}$ is an isomorphism by [4, Lemma 2]. Thus θ is a monomorphism, and hence $\mathrm{Hom}(P, F) \to \mathrm{Hom}(K, F)$ is epic, as required. $\square$

Recall that R is said to be a *QF ring* if R is left noetherian and ${}_RR$ is injective.

Proposition 2.8. *R is a QF ring if and only if every left R-module is FI-injective.*

Proof. It follows from the fact that R is a *QF* ring if and only if every (*FP*-)injective left R-module is projective. $\square$

Recall that R is called a *left IF ring* [4] if every injective left R-module is flat.

Proposition 2.9. *The following are equivalent for a ring R*:

(1) *R is a left IF ring.*
(2) *Every pure-injective left R-module is FI-injective.*
(3) *Every right R-module is FI-flat.*
(4) *Every finitely presented right R-module is FI-flat.*

Proof. (1) $\Rightarrow$ (2). Let M be any pure-injective left R-module. For any FP-injective left R-module N, there is a pure exact sequence $0 \to N \to E \to L \to 0$ with E injective. So N is flat since E is flat. On the other hand, there is an exact sequence $0 \to K \to P \to N \to 0$ with P projective. Note that the sequence is also pure since N is flat. Thus the sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to 0$ is exact, and so $\mathrm{Ext}^1(N, M) = 0$. Therefore, M is FI-injective.

(2) $\Rightarrow$ (3). Let M be a right R-module. Then M^+ is pure-injective, and so it is FI-injective by (2). Thus M is FI-flat by Remark 2.2(2).

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). Let E be an injective left R-module. Then $\mathrm{Tor}_1(M, E) = 0$ for any finitely presented right R-module M by (4). So E is flat. $\square$

We shall say that a right R-module M is *strongly FI-flat* if $\mathrm{Tor}_i(M, G) = 0$ for all FP-injective left R-modules G and all $i \geqslant 1$. Similarly, a left R-module N will be called *strongly FI-injective* if $\mathrm{Ext}^i(G, N) = 0$ for all FP-injective left R-modules G and all $i \geqslant 1$.

Theorem 2.10. *Let R be a left and right coherent ring. Consider the following conditions*:

(1) *FP-id*$(R_R) \leqslant 1$.
(2) *Every submodule of an FI-flat right R-module is FI-flat.*
(3) *Every FI-flat right R-module is strongly FI-flat.*
(4) *Every FI-injective left R-module is strongly FI-injective.*
(5) *Every quotient of an FI-injective left R-module is FI-injective.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftarrow$ (4) $\Leftarrow$ (5). (1) $\Rightarrow$ (5) *holds in case R is a left perfect ring.*

Proof. (1) $\Rightarrow$ (2). Let A be a submodule of an *FI*-flat right R-module B and M an *FP*-injective left R-module. Then one gets an exact sequence $\mathrm{Tor}_2(B/A, M) \to \mathrm{Tor}_1(A, M) \to \mathrm{Tor}_1(B, M) = 0$. On the other hand, there is a pure exact sequence $0 \to M \to \Pi(R_R)^+$ since $(R_R)^+$ is a cogenerator in ${}_R\mathcal{M}$. Thus we get a split exact sequence $(\Pi(R_R)^+)^+ \to M^+ \to 0$. Note that $fd((R_R)^+) = FP\text{-}id(R_R) \leqslant 1$ by [12, Theorem 2.2], and so $fd(\Pi(R_R)^+) \leqslant 1$ since R is right coherent. It follows that $FP\text{-}id((\Pi(R_R)^+)^+) = fd(\Pi(R_R)^+) \leqslant 1$ by [12, Theorem 2.1]. Hence $fd(M) = FP\text{-}id(M^+) \leqslant 1$. Thus $\mathrm{Tor}_2(B/A, M) = 0$, and so $\mathrm{Tor}_1(A, M) = 0$. Therefore, A is *FI*-flat.

(2) $\Rightarrow$ (3). Let M be an *FI*-flat right R-module. Then there is an exact sequence $0 \to K \to P \to M \to 0$ with P projective. So K is *FI*-flat by (2). Thus M is strongly *FI*-flat by induction.

(3) $\Rightarrow$ (1). Let M be a right R-module. Then there is an exact sequence $0 \to K \to P \to M \to 0$ with P projective. Note that K has a flat preenvelope $f : K \to F$ since R is left coherent. So f is a monomorphism, and we get an exact sequence $0 \to K \to F \to L \to 0$, where L is *FI*-flat by Proposition 2.7. Thus L is strongly *FI*-flat by (3), and so K is *FI*-flat. There is an induced exact sequence $0 = \mathrm{Tor}_2(P, (R_R)^+) \to \mathrm{Tor}_2(M, (R_R)^+) \to \mathrm{Tor}_1(K, (R_R)^+) = 0$. Thus $\mathrm{Tor}_2(M, (R_R)^+) = 0$ and hence $fd((R_R)^+) \leqslant 1$. So $FP\text{-}id(R_R) \leqslant 1$ by [12, Theorem 2.2].

(5) $\Rightarrow$ (4). Let M be an *FI*-injective left R-module. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. So L is *FI*-injective by (5). It is easy to check that M is strongly *FI*-injective by induction.

(4) $\Rightarrow$ (3) holds by Remark 2.2(2) and the standard isomorphism: $\mathrm{Ext}^n(N, M^+) \cong \mathrm{Tor}_n(M, N)^+$ for any right R-module M, any left R-module N and any $n \geqslant 1$ (see [18, p. 360]).

