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FP-PROJECTIVE DIMENSIONS#

Lixin Mao
Department of Mathematics, Nanjing University
Department of Basic Courses, Nanjing Institute of Technology, Nanjing, China

Nanqing Ding
Department of Mathematics, Nanjing University, Nanjing, China

We define a dimension, called an FP-projective dimension, for modules and rings. It measures how far away a finitely generated module is from being finitely presented, and how far away a ring is from being Noetherian. This dimension has nice properties when the ring in question is coherent. The relations between the FP-projective dimension and other homological dimensions are discussed.

Key Words: Coherent ring; *FP*-injective module; *FP*-projective dimension; *FP*-projective module; Noetherian ring.

Mathematics Subject Classification: Primary 16E10; Secondary 18G20.

1. INTRODUCTION

Let R be a ring and M a right R-module. Ng (1984) defined the finitely presented dimension $f.p.dim(M)$ of M as $\inf\{n :$ there exists an exact sequence $P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0$ of right R-modules, where each P_i is projective, and P_{n+1}, P_n are finitely generated$\}$. If no such sequence exists for any n, set $f.p.dim(M) = \infty$. The right finitely presented dimension $r.f.p.dim(R)$ of R is defined as $\sup\{f.p.dim(M) : M$ is a finitely generated right R-module$\}$. The dimension defined in this way has some nice properties, but no ring or finitely generated module can have finitely presented dimension 1 by Ng (1984), Proposition 1.5 and Corollary 1.6. To fill the gap, we shall introduce another kind of finitely presented dimension of modules and rings in this paper.

In Section 2, the definition and some general results are given. For a right R-module M, we define the *FP*-projective dimension $fpd(M)$ of M to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(M, N) = 0$ for any *FP*-injective right R-module N. If no such n exists, set $fpd(M) = \infty$. The right *FP*-projective dimension $rfpD(R)$ of a ring R is defined as $\sup\{fpd(M) : M$ is a finitely generated right R-module$\}$. M is called *FP*-projective if $fpd(M) = 0$, i.e., $\mathrm{Ext}^1(M, N) = 0$ for any *FP*-injective right

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Address correspondence to Lixin Mao, Department of Mathematics, Nanjing University, Nanjing 210093, P.R. China; E-mail: maolx2@hotmail.com

R-module N. The FP-projective dimension for modules and rings defined here is different from the finitely presented dimension Ng (1984), and the λ-dimension in Vasconcelos (1976), and it measures how far away a finitely generated module is from being finitely presented, and how far away a ring is from being Noetherian.

In Section 3, with the additional assumption of coherence, we show that the FP-projective dimenion has the properties that we expect of a "dimension." Let R be a right coherent ring. It is shown that $rfpD(R) = \sup\{fpd(M) : M$ is a cyclic right R-module$\} = \sup\{id(F) : F$ is an FP-injective right R-module$\} = \sup\{fpd(F) : F$ is an FP-injective right R-module$\}$. As corollaries, we have that R is a right Noetherian ring if and only if every FP-injective right R-module is FP-projective; and $rfpD(R) \leq 1$ if and only if for any pure submodule N of an injective right module M, the quotient M/N is injective. For a right semi-Artinian right coherent ring R, we prove that $rfpD(R) = \sup\{fpd(M) : M$ is a simple right R-module$\}$. If R and S are right coherent rings, then we get that $rfpD(R \otimes S) = \sup\{rfpD(R), rfpD(S)\}$. Let R be a commutative coherent ring and P any prime ideal of R, then $fpD(R_p) \leq fpD(R)$, where R_p is the localization of R at P.

In the last section, it is proven that $wD(R) = \sup\{pd(M) : M$ is an FP-projective right R-module$\}$ and $rD(R) \leq wD(R) + rfpD(R)$ for a right coherent ring R.

Throughout this paper, all rings are associative with identity and all modules are unitary. We write M_R (${}_RM$) to indicate a right (left) R-module. Let R be a ring and M, N be R-modules. $rD(R)$ ($wD(R)$) stands for the right (the weak) global dimension of R. $pd(M)$, $fd(M)$, and $id(M)$ denote the projective, flat, and injective dimensions of M, respectively. $\mathrm{Hom}(M, N)$ ($\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ ($\mathrm{Ext}^n_R(M, N)$) for an integer $n \geq 1$, and similary $M \otimes N$ ($\mathrm{Tor}_1(M, N)$) denotes $M \otimes_R N$ ($\mathrm{Tor}^R_1(M, N)$), unless otherwise specified. General background materials can be found in Anderson and Fuller (1974), Enochs and Jenda (2000), Rotman (1979), and Xu (1996).

2. DEFINITION AND GENERAL RESULTS

Recall that a right R-module M is called FP-injective (or absolutely pure) (Madox, 1967; Stenström, 1970) if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented right R-modules N.

Definition 2.1. Let R be a ring. For a right R-module M, let $fpd(M)$ denote the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(M, N) = 0$ for any FP-injective right R-module N and call $fpd(M)$ the FP-projective dimension of M. If no such n exists, set $fpd(M) = \infty$.

Put $rfpD(R) = \sup\{fpd(M) : M$ is a finitely generated right R-module$\}$ and call $rfpD(R)$ the right FP-projective dimension of R. Similarly, we have $lfpD(R)$ (when R is a commutative ring, we drop the unneeded letters r and l).

A right R-module M is called FP-projective if $fpd(M) = 0$, i.e., $\mathrm{Ext}^1(M, N) = 0$ for any FP-injective right R-module N.

Remarks 2.2. (1) It is clear that $fpd(M) \leq pd(M)$ for any right R-module M and $rfpD(R) \leq rD(R)$ for any ring R. It is also easy to see that a ring R is von Neumann regular if and only if $fpd(M) = pd(M)$ for any right R-module M if and only if every FP-projective right R-module is projective (flat).

(2) Enochs (1976) proved that a finitely generated right R-module M is finitely presented if and only if $\mathrm{Ext}^1(M, N) = 0$ for any FP-injective right R-module N. Thus $fpd(M)$ measures how far away a finitely generated right R-module M is from being finitely presented.

