Communications in Algebra®, 35: 2820–2837, 2007
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ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870701354017

GENERALIZED MORPHIC RINGS AND THEIR APPLICATIONS

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A ring R is called "left generalized morphic" if for every element a in R, there exists $b \in R$ such that $l(a) \cong R/Rb$, where $l(a)$ denotes the left annihilator of a in R. The aim of this article is to investigate these rings. Several examples are given. They include left morphic rings and left p.p. rings. As applications, some homological dimensions over these rings are defined and studied.

Key Words: Generalized morphic ring; P-Flat module; P-Injective module.

2000 Mathematics Subject Classification: 16E50; 16E10.

1. INTRODUCTION

Let R be a ring. An element a in R is called *left morphic* (Nicholson and Sánchez Campos, 2004a) if $l(a) \cong R/Ra$, where $l(a)$ denotes the left annihilator of a in R. The ring itself is called a *left morphic ring* if every element is left morphic. Left morphic rings were first introduced by Nicholson and Sánchez Campos (2004a) and were discussed in great detail there and in Nicholson and Sánchez Campos (2004b, 2005).

Our focus is on the case that the condition becomes $l(a) \cong R/Rb$ for some $b \in R$. We say that the ring R is *left generalized morphic* if every element satisfies this condition.

In Section 2, the definition and some general results are given. Examples of left generalized morphic rings include left morphic rings and left p.p. rings. It is shown that a ring R is left generalized morphic if and only if the exactness of $0 \to I \to {}_RR \to {}_RR$ of left R-modules implies that I is a principal left ideal. It is also shown that, if R is a left P-injective ring, then R is left generalized morphic if and only if $(R/aR)^*$ is cyclic for every torsionless right R-module of the form R/aR with $a \in R$. Let $\mathscr{PP}$ ($\mathscr{PI}$) be the class of P-projective (P-injective) left R-modules. We prove that $(\mathscr{PP}, \mathscr{PI})$ is a hereditary cotorsion theory if R is left generalized morphic.

Received February 18, 2006; Accepted May 23, 2006. Communicated by T. Albu.
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Section 3 is devoted to some homological dimensions over generalized morphic rings. Let R be a ring. The P-injective dimension $P\text{-}id(M)$ of ${}_RM$ is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(R/Ra, M) = 0$ for all $a \in R$. If no such n exists, set $P\text{-}id(M) = \infty$. $l.P\text{-}i\dim(R)$ is defined as $\sup\{P\text{-}id(M) \mid M \in R\text{-Mod}\}$. The P-flat dimension $P\text{-}fd(M)$ of M_R is defined to be the least non-negative integer n such that $\mathrm{Tor}_{n+1}(M, R/Ra) = 0$ for all $a \in R$. If no such n exists, set $P\text{-}fd(M) = \infty$. $r.P\text{-}f\dim(R)$ is defined to be $\sup\{P\text{-}fd(M) \mid M \in \text{Mod-}R\}$. We shall see that $l.P\text{-}i\dim(R)$ and $r.P\text{-}f\dim(R)$ measure how far away a ring R is from being a von Neumann regular ring. If the ring in question is generalized morphic, the homological dimensions defined here have the properties that we expect of a "dimension". It is proven that, if R is left generalized morphic, then $l.P\text{-}i\dim(R) = r.P\text{-}f\dim(R) = \sup\{P\text{-}id(M) \mid {}_RM \in \mathcal{PP}\} = \sup\{P\text{-}fd(M) \mid M_R \text{ is cyclic}\} = \sup\{pd(M) \mid {}_RM \in \mathcal{PP}\} = \sup\{pd(R/Ra) \mid a \in R\} = \sup\{fd(R/Ra) \mid a \in R\}$. As a corollary, we have that a ring R is unit regular if and only if R is a left p.p. and right morphic ring.

Next we recall some known notions and facts needed in the sequel.

Given a class $\mathcal{L}$ of R-modules, we denote by $\mathcal{L}^{\perp} = \{C : \mathrm{Ext}^1(L, C) = 0$ for all $L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by ${}^{\perp}\mathcal{L} = \{C : \mathrm{Ext}^1(C, L) = 0$ for all $L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following Enochs and Jenda (2000), we say that a homomorphism $\phi : M \to C$ is a *$\mathcal{C}$-preenvelope* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(\phi, C')$: $\mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a *$\mathcal{C}$-envelope* if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. A monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a *special $\mathcal{C}$-preenvelope of M* if coker $(\alpha) \in {}^{\perp}\mathcal{C}$. Dually we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a *cotorsion theory* if $\mathcal{F} = {}^{\perp}\mathcal{C}$ and $\mathcal{C} = \mathcal{F}^{\perp}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *complete* (*perfect*) provided that every R-module has a special $\mathcal{C}$-preenvelope and a special $\mathcal{F}$-precover (a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover). A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be *hereditary* if $\mathrm{Ext}^i(F, C) = 0$ for all $i \geq 1$ and all $F \in \mathcal{F}$ and $C \in \mathcal{C}$, or equivalently, if whenever $0 \to C' \to C \to C'' \to 0$ is exact with $C', C \in \mathcal{C}$ then C'' is also in $\mathcal{C}$. See Enochs and Jenda (2000), Enochs et al. (1998), García Rozas (1999), Trlifaj (2000), and Xu (1996) for more details about covers, envelopes and cotorsion theories.

Throughout this article, R is an associative ring with identity and all modules are unitary R-modules. R-Mod (Mod-R) is the category of all left (right) R-modules. We write ${}_RM$ (M_R) to indicate that M is a left (right) R-module. For an R-module M, $M^* = \mathrm{Hom}_R(M, R)$ stands for the dual module, and the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. $pd(M)$ and $id(M)$ denote the projective and injective dimensions of M, respectively. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ ($\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ ($\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ ($\mathrm{Tor}_n(M, N)$) stands for $M \otimes_R N$ ($\mathrm{Tor}^R_n(M, N)$) for an integer $n \geq 1$. General background materials can be found in Anderson and Fuller (1974), Lam (1999), Rotman (1979), and Wisbauer (1991).

2. GENERALIZED MORPHIC RINGS

We start with the following definition.

Definition 2.1. Let R be a ring. An element a in R is called *left generalized morphic* if $l(a) \cong R/Rb$ for some $b \in R$. The ring itself is said to be *left generalized morphic* if every element is left generalized morphic.

