GENERALIZED LIFTING MODULES

YONGDUO WANG AND NANQING DING

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We introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given left module. We also introduce the notion of SSRS-modules. It is shown that (1) if M is an amply supplemented module and $0 \to N' \to N \to N'' \to 0$ an exact sequence, then M is N-lifting if and only if it is N'-lifting and N''-lifting; (2) if M is a Noetherian module, then M is lifting if and only if M is R-lifting if and only if M is an amply supplemented SSRS-module; and (3) let M be an amply supplemented SSRS-module such that $\mathrm{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where K is a radical module and K' is a lifting module.

1. Introduction and preliminaries

Extending modules and their generalizations have been studied by many authors (see [2, 3, 8, 7]). The motivation of the present discussion is from [2, 8], where the concepts of extending modules and (quasi-)continuous modules with respect to a given module and CESS-modules were studied, respectively. In this paper, we introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given module and SSRS-modules. It is shown that (1) if $0 \to N' \to N \to N'' \to 0$ is an exact sequence and M an amply supplemented module, then M is N-lifting if and only if it is both N'-lifting and N''-lifting; (2) if M is a Noetherian module, then M is lifting if and only if M is R-lifting if and only if M is an amply supplemented SSRS-module; and (3) let M be an amply supplemented SSRS-module such that $\mathrm{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where K is a radical module and K' is a lifting module.

Throughout this paper, R is an associative ring with identity and all modules are unital left R-modules. We use $N \le M$ to indicate that N is a submodule of M. As usual, $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ stand for the Jacobson radical and the socle of a module M, respectively.

Let M be a module and $S \le M$. S is called *small* in M (notation $S \ll M$) if $M \ne S + T$ for any proper submodule T of M. Let N and L be submodules of M, N is called a *supplement* of L in M if $N + L = M$, and N is minimal with respect to this property. Equivalently,

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$M = N + L$ and $N \cap L \ll N$. N is called a *supplement submodule* if N is a supplement of some submodule of M. M is called an *amply supplemented* module if for any two submodules A and B of M with $A + B = M$, B contains a supplement of A. M is called a *weakly supplemented module* (see [5]) if for each submodule A of M there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll M$. Let $B \leq A \leq M$. If $A/B \ll M/B$, then B is called a *coessential submodule* of A and A is called a *coessential extension* of B in M. A submodule A of M is called *coclosed* if A has no proper coessential submodules in M. Following [5], B is called an *s-closure* of A in M if B is a coessential submodule of A and B is coclosed in M.

Let M be a module. M is called a *lifting module* (or satisfies (D_1)) (see [9]) if for every submodule A of M, there exists a direct summand K of M such that $K \leq A$ and $A/K \ll M/K$, equivalently, M is amply supplemented and every supplement submodule of M is a direct summand. M is called *discrete* if M is lifting and has the following condition.

(D_2) If $A \leq M$ such that M/A is isomorphic to a direct summand of M, then A is a summand of M.

M is called *quasidiscrete* if M is lifting and has the following condition:

(D_3) For each pair of direct summands A and B of M with $A + B = M$, $A \cap B$ is a direct summand of M. For more details on these concepts, see [9].

Lemma 1.1 (see [12, 19.3]). *Let M be a module and $K \leq L \leq M$.*

(1) *$L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*

(2) *If M' is a module and $\phi : M \to M'$ a homomorphism, then $\phi(L) \ll M'$ whenever $L \ll M$.*

Lemma 1.2 (see Lemma 1.1 in [5]). *Let M be a weakly supplemented module and $N \leq M$. Then the following statements are equivalent.*

(1) *N is a supplement submodule of M.*

(2) *N is coclosed in M.*

(3) *For all $X \leq N$, $X \ll M$ implies $X \ll N$.*

Lemma 1.3 (see Proposition 1.5 in [5]). *Let M be an amply supplemented module. Then every submodule of M has an s-closure.*

Lemma 1.4 (see [12, 41.7]). *Let M be an amply supplemented module. Then every coclosed submodule of M is amply supplemented.*

2. Relative lifting modules

To define the concepts of relative lifting and (quasi-)discrete modules, we dualize the concepts of relative extending and (quasi-)continuous modules introduced in [8] in this section. We start with the following.