Proposition 2.3. *Let R be a ring and M a right R-module. Then $fpd(M) \leq f.p.dim(M)$.*

Proof. We may assume $f.p.dim(M) = n < \infty$. Then there exists an exact sequence

$$P_{n+1} \to P_n \to \cdots \to P_0 \to M \to 0$$

of right R-modules, where each P_i is projective, and P_{n+1}, P_n are finitely generated. Let $K_{n-1} = \mathrm{coker}(P_{n+1} \to P_n)$, then we have the exact sequence

$$0 \to K_{n-1} \to P_{n-1} \cdots \to P_0 \to M \to 0$$

with K_{n-1} finitely presented. Thus $\mathrm{Ext}^{n+1}(M, N) \cong \mathrm{Ext}^1(K_{n-1}, N) = 0$ for any FP-injective right R-module N, and so $fpd(M) \leq n$, as required. □

Corollary 2.4. *Let R be a ring. Then $rfpD(R) \leq r.f.p.dim(R)$.*

Remark 2.5. The inequalities in Proposition 2.3 and Corollary 2.4 may be strict. In fact, let M be a nonfinitely generated projective module, then $fpd(M) = 0$, while $f.p.dim(M) = 1$ by Ng (1984), Proposition 1.2. On the other hand, let

$$R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{Q} \end{pmatrix}.$$

Then R is a right hereditary ring that is not right Noetherian (cf. Anderson and Fuller, 1974, Example 28.12), thus $r.f.p.dim(R) = 2$ by the remark just before Ng (1984), Proposition 1.7. However, $rfpD(R) = 1$. Clearly, the FP-projective dimension defined here is different from the finitely presented dimension in Ng (1984).

Proposition 2.6. *For any ring R the following are equivalent*:

(1) $rfpD(R) = 0$;
(2) *R is right Noetherian*;
(3) *Every finitely generated right R-module is finitely presented*;
(4) *Every cyclic right R-module is finitely presented*;
(5) *Every right R-module is FP-projective*;
(6) *Every FP-injective right R-module is injective*;
(7) *Every direct limit of FP-projective right R-modules is FP-projective.*

Proof. (1) ⇔ (2) ⇔ (3), (5) ⇒ (7) and (6) ⇒ (5) ⇒ (3) ⇒ (4) are obvious.

(4) ⇒ (6). Let N be an FP-injective right R-module and I a right ideal of R. Then $\mathrm{Ext}^1(R/I, N) = 0$ by (4). Thus N is injective, as desired.

(7) $\Rightarrow$ (5). Note that every right R-module is a direct limit of finitely presented right R-modules. Therefore (5) follows from the fact that every finitely presented right R-module is FP-projective. $\square$

Remarks 2.7. (1) By Proposition 2.6, $rfpD(R)$ measures how far away a ring is from being right Noetherian. It is well known that right Noetherian rings need not be left Noetherian, so $rfpD(R) \neq lfpD(R)$ in general.

(2) Let R be a commutative ring. The λ-dimension $\lambda_R(M)$ of an R-module M and the λ-dimension $\lambda\text{-}dim(R)$ of the ring R have been widely studied (see Couchot, 2003; Vasconcelos, 1976). It is well known that R is Noetherian if and only if $\lambda\text{-}dim(R) = 0$, and R is coherent if and only if $\lambda\text{-}dim(R) \leq 1$. However the λ-dimension is completely different from the FP-projective dimension defined here. In fact, take M to be a finitely presented R-module, then $\lambda_R(M) \geq 1$, but $fpd(M) = 0$. In addition, we can choose a commutative von Neumann regular ring R of global dimension 2 by Pierce (1967), Corollary 5.2, then $fpD(R) = D(R) = 2$ by Remark 2.2, while $\lambda\text{-}dim(R) = 1$.

(3) Recall that a right R-module M is said to be pure projective if for every pure exact sequence $0 \to T \to N \to N/T \to 0$, the sequence $\mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N/T) \to 0$ is exact. By Dauns (1994), Theorem 18-2.10, M is pure projective if and only if M is a direct summand of a direct sum of finitely presented modules. Clearly, pure projective modules are FP-projective, but the converse is not true. In fact, Azumaya and Facchini (1989), Proposition 5, assert that if every right R-module is pure projective, then R must be right Artinian. Take R to be a right Noetherian ring which is not right Artinian, then there exists an FP-projective right R-module which is not pure projective.

Let M be a right R-module. Recall that a homomorphism $\phi : M \to F$, where F is FP-injective, is called an FP-injective preenvelope of M (see Enochs and Jenda, 2000) if for any homomorphism $f : M \to F'$, where F' is FP-injective, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F, when $F' = F$ and $f = \phi$, the FP-injective preenvelope ϕ is called an FP-injective envelope of M. Clearly, ϕ is a monomorphism. FP-projective (pre)covers of M can be defined dually. By Enochs and Jenda (2000), Proposition 6.2.4, every R-module has an FP-injective preenvelope.

Remark 2.8. Denote by $\mathcal{FP}\text{-}proj$ ($\mathcal{FP}\text{-}inj$) the class of FP-projective (FP-injective) right R-modules. Then ($\mathcal{FP}\text{-}proj$, $\mathcal{FP}\text{-}inj$) is a cotorsion theory that is cogenerated by the representative set of all finitely presented R-modules (cf. Enochs and Jenda, 2000, Definiton 7.1.2). We note that the concept of FP-projective modules coincides with that of *finitely covered* modules introduced by Trlifaj (see Trlifaj, 2000, Definition 3.3 and Theorem 3.4). By Enochs and Jenda (2000), Theorem 7.4.1 and Definition 7.1.5, every R-module has a special FP-injective preenvelope, i.e., there is an exact sequence $0 \to M \to F \to L \to 0$, where F is FP-injective and L is FP-projective; and every R-module has a special FP-projective precover, i.e., there is an exact sequence $0 \to K \to F \to M \to 0$, where F is FP-projective and K is FP-injective (note that every R-module has a pure projective precover by Enochs and Jenda, 2000, Example 8.3.2). However, FP-injective envelopes may not exist in general (see Trlifaj, 2000, Theorem 4.9). We observe that, if $\alpha : M \to F$ is an *FP*-injective envelope of M, then $\mathrm{coker}(\alpha)$ is *FP*-projective by Enochs and Jenda (2000), Proposition 7.2.4, and if $\beta : F \to M$ is an *FP*-projective cover of M, then $\ker(\beta)$ is *FP*-injective by Enochs and Jenda (2000), Proposition 7.2.3.

Recall that a ring R is called right self-*FP*-injective if R_R is an *FP*-injective module. We end this section with the following characterizations of *FP*-projective R-modules.