Lemma 2.2. *The following are equivalent for an element a in R:*

(1) *a is left generalized morphic;*
(2) *There exist $b, c \in R$ such that $l(a) = Rb$ and $l(b) = Rc$;*
(3) *There exist $b, c \in R$ such that $l(a) \cong Rb$ and $l(b) = Rc$.*

Proof. (1) $\Rightarrow$ (2). By (1), there is $c \in R$ such that $l(a) \cong R/Rc$. Let $\sigma : R/Rc \to l(a)$ be an isomorphism, and put $b = \sigma(1 + Rc)$. Then $Rb = \mathrm{im}(\sigma) = l(a)$ because σ is onto, and $l(b) = Rc$ because σ is one-to-one.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) follows since $l(a) \cong Rb \cong R/l(b) = R/Rc$. □

Corollary 2.3. *A ring R is a left generalized morphic ring if and only if $l(a)$ is a principal left ideal for each $a \in R$.*

Example 2.4. 1) Clearly, each left morphic ring is left generalized morphic, and the converse is false. In fact, every principal left ideal ring (every left ideal is principal) is a left generalized morphic ring by Corollary 2.3, but it need not be left morphic. For example, the ring $\mathbb{Z}$ of integers is generalized morphic but not morphic.

2) Left p.p. rings (Endo, 1960) (principal left ideals are all projective) are left generalized morphic. In particular, all domains are generalized morphic, and hence generalized morphic rings need not be coherent (a ring R is called *left coherent* if every finitely generated left ideal is finitely presented).

3) Let V be a vector space of countably infinite dimension over a field, then $R = \mathrm{End}(V)$ is a von Neumann regular ring. So R is generalized mophic, but it is not a morphic ring by the proof of Nicholson and Sánchez Campos (2004a, Example 25).

An elementary argument using condition (2) in Lemma 2.2 shows that a direct product of rings is left generalized morphic if and only if each component is left generalized morphic.

Recall that a ring R is called *left* $(1, 1)$*-coherent* (Zhang et al., 2005) in case each principal left ideal of R is finitely presented. Clearly, a left generalized morphic ring is left $(1, 1)$-coherent, but the converse is false as shown by the following example.

Example 2.5. Let $R = \mathbb{Z}[x_1, x_2]$, the ring of polynomials in 2 indeterminates over $\mathbb{Z}$, then R is a coherent ring. Now let I be a left ideal of R generated by x_1^2, x_1x_2, then R/I is a left coherent ring by Glaz (1989). In particular, R/I is a $(1, 1)$-coherent ring. However, the left annihilator of $x_1 + I$ in R/I is generated by $x_1 + I, x_2 + I$, but it is not a principal left ideal. So R/I is not left generalized morphic.

Next we construct a right generalized morphic ring which is not left generalized morphic. This example is taken from Lam (1999, Examples 4.46(e)).

Example 2.6. Let $L = \mathbb{Q}(x_2, x_3, \ldots)$ be a subfield of $K = \mathbb{Q}(x_1, x_2, x_3, \ldots)$ with $\mathbb{Q}$ the field of rational numbers, and there exists a field isomorphism $\varphi : K \to L$. We define a ring by taking $R = K \times K$ with multiplication

$$(x, y)(x', y') = (xx', \varphi(x)y' + yx'), \qquad \text{where } x, y, x', y' \in K.$$

Let $a = (0, 1)$, then $l(a) = (0, K)$ is not a cyclic left ideal. On the other hand, it is easy to see that R has exactly three right ideals, (0), R, and $(0, K) = (0, 1)R$. Therefore, R is right generalized morphic.

Recall that a ring is called *left special* (Nicholson and Sánchez Campos, 2004a) in case it is left morphic, local, and the Jacobson radical J is nilpotent. Here we can construct a left and right artinian, left special ring that is not right generalized morphic. The example traces back to Björk (1970).

Example 2.7. Let F be a field with an isomorphism $x \mapsto \bar{x}$ from F to a subfield $\overline{F} \neq F$. Let R denote the left F-space on basis $\{1, c\}$ where $c^2 = 0$ and $cx = \bar{x}c$ for all $x \in F$. Then R is a left and right artinian, left special ring by Björk (1970, Example) and Nicholson and Sánchez Campos (2004a, Example 11). But R is not right generalized morphic. Otherwise, assume R is right generalized morphic. Let $a \in J = Rc = Fc$, then $J \subseteq r(a) \neq R$ since $J^2{=}0$ where $r(a)$ is the right annihilator of a in R. Note that R is local, and so $J = r(a)$. It follows that $J = bR$ for some $b \in R$ by assumption, and hence there is $u \in R$ such that $b = uc$. Since $b \neq 0$, $u \notin J$. So u is a unit. Thus $cR = u^{-1}bR = u^{-1}J = J = Fc$. But $cR = \overline{F}c$, and so $\overline{F}c = Fc$, which implies $\overline{F} = F$, a contradiction.

Proposition 2.8. *Let R be a local ring and J nilpotent. Then R is left morphic if and only if R is left generalized morphic.*

Proof. Necessity is clear.

Conversely, let $J^n = 0$ but $J^{n-1} \neq 0$ for some $n \geq 1$. Choose $0 \neq a \in J^{n-1}$, then $J \subseteq l(a) \neq R$, and so $J = l(a)$. Since R is left generalized morphic, $l(a) = Rb$ for some $b \in R$. Note that $b \in J$, then $b^n{=}0$. Thus R is left morphic by Nicholson and Sánchez Campos (2004a, Theorem 9(2)). □

Theorem 2.9. *The following are equivalent for a ring R:*

(1) *R is a left generalized morphic ring;*
(2) *If $0 \to I \to {}_RR \to {}_RR$ is an exact sequence of left R-modules, then I is a principal left ideal;*
(3) *$(R/aR)^*$ is a principal left ideal for any $a \in R$.*

Proof. (1) $\Rightarrow$ (2) Let I be a left ideal of R such that $0 \to I \to R \to R$ is exact. Then we have a monomorphism $\varphi : R/I \to R$, and so $R/I \cong Ra$ where $a = \varphi(1_R + I)$. Thus $I = l(a)$ is a principal left ideal by (1).

(2) $\Rightarrow$ (3) For any $a \in R$, there are two exact sequences

$$0 \to r(a) \to R_R \to aR \to 0 \quad \text{and} \quad 0 \to aR \to R_R \to R/aR \to 0.$$

Applying $\mathrm{Hom}(-, R)$ to the sequences above, we have the exactness of the following sequences

$$0 \to (aR)^* \to {}_RR \to (r(a))^* \quad \text{and} \quad 0 \to (R/aR)^* \to {}_RR \to (aR)^*.$$

It follows that the sequence $0 \to (R/aR)^* \to {}_RR \to {}_RR$ is exact, and hence $(R/aR)^*$ is a principal left ideal for any $a \in R$ by hypothesis.