Let N and M be modules. We define the family

$$\$(N,M) = \left\{A \leq M \mid \exists X \leq N,\ \exists f \in \operatorname{Hom}(X,M),\ \ni \frac{A}{f(X)} \ll \frac{M}{f(X)}\right\}. \tag{2.1}$$

Proposition 2.1. *$\$(N,M)$ is closed under small submodules, isomorphic images, and coessential extensions.*

Proof. We only show that $\$(N,M)$ is closed under coessential extensions. Let $A \in \$(N,M)$, $A \leq A' \leq M$, and $A'/A \ll M/A$. There exist $X \leq N$ and $f \in \operatorname{Hom}(X,M)$ such that $f(X) \leq A$ and $A/f(X) \ll M/f(X)$ since $A \in \$(N,M)$. Note that $A'/A \ll M/A$, so $A'/f(X) \ll M/f(X)$ by Lemma 1.1(1). Thus $A' \in \$(N,M)$. □

Lemma 2.2. *Let $A \in \$(N,M)$ and A be coclosed in M. Then $B \in \$(N,M)$ for any submodule B of A.*

Proof. There exist $X \leq N$ and $f \in \operatorname{Hom}(X,M)$ such that $f(X) \leq A$ and $A/f(X) \ll M/f(X)$ by hypothesis. Since A is coclosed in M, $f(X) = A$. Let B be any submodule of A and $Y = f^{-1}(B) \leq X \leq N$. Then $f|_Y : Y \to M$ is a homomorphism such that $f|_Y(Y) = B$ for $f(X) = A$. Clearly $B/f|_Y(Y) \ll M/f|_Y(Y)$. Therefore $B \in \$(N,M)$. □

Lemma 2.3. *Let $C \leq A \leq B \leq M$ and A be a coessential submodule of B. If C is an s-closure of A, then it is also an s-closure of B.*

Proof. It is clear by Lemma 1.1(1). □

Proposition 2.4. *Let M be an amply supplemented module. Then every A in $\$(N,M)$ has an s-closure $\overline{A}$ in $\$(N,M)$.*

Proof. Since $A \in \$(N,M)$, there exist $X \leq N$ and $f \in \operatorname{Hom}(X,M)$ such that $A/f(X) \ll M/f(X)$. Note that M is amply supplemented, and so $f(X)$ has an s-closure $\overline{A}$ in M by Lemma 1.3. Thus $\overline{A}$ is also an s-closure of A by Lemma 2.3. The verification for $\overline{A} \in \$(N,M)$ is analogous to that for $B \in \$(N,M)$ in Lemma 2.2. □

Let N be a module. Consider the following conditions for a module M.

($\$(N,M)$-$D_1$) For every submodule $A \in \$(N,M)$, there exists a direct summand K of M such that $K \leq A$ and $A/K \ll M/K$.

($\$(N,M)$-$D_2$) If $A \in \$(N,M)$ such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

($\$(N,M)$-$D_3$) If A and L are direct summands of M with $A \in \$(N,M)$ and $A + L = M$, then $A \cap L$ is a direct summand of M.

Definition 2.5. Let N be a module. A module M is said to be N-lifting, N-discrete, or N-quasidiscrete if M satisfies $\$(N,M)$-$D_1$, $\$(N,M)$-$D_1$ and $\$(N,M)$-$D_2$ or $\$(N,M)$-$D_1$ and $\$(N,M)$-$D_3$, respectively.

One easily obtains the hierarchy: M is N-discrete $\Rightarrow$ M is N-quasidiscrete$\Rightarrow$ M is N-lifting. Clearly, the notion of relative discreteness generalizes the concept of discreteness. For any module N, lifting modules are N-lifting. But the converse is not true as shown in the following examples.

Example 2.6. Since, for any module M, $\$(0,M) = \{A \mid A \ll M\}$ and 0 is a direct summand of M such that $A/0 \ll M/0$ for any $A \in \$(0,M)$, all modules are 0-lifting. However, the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ is not lifting since the supplement submodule $\langle(1,2)\rangle$

$(\langle(1,2)\rangle$ is a supplement of $\langle(1,1)\rangle)$ and is not a direct summand of it though it is amply supplemented.