(1) $\Rightarrow$ (5). Suppose that R is a left perfect ring. Then the projective (flat) dimension of any *FP*-injective left R-module is at most 1 by the proof of (1) $\Rightarrow$ (2). So (5) holds. $\square$

3. *FP*-injective dimensions and the left derived functors of Hom

As is mentioned in the introduction, if R is a left coherent ring, then $\mathrm{Hom}(-,-)$ is left balanced on ${}_R\mathcal{M} \times {}_R\mathcal{M}$ by $\mathcal{FI} \times \mathcal{FI}$. Let $\mathrm{Ext}_n(-,-)$ denote the nth left derived functor of $\mathrm{Hom}(-,-)$ with respect to the pair $\mathcal{FI} \times \mathcal{FI}$. Then, for two left R-modules M and N, $\mathrm{Ext}_n(M, N)$ can be computed using a right $\mathcal{FI}$-resolution of M or a left $\mathcal{FI}$-resolution of N.

Let $0 \to M \xrightarrow{g} F^0 \xrightarrow{f} F^1 \to \cdots$ be a right $\mathcal{FI}$-resolution of M. Applying $\mathrm{Hom}(-, N)$, we obtain the deleted complex $\cdots \to \mathrm{Hom}(F^1, N) \xrightarrow{f^*} \mathrm{Hom}(F^0, N) \to 0$. Then $\mathrm{Ext}_n(M, N)$ is exactly the nth homology of the complex above. There is a canonical map

$$\sigma : \mathrm{Ext}_0(M, N) = \mathrm{Hom}(F^0, N)/\mathrm{im}(f^*) \to \mathrm{Hom}(M, N)$$

defined by $\sigma(\alpha + \mathrm{im}(f^*)) = \alpha g$ for $\alpha \in \mathrm{Hom}(F^0, N)$.

Proposition 3.1. *Let R be a left coherent ring. The following are equivalent for a left R-module M:*

(1) *M is FP-injective.*
(2) *The canonical map $\sigma : \mathrm{Ext}_0(M, N) \to \mathrm{Hom}(M, N)$ is an epimorphism for any left R-module N.*
(3) *The canonical map $\sigma : \mathrm{Ext}_0(M, M) \to \mathrm{Hom}(M, M)$ is an epimorphism.*

Proof. (1) $\Rightarrow$ (2) is obvious by letting $F^0 = M$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). By (3), there exists $\alpha \in \mathrm{Hom}(F^0, M)$ such that $\sigma(\alpha + \mathrm{im}(f^*)) = \alpha g = 1_M$. Thus M is isomorphic to a direct summand of F^0, and hence it is *FP*-injective. $\square$

Corollary 3.2. *The following are equivalent for a left coherent ring R:*

(1) *${}_R R$ is FP-injective.*
(2) *The canonical map $\sigma : \mathrm{Ext}_0({}_R R, N) \to \mathrm{Hom}({}_R R, N)$ is an epimorphism for any left R-module N.*
(3) *The canonical map $\sigma : \mathrm{Ext}_0({}_R R, {}_R R) \to \mathrm{Hom}({}_R R, {}_R R)$ is an epimorphism.*
(4) *Every (finitely presented) left R-module has an epic FP-injective cover.*
(5) *Every (finitely presented) right R-module has a monic flat preenvelope.*
(6) *Every (finitely presented) right R-module is a submodule of a flat right R-module.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Proposition 3.1.

(1) $\Rightarrow$ (4). Let M be a left R-module, then M has an *FP*-injective cover g. On the other hand, there is an exact sequence $F \to M \to 0$ with F free. Since F is *FP*-injective by (1), g is an epimorphism.

(4) $\Rightarrow$ (1). Let $f : N \to_R R$ be an epic *FP*-injective cover. Then ${}_R R$ is isomorphic to a direct summand of N, and so ${}_R R$ is *FP*-injective.

(1) $\Rightarrow$ (5). Note that R is a right *IF* ring by [4, Theorem 1], and so (5) follows.

(5) $\Rightarrow$ (1) is clear by [14, Theorem 2.3] since every finitely presented right R-module is torsionless.

(5) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (5) follows since R is a left coherent ring. $\square$

Proposition 3.3. *Let R be a left coherent ring. Then the following are equivalent for a left R-module M:*

(1) *right $\mathcal{FI}$-dim $M \leqslant 1$.*
(2) *The canonical map $\sigma : \mathrm{Ext}_0(M, N) \to \mathrm{Hom}(M, N)$ is a monomorphism for any left R-module N.*

Proof. (1) $\Rightarrow$ (2). By (1), M has a right $\mathcal{FI}$-resolution $0 \to M \to F^0 \to F^1 \to 0$. Thus we get an exact sequence $0 \to \mathrm{Hom}(F^1, N) \to \mathrm{Hom}(F^0, N) \to \mathrm{Hom}(M, N)$ for any left R-module N. Hence σ is a monomorphism.