(3) $\Rightarrow$ (1) follows from the fact that $l(a) \cong (R/aR)^*$ for all $a \in R$. □

Recall that a left R-module M is called *P-injective* (Nicholson and Yousif, 1995) provided that $\mathrm{Ext}^1(R/Ra, M) = 0$ for all $a \in R$. A ring R is said to be *left P-injective* if ${}_RR$ is P-injective. A left morphic ring is right P-injective by Nicholson and Sánchez Campos (2004a, Theorem 24), but a generalized morphic ring need not be P-injective. For instance, let K be a field, then the polynomial ring $K[x]$ is generalized morphic but not P-injective by Nicholson and Yousif (1995, Example 5, p. 78).

Proposition 2.10. *Let R be a left P-injective ring, then the following are equivalent*:

(1) *R is a left generalized morphic ring*;
(2) *$(R/aR)^*$ is cyclic for every torsionless right R-module of the form R/aR with $a \in R$. Moreover, if R is also a right P-injective ring, then the above conditions are equivalent to*:
(3) *Every torsionless right R-module of the form R/aR embeds in the regular module R_R with $a \in R$.*

Proof. (1) $\Rightarrow$ (2) is clear by Theorem 2.9.

(2) $\Rightarrow$ (1) Let $a \in R$. Then the exact sequence $0 \to Ra \xrightarrow{i} {}_RR \to R/Ra \to 0$ gives rise to the exactness of

$$0 \to (R/Ra)^* \to R_R \xrightarrow{i^*} (Ra)^* \to \mathrm{Ext}^1(R/Ra, R) = 0$$

since R is left P-injective. On the other hand, there is an exact sequence

$$0 \to l(a) \to {}_RR \xrightarrow{f} Ra \to 0$$

which induces the exactness of

$$0 \to (Ra)^* \xrightarrow{f^*} R_R \to C_R \to 0$$

where $C = \operatorname{coker}(f^*)$. Since i^* is epic and f^* is monic, $(Ra)^* \cong bR$ for some $b \in R$. Then $C \cong R/bR$ is a torsionless right R-module (for $C \subseteq (l(a))^*$). Note that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & l(a) & \longrightarrow & R & \longrightarrow & Ra & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^* & \longrightarrow & R^{**} & \longrightarrow & (Ra)^{**}. & & \end{array}$$

Thus $l(a) \cong C^* \cong (R/bR)^*$ is cyclic by (2). So R is a left generalized morphic ring by Corollary 2.3.

(1) $\Rightarrow$ (3) Let $a \in R$ and R/aR be a torsionless right R-module, then $(R/aR)^*$ is a principal left ideal by Theorem 2.9. It follows that there is an exact sequence $R \to (R/aR)^* \to 0$ which induces the exactness of $0 \to (R/aR)^{**} \to R$. Since R/aR is torsionless, there exists a monomorphism $i : R/aR \to R$.

(3) $\Rightarrow$ (2) Let $a \in R$ and R/aR be a torsionless right R-module. By hypothesis, there is an exact sequence

$$0 \to R/aR \to R_R \to C_R \to 0 \qquad (*)$$

which induces the exact sequence

$$0 \to C^* \to {}_RR \to (R/aR)^* \to \operatorname{Ext}^1(C, R).$$

Note that $C \cong R/bR$ for some $b \in R$ by the exact sequence $(*)$. Since R is right P-injective, $\operatorname{Ext}^1(C, R) = 0$. So $(R/aR)^*$ is cyclic. $\square$

Definition 2.11. A left R-module N is said to be *P-projective* if $\operatorname{Ext}^1(N, M) = 0$ for any P-injective left R-module M. A right R-module M is called *P-flat* in case $0 \to M \otimes_R Ra \to M \otimes_R R$ is exact for any $a \in R$.

Remark 2.12. Denote by $\mathcal{PP}$ ($\mathcal{PI}$) the class of P-projective (P-injective) left R-modules, and by $\mathcal{PF}$ the class of P-flat right R-modules. We note that the concepts of P-injective, P-projective, and P-flat modules coincide with those of divisible modules, cyclically covered modules, and torsion-free modules, respectively (see Trlifaj, 2000). But the concept of P-projective modules is different from that of P-projective modules introduced by Chen (1996). P-flat modules are also called "(1,1)-flat" in Zhang et al. (2005). Note that a left R-module M is P-projective if and only if M is a direct summand in a left R-module N such that N is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$ and $N_{\alpha+1}/N_\alpha \cong R/Ra$ for all $\alpha < \lambda$ by Trlifaj (2000, Definition 3.3).

By definition, we have the following lemma.

Lemma 2.13. *Let $\{P_j\}_{j \in J}$ be a family of left (right) R-modules. Then $\bigoplus_{j \in J} M_j$ is P-projective (P-flat) if and only if each M_j is P-projective (P-flat).*

Lemma 2.14. *Let R be a ring, then the following hold*:

(1) (*Trlifaj*, 2000, *Theorem* 3.4) $(\mathcal{PP}, \mathcal{PI})$ *is a complete cotorsion theory*;
(2) (*Mao and Ding*, 2006, *Theorem* 2.3) $(\mathcal{PF}, \mathcal{PF}^{\perp})$ *is a perfect cotorsion theory.*

In general, a cotorsion theory need not be hereditary, but we have the following proposition.

Proposition 2.15. *If R is a left generalized morphic ring, then the following hold*:

(1) $(\mathcal{PP}, \mathcal{PI})$ *is a hereditary cotorsion theory*;
(2) $(\mathcal{PF}, \mathcal{PF}^{\perp})$ *is a hereditary cotorsion theory.*

Proof. (1) Let $a \in R$ and M be a P-injective left R-module. The exactness of

$$0 \to Ra \to {}_RR \to R/Ra \to 0$$

gives rise to

$$\mathrm{Ext}^i(Ra, M) \cong \mathrm{Ext}^{i+1}(R/Ra, M), \qquad i = 1, 2, \ldots.$$

If R is a left generalized morphic ring, then $Ra \cong R/l(a) \cong R/Rb$ for some $b \in R$, and so $\mathrm{Ext}^1(Ra, M) = 0$. Hence $\mathrm{Ext}^2(R/Ra, M) = 0$.