Example 2.7. Let M be a module with zero socle and S a simple module. Then M is S-lifting since $\$(S,M)$ is a family only containing all small submodules of M. So all torsion-free $\mathbb{Z}$-modules are S-lifting for any simple $\mathbb{Z}$-module S (see [12, Exercise 21.17]). In particular, ${}_{\mathbb{Z}}\mathbb{Z}$ and ${}_{\mathbb{Z}}\mathbb{Q}$ are S-lifting for any simple $\mathbb{Z}$-module, but each one is not a lifting module.

Lemma 2.8. *Let M be a module. Then $\$(M,M) = \{A \mid A \leq M\} = \bigcup_{N\in R\text{-Mod}} \$(N,M)$, where R-Mod denotes the category of left R-module.*

Proof. It is straight forward. □

Proposition 2.9. *Let M be a module. Then M is lifting or (quasi-)discrete if and only if M is M-lifting or M-(quasi-)discrete if and only if M is N-lifting or N-(quasi-)discrete for any module N.*

Proof. It is clear by Lemma 2.8. □

Proposition 2.10. *Let M be an amply supplemented module. Then the condition $\$(N,M)$-$D_1$ is inherited by coclosed submodules of M.*

Proof. Let M satisfy $\$(N,M)$-$D_1$ and H be a coclosed submodule of M. H is amply supplemented by Lemma 1.4. For any $A \in \$(N,H)$, A has an s-closure $\overline{A} \in \$(N,H)$ in H by Proposition 2.4. Since $\overline{A} \in \$(N,H) \subseteq \$(N,M)$ and M satisfies $\$(N,M)$-$D_1$, there is a direct summand K of M such that $K \leq \overline{A}$ and $\overline{A}/K \ll M/K$. By Lemma 1.2, $\overline{A}/K \ll H/K$. Now $\overline{A} = K$ since $\overline{A}$ is coclosed in H. Thus H satisfies $\$(N,H)$-$D_1$. □

Corollary 2.11. *Let M be an amply supplemented module. Then the condition $\$(N,M)$-$D_1$ is inherited by direct summands of M.*

Proposition 2.12. *Let M be an amply supplemented module. Then $\$(N,M)$-$D_i$ $(i = 2,3)$ is inherited by direct summands of M.*

Proof. (1) Let M satisfy $\$(N,M)$-$D_2$ and H be a direct summand of M. We will show that H satisfies $\$(N,H)$-$D_2$.

Let $A \in \$(N,H) \subseteq \$(N,M)$ and H/A is isomorphic to a direct summand of H. Since H is a direct summand of M, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M/A = (H \oplus H')/A \simeq (H/A) \oplus H'$, and so M/A is isomorphic to a direct summand of M. A is a direct summand of M since M satisfies $\$(N,M)$-$D_2$, and hence A is a direct summand of H.

(2) Let $A \in \$(N,H) \subseteq \$(N,M)$ and A, L be direct summands of H with $A + L = H$. We will show that $A \cap L$ is a direct summand of H. Since H is a direct summand of M, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M = (A + L) \oplus H' = A + (L \oplus H')$. Now $A \cap (L \oplus H')$ is a direct summand of M since M satisfies $\$(N,M)$-$D_3$. Note that $A \cap (L \oplus H') = A \cap L$, so $A \cap L$ is a direct summand of H. □

Theorem 2.13. *Let M be an amply supplemented module and $A \in \$(N,M)$ a direct summand of M. If M is N-(quasi-)discrete, then A is (quasi-)discrete.*

Proof. The proof follows from Lemma 2.2, Corollary 2.11, and Proposition 2.12. □

Proposition 2.14. *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence. Then $\$(N',M) \cup \$(N'',M) \subseteq \$(N,M)$. Therefore, if M is N-lifting (resp., (quasi-)discrete), then M is N'-lifting and N''-lifting (resp., (quasi-)discrete).*

Proof. Without loss of generality we can assume that $N' \leq N$ and $N'' = N/N'$. By definition, $N' \leq N$ implies $\$(N',M) \subseteq \$(N,M)$. Next, let $A_2 \in \$(N'',M)$. Then there exist $X \leq N'' = N/N'$ and $f \in \operatorname{Hom}(X,M)$ such that $A_2/f(X) \ll M/f(X)$. Write $X = Y/N'$, $Y \leq N$ and let $\delta : Y \to Y/N'$ be the canonical homomorphism. It is clear that $g = f\delta \in \operatorname{Hom}(Y,M)$ and $g(Y) = f(X)$, hence $A_2/g(Y) \ll M/g(Y)$. Thus $A_2 \in \$(N,M)$. Therefore $\$(N',M) \cup \$(N'',M) \subseteq \$(N,M)$. The rest is obvious. □

Dual to [8, Proposition 2.7], we have the following.