(2) $\Rightarrow$ (1). Consider the exact sequence $0 \to M \to F^0 \to L^1 \to 0$, where $M \to F^0$ is an *FP*-injective preenvelope. We only need to show that L^1 is *FP*-injective. By [11, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & \mathrm{Ext}_0\big(L^1, L^1\big) & \longrightarrow & \mathrm{Ext}_0\big(F^0, L^1\big) & \longrightarrow & \mathrm{Ext}_0\big(M, L^1\big) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_3} & & \\ 0 & \longrightarrow & \mathrm{Hom}\big(L^1, L^1\big) & \longrightarrow & \mathrm{Hom}\big(F^0, L^1\big) & \longrightarrow & \mathrm{Hom}\big(M, L^1\big). & & \end{array}$$

Note that σ_2 is an epimorphism by Proposition 3.1 and σ_3 is a monomorphism by (2). Hence σ_1 is an epimorphism by the Snake Lemma [18, Theorem 6.5]. Thus L^1 is *FP*-injective by Proposition 3.1, and so (1) follows. $\square$

Let $wD(R)$ denote the weak global dimension of a ring R. We have the following lemma which will be needed frequently.

Lemma 3.4. *Let R be a left coherent ring. Then*

(1) *right $\mathcal{FI}$-dim $M = FP$-$id(M)$ for any left R-module M;*
(2) *$wD(R) = l.FP$-dim$(R) = gl$ right $\mathcal{FI}$-dim$_R\,\mathcal{M}$.*

Proof. (1) It is clear that FP-$id(M) \leqslant$ right $\mathcal{FI}$-dim M. Conversely, we may assume that FP-$id(M) = n < \infty$. Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1}$ be a partial right $\mathcal{FI}$-resolution of M. Then we get an exact sequence $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to L \to 0$. Therefore, L is *FP*-injective by [19, Lemma 3.1], and so right $\mathcal{FI}$-dim $M \leqslant n$, as desired.

(2) follows from [19, Theorem 3.3] and (1). $\square$

Proposition 3.5. *The following are equivalent for a left coherent ring R:*

(1) *FP-$id({}_RR) \leqslant 1$.*
(2) *The canonical map $\sigma : \mathrm{Ext}_0({}_RR, N) \to \mathrm{Hom}({}_RR, N)$ is a monomorphism for any left R-module N.*
(3) *Every finitely presented FI-flat right R-module has a monic flat preenvelope.*

Proof. (1) $\Leftrightarrow$ (2) holds by Proposition 3.3 and Lemma 3.4.

(1) $\Rightarrow$ (3). Let M be a finitely presented *FI*-flat right R-module. Then M is cokernel of a flat preenvelope $K \to F^0$ of a right R-module K by Proposition 2.7(2). Thus we have a right $\mathcal{F}lat$-resolution

$$0 \to K \to F^0 \to F^1 \to \cdots,$$

where $M = \mathrm{coker}(K \to F^0)$ and $\mathcal{F}lat$ is the class of all flat right R-modules. This resolution is exact at F^0 by (1) and [11, Theorem 8.4.31], and hence M has a monic flat preenvelope.

(3) $\Rightarrow$ (1). Let $0 \to M \xrightarrow{f} F^0 \to F^1 \to \cdots$ be a right $\mathcal{P}roj_{fg}$-resolution of a finitely presented right R-module M, where $\mathcal{P}roj_{fg}$ is the class of all finitely generated projective right R-modules. Then coker(f) is a finitely presented *FI*-flat right R-module by Proposition 2.7(1), and hence it

has a monic flat preenvelope by (3). It follows that the above complex is exact at F^k for $k \geqslant 0$. So (1) holds by [11, Theorem 8.4.31]. □

Lemma 3.6. *Let $\mathcal{C}$ be a class of R-modules and M an R-module.*

(1) *If $F \to M$ and $G \to M$ are $\mathcal{C}$-precovers with kernels K and L, respectively, then $K \oplus G \cong L \oplus F$.*
(2) *If $M \to F$ and $M \to G$ are $\mathcal{C}$-preenvelopes with cokernels K and L, respectively, then $K \oplus G \cong L \oplus F$.*

Proof. (1) follows from [11, Lemma 8.6.3]. The proof of (2) is dual to that of [11, Lemma 8.6.3]. □

Theorem 3.7. *The following are equivalent for a left coherent ring R:*

(1) *R is a left semihereditary ring (i.e. $l.FP$-$\mathrm{dim}(R) \leqslant 1$).*
(2) *$\sigma : \mathrm{Ext}_0(M, N) \to \mathrm{Hom}(M, N)$ is monic for all left R-modules M and N.*
(3) *Every left R-module has a monic FP-injective cover.*
(4) *Every FI-injective left R-module is injective.*
(5) *Every FI-injective left R-module is FP-injective.*
(6) *Every (finitely presented) FI-flat right R-module is flat.*
(7) *Every right R-module has an epic flat (pre)envelope.*
(8) *Every finitely presented right R-module has an epic flat (pre)envelope.*
(9) *The kernel of any FP-injective (pre)cover of a left R-module is FP-injective.*
(10) *The cokernel of any FP-injective preenvelope of a left R-module is FP-injective.*
(11) *The cokernel of any flat preenvelope of a right R-module is flat.*
(12) *The kernel of any flat (pre)cover of a right R-module is flat.*

Proof. (1) ⇔ (2) holds by Proposition 3.3 and Lemma 3.4.

(1) ⇒ (4) follows from Proposition 2.3 and Lemma 3.4.

(4) ⇒ (5) is trivial.

(5) ⇒ (6). Let M be an *FI*-flat right R-module. Then M^+ is *FI*-injective by Remark 2.2(2), and hence M^+ is *FP*-injective by (5). So M is flat by [12, Theorem 2.1].