Proposition 2.9. *Let R be a right self-FP-injective ring. If M is a right R-module, then the following are equivalent*:

(1) *M is FP-projective*;
(2) *M is projective with respect to every exact sequence $0 \to A \to B \to C \to 0$, where A is FP-injective*;
(3) *For every exact sequence $0 \to K \to F \to M \to 0$, where F is FP-injective, $K \to F$ is an FP-injective preenvelope of K*;
(4) *M is a cokernel of an FP-injective preenvelope $K \to F$ with F projective.*

Proof. (1) $\Rightarrow$ (2) Let $0 \to A \to B \to C \to 0$ be an exact sequence, where A is *FP*-injective. Then $\mathrm{Ext}^1(M, A) = 0$ by (1). Thus $\mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is exact, and (2) holds.

(2) $\Rightarrow$ (1) For every *FP*-injective right R-module N, there is a short exact sequence $0 \to N \to E \to L \to 0$ with E injective, which induces an exact sequence $\mathrm{Hom}(M, E) \to \mathrm{Hom}(M, L) \to \mathrm{Ext}^1(M, N) \to 0$. Since $\mathrm{Hom}(M, E) \to \mathrm{Hom}(M, L) \to 0$ is exact by (2), we have $\mathrm{Ext}^1(M, N) = 0$, and (1) follows.

(1) $\Rightarrow$ (3) is easy to verify.

(3) $\Rightarrow$ (4) Let $0 \to K \to P \to M \to 0$ be an exact sequence with P projective. Note that P is *FP*-injective by hypothesis, thus $K \to P$ is an *FP*-injective preenvelope.

(4) $\Rightarrow$ (1) By (4), there is an exact sequence $0 \to K \to P \to M \to 0$, where $K \to P$ is an *FP*-injective preenvelope with P projective. It gives rise to the exactness of $\mathrm{Hom}(P, N) \to \mathrm{Hom}(K, N) \to \mathrm{Ext}^1(M, N) \to 0$ for each *FP*-injective right R-module N. Note that $\mathrm{Hom}(P, N) \to \mathrm{Hom}(K, N) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(M, N) = 0$, as desired. □

3. THE *FP*-PROJECTIVE DIMENSION OVER COHERENT RINGS

Recall that a ring R is called right coherent if every finitely generated right ideal of R is finitely presented.

Proposition 3.1. *Let R be a right coherent ring. For any right R-module M and an integer $n \geq 0$, the following are equivalent*:

(1) *$fpd(M) \leq n$*;
(2) $\mathrm{Ext}^{n+1}(M, N) = 0$ *for any FP-injective right R-module N*;

(3) $\mathrm{Ext}^{n+j}(M, N) = 0$ *for any FP-injective right R-module N and* $j \geq 1$;
(4) *There exists an exact sequence* $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, *where each* P_i *is FP-projective.*

Proof. (3) $\Rightarrow$ (1) is obvious.

(2) $\Rightarrow$ (3) For any FP-injective right R-module N, there is a short exact sequence $0 \to N \to E \to L \to 0$, where E is injective. Then the sequence $\mathrm{Ext}^{n+1}(M, L) \to \mathrm{Ext}^{n+2}(M, N) \to \mathrm{Ext}^{n+2}(M, E) = 0$ is exact. Note that L is FP-injective by Stenström (1970), Lemma 3.1, so $\mathrm{Ext}^{n+1}(M, L) = 0$ by (2). Hence $\mathrm{Ext}^{n+2}(M, N) = 0$, and (3) follows by induction.
The proof of (1) $\Rightarrow$ (2) is similar to that of (2) $\Rightarrow$ (3).

(1) $\Leftrightarrow$ (4) is straightforward. □

The proof of the next proposition is standard homological algebra.

Proposition 3.2. *Let R be a right coherent ring,* $0 \to A \to B \to C \to 0$ *an exact sequence of right R-modules. If two of* $fpd(A)$, $fpd(B)$, *and* $fpd(C)$ *are finite, so is the third. Moreover,*

(1) $fpd(B) \leq sup\{fpd(A), fpd(C)\}$.
(2) $fpd(A) \leq sup\{fpd(B), fpd(C) - 1\}$.
(3) $fpd(C) \leq sup\{fpd(B), fpd(A) + 1\}$.

Corollary 3.3. *Let R be a right coherent ring.*

(1) *If* $0 \to A \to B \to C \to 0$ *is an exact sequence of right R-modules, where* $0 < fpd(A) < \infty$ *and B is FP-projective, then* $fpd(C) = fpd(A) + 1$.
(2) $rfpD(R) = n$ *if and only if* $sup\{fpd(I) : I \text{ is any right ideal of } R\} = n - 1$ *for any integer* $n \geq 2$.

Proof. (1) is true by Proposition 3.2.

(2) For a right ideal I of R, consider the exact sequence $0 \to I \to R \to R/I \to 0$. Then (2) follows from (1). □

Theorem 3.4. *Let R be a right coherent ring, then the following are identical:*

(1) $rfpD(R)$;
(2) $sup\{fpd(M) : M \text{ is a cyclic right } R\text{-module}\}$;
(3) $sup\{fpd(M) : M \text{ is any right } R\text{-module}\}$;
(4) $sup\{id(F) : F \text{ is an FP-injective right } R\text{-module}\}$;
(5) $sup\{fpd(F) : F \text{ is an FP-injective right } R\text{-module}\}$.

Proof. (2) $\leq$ (1) $\leq$ (3) and (5) $\leq$ (3) are obvious.

(3) $\leq$ (4) We may assume $\sup\{id(F) : F$ is an FP-injective right R-module$\} = m < \infty$. Let M be any right R-module and N any FP-injective right R-module. Since $id(N) \leq m$, it follows that $\mathrm{Ext}^{m+1}(M, N) = 0$. Hence $fpd(M) \leq m$.

(4) $\leq$ (2) We may assume $\sup\{fpd(M) : M$ is a cyclic right R-module$\} = n < \infty$. Let N be an FP-injective right R-module and I any right ideal, then $fpd(R/I) \leq n$. By Proposition 3.1, $\mathrm{Ext}^{n+1}(R/I, N) = 0$, and so $id(N) \leq n$.