Theorem 2.15. *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence and M an amply supplemented module. Then M is N-lifting if and only if it is both N'-lifting and N''-lifting.*

Proof. Let M be N-lifting. Then it is both N'-lifting and N''-lifting by Proposition 2.14. Conversely suppose that M is both N'-lifting and N''-lifting. For any submodule $A \in \$(N,M)$, A has an s-closure $\overline{A} \in \$(N,M)$ by Proposition 2.4. Since $\overline{A} \in \$(N,M)$, there exist $X \leq N$ and $f \in \operatorname{Hom}(X,M)$ such that $\overline{A}/f(X) \ll M/f(X)$. Since $\overline{A}$ is coclosed in M, $f(X) = \overline{A}$. Write $Y = X \cap N' \leq N'$ and $f|_Y : Y \to M$ is a homomorphism, then $f(Y) \leq f(X) = \overline{A}$. Let $\overline{f(Y)}$ be an s-closure of $f(Y)$ in $\overline{A}$ (for $\overline{A}$ is amply supplemented). Thus we conclude that $f(Y)/\overline{f(Y)} \ll M/\overline{f(Y)}$ and $\overline{f(Y)} \in \$(N',M)$. Since M is N'-lifting, there exists a direct summand K of M such that $\overline{f(Y)}/K \ll M/K$. It is easy to see $\overline{f(Y)}$ is coclosed in M, hence $\overline{f(Y)} = K$ is a direct summand of M. Write $M = \overline{f(Y)} \oplus K'$, $K' \leq M$ and $\overline{A} = \overline{A} \cap M = \overline{f(Y)} \oplus (\overline{A} \cap K')$. Define $h : W = (X + N')/N' \to M$ by $h(x + N') = \pi f(x)$, where $\pi : \overline{A} \to \overline{A} \cap K'$ denotes the canonical projection. It is clear that $h(W) = \overline{A} \cap K'$, thus $(\overline{A} \cap K')/h(W) \ll M/h(W)$, and hence $(\overline{A} \cap K') \in \$(N'',M)$. Since M is N''-lifting, there exists a direct summand K'' of M such that $(\overline{A} \cap K')/K'' \ll M/K''$. Since $\overline{A} \cap K'$ is coclosed in M, $\overline{A} \cap K' = K''$. Now $\overline{A} \cap K'$ is a direct summand of K'. Thus $\overline{A}$ is a direct summand of M. It follows that M is N-lifting. □

Corollary 2.16. *Let M be an amply supplemented module. If M is N_i-lifting for $i = 1, 2, \ldots, n$ and $N = \bigoplus_i^n N_i$, then M is N-lifting.*

Corollary 2.17. *Let M be an amply supplemented module. Then M is lifting if and only if M is N-lifting and M/N-lifting for every submodule N of M if and only if M is N-lifting and M/N-lifting for some submodule N of M.*

Recall that a module M is said to be *distributive* if $N \cap (K + L) = (N \cap K) + (N \cap L)$ for all submodules N, K, L of M. A module M has *SSP* (see [4]) if the sum of any pair of direct summands of M is a direct summand of M.