(6) ⇒ (8). Let M be a finitely presented right R-module. Then M has a flat preenvelope $f : M \to F$ with F finitely generated and projective. It is easy to see that the inclusion $i : \mathrm{im}(f) \to F$ is a flat preenvelope. Thus $F/\mathrm{im}(f)$ is finitely presented and *FI*-flat by Proposition 2.7(1), and hence it is flat by (6). It follows that $\mathrm{im}(f)$ is flat, and $M \to \mathrm{im}(f)$ is an epic flat (pre)envelope.

(8) ⇒ (7). Let M be any right R-module. Then $M = \varinjlim M_i$ with M_i finitely presented for each i. By (8), each M_i has an epic flat (pre)envelope $M_i \to F_i$. It is easy to see that $\{F_i\}$ is a direct system and $M \to \varinjlim F_i$ is an epic flat (pre)envelope.

(1) ⇒ (3). Let M be a left R-module. Then M has an *FP*-injective cover $f : N \to M$. Note that $\mathrm{im}(f)$ is *FP*-injective by (1) and [15, Theorem 2]. So the inclusion $\mathrm{im}(f) \to M$ is a monic *FP*-injective cover.

(3) ⇒ (9). Let $f : F \to M$ be an *FP*-injective precover of a left R-module M and $K = \ker(f)$. Since there exists a monic *FP*-injective cover $g : G \to M$ by (3), we have $K \oplus G \cong F$ by Lemma 3.6(1). So K is *FP*-injective.

(9) $\Rightarrow$ (1). It is enough to show that any quotient of an *FP*-injective left R-module is *FP*-injective. Let M be a quotient of an *FP*-injective left R-module. Note that M has an *FP*-injective cover $f : F \to M$. So f is an epimorphism. Since $\ker(f)$ is *FP*-injective by (9), M is *FP*-injective by [19, Lemma 3.1] (for R is a left coherent ring).

(1) $\Leftrightarrow$ (10) follows from Lemma 3.4.

(7) $\Rightarrow$ (11). The proof is dual to that of (3) $\Rightarrow$ (9).

(11) $\Rightarrow$ (1). By a proof dual to that of (9) $\Rightarrow$ (1), we can show that any submodule of a flat right R-module is flat. Thus R is a left semihereditary ring.

(1) $\Leftrightarrow$ (12) is obvious. $\square$

Remark 3.8. We note that the equivalences of (1), (3), (7) and (8) were known earlier (see [1,3,6,8,17]).

As an immediate consequence of the above theorem, we have the following result which was proven in a different way by Enochs and Jenda (see [9, Corollary 2.4]).

Corollary 3.9. *Let R be a left noetherian ring. Then R is a left hereditary ring if and only if every copure injective left R-module is injective.*

Proposition 3.10. *Let R be a left coherent ring and an integer $n \geqslant 2$. The following are equivalent for a left R-module M:*

(1) *right* $\mathcal{FI}$-$\dim M \leqslant n$.
(2) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules N and all $k \geqslant -1$.*
(3) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules N.*

Proof. (1) $\Rightarrow$ (2). Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^n \to 0$ be a right $\mathcal{FI}$-resolution of M, which induces an exact sequence

$$0 \to \mathrm{Hom}\big(F^n, N\big) \to \mathrm{Hom}\big(F^{n-1}, N\big) \to \mathrm{Hom}\big(F^{n-2}, N\big)$$

for any left R-module N. Hence $\mathrm{Ext}_n(M, N) = \mathrm{Ext}_{n-1}(M, N) = 0$. Note that it is clear that $\mathrm{Ext}_{n+k}(M, N) = 0$ for all $k \geqslant 1$. Then (2) holds.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $0 \to M \to F^0 \to F^1 \to \cdots$ be a right $\mathcal{FI}$-resolution of M with $L^n = \mathrm{coker}(F^{n-2} \to F^{n-1})$. We only need to show that L^n is *FP*-injective. In fact, we have the following exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & F^0 \arrow[r] & \cdots \arrow[r] & F^{n-2} \arrow[r, "f"] & F^{n-1} \arrow[rr, "g"] \arrow[dr, "\pi"'] & & F^n \arrow[r] & \cdots \\
& & & & & & L^n \arrow[ur, "\lambda"'] \arrow[dr] & & \\
& & & & & 0 \arrow[ur] & & 0. &
\end{tikzcd}$$

By (3), $\mathrm{Ext}_{n-1}(M, L^n) = 0$. Thus the sequence

$$\mathrm{Hom}(F^n, L^n) \xrightarrow{g^*} \mathrm{Hom}(F^{n-1}, L^n) \xrightarrow{f^*} \mathrm{Hom}(F^{n-2}, L^n)$$

is exact. Since $f^*(\pi) = \pi f = 0$, $\pi \in \ker(f^*) = \mathrm{im}(g^*)$. Thus there exists $h \in \mathrm{Hom}(F^n, L^n)$ such that $\pi = g^*(h) = hg = h\lambda\pi$, and hence $h\lambda = 1$ since π is epic. Therefore L^n is *FP*-injective. $\square$

Corollary 3.11. *The following are equivalent for a left coherent ring R and an integer $n \geqslant 2$:*

(1) $l.FP\text{-}\mathrm{dim}(R) \leqslant n$.
(2) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M, N and all $k \geqslant -1$.*
(3) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M and N.*

Proof. It follows from Lemma 3.4 and Proposition 3.10. $\square$

Lemma 3.12. *The following are true for any ring R:*

(1) *A left R-module N is FP-injective if and only if, for every pure-injective left R-module G, every homomorphism $f : N \to G$ factors through an injective left R-module.*
(2) *If M is a pure-injective left R-module, and $f : F \to M$ is an FP-injective cover of M, then F is injective.*

Proof. (1) "Only if" part. There is an exact sequence $0 \to N \xrightarrow{i} E \to L \to 0$ with E injective. Since the exact sequence is pure, there exists $g : E \to G$ such that $gi = f$, as required.