Let $0 \to N' \to N \to N'' \to 0$ be exact with N', N P-injective, then we have an exact sequence

$$\mathrm{Ext}^1(R/Ra, N) \to \mathrm{Ext}^1(R/Ra, N'') \to \mathrm{Ext}^2(R/Ra, N').$$

The first term is zero by definition, and the last term is zero by the preceding proof. Thus $\mathrm{Ext}^1(R/Ra, N'') = 0$, and so N'' is P-injective. This completes the proof.

The proof of (2) is analogous to that of (1). □

As an application of Proposition 2.15, we end this section with the following proposition.

Proposition 2.16. *Let R be a left generalized morphic and left P-injective ring. If ${}_RM$ is P-projective with finite projective dimension, then ${}_RM$ is projective.*

Proof. Suppose ${}_RM$ is P-projective with $pd(M) = n < \infty$. Then M admits a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Note that R is left P-injective and $\mathcal{PI}$ is closed under direct sums and direct summands, then each P_i is P-injective for any $0 \leq i \leq n$. Since $\mathcal{PI}$ is closed under cokernels of monomorphisms by Proposition 2.15(1), $K_{n-1} = \mathrm{coker}(P_n \to P_{n-1})$ is P-injective. It follows that $K_1 = \mathrm{coker}(P_2 \to P_1)$ is also P-injective by induction. Thus $\mathrm{Ext}^1(M, K_1) = 0$ by hypothesis, and hence the sequence $0 \to K_1 \to P_0 \to M \to 0$ is split. So M is projective, as required. □

3. HOMOLOGICAL DIMENSIONS OVER GENERALIZED MORPHIC RINGS

In this section, we study some homological dimensions over generalized morphic rings.

Definition 3.1. Let R be a ring. The P-injective dimension of ${}_RM$, denoted by $P\text{-}id(M)$, is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(R/Ra, M) = 0$ for all $a \in R$. If no such n exists, set $P\text{-}id(M) = \infty$.

$l.P\text{-}i\dim(R)$ is defined as $\sup\{P\text{-}id(M) | M \text{ is a left } R\text{-module}\}$.

The P-flat dimension of M_R, denoted by $P\text{-}fd(M)$, is defined to be the least non-negative integer n such that $\mathrm{Tor}_{n+1}(M, R/Ra) = 0$ for all $a \in R$. If no such n exists, set $P\text{-}fd(M) = \infty$.

$r.P\text{-}f\dim(R)$ is defined to be $\sup\{P\text{-}fd(M) | M \text{ is a right } R\text{-module}\}$.

Remark 3.2.

(1) It is clear that M is P-injective (P-flat) if and only if $P\text{-}id(M) = 0$ ($P\text{-}fd(M) = 0$).

(2) In general, $r.P\text{-}f\dim(R) \leq l.P\text{-}i\dim(R)$. In fact, suppose $l.P\text{-}i\dim(R) = n < \infty$. Let $M \in \text{Mod-}R$, then there is an integer $m \leq n$ such that $(\mathrm{Tor}_{m+1}(M, R/Ra))^+ \cong \mathrm{Ext}^{m+1}(R/Ra, M^+) = 0$ for all $a \in R$. Thus $\mathrm{Tor}_{m+1}(M, R/Ra) = 0$ for all $a \in R$, and hence $P\text{-}fd(M) \leq m \leq n$. It follows that $r.P\text{-}f\dim(R) \leq l.P\text{-}i\dim(R)$.

Proposition 3.3. *The following are equivalent for a ring R:*

(1) $r.P\text{-}f\dim(R) = 0$;
(2) $l.P\text{-}i\dim(R) = 0$;
(3) *R is a von Neumann regular ring*;
(4) *All left R-modules are P-injective*;
(5) *All right R-modules are P-flat*;
(6) *Every P-projective left R-module is projective.*

Proof. (3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) $\Rightarrow$ (6), (1) $\Leftrightarrow$ (5), and (2) $\Leftrightarrow$ (4) are trivial.

(6) $\Rightarrow$ (3) Let $a \in R$. Then R/Ra is projective by (6), and so Ra is a direct summand of R. Hence (3) follows.

(1) $\Rightarrow$ (3) Let $r.P\text{-}f\dim(R) = 0$. Then $\mathrm{Tor}_1(M, R/Ra) = 0$ for all right R-modules M and all $a \in R$, and so R/Ra is flat for all $a \in R$. Note that R/Ra is finitely presented, so R/Ra is projective for all $a \in R$. It follows that R is von Neumann regular. □

Remark 3.4. Proposition 3.3 shows that $l.P\text{-}i\dim(R)$ and $r.P\text{-}f\dim(R)$ measure how far away a ring R is from being a von Neumann regular ring.

In what follows, we shall see that the homological dimensions defined above have the properties that we expect of a "dimension" if the ring in question is generalized morphic.

Lemma 3.5. *If R is a left generalized morphic ring and $a \in R$, then there is an exact sequence*:

$$\cdots \to R_n \to R_{n-1} \to \cdots \to R_0 \to R/Ra \to 0$$

with $R_i = R$, $i = 0, 1, \ldots, n, \ldots$.

Lemma 3.6. *Let R be a left generalized morphic ring, then the following are equivalent for a left R-module M and an integer $n \geq 0$:*

(1) *P-$id(M) \leq n$;*
(2) $\operatorname{Ext}^{n+1}(R/Ra, M) = 0$ *for all $a \in R$;*
(3) $\operatorname{Ext}^{n+j}(R/Ra, M) = 0$ *for all $a \in R$ and $j \geq 1$;*
(4) $\operatorname{Ext}^{n+1}(P, M) = 0$ *for all $P \in \mathscr{PP}$;*
(5) $\operatorname{Ext}^{n+j}(P, M) = 0$ *for all $P \in \mathscr{PP}$ and $j \geq 1$;*
(6) *If the sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$ is exact with $E^0, E^1, \ldots, E^{n-1}$ P-injective, then E^n is P-injective;*
(7) *There is an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$ with $E^0, E^1, \ldots, E^n$ P-injective.*

Proof. (1) $\Rightarrow$ (2) Let ${}_RM$ be a left R-module and $a \in R$, then P-$id(M) = m \leq n$ by (1). If $m = n$, then $\operatorname{Ext}^{n+1}(R/Ra, M) = 0$. Otherwise, since R is left generalized morphic, by Lemma 3.5, there exists an exact sequence

$$\cdots \to R_{n-m} \to R_{n-m-1} \to \cdots \to R_0 \to R/Ra \to 0 \qquad (**)$$

with each $R_i = R$. Clearly, $K_{n-m} = \operatorname{im}(R_{n-m} \to R_{n-m-1})$ is principal, and so $K_{n-m} \cong R/Rb$ for some $b \in R$. Thus $\operatorname{Ext}^{n+1}(R/Ra, M) \cong \operatorname{Ext}^{m+1}(K_{n-m}, M) = 0$ by (1).