(3) $\leq$ (5) We may assume that $\sup\{fpd(F) : F$ is an FP-injective right R-module$\} = n < \infty$. Let M be any right R-module. By Remark 2.8, there is a short exact sequence $0 \to M \to F \to L \to 0$, where F is FP-injective and L is FP-projective. Thus $fpd(M) \leq fpd(F) \leq n$. This completes the proof. □

Corollary 3.5. *Let R be a right coherent ring. Then the following are equivalent for an integer $n \geq 0$:*

(1) $rfpD(R) \leq n$;
(2) $id(M) \leq n$ *for all FP-injective right R-modules M*;
(3) $fpd(M) \leq n$ *for all FP-injective right R-modules M*;
(4) $id(M) \leq n$ *for all right R-modules M that are both FP-projective and FP-injective, and* $rfpD(R) < \infty$;
(5) $fpd(M) \leq n$ *for all injective right R-modules M, and* $rfpD(R) < \infty$;
(6) $\mathrm{Ext}^{n+1}(M, N) = 0$ *for all FP-injective right R-modules M and N*;
(7) $\mathrm{Ext}^{n+j}(M, N) = 0$ *for all FP-injective right R-modules M, N and* $j \geq 1$.

Proof. By Theorem 3.4, it suffices to show that (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3).

(4) $\Rightarrow$ (2) Let M be any FP-injective right R-module. Since $rfpD(R) < \infty$, $fpd(M) = m$ for a nonnegative integer m by Theorem 3.4 (4). Note that every right R-module has a special FP-projective precover, then there exists an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is both FP-projective and FP-injective. Since $id(P_i) \leq n$ by (4), $id(M) \leq n$.

(5) $\Rightarrow$ (3) Let M be any FP-injective right R-module. Since $rfpD(R) < \infty$, $id(M) = m$ for an integer $m \geq 0$ by Theorem 3.4 (5). Hence M admits an injective resolution

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \to E^m \to 0.$$

Note that $fpd(E^i) \leq n$ for each E^i by (5), so $fpd(M) \leq n$ by Proposition 3.2. □

In what follows, $\sigma_M : M \to E(M)(\epsilon_M : FP(M) \to M)$ denotes the injective envelope (FP-projective cover) of a right R-module M. Recall that an injective envelope $\sigma_M : M \to E(M)$ has the unique mapping property (see Ding, 1996) if for any homomorphism $f : M \to N$ with N injective, there exists a unique homomorphism $g : E(M) \to N$ such that $g\sigma_M = f$. The concept of an FP-projective cover (FP-injective envelope) with the unique mapping property can be defined similarly.

Corollary 3.6. *Let R be a right coherent ring. Then the following are equivalent*:

(1) *R is a right Noetherian ring*;
(2) *Every FP-injective right R-module is FP-projective*;

(3) *$rfpD(R) < \infty$, and every injective right R-module is FP-projective*;
(4) $\mathrm{Ext}^1(M, N) = 0$ *for all FP-injective right R-modules M and N*;
(5) *Every FP-injective right R-module has an injective envelope with the unique mapping property*;
(6) *Every FP-injective right R-module has an FP-projective cover with the unique mapping property.*

Proof. It is enough to show that (5) $\Rightarrow$ (1) and (6) $\Rightarrow$ (2).

(5) $\Rightarrow$ (1) Let M be any FP-injective right R-module. There is the following exact commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\sigma_M"] \arrow[drr, "0"'] & E(M) \arrow[r, "\gamma"] \arrow[dr, "\sigma_L\gamma"] & L \arrow[r] \arrow[d, "\sigma_L"] & 0 \\
 & & & E(L) &
\end{tikzcd}$$

Note that $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$, so $\sigma_L\gamma = 0$ by (5). Therefore $L = \mathrm{im}(\gamma) \subseteq \ker(\sigma_L) = 0$, and hence M is injective. Thus (1) follows.

(6) $\Rightarrow$ (2) Let M be any FP-injective right R-module. There is the following exact commutative diagram

$$\begin{tikzcd}
 & FP(K) \arrow[d, "\epsilon_K"'] \arrow[dr, "\alpha\epsilon_K"] \arrow[drr, "0"] & & & \\
0 \arrow[r] & K \arrow[r, "\alpha"'] \arrow[d] & FP(M) \arrow[r, "\epsilon_M"'] & M \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

where K is FP-injective by Remark 2.8. Note that $\epsilon_M\alpha\epsilon_K = 0 = \epsilon_M 0$, so $\alpha\epsilon_K = 0$ by (6). Therefore $K = \mathrm{im}(\epsilon_K) \subseteq \ker(\alpha) = 0$, and so M is FP-projective, as required. □

It is known that a ring R is right coherent if and only if for any pure submodule N of an FP-injective right R-module M, the quotient M/N is FP-injective (see Wisbauer, 1991, 35.9, p. 302). Here we have the following

Proposition 3.7. *Let R be a right coherent ring. Then the following are equivalent*:

(1) $rfpD(R) \leq 1$;
(2) *For any pure submodule N of an injective right module M, the quotient M/N is injective*;
(3) *Every submodule of an (FP-)projective right R-module is FP-projective*;
(4) *Every right ideal of R is FP-projective.*

Proof. (1) ⇒ (2) Let N be a pure submodule of an injective right module M. Then N is FP-injective, and so $id(N) \leq 1$ by Theorem 3.4 (4). Thus the exactness of $0 \to N \to M \to M/N \to 0$ implies the injectivity of M/N.

(2) ⇒ (1) Suppose N is an FP-injective right R-module. Then N is a pure submodule of its injective envelope $E(N)$, and hence $E(N)/N$ is injective by (2). Therefore $id(N) \leq 1$, and so (1) follows from Theorem 3.4 (4).

(3) ⇒ (4) is trivial.

(1) ⇒ (3) Let N be a submodule of an FP-projective right R-module M. Then, for any FP-injective right R-module L, we get an exact sequence

$$0 = \mathrm{Ext}^1(M, L) \to \mathrm{Ext}^1(N, L) \to \mathrm{Ext}^2(M/N, L).$$

Note that the last term is zero by (1), hence $\mathrm{Ext}^1(N, L) = 0$, and (3) follows.

(4) ⇒ (1) Let I be a right ideal of R. The exact sequence $0 \to I \to R \to R/I \to 0$ implies $fpd(R/I) \leq 1$ by Proposition 3.1. So (1) follows from Theorem 3.4 (2). □

It is well known that for a right coherent ring R, the dual module $\mathrm{Hom}(M, R)$ of any finitely presented left R-module M is finitely presented. Here we have the following

Corollary 3.8. *If R is a right coherent ring with $rfpD(R) \leq 1$, then the dual module $\mathrm{Hom}(M, R)$ of any finitely generated left R-module M is FP-projective.*

In addition, if R is also left coherent, then the following are equivalent:

(1) *Every flat right R-module is FP-projective*;
(2) *M^+ is right FP-projective for any (FP-)injective left R-module M, where M^+ denotes the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$*;
(3) *N^{++} is right FP-projective for any flat right R-module N.*

Proof. Let M be a finitely generated left R-module. Then there exists an exact sequence $P \to M \to 0$ with P finitely generated projective. So we have a right R-module exact sequence $0 \to \mathrm{Hom}(M, R) \to \mathrm{Hom}(P, R)$. Note that $\mathrm{Hom}(P, R)$ is projective, therefore $\mathrm{Hom}(M, R)$ is FP-projective by Proposition 3.7.