"If" part. It is enough to show that the exact sequence $0 \to N \xrightarrow{i} E(N) \to L \to 0$ is pure. Let H be any right R-module. Then H^+ is pure-injective. For any $f : N \to H^+$, there exist an injective left R-module Q and $g : N \to Q$ and $h : Q \to H^+$ such that $f = hg$ by hypothesis. Thus there exists $\alpha : E(N) \to Q$ such that $g = \alpha i$, and so $f = (h\alpha)i$. Therefore we get an exact sequence $\mathrm{Hom}(E(N), H^+) \to \mathrm{Hom}(N, H^+) \to 0$, which gives the exactness of the sequence $(H \otimes E(N))^+ \to (H \otimes N)^+ \to 0$. It follows that $0 \to H \otimes N \to H \otimes E(N)$ is exact. So N is *FP*-injective.

(2) By (1), there exist an injective left R-module E and $g : F \to E$ and $h : E \to M$ such that $f = hg$. So there exists $\varphi : E \to F$ such that $f\varphi = h$ since f is a cover. Therefore $f\varphi g = f$ and hence φg is an isomorphism. It follows that F is isomorphic to a direct summand of E, and so F is injective. $\square$

Lemma 3.13. *Let R be a left coherent ring. If M is an FI-injective left R-module, then there exists an FP-injective cover $N \to M$ with N injective.*

Proof. M has an *FP*-injective cover $f : N \to M$ since R is left coherent. Consider the short exact sequence $0 \to N \xrightarrow{i} E \to L \to 0$ with E injective. Note that L is *FP*-injective by [19, Lemma 3.1] since R is left coherent. So there exists $g : E \to M$ such that $gi = f$ since M is *FI*-injective. Thus there exists $h : E \to N$ such that $fh = g$ since f is a cover. Therefore $fhi = f$, and hence hi is an isomorphism. It follows that N is injective, as desired. $\square$

Corollary 3.14. *Let R be a left coherent ring. If M is a pure-injective left R-module, then M has a minimal left $\mathcal{FI}$-resolution $\cdots \to F_{n-2} \to F_{n-3} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i injective.*

Proof. By Lemma 3.12, M has an FP-injective cover $f : F_0 \to M$ with F_0 injective. Note that $\ker(f)$ is FI-injective by Proposition 2.4. Hence $\ker(f)$ has an FP-injective cover $g : F_1 \to \ker(f)$ with F_1 injective by Lemma 3.13. Note that $\ker(g)$ is FI-injective by Proposition 2.4. So we can continue the above process to get the desired minimal left $\mathcal{FI}$-resolution of M. $\square$

Theorem 3.15. *Let R be a left coherent ring. Consider the following conditions for a left R-module N and an integer $n \geqslant 2$:*

(1) *left* $\mathcal{FI}\text{-dim}\, N \leqslant n-2$.
(2) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M and all $k \geqslant -1$.*
(3) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *The converses hold if N is pure-injective.*

Proof. (1) $\Rightarrow$ (2). By (1), N has a left $\mathcal{FI}$-resolution

$$0 \to F_{n-2} \to \cdots \to F_1 \to F_0 \to N \to 0.$$

Then we have the following complex

$$0 \to \mathrm{Hom}(M, F_{n-2}) \to \mathrm{Hom}(M, F_{n-3}) \to \cdots \to \mathrm{Hom}(M, F_0) \to 0$$

for any left R-module M. Hence $\mathrm{Ext}_{n+k}(M, N) = 0$ for all $k \geqslant -1$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Since N is pure-injective, N has a minimal left $\mathcal{FI}$-resolution:

$$\cdots \to F_n \xrightarrow{f} F_{n-1} \xrightarrow{g} F_{n-2} \xrightarrow{h} F_{n-3} \xrightarrow{j} \cdots \to F_1 \to F_0 \to N \to 0$$

with each F_i injective by Corollary 3.14. Put $K = \ker(g)$, $H = F_{n-1}/K$. Let $\lambda : K \to F_{n-1}$ be the inclusion and $\pi : F_{n-1} \to H$ the canonical projection. Then there exists $p : F_n \to K$ such that $f = \lambda p$ and there exists a monomorphism $\alpha : H \to F_{n-2}$ such that $g = \alpha\pi$. Put $L = F_{n-2}/\operatorname{im}(\alpha)$ and let $\beta : F_{n-2} \to L$ be the canonical projection. Then there exists a homomorphism $i : L \to F_{n-3}$ such that $h = i\beta$. So we have the following commutative diagram:

$$\begin{tikzcd}
F_n \arrow[rr, "f"] \arrow[dr, "p"] & & F_{n-1} \arrow[rr, "g"] \arrow[dr, "\pi"] & & F_{n-2} \arrow[rr, "h"] \arrow[dr, "\beta"] & & F_{n-3} \\
& K \arrow[ur, "\lambda"] \arrow[dr] & & H \arrow[ur, "\alpha"] \arrow[dr] & & L \arrow[ur, "i"] \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0.
\end{tikzcd}$$