(2) $\Rightarrow$ (3) holds by induction.

(3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are trivial.

(2) $\Rightarrow$ (6) Let $0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$ be an exact sequence with $E^0, E^1, \ldots, E^{n-1}$ P-injective. Note that P-$id(E^i) = 0$, then $\operatorname{Ext}^k(R/Ra, E^i) = 0$ for all $k \geq 1$ and for all $a \in R$ since (1) $\Leftrightarrow$ (3), $i = 0, 1, \ldots, n-1$. Put $C^i = \ker(E^i \to E^{i+1})$, $i = 1, 2, \ldots, n-1$. Then we have isomorphisms

$$\operatorname{Ext}^1(R/Ra, E^n) \cong \operatorname{Ext}^2(R/Ra, C^{n-1}) \cong \cdots \cong \operatorname{Ext}^n(R/Ra, C^1) \cong \operatorname{Ext}^{n+1}(R/Ra, M).$$

Thus $\operatorname{Ext}^1(R/Ra, E^n) = 0$ by (2), and so E^n is P-injective.

(6) $\Rightarrow$ (7) follows from the fact that there exists an injective resolution for each left R-module.

(7) $\Rightarrow$ (5) By (7), there is an exact sequence

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0$$

with $E^0, E^1, \ldots, E^n$ P-injective. Since $\mathrm{Ext}^{n+j}(P, M) \cong \mathrm{Ext}^j(P, E^n)$ for any $P \in \mathcal{PP}$ and E^n is P-injective, (5) follows.

(5) $\Rightarrow$ (4) is obvious. □

The proof of the next corollary is standard homological algebra.

Corollary 3.7. *Let R be a left generalized morphic ring and $0 \to A \to B \to C \to 0$ an exact sequence of left R-modules. If two of P-$id(A)$, P-$id(B)$, P-$id(C)$ are finite, so is the third. Moreover*:

(1) $P\text{-}id(B) \leq sup\{P\text{-}id(A), P\text{-}id(C)\}$;
(2) $P\text{-}id(A) \leq sup\{P\text{-}id(B), P\text{-}id(C) + 1\}$;
(3) $P\text{-}id(C) \leq sup\{P\text{-}id(B), P\text{-}id(A) - 1\}$.

In particular, if $B = A \oplus C$ then P-$id(B)$= $sup\{P\text{-}id(A), P\text{-}id(C)\}$.

The following easy result may be viewed as the dual of Lemma 3.6.

Lemma 3.8. *Let R be a left generalized morphic ring. The following are equivalent for any right R-module M and an integer $n \geq 0$:*

(1) $P\text{-}fd(M) \leq n$;
(2) $\mathrm{Tor}_{n+1}(M, R/Ra) = 0$ *for all* $a \in R$;
(3) $\mathrm{Tor}_{n+j}(M, R/Ra) = 0$ *for all* $a \in R$ *and* $j \geq 1$;
(4) $\mathrm{Ext}^{n+1}(M, N) = 0$ *for all* $N \in \mathcal{PF}^{\perp}$;
(5) $\mathrm{Ext}^{n+j}(M, N) = 0$ *for all* $N \in \mathcal{PF}^{\perp}$ *and* $j \geq 1$;
(6) *If the sequence* $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ *is exact with* $F_0, F_1, \ldots, F_{n-1}$ *P-flat, then* F_n *is P-flat*;
(7) *There is an exact sequence* $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ *with* $F_0, F_1, \ldots, F_n$ *P-flat.*

Proposition 3.9. *Let R be a ring and M a right R-module, then P-$fd(M) = P$-$id(M^+)$.*

Proof. This follows from the standard isomorphism

$$\mathrm{Tor}_{m+1}(M, R/Ra)^+ \cong \mathrm{Ext}^{m+1}(R/Ra, M^+).$$ □

Proposition 3.10. *If R is a left generalized morphic ring, then P-$id(M) = P$-$fd(M^+)$ for any left R-module M.*

Proof. Since R is a left generalized morphic ring,

$$\mathrm{Tor}_1(M^+, R/Ra) \cong \mathrm{Ext}^1(R/Ra, M)^+ \qquad (*)$$

for any $a \in R$ by Lemma 3.5, Rotman (1979, Theorem 9.51) and the remark following it. So we have that a left R-module is P-injective if and only if its character module is P-flat.

Let

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots E_{m-1} \longrightarrow E_m \longrightarrow \cdots$$

be an injective resolution of M and set $C_i = \mathrm{im}(E_{i-1} \to E_i)$ for all $i \geq 1$, then we get an exact sequence

$$0 \longrightarrow C_m^+ \longrightarrow E_{m-1}^+ \longrightarrow \cdots \longrightarrow E_1^+ \longrightarrow E_0^+ \longrightarrow M^+ \longrightarrow 0,$$

where each $E_i^+ (i = 0, 1, \ldots, m-1)$ is P-flat by the foregoing proof.

Suppose $P\text{-}fd(M^+) = m < \infty$, then $\mathrm{Tor}_1(C_m^+, R/Ra) \cong \mathrm{Tor}_{m+1}(M^+, R/Ra) = 0$ for all $a \in R$. Since $\mathrm{Tor}_1(C_m^+, R/Ra) \cong \mathrm{Ext}^1(R/Ra, C_m)^+$, $\mathrm{Ext}^1(R/Ra, C_m) = 0$ for all $a \in R$. It follows that $P\text{-}id(M) \leq m$.

Conversely, if $P\text{-}id(M) = n < \infty$, then $\mathrm{Ext}^1(R/Ra, C_n) \cong \mathrm{Ext}^{n+1}(R/Ra, M_R) = 0$ for all $a \in R$. It follows that C_n^+ is P-flat, and so $\mathrm{Tor}_1(C_n^+, R/Ra) = 0$. Thus $\mathrm{Tor}_{n+1}(M^+, R/Ra) = 0$, and hence $P\text{-}fd(M^+) \leq n$. □

We are now in a position to prove the following theorem.