If R is also left coherent, then (1) ⇒ (2) ⇒ (3) are clear.

(3) ⇒ (1) Let N be any flat right R-module. There exists an exact sequence $0 \to N \to N^{++}$. Since $rfpD(R) \leq 1$ and N^{++} is right FP-projective by (3), we have that N is FP-projective by Proposition 3.7. □

Corollary 3.9. *Let R be a commutative hereditary ring. Then $\mathrm{Tor}_1(M, N)$ is FP-projective for any R-module M and any FP-projective R-module N.*

Proof. For any R-module M, there is an exact sequence $0 \to P_1 \to P_0 \to M \to 0$, with P_0 and P_1 projective by hypothesis, which induces an exact sequence $0 \to \mathrm{Tor}_1(M, N) \to P_1 \otimes N$. It is easy to see that $P_1 \otimes N$ is FP-projective (for N is FP-projective). Thus $\mathrm{Tor}_1(M, N)$ is FP-projective by Proposition 3.7. □

A ring R is called right semi-Artinian if every nonzero cyclic right R-module has a nonzero socle. The following proposition shows that we may compute the FP-projective dimension of a semi-Artinian coherent ring using just the FP-projective dimensions of simple modules.

Proposition 3.10. *If R is a right semi-Artinian right coherent ring, then $rfpD(R) = sup\{fpd(M)$: M is a simple right R-module$\}$.*

Proof. It suffices to show that $rfpD(R) \leq \sup\{fpd(M) : M$ is a simple right R-module$\}$. We may assume that $\sup\{fpd(M): M$ is a simple right R-module$\} = n < \infty$. Let N be an FP-injective right R-module and I a maximal right ideal of R. Consider the injective resolution of N

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots .$$

Write $L = \mathrm{coker}(E^{n-2} \to E^{n-1})$. Then $\mathrm{Ext}^1(R/I, L) = \mathrm{Ext}^{n+1}(R/I, N) = 0$ by Proposition 3.1. Therefore L is injective by Smith (1981), Lemma 4, since R is right semi-Artinian. So $id(N) \leq n$, and hence $rfpD(R) \leq n$ by Theorem 3.4. □

Corollary 3.11. *Let R be a right coherent ring.*

(1) *If R is right semi-Artinian, then R is a right Noetherian ring if and only if every simple right R-module is finitely presented.*
(2) *If R is a left perfect ring with Jacobson radical J, then $rfpD(R) = fpd(R/J)$, where R/J is considered as a right R-module.*

Proof. (1) follows from Proposition 3.10.

(2) Note that R is left perfect if and only if R is right semi-Artinian and semilocal (cf. Stenström, 1975). (2) follows immediately since every simple right R-module is the direct summand of the right R-module R/J by Kasch (1982), Theorem 9.3.4. □

Proposition 3.12. *Let J be the Jacobson radical of a ring R. Then the following are equivalent:*

(1) *R is a left perfect right coherent ring with $rfpD(R) < \infty$, and R/J (as a right R-module) embeds in an FP-projective right R-module;*
(2) *R is a right Artinian ring.*

Proof. (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2) Since R is a left perfect and right coherent ring, then $rfpD(R) = fpd(R/J) = n < \infty$ by Corollary 3.11. We claim that $n = 0$. Otherwise, let $\alpha : R/J \to F$ be the embedding, where F is FP-projective. Thus the exactness of the sequence $0 \to R/J \to F \to L \to 0$ implies that $fpd(L) = fpd(R/J) + 1 = n + 1$ by Corollary 3.3. However, $fpd(L) \leq rfpD(R) = n$, this is impossible. Thus R is right Noetherian and hence right Artinian. □

To prove the next main result, we need the following three lemmas.

Lemma 3.13. *If $\varphi : R \to S$ is a surjective ring homomorphism, both ${}_RS$ and S_R are projective. Let M be a right S-module* (*and hence a right R-module*), *then M is a finitely presented right S-module if and only if M is a finitely presented right R-module.*

Proof. ($S \Rightarrow R$). If M is a finitely presented right S-module, then there is an exact sequence $0 \to K \to P \to M \to 0$ of right S-modules, where K and P are finitely generated, and P is projective. It is easy to see that K is a finitely generated right R-module and P is a finitely generated projective right R-module (for S_R is projective). Therefore M is a finitely presented right R-module.

($S \Leftarrow R$). If M is a finitely presented right R-module, then there is an exact sequence $0 \to K \to P \to M \to 0$ of right R-modules, where K is finitely generated and P is finitely generated projective. Since ${}_RS$ is projective, we have the exact sequence

$$0 \to K \otimes_R S_S \to P \otimes_R S_S \to M \otimes_R S_S \to 0.$$

Note that $K \otimes_R S_S$ is a finitely generated right S-module, $P \otimes_R S_S$ is a finitely generated projective right S-module, and $M \otimes_R S_S \cong M_S$. Thus M_S is a finitely presented right S-module. □

The following fact can be verified easily, so we omit its proof.

Lemma 3.14. *Let R and S be rings. Every right $(R \oplus S)$-module M has a unique decomposition that $M = A \oplus B$, where $A = M(R, 0)$ is a right R-module and $B = M(0, S)$ is a right S-module via $xr = x(r, 0)$ for $x \in A$, $r \in R$, and $ys = y(0, s)$ for $y \in B$, $s \in S$.*

Lemma 3.15. *Let R and S be rings. If M is a right R-module* (*and hence a right* $(R \oplus S)$*-module*), *then M is an FP-projective right R-module if and only if M is an FP-projective right $(R \oplus S)$-module.*

Proof. ($R \Rightarrow R \oplus S$). Suppose M is an *FP*-projective right R-module. By Remark 2.8, M is finitely covered, i.e., M is a direct summand in a right R-module N such that N is a union of a continuous chain $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is a finitely presented right R-module for all $\alpha < \lambda$ (see Trlifaj, 2000, Definition 3.3). Note that $N_{\alpha+1}/N_\alpha$ is a finitely presented right $(R \oplus S)$-module for all $\alpha < \lambda$ by Lemma 3.13. Thus M is also an *FP*-projective right $(R \oplus S)$-module.