By (3), $\mathrm{Ext}_{n-1}(K, N) = 0$. Thus the sequence

$$\mathrm{Hom}(K, F_n) \xrightarrow{f_*} \mathrm{Hom}(K, F_{n-1}) \xrightarrow{g_*} \mathrm{Hom}(K, F_{n-2})$$

is exact. Since $g_*(\lambda) = g\lambda = 0$, $\lambda \in \ker(g_*) = \mathrm{im}(f_*)$. So $\lambda = f_*(l) = fl$ for some $l \in \mathrm{Hom}(K, F_n)$. But $f = \lambda p$, and hence $\lambda = \lambda pl$. Thus $pl = 1$ since λ is monic, and so K is injective. It follows that H and L are injective. We claim that the complex

$$0 \to L \xrightarrow{i} F_{n-3} \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a left $\mathcal{FI}$-resolution of N. In fact, it is enough to show that the complex

$$0 \longrightarrow \mathrm{Hom}(G, L) \xrightarrow{i_*} \mathrm{Hom}(G, F_{n-3}) \xrightarrow{j_*} \mathrm{Hom}(G, F_{n-4})$$

is exact for any *FP*-injective left R-module G. Note that we have the following exact commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(G, F_{n-1}) \arrow[rr, "g_*"] \arrow[dr, "\pi_*"] & & \mathrm{Hom}(G, F_{n-2}) \arrow[rr, "h_*"] \arrow[dr, "\beta_*"] & & \mathrm{Hom}(G, F_{n-3}) \\
& \mathrm{Hom}(G, H) \arrow[ur, "\alpha_*"] \arrow[dr] & & \mathrm{Hom}(G, L) \arrow[ur, "i_*"] \arrow[dr] & \\
0 \arrow[ur] & & 0 & & 0.
\end{tikzcd}$$

So $\ker(i_*\beta_*) = \ker(h_*) = \mathrm{im}(g_*) = \mathrm{im}(\alpha_*\pi_*) = \mathrm{im}(\alpha_*) = \ker(\beta_*)$. Let $\theta \in \ker(i_*)$. Since β_* is epic, $\theta = \beta_*(\gamma)$ for some $\gamma \in \mathrm{Hom}(G, F_{n-2})$. Thus $i_*\beta_*(\gamma) = 0$, and hence $\theta = \beta_*(\gamma) = 0$. It follows that i_* is monic. On the other hand, $\ker(j_*) = \mathrm{im}(h_*) = \mathrm{im}(i_*)$. So we obtain the desired exact sequence. This completes the proof. □

Corollary 3.16. *Consider the following conditions for a left coherent ring R and an integer $n \geqslant 2$:*

(1) *gl left* $\mathcal{FI}$-$\mathrm{dim}_R\, \mathcal{M} \leqslant n - 2$.
(2) $l.FP$-$\mathrm{dim}(R) \leqslant n$.
(3) *left* $\mathcal{FI}$-$\dim N \leqslant n - 2$ *for all pure-injective left R-modules N.*
(4) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M, all pure-injective left R-modules N and all $k \geqslant -1$.*
(5) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M and all pure-injective left R-modules N.*

Then (1) ⇒ (2) ⇒ (3) ⇔ (4) ⇔ (5).

Proof. It follows from Corollary 3.11 and Theorem 3.15. □

Lemma 3.17. *Let R be a left coherent ring. If M is a pure-injective left R-module, then $id(M) \leqslant n$ $(n \geqslant 0)$ if and only if for the minimal left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of any pure-injective left R-module N,* $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ *is an epimorphism.*

Proof. The proof is modeled on that of [11, Lemma 8.4.34].

We will proceed by induction on n. Let $n = 0$. If M is injective, it is clear that $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, K_0)$ is an epimorphism. Conversely, put $N = M$. Then $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, M)$ is an epimorphism, and so M is injective.

Let $n \geqslant 1$. There is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Then we have the following exact commutative diagrams:

$$\begin{tikzcd}
\mathrm{Hom}(E, F_n) \arrow[r] \arrow[d] & \mathrm{Hom}(E, K_n) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}(M, F_n) \arrow[r] \arrow[d] & \mathrm{Hom}(M, K_n) & \\
0 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(L, K_n) \arrow[r] \arrow[d] & \mathrm{Hom}(L, F_{n-1}) \arrow[r] \arrow[d] & \mathrm{Hom}(L, K_{n-1}) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(E, K_n) \arrow[r] \arrow[d] & \mathrm{Hom}(E, F_{n-1}) \arrow[r] \arrow[d] & \mathrm{Hom}(E, K_{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(M, K_n) \arrow[r] & \mathrm{Hom}(M, F_{n-1}) \arrow[r] \arrow[d] & \mathrm{Hom}(M, K_{n-1}). & \\
 & & 0 & &
\end{tikzcd}$$

Note that L is pure-injective by [13, Lemma 3.2.10]. Thus $id(M) \leqslant n$ if and only if $id(L) \leqslant n-1$ if and only if $\mathrm{Hom}(L, F_{n-1}) \to \mathrm{Hom}(L, K_{n-1})$ is an epimorphism by induction if and only if $\mathrm{Hom}(E, K_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the first diagram. □

Theorem 3.18. *Let R be a left coherent ring. Then the following are equivalent for an integer $n \geqslant 2$:*

(1) $l.FP\text{-}\dim(R) \leqslant n$.