Theorem 3.11. *The following are identical for a left generalized morphic ring R:*

(1) $l.P\text{-}i\dim(R)$;
(2) $r.P\text{-}f\dim(R)$;
(3) $\sup\{P\text{-}id(M) \mid {}_RM \in \mathcal{PP}\}$;
(4) $\sup\{P\text{-}fd(M) \mid M_R \in \mathcal{PF}^{\perp}\}$;
(5) $\sup\{P\text{-}fd(M) \mid M_R \textit{ is cyclic}\}$;
(6) $\sup\{pd(M) \mid {}_RM \in \mathcal{PP}\}$;
(7) $\sup\{pd(R/Ra) \mid a \in R\}$;
(8) $\sup\{fd(R/Ra) \mid a \in R\}$;
(9) $\sup\left\{id(M) \mid M_R \in \mathcal{PF}^{\perp}\right\}$.

Proof. (1) = (2) Suppose that $r.P\text{-}f\dim(R) = n < \infty$. Then for any left R-module M, $P\text{-}id(M) = P\text{-}fd(M^+) \leq n$ by Proposition 3.10, and so $l.P\text{-}i\dim(R) \leq n$.

Conversely, suppose $l.P\text{-}i\dim(R) = n < \infty$. Then $P\text{-}fd(N) = P\text{-}id(N^+) \leq n$ for any right R-module N by Proposition 3.9, and hence (2) $\leq$ (1).

(1) $\geq$ (3), (2) $\geq$ (5), and (6) $\geq$ (7) $\geq$ (8) are obvious.

(3) $\geq$ (6) Assume $\sup\{P\text{-}id(M) \mid {}_RM \in \mathcal{PP}\} = n$. Let M and N be two P-projective left R-modules. M admits a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

N admits a $\mathcal{PI}$-resolution by Lemma 3.6

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0,$$

where $E^0, E^1, \ldots, E^n$ are P-injective. Therefore we form the following double complex:

$$
\begin{array}{ccccccccc}
 & 0 & & 0 & & & & 0 & \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^n) & \to & \mathrm{Hom}(P_0, E^n) & \to & \cdots & \to & \mathrm{Hom}(P_n, E^n) & \to \cdots \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
 & \vdots & & \vdots & & & & \vdots & \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^1) & \to & \mathrm{Hom}(P_0, E^1) & \to & \cdots & \to & \mathrm{Hom}(P_n, E^1) & \to \cdots \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^0) & \to & \mathrm{Hom}(P_0, E^0) & \to & \cdots & \to & \mathrm{Hom}(P_n, E^0) & \to \cdots \\
 & \uparrow & & \uparrow & & & & \uparrow & \\
 & 0 & \to & \mathrm{Hom}(P_0, N) & \to & \cdots & \to & \mathrm{Hom}(P_n, N) & \to \cdots \\
 & & & \uparrow & & & & \uparrow & \\
 & & & 0 & & & & 0. &
\end{array}
$$

Note that all rows are exact except for the bottom row since M is P-projective and all E^i are P-injective, also note that all columns are exact except for the left column since all P_i are projective.

Using a spectral sequence argument, we know that the two complexes

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \cdots \to \mathrm{Hom}(P_n, N) \to \cdots$$

and

$$0 \to \mathrm{Hom}(M, E^0) \to \mathrm{Hom}(M, E^1) \to \cdots \to \mathrm{Hom}(M, E^n) \to 0$$

have isomorphic homology groups. Thus $\mathrm{Ext}^{n+j}(M, N) = 0$ for P-projective left R-modules M, N and $j \geq 1$. We claim that $pd(M) \leq n$. In fact, for any left R-module L, there exists an exact sequence $0 \to K \to F \to L \to 0$, where F is P-projective and K is P-injective by Lemma 2.14(1). Thus we have the exactness of

$$\mathrm{Ext}^{n+1}(M, F) \to \mathrm{Ext}^{n+1}(M, L) \to \mathrm{Ext}_R^{n+2}(M, K).$$

Since $\mathrm{Ext}^{n+2}(M, K) = 0$ by Proposition 2.15 (1) and $\mathrm{Ext}^{n+1}(M, F) = 0$ by the foregoing proof, $\mathrm{Ext}^{n+1}(M, L) = 0$. So $pd(M) \leq n$.

(8) $\geq$ (9) Suppose that $\sup\{fd(R/Ra) | a \in R\} = n < \infty$ and N is a right R-module, then $\mathrm{Tor}_{n+1}(N, R/Ra) = 0$ for all $a \in R$ by hypothesis. Thus $\mathrm{Ext}^{n+1}(N,$

$M) = 0$ for any $M_R \in \mathscr{PF}^{\perp}$ by Lemma 3.8. Since N is arbitrary, $id(M) \leq n$, as required.

(9) $\geq$ (4) Suppose $\sup\{id(M) | M_R \in \mathscr{PF}^{\perp}\} = n < \infty$, then $\mathrm{Ext}^{n+1}(M, N) = 0$ for any $M_R, N_R \in \mathscr{PF}^{\perp}$. It follows that $P\text{-}fd(M) \leq n$ by Lemma 3.8. So (4) holds.

(4) $\geq$ (2) Assume that $\sup\{P\text{-}fd(M) | M_R \in \mathscr{PF}^{\perp}\} = n < \infty$. For any right R-module M, there exists an exact sequence $0 \to M \to F \to C \to 0$, where $F \in \mathscr{PF}^{\perp}$ and C is P-flat by Lemma 2.14 (2). Thus $\mathrm{Tor}_{n+1}(M, R/Ra) \cong \mathrm{Tor}_{n+1}(F, R/Ra) = 0$ for all $a \in R$ by hypothesis and Lemma 3.8, and so $P\text{-}fd(M) \leq n$, as desired.

(5) $\geq$ (8) Suppose $\sup\{P\text{-}fd(M) | M_R \text{ is cyclic}\} = n$. Let I be a right ideal of R, then $\mathrm{Tor}_{n+1}(R/I, R/Ra) = 0$ for all $a \in R$ by Lemma 3.8. Thus $fd(R/Ra) \leq n$ for all $a \in R$, and so (8) $\leq$ (5). ☐

Recall that a ring R is called *left p.f.* (Jøndrup, 1971) in case every principal left ideal of R is flat.

Corollary 3.12. *The following are equivalent for a left generalized morphic ring R:*

(1) *R is a left p.p. ring*;
(2) $r.P\text{-}f\dim(R) \leq 1$;
(3) $l.P\text{-}i\dim(R) \leq 1$;
(4) *R is a left p.f. ring*;
(5) *Every quotient module of a P-injective left R-module is P-injective*;
(6) *Every submodule of a P-flat right R-module is P-flat.*

Remark 3.13. (1) It is well known that a left p.p. ring need not be a right p.p. ring, so $l.P\text{-}i\dim(R) \neq r.P\text{-}i\dim(R)$ and $l.P\text{-}f\dim(R) \neq r.P\text{-}f\dim(R)$ in general. Therefore, for a ring R, the P-flat dimension of R is different from the weak global dimension of R.