Corollary 2.18. *Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence and let M be a distributive and amply supplemented module with SSP. If M is both N'-quasidiscrete and N''-quasidiscrete, then M is N-quasidiscrete.*

Proof. We only need to show that M satisfies $\$(N,M)$-$D_3$ when M satisfies $\$(N',M)$-$D_3$ and $\$(N'',M)$-$D_3$ by Theorem 2.15. Let $A \in \$(N,M)$ and A, H be direct summands of M with $A+H=M$. We know that $A = A_1 \oplus A_2$, where $A_1 \in \$(N',M)$, $A_2 \in \$(N'',M)$ from the proof of Theorem 2.15. Since M is a distributive module with SSP, $A_1 \cap H$ and $A_2 \cap H$ are direct summands of M. This implies that $A \cap H$ is a direct summand of M. Thus M satisfies $\$(N,M)$-$D_3$. □

3. SSRS-modules

In [2], a module is called a *CESS-module* if every complement with essential socle is a direct summand. As a dual of CESS-modules, the concept of SSRS-modules is given in this section. It is proven that: (1) let M be an amply supplemented SSRS-module such that $\mathrm{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where K is a radical module and K' is a lifting module; (2) let M be a finitely generated amply supplemented module, then M is an SSRS-module if and only if M/K is a lifting module for every coclosed submodule K of M.

Definition 3.1. A module is called an *SSRS-module* if every supplement with small radical is a direct summand.

Lifting modules are SSRS-modules, but the converse is not true. For example, ${}_{\mathbb{Z}}\mathbb{Z}$ is an SSRS-module which is not a lifting module.

Proposition 3.2. *Let M be an SSRS-module. Then any direct summand of M is an SSRS-module.*

Proof. Let K be a direct summand of M and N a supplement submodule of K such that $\mathrm{Rad}(N) \ll N$. Let N be a supplement of L in K, that is, $N+L=K$ and $N \cap L \ll N$. Since K is a direct summand of M, there exists $K' \leq M$ such that $M = K \oplus K'$. Note that $M = N + (L \oplus K')$ and $N \cap (L \oplus K') = N \cap L \ll N$. Therefore N is a supplement of $L \oplus K'$ in M. Thus N is a direct summand of M since M is an SSRS-module. So N is a direct summand of K. The proof is complete. □

Proposition 3.3. *Let M be a weakly supplemented SSRS-module and K a coclosed submodule of M. Then K is an SSRS-module.*

Proof. It follows from the assumption and [4, Lemma 2.6(3)]. □

Proposition 3.4. *Let M be an amply supplemented module. Then M is an SSRS-module if and only if for every submodule N with small radical, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$.*

Proof. "⇐." Let N be a supplement submodule with small radical. By assumption, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. Since N is coclosed in M, $N = K$. Thus N is a direct summand of M.

"⇒." Let $N \leq M$ with $\mathrm{Rad}(N) \ll N$. There exists an s-closure $\overline{N}$ of N since M is amply supplemented. Since $\mathrm{Rad}(N) \ll M$ (for $\mathrm{Rad}(N) \ll N$) and $\mathrm{Rad}(\overline{N}) \leq \mathrm{Rad}(N)$,

$\mathrm{Rad}(\overline{N}) \ll \overline{N}$ and $\overline{N}$ is a supplement submodule by Lemma 1.2. Therefore $\overline{N}$ is a direct summand of M by assumption. This completes the proof. □

Corollary 3.5. *Let M be an amply supplemented SSRS-module. Then every simple submodule of M is either a direct summand or a small submodule of M.*

Proposition 3.6. *Let M be an amply supplemented module. Then M is an SSRS-module if and only if for every submodule N of M, every s-closure of N with small radical is a lifting module and a direct summand of M.*

Proof. It is straight forward. □

Proposition 3.7. *Let M be an amply supplemented SSRS-module. Then $M = K \oplus K'$, where K is semisimple and K' has small socle.*

Proof. For $\mathrm{Soc}(M)$, there exists a direct summand K of M such that $\mathrm{Soc}(M)/K \ll M/K$ by Proposition 3.4. It is easy to see that K is semisimple. Since K is a direct summand of M, there exists $K' \leq M$ such that $M = K \oplus K'$. Note that $\mathrm{Soc}(M) = \mathrm{Soc}(K) \oplus \mathrm{Soc}(K')$. So $\mathrm{Soc}(M)/K = (K \oplus \mathrm{Soc}(K'))/K \ll M/K = (K \oplus K')/K$. Thus $\mathrm{Soc}(K') \ll K'$. □

Recall that a module M is called a *radical module* if $\mathrm{Rad}(M) = M$. Dual to [2, Theorem 2.6], we have the following.