($R \oplus S \Rightarrow R$). Suppose M is an *FP*-projective right $(R \oplus S)$-module. Then there exist right $(R \oplus S)$-modules N and Q such that $M \oplus Q = N$, and N is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is a finitely presented right $(R \oplus S)$-module for all $\alpha < \lambda$. In view of Lemma 3.14, we may assume $Q = Q^1 + Q^2$, $N_\alpha = N_\alpha^1 \oplus N_\alpha^2$, where Q^1 and N_Q^1 are right R-modules, Q^2 and N_α^2 are right S-modules. Then

$$N = \bigcup_{\alpha<\lambda} N_\alpha = \bigcup_{\alpha<\lambda} \left(N_\alpha^1 \oplus N_\alpha^2\right) = \Big(\bigcup_{\alpha<\lambda} N_\alpha^1\Big) \oplus \Big(\bigcup_{\alpha<\lambda} N_\alpha^2\Big).$$

Note that M is a right R-module, and so $M \oplus Q^1 = \bigcup_{\alpha<\lambda} N_\alpha^1$ by Lemma 3.14. In addition,

$$N_{\alpha+1}/N_\alpha = \left(N_{\alpha+1}^1 \oplus N_{\alpha+1}^2\right)/\left(N_\alpha^1 \oplus N_\alpha^2\right) \cong \left(N_{\alpha+1}^1\ /N_\alpha^1\right) \oplus \left(N_{\alpha+2}^2/N_\alpha^2\right),$$

which implies that each $N_{\alpha+1}^1/N_\alpha^1$ is fintiely presented as a right $(R \oplus S)$-module, and so it is finitely presented as a right R-module by Lemma 3.13. Therefore, M is an FP-projective right R-module. □

We are now in a position to prove the following

Theorem 3.16. *Let R and S be right coherent rings. Then*

$$rfpD(R \oplus S) = sup\{rfpD(R), rfpD(S)\}.$$

Proof. The proof is motivated by that of Ng (1984), Theorem 2.11.

We first show that $rfpD(R \oplus S) \leq sup\{rfpD(R), rfpD(S)\}$. We may assume that $rfpD(R) = m < \infty$, $rfpD(S) = n < \infty$, and $m \geq n$. Let M be a right $(R \oplus S)$-module. Then $M = A \oplus B$, where A is a right R-module and B is a right S-module. Since $fpd(A) \leq m$, $fpd(B) \leq n \leq m$, by Proposition 3.1 there exist two exact sequences

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

and

$$0 \to Q_m \to Q_{m-1} \to \cdots \to Q_1 \to Q_0 \to B \to 0$$

of right R-modules and right S-modules, respectively, where each P_i is an FP-projective right R-module, and each Q_i is an FP-projective right S-module. Regarding these as exact sequences of right $(R \oplus S)$-modules, we have an exact sequence of right $(R \oplus S)$-modules

$$0 \to P_m \oplus Q_m \to P_{m-1} \oplus Q_{m-1} \to \cdots \to P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to A \oplus B \to 0.$$

Note that each $P_i \oplus Q_i$ is an FP-projective right $(R \oplus S)$-module by Lemma 3.15. Thus $fpd(M_{R\oplus S}) \leq m$, and hence $rfpD(R \oplus S) \leq sup\{rfpD(R), rfpD(S)\}$.

Next we prove that $\sup\{rfpD(R), rfpD(S)\} \leq rfpD(R \oplus S)$. Suppose that $rfpD(R \oplus S) = k < \infty$, and $rfpD(R) > k$. Then there is a right R-module M with $fpd(M) > k$. Note that M may be regarded as a right $(R \oplus S)$-module, so $fpd(M_{R\oplus S}) \leq rfpD(R \oplus S) = k$. Thus there exists an exact sequence

$$0 \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of right $(R \oplus S)$-modules, where each P_i is an FP-projective right $(R \oplus S)$-module. By Lemma 3.14, we may assume $P_i = A_i \oplus B_i$, where A_i is a right R-module and B_i

is a right S-module, $i = 0, 1, \ldots, k$. Since M is a right R-module, we have the exact sequence

$$0 \to A_k \to A_{k-1} \to \cdots \to A_1 \to A_0 \to M \to 0$$

of right R-modules. Note that each A_i is an FP-projective right $(R \oplus S)$-module, and so an FP-projective right R-module by Lemma 3.15, whence $fpd(M_R) \leq k$, a contradiction. Thus $\sup\{rfpD(R), rfpD(S)\} \leq rfpD(R \oplus S)$. The proof is complete. □

Remark 3.17. Theorem 3.16 shows that $rfpD\big(\bigoplus_{i=1}^{n} R_i\big) = \sup_{1\leq i\leq n}\{rfpD(R_i)\}$ if each R_i is right coherent. In particular, we have the well-known result that $\bigoplus_{i=1}^{n} R_i$ is right Noetherian if and only if each R_i is right Noetherian, $i = 1, 2, \ldots, n$. However, $rfpD\big(\bigoplus_{i=1}^{\infty} R_i\big) \neq \sup_{i\geq 1}\{rfpD(R_i)\}$ in general. For example, $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ is not Noetherian, where $\mathbb{Z}_2$ is the field of two elements.

The proof of the next main result requires a lemma.

Lemma 3.18. *Let R and S be rings. Suppose ${}_SL_R$ is an S-R-bimodule, L_R is flat, and ${}_SL$ is finitely generated projective.*

(1) *If M is a finitely presented left R-module, then ${}_SL \otimes_R M$ is a finitely presented left S-module.*
(2) *If M is an FP-projective left R-module, then ${}_SL \otimes_R M$ is an FP-projective left S-module.*

Proof. (1) is straightforward.