(2) Let R be a left p.p. ring that is not a right p.p. ring. Since R is left p.f., it is right p.f. by Jøndrup (1971, Theorem 2.2). Note that R is left generalized morphic, but it is not right generalized morphic. Otherwise, R is right p.p. by Corollary 3.12, a contradiction. So Corollary 3.12 also implies that generalized morphic rings are not left-right symmetric.

(3) Note that if R is a left morphic ring, then $r.P\text{-}f\dim(R)$ (or $l.P\text{-}i\dim(R)$) is infinite or R is a left p.p. ring. In fact, for any a in R, the exactness of the following sequences

$$0 \to l(a) \to R \to Ra \to 0 \quad (1) \quad \text{and} \quad 0 \to Ra \to R \to l(a) \cong R/Ra \to 0 \quad (2)$$

induces an exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & R \arrow[rr] \arrow[dr] & & R \arrow[rr] \arrow[dr] & & R \arrow[rr] \arrow[dr] & & \cdots \arrow[rr] \arrow[dr] & & R \arrow[r] & R/Ra \arrow[r] & 0. \\
& & l(a) \arrow[ur] & & Ra \arrow[ur] & & l(a) \arrow[ur] & & Ra \arrow[ur] & & &
\end{tikzcd} \quad (3)$$

Suppose $r.P\text{-}f\dim(R) < \infty$. Let $a \in R$. Then $pd(R/Ra) < \infty$ by Theorem 3.11, and so Ra or $l(a)$ is projective by the exactness of (3). If $l(a)$ is projective, then the exactness of (2) implies that Ra is projective. Thus Ra is always projective, and so R is a left p.p. ring.

Proposition 3.14. *A ring R is von Neumann regular if and only if R is a left generalized morphic and left P-injective ring with* $\sup\{pd(Ra)|a \in R\} < \infty$.

Proof. Necessity is clear.

Conversely, let $\sup\{pd(Ra)|a \in R\} < \infty$. For any left R-module M, there is an exact sequence $0 \to M \to E \to C \to 0$, where E is P-injective and C is P-projective. Note that $pd(C) \leq \sup\{pd(R/Ra)|a \in R\} \leq \sup\{pd(Ra)|a \in R\} + 1 < \infty$ by Theorem 3.11, and hence C is projective by Proposition 2.16. Thus $0 \to M \to E \to C \to 0$ splits, and so M is P-injective, as required. □

Corollary 3.15. *A ring R is von Neumann regular if and only if R is a left p.p. and left P-injective ring.*

An element a in a ring R is called *unit regular* if $aba = a$ for some unit $b \in R$, and the ring R is called a *unit regular ring* if every element is unit regular. Nicholson and Sánchez Campos (2004a) proved that every unit regular ring is left and right morphic and raised the question whether a left and right morphic ring with the Jacobson radical $J = 0$ is unit regular. This is shown to be false in general by Chen et al. (2005, Example 0.1). But we have the following corollary.

Corollary 3.16. *A ring R is unit regular if and only if R is a left p.p. and right morphic ring.*

Proof. Necessity is obvious. Sufficiency follows from Corollary 3.15 and Nicholson and Sánchez Campos (2004a, Theorem 24 and Proposition 5). □

Next, we consider approximations by modules of finite homological dimensions. For a fixed non-negative integer n, denote by $\mathcal{PI}_n$ the class of all left R-modules of P-injective dimension $\leq n$, then we get the following proposition.

Proposition 3.17. *If R is left generalized morphic, then $({}^{\perp}\mathcal{PI}_n, \mathcal{PI}_n)$ is a complete hereditary cotorsion theory.*

Proof. Let M be a left R-module, then $M \in \mathcal{PI}_n$ if and only if $\mathrm{Ext}^{n+1}(R/Ra, M) = 0$ for all $a \in R$ by Lemma 3.6. And the latter is equivalent to $\mathrm{Ext}^1(K_a, M) = 0$ with K_a the nth syzygy module (in a projective resolution) of R/Ra. So $\mathcal{PI}_n = \left(\bigoplus_{a\in R} K_a\right)^{\perp}$, then $({}^{\perp}\mathcal{PI}_n, \mathcal{PI}_n)$ is a complete cotorsion theory by Enochs and Jenda (2000, Theorem 7.4.1). The fact that $({}^{\perp}\mathcal{PI}_n, \mathcal{PI}_n)$ is hereditary can be proven in a way similar to that of Proposition 2.15. □

Lemma 3.18. *Let R be a left generalized morphic ring. Then $\mathcal{PI}_n$ is closed under direct sums.*

Proof. Note that $\mathcal{PI}$ is closed under direct sums, and so the result follows from Lemma 3.6. □

Theorem 3.19. *Let R be a left generalized morphic ring. If $l.P\text{-}i\dim(R) < \infty$, then the following are equivalent for a fixed non-negative integer n:*

(1) *$l.P\text{-}i\dim(R) \leq n$;*
(2) *$sup\{pd(M)|_R M$ is both P-projective and P-injective$\} \leq n$;*
(3) *$sup\{P\text{-}id(M)|_R M$ is projective$\} \leq n$;*
(4) *Every left R-module in ${}^{\perp}\mathcal{PI}_n$ is projective.*
Moreover if $n \geq 1$, then the above conditions are also equivalent to:
(5) *Every left R-module (in ${}^{\perp}\mathcal{PI}_{n-1}$) has a monic $\mathcal{PI}_{n-1}$-cover.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 3.11.

(2) $\Rightarrow$ (3) Let M be a projective left R-module. We shall show that $\mathrm{Ext}^{n+1}(N, M) = 0$ for any P-projective left R-module N. In fact, we may assume $P\text{-}id(N) = m < \infty$ since $l.P\text{-}i\dim(R) < \infty$. By Lemmas 2.14 and 3.6, there is an exact sequence

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{m-1} \longrightarrow E^m \longrightarrow 0,$$

with each E^i both P-injective and P-projective, $i = 0, 1, \ldots, m$. Since $pd(E^i) \leq n$ for all $1 \leq i \leq m$ by (2), $pd(N) \leq n$. Thus $\mathrm{Ext}^{n+1}(N, M) = 0$. It follows that $P\text{-}id(M) \leq n$. So (3) holds.