Theorem 3.8. *Let M be an amply supplemented SSRS-module such that* $\mathrm{Rad}(M)$ *is finitely generated. Then $M = K \oplus K'$, where K is a radical module and K' is a lifting module.*

Proof. $\mathrm{Rad}(\mathrm{Rad}(M)) \ll \mathrm{Rad}(M)$ since $\mathrm{Rad}(M)$ is finitely generated. There exists a direct summand K of M such that $\mathrm{Rad}(M)/K \ll M/K$ by Proposition 3.4. Since K is a direct summand of M, there exists $K' \leq M$ such that $M = K \oplus K'$. Note that $\mathrm{Rad}(M) = \mathrm{Rad}(K) \oplus \mathrm{Rad}(K')$. Therefore $K = K \cap \mathrm{Rad}(M) = \mathrm{Rad}(K)$ and $\mathrm{Rad}(M)/K = (\mathrm{Rad}(K) \oplus \mathrm{Rad}(K'))/K \ll M/K = (K \oplus K')/K$. Thus $\mathrm{Rad}(K) = K$ and $\mathrm{Rad}(K') \ll K'$.

Next, we show that K' is a lifting module. K' is amply supplemented since it is a direct summand of M. So we only prove that every supplement submodule of K' is a direct summand of K'. Let N be a supplement submodule of K'. By Lemma 1.2 and $\mathrm{Rad}(K') \ll K'$, we know that $\mathrm{Rad}(N) \ll N$. N is a direct summand of K' since K' is an SSRS-module by Proposition 3.2. The proof is complete. □

Corollary 3.9. *Let M be an amply supplemented module with small radical. Then M is an SSRS-module if and only if M is a lifting module.*

Theorem 3.10. *Let M be a finitely generated amply supplemented module. Then the following statements are equivalent.*

(1) *M is an SSRS-module.*

(2) *M is a lifting module.*

(3) *M/K is a lifting module for every coclosed submodule K of M.*

Proof. (1)$\Leftrightarrow$(2) follows from Corollary 3.9.

(3)$\Rightarrow$(1) is clear.

(1)$\Rightarrow$(3) we only prove that any supplement submodule of M/K is a direct summand. Let A/K be a supplement submodule of M/K. A is coclosed in M since A/K is coclosed in

M/K and K is coclosed in M. $\mathrm{Rad}(A) \ll A$ since M is finitely generated and A is coclosed in M. A is a direct summand of M by assumption. Thus A/K is a direct summand of M/K. □

LEMMA 3.11. *Let M be a module. Then the following statements are equivalent.*
(1) *For every cyclic submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$.*
(2) *For every finitely generated submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$.*

Proof. See [12, 41.13]. □

COROLLARY 3.12. *Let M be a Noetherian module. Then the following statements are equivalent.*
(1) *M is R-lifting.*
(2) *M is F-lifting, for any free module F.*
(3) *M is lifting.*
(4) *M is an amply supplemented SSRS-module.*

Proof. It is easy to see that $\$(R,M)$ and $\$(F,M)$ are closed under cyclic submodules. The rest follows immediately from Theorem 3.10 and Lemma 3.11. □

COROLLARY 3.13. *Let R be a left perfect (semiperfect) ring. Then every SSRS-module (finitely generated SSRS-module) is a lifting module.*

Proof. It follows from the fact that every module over a left perfect ring has small radical, [11, Theorems 1.6 and 1.7] and Corollary 3.9. □

A module M is *uniserial* (see [6]) if its submodules are linearly ordered by inclusion and it is *serial* if it is a direct sum of uniserial submodules. A ring R is *right (left) serial* if the right (left) R-module $R_R({}_RR)$ is serial and it is serial if it is both right and left serial.

COROLLARY 3.14. *The following statements are equivalent for a ring R with radical J.*
(1) *R is an artinian serial ring and $J^2 = 0$.*
(2) *R is a left semiperfct ring and every finitely generated module is an SSRS-module.*
(3) *R is a left perfect ring and every module is an SSRS-module.*

Proof. It holds by [6, Theorem 3.15], [10, Theorem 1 and Proposition 2.13], and Corollary 3.13. □

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Yongduo Wang: Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, China
E-mail address: ydwang333@sohu.com

Nanqing Ding: Department of Mathematics, Nanjing University, Nanjing 210093, China
E-mail address: nqding@nju.edu.cn