(2) Since M is an FP-projective left R-module, M is a direct summand in a left R-module N such that N is a union of a continuous chain $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is a finitely presented left R-module for all $\alpha < \lambda$. Since L_R is flat, the short exact sequence

$$0 \longrightarrow N_\alpha \xrightarrow{i_\alpha} N_{\alpha+1} \longrightarrow N_{\alpha+1}/N_\alpha \longrightarrow 0$$

gives rise to the exactness of the sequence

$$0 \longrightarrow {}_SL \otimes_R N_\alpha \xrightarrow{1\otimes i_\alpha} {}_SL \otimes_R N_{\alpha+1} \longrightarrow {}_SL \otimes_R (N_{\alpha+1}/N_\alpha) \longrightarrow 0.$$

By (1), ${}_SL \otimes_R (N_{\alpha+1}/N_\alpha)$ is a finitely presented left S-module. Regarding each $1 \otimes i_\alpha$ as an inclusion map, then ${}_SL \otimes_R N_\alpha$ is a submodule of ${}_SL \otimes_R N_{\alpha+1}$. Thus ${}_SL \otimes_R M$ is a direct summand in a left S-module ${}_SL \otimes_R N$ such that ${}_SL \otimes_R N$ is a union of a continuous chain $({}_SL \otimes_R N_\alpha : \alpha < \lambda)$, for a cardinal λ, ${}_SL \otimes_R N_0 = 0$, and $({}_SL \otimes_R N_{\alpha+1})/({}_SL \otimes_R N_\alpha) \cong {}_SL \otimes_R (N_{\alpha+1}/N_\alpha)$ is a finitely presented left S-module for all $\alpha < \lambda$. That is to say, ${}_SL \otimes_R M$ is an FP-projective left S-module. □

Theorem 3.19. *Let R be a commutative coherent ring. If P is any prime ideal of R, then $fpD(R_p) \leq fpD(R)$, where R_p is the localization of R at P.*

Proof. We may assume $fpD(R) = n < \infty$. Let M be any R_p-module, then M may be viewed as an R-module and so $fpd(M_R) \leq n$. Thus there exists an FP-projective resolution of M_R

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

which induces an R_p-module exact sequence

$$0 \to (F_n)_P \to (F_{n-1})_P \to \cdots \to (F_1)_P \to (F_0)_P \to M_P \to 0.$$

Note that each $(F_i)_P$ is an FP-projective R_P-module by Lemma 3.18, $i = 0, 1, \ldots, n$, it follows that $fpd(M_P)_{R_P} \leq n$. Since $M_P \cong M$ as R_P-modules, $fpd(M) \leq n$. Thus $fpD(R_P) \leq n$, as required. □

Remark 3.20. The theorem above gives the well-known result that any localization of a Noetherian ring is again Noetherian. However, in general, $rfpD(R) \neq \sup\{fpD(R_P) : P$ is a prime ideal of $R\}$. In fact, take R to be a commutative non-Noetherian ring whose localization with respect to any prime ideal is Noetherian, e.g. the direct product of countably many copies of $\mathbb{Z}_2$. Then $rfpD(R) > 0$, while $fpD(R_P) = 0$ for all prime ideals P of R.

Proposition 3.21. *Let R be a right coherent and right self-FP-injective ring. Then*

(1) $rfpD(R) \geq sup\{id(M) : M$ *is a right R-module with* $fd(M) < \infty\}$
$\geq sup\{id(M) : M$ *is a flat right R-module*$\}$.
(2) *If R is a two-sided coherent and two-sided self-FP-injective ring, then the equalities in* (1) *hold.*

Proof. (1) Write $rfpD(R) = n$. Let M be a right R-module with $fd(M) = m < \infty$. Then we have an exact sequence

$$0 \to F_m \to F_{m-1} \to \cdots F_1 \to F_0 \to M \to 0.$$

Note that every F_i is FP-injective by Stenström (1970), Lemma 4.1, so $id(F_i) \leq n$ by Theorem 3.4, whence $id(M) \leq n$, as desired. The second inequality is trivial.

(2) If R is a two-sided coherent and two-sided self-FP-injective ring, then every FP-injective right R-module is flat by Stenström (1970), Proposition 4.2, thus

$$rfpD(R) \leq \sup\{id(M) : M \text{ is a flat right } R\text{-module}\}$$

and (2) follows from (1). □

4. RELATIONS WITH OTHER HOMOLOGICAL DIMENSIONS

It is well known that if R is a right coherent ring, then $fd(M) = pd(M)$ for any finitely presented right R-module M (see Jones and Teply, 1982, Lemma 5). Now we have

Proposition 4.1. *Let R be a right coherent and right self-FP-injective ring. If M is an FP-projective right R-module, then $fd(M) = pd(M)$.*

Proof. It is clear that $fd(M) \leq pd(M)$. Conversely, we may suppose that $fd(M) = n < \infty$. There is an exact sequence

$$0 \to F_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0$$

with $P_0, P_1, \ldots, P_{n-1}$ projective. Since $fd(M) = n$, F_n is flat. Hence we have a pure exact sequence $0 \to K \to P \to F_n \to 0$ of right R-modules with P projective. Since P is FP-injective by hypothesis, so is K. Note that F_n is FP-projective (for M is FP-projective). Thus the short exact sequence $0 \to K \to P \to F_n \to 0$ splits, and hence F_n is projective. So $pd(M) \leq n$, as desired. $\square$

Let R be a ring and M a right R-module. Following Stenström (1970), *FP-inj.dim*(M) denotes the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for every finitely presented module F, and *r.FP-dim*$(R) = \sup\{$*FP-inj.dim*$(M) : M$ is a right R-module$\}$.

Theorem 4.2. *Let R be a right coherent ring. Then the following are identical:*

(1) $wD(R)$;
(2) *r.FP-dim*(R);
(3) *sup*$\{pd(M) : M$ *is a finitely presented right R-module*$\}$;
(4) *sup*$\{pd(M) : M$ *is an FP-projective right R-module*$\}$;
(5) *sup*$\{fd(M) : M$ *is an FP-projective right R-module*$\}$;
(6) *sup*$\{$*FP-inj.dim*$(M) : M$ *is an FP-projective right R-module*$\}$.

Proof. (1) = (2) = (3) = (6) follow from Stenström (1970), Theorem 3.3.

(3) $\leq$ (4) is trivial.