(3) $\Rightarrow$ (1) Let N be a left R-module and $0 \to K \to F \to N \to 0$ an exact sequence, where F is P-projective and K is P-injective, then $pd(F) = m < \infty$ by Theorem 3.11 and hypothesis. Hence there is a projective resolution of F

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0.$$

Note that $P\text{-}id(P_i) \leq n$ by (3), $i = 0, 1, \ldots, m$, and so $P\text{-}id(F) \leq n$ by Corollary 3.7(3). Therefore $P\text{-}id(N) \leq n$, as desired.

(1) $\Leftrightarrow$ (4) comes from Proposition 3.17 and the fact that $(\mathcal{P}roj, R\text{-Mod})$ is a cotorsion theory, where $\mathcal{P}roj$ is the class of projective left R-modules.

(1) $\Rightarrow$ (5) Let M be a left R-module. Write $E = \sum\{N \leq M : N \in \mathcal{PI}_{n-1}\}$ and $L = \bigoplus\{N \leq M : N \in \mathcal{PI}_{n-1}\}$. Then there exists an exact sequence $0 \to K \to L \to E \to 0$. Note that $K \in \mathcal{PI}_n$ since $l.P\text{-}i\dim(R) \leq n$ and $L \in \mathcal{PI}_{n-1}$ by Lemma 3.18, we have $E \in \mathcal{PI}_{n-1}$ by Corollary 3.7(3). Next we prove that the inclusion $i : E \to M$ is a $\mathcal{PI}_{n-1}$-cover of M. Let $\psi : E' \to M$ with $E' \in \mathcal{PI}_{n-1}$ be a left R-homomorphism. Note that $\psi(E') \leq E$ by the proof above. Define $\zeta : E' \to E$ via $\zeta(x) = \psi(x)$ for $x \in E'$. Then $i\zeta = \psi$, and so $i : E \to M$ is a $\mathcal{PI}_{n-1}$-precover of M. In addition, it is clear that the identity map 1_E of E is the only homomorphism $g : E \to E$ such that $ig = i$, and hence (5) follows.

(5) $\Rightarrow$ (1) Let M be a left R-module, then there exists an exact sequence $0 \to M \to E \to C \to 0$ with $E \in \mathcal{PI}_{n-1}$ and $C \in {}^{\perp}\mathcal{PI}_{n-1}$. Since C has a monic $\mathcal{PI}_{n-1}$-

cover $\phi : E' \to C$, there is $\alpha : E \to E'$ such that the following exact diagram is commutative:

$$\begin{tikzcd} & & & E' \arrow[d, "\phi"] & \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r] \arrow[ur, "\alpha"] & C \arrow[r] & 0. \end{tikzcd}$$

Thus ϕ is epic, and hence it is an isomorphism. Therefore $C \in \mathcal{PI}_{n-1}$, then $M \in \mathcal{PI}_n$ follows from Corollary 3.7(2), as desired. $\square$

Let $\mathcal{C}$ be a class of left R-modules and M a left R-module. Recall that a $\mathcal{C}$-cover $\phi : F \to M$ is said to have the unique mapping property (Ding, 1996) if for any homomorphism f: $F' \to M$ with $F' \in \mathcal{C}$, there is a unique homomorphism $g : F' \to F$ such that $\phi g = f$.

We conclude this article by the following theorem.

Theorem 3.20. *The following are equivalent for a ring R:*

(1) *Every left R-module is P-projective*;
(2) *Every cyclic left R-module is P-projective*;
(3) *Every P-injective left R-module is injective*;
(4) *Every nonzero left R-module has a nonzero P-projective submodule.*
Moreover if R is a left generalized morphic ring, then the above statements are also equivalent to:
(5) *Every P-injective left R-module is P-projective*;
(6) *Every* (P-*injective*) *left R-module has a $\mathcal{PP}$-cover with the unique mapping property.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) follows from Lemma 2.14 and Mao and Ding (2006, Corollary 2.5).

(1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (6) are trivial.

(4) $\Rightarrow$ (3) Let E be a P-injective left R-module. Suppose we have the diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & A \arrow[r, "i"] \arrow[u, "f"] & B. \end{tikzcd}$$

For notational convenience, let us assume i is an inclusion map. It suffices to show that there is $g : B \to E$ that extends f. Now we approximate a map g by looking at all modules between A and B that do possess an extension of f. More precisely, let $\mathbb{C}$ consist of all pairs (A', g'), where $A \subseteq A' \subseteq B$ and $g' : A' \to E$ extends f. Note that $\mathbb{C} \neq \emptyset$, for (A, f) in $\mathbb{C}$. Partially order $\mathbb{C}$ by saying $(A', g') \leq (A'', g'')$ if $A' \subseteq A''$ and g'' extends g'. By Zorn's Lemma, there is a maximal pair (A_0, g_0) in $\mathbb{C}$. If $A_0 = B$, we are done.

Assume $A_0 \neq B$, then $B/A_0 \neq 0$. By (4), there is a nonzero P-projective submodule C/A_0 of B/A_0. Note that there is a exact sequence $0 \to A_0 \to C \to$

$C/A_0 \to 0$ with C/A_0 P-projective, and so there exits a map $h : C \to E$ extending g_0 since E is P-injective. It is easy to check that (C, h) belongs to $\mathbb{C}$ and is larger than the maximal pair (A_0, g_0), a contradiction. Therefore $A_0 = B$ and E is injective, as required.

(6) $\Rightarrow$ (5) Let M be a P-injective left R-module. Then, by (6), M has a $\mathcal{PP}$-cover $\delta_M : P(M) \to M$ with the unique mapping property. Let now $K = \ker(\delta_M)$ and $\delta_K : P(K) \to K$ be a $\mathcal{PP}$-cover of K. Then there is the following exact commutative diagram:

$$\begin{tikzcd} & P(K) \arrow[d, "\delta_K"] \arrow[drr, dotted, "0"] & & & \\ 0 \arrow[r] & K \arrow[r, "i"'] & P(M) \arrow[r, "\delta_M"'] & M \arrow[r] & 0. \end{tikzcd}$$

Note that $\delta_M i \delta_K = 0 = \delta_M 0$, and so $i\delta_K = 0$ by assumption. Therefore $\delta_K = 0$, and hence $M \in \mathcal{PP}$. This completes the proof. $\square$

ACKNOWLEDGMENT

This research was partially supported by SRFDP (No. 20050284015) and NSFC (No. 10331030).

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