(2) *left* $\mathcal{FI}$-dim $N \leqslant n-2$ *for all pure-injective left* R*-modules* N.
(3) $\mathrm{Ext}_{n+k}(M,N)=0$ *for all left* R*-modules* M*, all pure-injective left* R*-modules* N *and all* $k \geqslant -1$.
(4) $\mathrm{Ext}_{n-1}(M,N)=0$ *for all left* R*-modules* M *and all pure-injective left* R*-modules* N.
(5) $\mathrm{Ext}_{n+k}(M,N)=0$ *for all pure-injective left* R*-modules* M *and* N*, and all* $k \geqslant -1$.
(6) $\mathrm{Ext}_{n-1}(M,N)=0$ *for all pure-injective left* R*-modules* M *and* N.
(7) *For the minimal left* $\mathcal{FI}$*-resolution* $\cdots \to F_n \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to N \to 0$ *of any pure-injective left* R*-module* N*,* $\mathrm{Hom}(M,F_n) \to \mathrm{Hom}(M,K_n)$ *is an epimorphism for any pure-injective left* R*-module* M.

Proof. (1) ⇒ (2) ⇒ (3) hold by Corollary 3.16.

(3) ⇒ (4) ⇒ (6), and (3) ⇒ (5) ⇒ (6) are trivial.

(6) ⇒ (7). Let M and N be pure-injective left R-modules and $\cdots \to F_n \xrightarrow{f} F_{n-1} \xrightarrow{g} F_{n-2} \to \cdots \to F_0 \to N \to 0$ be the minimal left $\mathcal{FI}$-resolution of N. Then the sequence

$$\mathrm{Hom}(M,F_n) \xrightarrow{f_*} \mathrm{Hom}(M,F_{n-1}) \xrightarrow{g_*} \mathrm{Hom}(M,F_{n-2})$$

is exact since $\mathrm{Ext}_{n-1}(M,N)=0$. Note that the sequence

$$0 \longrightarrow \mathrm{Hom}(M,K_n) \longrightarrow \mathrm{Hom}(M,F_{n-1}) \xrightarrow{g_*} \mathrm{Hom}(M,F_{n-2})$$

is exact. It is easy to see that the sequence $\mathrm{Hom}(M,F_n) \to \mathrm{Hom}(M,K_n) \to 0$ is exact.

(7) ⇒ (1) follows from [20, Theorem 3.3.2], Lemmas 3.17 and 3.4. □

Recall that a homomorphism $\phi: M \to C$ with $C \in \mathcal{C}$ is said to a $\mathcal{C}$*-envelope with the unique mapping property* [5] if for any homomorphism $f: M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g: C \to C'$ such that $g\phi = f$. Dually we have the definition of a $\mathcal{C}$*-cover with the unique mapping property*.

It has been proven that R is a left coherent ring and $l.FP$-dim$(R) \leqslant 2$ if and only if every right R-module has a flat envelope with the unique mapping property (see [2]). Now we have

Theorem 3.19. *The following are equivalent for a ring* R:

(1) R *is left coherent and* $l.FP$-dim$(R) \leqslant 2$.
(2) *Every left* R*-module has an FP-injective cover with the unique mapping property.*
(3) R *is left coherent and* $\mathrm{Ext}_1(M,N)=0$ *for all left* R*-modules* M *and* N.
(4) R *is left coherent and* $\mathrm{Ext}_k(M,N)=0$ *for all left* R*-modules* M*,* N *and all* $k \geqslant 1$.
(5) R *is left coherent and every finitely presented FI-flat right* R*-module has an epic flat (pre)envelope.*

Proof. (1) ⇔ (3) ⇔ (4) follow from Corollary 3.11.

(1) ⇒ (5). Let M be a finitely presented *FI*-flat right R-module. By the proof of (1) ⇒ (3) in Proposition 3.5, we can construct a right $\mathcal{F}lat$-resolution of a right R-module K:

$$0 \longrightarrow K \xrightarrow{f} F^0 \xrightarrow{g} F^1 \longrightarrow \cdots$$

such that $\mathrm{coker}(f) = M$. Note that the complex is exact at F^i for $i \geqslant 1$ by (1) and [11, Theorem 8.4.31]. So we get an exact sequence

$$0 \longrightarrow \mathrm{im}(g) \longrightarrow F^1 \longrightarrow F^2 \longrightarrow L^3 \longrightarrow 0.$$

Thus $\mathrm{im}(g)$ is flat since $fd(L^3) \leqslant 2$. It follows that $M \to \mathrm{im}(g)$ is an epic flat (pre)envelope.