(4) $\leq$ (2) Let M be an FP-projective right R-module. It is enough to show that $pd(M) \leq$ *r.FP-dim*(R). We may assume that *r.FP-dim*$(R) = n < \infty$. M admits a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0.$$

Let N be any right R-module. We have *FP-inj.dim*$(N) \leq n$, thus by Stenström (1970), Lemma 3.1, there is an exact sequence

$$0 \to N \to E^0 \to E^1 \to \cdots E^{n-1} \to E^n \to 0,$$

where $E^0, E^1, \ldots, E^n$ are *FP*-injective. Therefore we form a double complex

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & & & 0 & \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^n) & \to & \mathrm{Hom}(P_0, E^n) & \to & \cdots & \to & \mathrm{Hom}(P_n, E^n) & \to \cdots \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
 & \vdots & & \vdots & & & & \vdots & \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^1) & \to & \mathrm{Hom}(P_0, E^1) & \to & \cdots & \to & \mathrm{Hom}(P_n, E^1) & \to \cdots \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^0) & \to & \mathrm{Hom}(P_0, E^0) & \to & \cdots & \to & \mathrm{Hom}(P_n, E^0) & \to \cdots \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
 & 0 & \to & \mathrm{Hom}(P_0, N) & \to & \cdots & \to & \mathrm{Hom}(P_n, N) & \to \cdots \\
 & & & \uparrow & & & & \uparrow & \\
 & & & 0 & & & & 0 &
\end{array}$$

Note that all rows are exact except for the bottom row since M is *FP*-projective and all E^i are *FP*-injective, also note that all columns are exact except for the left column since all P_i are projective.
Using a spectral sequence argument, we know that the two complexes

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \cdots \to \mathrm{Hom}(P_n, N) \to \cdots$$

and

$$0 \to \mathrm{Hom}(M, E^0) \to \mathrm{Hom}(M, E^1) \to \cdots \to \mathrm{Hom}(M, E^n) \to 0$$

have isomorphic homology groups. Thus $\mathrm{Ext}^{n+j}(M, N) = 0$ for all $j \geq 1$. Hence $pd(M) \leq n$.

(1) = (5) follows from the fact that $wD(R) = \sup\{fd(M) : M$ is a finitely presented right R-module$\} \leq \sup\{fd(M) : M$ is an *FP*-projective right R-module$\} \leq wD(R)$. □

Corollary 4.3. *Let R be a ring. Then the following are equivalent*:

(1) *R is von Neumann regular*;
(2) *R is right coherent and every FP-projective right R-module is FP-injective*;
(3) *Every right R-module has an FP-injective envelope with the unique mapping property*;
(4) *R is right coherent and every FP-projective right R-module has an FP-injective envelope with the unique mapping property.*

Proof. This follows from Theorem 4.2 and an argument similar to that in Corollary 3.6. □

It is proved that, if R is a Prüfer domain, then an R-module M is *FP*-projective if and only if $pd(M) \leq 1$ (see Fuchs and Salce, 2001, Theorem 6.5, p. 217). Removing the commutative domain condition, we have the following result.

Corollary 4.4. *A ring R is right semihereditary if and only if $pd(M) \leq 1$ for every FP-projective right R-module M.*

Proof. Since R is right semihereditary if and only if R is right coherent and $wD(R) \leq 1$, the necessity follows from Theorem 4.2. Conversely, for a finitely generated right ideal I of R, consider the exact sequence $0 \to I \to R \to R/I \to 0$. Since $pd(R/I) \leq 1$ by hypothesis, I is projective. So R is a right semihereditary ring. $\square$

It is known that $rD(R) = wD(R)$ if R is a right Noetherian ring (see Rotman, 1979, Theorem 9.22), and $rD(R) = rfpD(R)$ if R is a von Neumann regular ring by Remark 2.2. In general, we have

Proposition 4.5. *Let R be a right coherent ring. Then*

$$rD(R) \leq wD(R) + rfpD(R).$$

Proof. We may assume without loss of generality that both $rfpD(R)$ and $wD(R)$ are finite. Let $rfpD(R) = m < \infty$ and $wD(R) = n < \infty$. Suppose M is a right R-module, then $fpd(M) \leq m$ by Theorem 3.4. So M admits an *FP*-projective resolution

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is *FP*-projective, $i = 0, 1, \ldots, m$. Let $K_i = \ker(P_i \to P_{i-1})$, $i = 0, 1, 2, \ldots, m-1, P_{-1} = M$, $K_{m-1} = P_m$. Then we have the following short exact sequences

$$0 \to P_m \to P_{m-1} \to K_{m-2} \to 0,$$
$$0 \to K_{m-2} \to P_{m-2} \to K_{m-3} \to 0,$$
$$\vdots$$
$$0 \to K_0 \to P_0 \to M \to 0.$$

Note that $\sup\{pd(M) : M \text{ is an } FP\text{-projective right } R\text{-module}\} = wD(R) = n$ by Theorem 4.2. It follows that $pd(K_{m-2}) \leq 1 + n, pd(K_{m-3}) \leq 2 + n, \ldots, pd(M) \leq m + n$, and hence $rD(R) \leq m + n$. This completes the proof. $\square$

We conclude this paper with the following remark.

Remark 4.6. For convenience, we write (a, b, c) for the class of rings R with $wD(R) = a, rD(R) = b$ and $rfpD(R) = c$ and call a ring R an (a, b, c)-ring if $R \in (a, b, c)$. It is easy to see that the class of semisimple Artinian rings $= (0, 0, 0)$, the class of von Neumann regular rings $= \bigcup_{m \geq 0}(0, m, m)$, the class of right Noetherian rings $= \bigcup_{m \geq 0}(m, m, 0)$, and the class of right hereditary rings $= (0, 0, 0) \cup (1, 1, 0) \cup (0, 1, 1) \cup (1, 1, 1)$, and we also note that the class of right semihereditary rings $= (0, a, a) \cup ((1, b, b) \cup (1, c, c-1)) \cap \mathbb{C}, a \geq 0, b \geq 1, c \geq 1$ by Proposition 4.5, where $\mathbb{C}$ denotes the class of right coherent rings.

We observe that if R is a right coherent (a_1, b_1, c_1)-ring and S is a right coherent (a_2, b_2, c_2)-ring, then $R \oplus S$ is a right coherent $(\sup(a_1, a_2), \sup(b_1, b_2),$

sup(c_1, c_2))-ring by Theorem 3.16. Let m and n be integers with $m \geq n > 0$. We claim that (m, m, n)-rings and (n, m, m)-rings always exist. In fact, the polynomial ring of m indeterminates over a field is an $(m, m, 0)$-ring. By Pierce (1967), Corollary 5.2, there exists a von Neumann regular ring of global dimension n. This is a $(0, n, n)$-ring. The direct sum of an $(m, m, 0)$-ring and a $(0, n, n)$-ring is an (m, m, n)-ring, and the direct sum of an $(n, n, 0)$-ring and a $(0, m, m)$-ring gives an (n, m, m)-ring. This fact also shows that the inequality in Proposition 4.5 may be strict.

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