(5) $\Rightarrow$ (1). For any finitely presented right R-module M, there is a right $\mathcal{P}roj_{fg}$-resolution $0 \to M \to P^0 \to P^1 \to 0$ with P^0 and P^1 finitely generated and projective by (5) and Proposition 2.7(1). Thus *gl* right $\mathcal{P}roj_{fg}$-$\dim \mathcal{M}_{R_{fp}} \leqslant 1$, and so $l.FP\text{-}\dim(R) \leqslant 3$ by [11, Corollary 8.4.28]. Hence $l.FP\text{-}\dim(R) = FP\text{-}id({}_R R)$ by [19, Proposition 3.5]. On the other hand, let $0 \to M \to F^0 \to F^1 \to \cdots$ be any right $\mathcal{P}roj_{fg}$-resolution of a finitely presented right R-module M. Since L^1 has an epic flat preenvelope $L^1 \to G$ by (5) and $L^1 \to F^1$ is a flat preenvelope with $L^2 = \mathrm{coker}(L^1 \to F^1)$, we have $G \oplus L^2 \cong F^1$ by Lemma 3.6(2). Hence L^2 is finitely generated and projective. It follows that the above complex is exact at F^i for $i \geqslant 1$, and so $FP\text{-}id({}_R R) \leqslant 2$ by [11, Theorem 8.4.31]. Therefore, $l.FP\text{-}\dim(R) \leqslant 2$.

(1) $\Rightarrow$ (2). Let M be any left R-module. Then M has an FP-injective cover $f : F \to M$. It is enough to show that, for any FP-injective left R-module G and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\mathrm{im}(g) \to M$ such that $\beta\pi = f$ since $\mathrm{im}(g) \subseteq \ker(f)$, where $\pi : F \to F/\mathrm{im}(g)$ is the natural map. Since $l.FP\text{-}\dim(R) \leqslant 2$, $F/\mathrm{im}(g)$ is FP-injective. Thus there exists $\alpha : F/\mathrm{im}(g) \to F$ such that $\beta = f\alpha$, and so we get the commutative diagram with an exact row:

$$\begin{tikzcd}
0 \arrow[r] & \ker(g) \arrow[r, "i"] & G \arrow[r, "g"] \arrow[dr, "0"'] & F \arrow[r, shift left, "\pi"] \arrow[d, "f"] & F/\mathrm{im}(g) \arrow[l, shift left, "\alpha"] \arrow[dl, "\beta"] \arrow[r] & 0. \\
 & & & M & &
\end{tikzcd}$$

Thus $f\alpha\pi = f$, and hence $\alpha\pi$ is an isomorphism. Therefore, π is monic, and so $g = 0$.

(2) $\Rightarrow$ (1). We first prove that R is a left coherent ring. Let $\{C_i, \varphi^i_j\}$ be a direct system with each C_i FP-injective. By hypothesis, $\varinjlim C_i$ has an FP-injective cover $\alpha : E \to \varinjlim C_i$ with the unique mapping property. Let $\alpha_i : C_i \to \varinjlim C_i$ satisfy $\alpha_i = \alpha_j \varphi^i_j$ whenever $i \leqslant j$. Then there exists $f_i : C_i \to E$ such that $\alpha_i = \alpha f_i$ for any i. It follows that $\alpha f_i = \alpha f_j \varphi^i_j$, and so $f_i = f_j \varphi^i_j$ whenever $i \leqslant j$. Therefore, by the definition of direct limits, there exists $\beta : \varinjlim C_i \to E$ such that the following diagram is commutative:

$$\begin{tikzcd}
\varinjlim C_i \arrow[rr, "\beta"] & & E \arrow[rr, "\alpha"] & & \varinjlim C_i. \\
 & & C_i \arrow[ull, "\alpha_i"'] \arrow[u, "f_i"] \arrow[urr, "\alpha_i"] \arrow[d, "\varphi^i_j"] & & \\
 & & C_j \arrow[uull, "\alpha_j"] \arrow[uu, bend right, "f_j"'] \arrow[uurr, "\alpha_j"'] & &
\end{tikzcd}$$

Thus $f_i = \beta\alpha_i$, and so $(\alpha\beta)\alpha_i = \alpha(\beta\alpha_i) = \alpha f_i = \alpha_i$ for any i. Therefore $\alpha\beta = 1_{\varinjlim C_i}$ by the definition of direct limits, and hence $\varinjlim C_i$ is a direct summand of E. So $\varinjlim C_i$ is FP-injective. Thus R is a left coherent ring by [19, Theorem 3.2].

Next we prove that $l.FP\text{-}\dim(R) \leqslant 2$. Let M be any left R-module. Then M has an FP-injective cover $f : F \to M$ with the unique mapping property. So $0 \to F \to M \to 0$ is a left $\mathcal{FI}$-resolution. Thus gl left $\mathcal{FI}\text{-}\dim_R \mathcal{M} = 0$, and hence $l.FP\text{-}\dim(R) \leqslant 2$ by Corollary 3.16. □

We conclude the paper with the following

Remark 3.20. It would be interesting to compare the results of Corollary 3.2, Proposition 3.5, Theorems 3.7 and 3.19. Let R be a left coherent ring. Then $_RR$ is FP-injective (respectively $FP\text{-}id(_RR) \leqslant 1$) if and only if every finitely presented (respectively finitely presented FI-flat) right R-module has a flat preenvelope which is a monomorphism by Corollary 3.2 and Proposition 3.5; R is left semihereditary (respectively $l.FP\text{-}\dim(R) \leqslant 2$) if and only if every finitely presented (respectively finitely presented FI-flat) right R-module has a flat preenvelope which is an epimorphism by Theorems 3.7 and 3.19. On the other hand, in view of Theorem 3.7(6), R is von Neumann regular (respectively left semihereditary) if and only if every finitely presented (respectively finitely presented FI-flat) right R-module has a flat preenvelope which is an isomorphism. This observation may be viewed as an illustration of the usefulness of FI-flat modules.

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