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Global dimension and left derived functors of Hom

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Abstract It is well known that the right global dimension of a ring R is usually computed by the right derived functors of Hom and the left projective resolutions of right R-modules. In this paper, for a left coherent and right perfect ring R, we characterize the right global dimension of R, from another point of view, using the left derived functors of Hom and the right projective resolutions of right R-modules. It is shown that $rD(R) \leqslant n$ $(n \geqslant 2)$ if and only if the gl right $\mathcal{P}roj$-dim $\mathcal{M_R} \leqslant n-2$ if and only if $\mathrm{Ext}_{n-1}(N, M) = 0$ for all right R-modules N and M if and only if every $(n-2)$th $\mathcal{P}roj$-cosyzygy of a right R-module has a projective envelope with the unique mapping property. It is also proved that $rD(R) \leqslant n$ $(n \geqslant 1)$ if and only if every $(n-1)$th $\mathcal{P}roj$-cosyzygy of a right R-module has an epic projective envelope if and only if every nth $\mathcal{P}roj$-cosyzygy of a right R-module is projective. As corollaries, the right hereditary rings and the rings R with $rD(R) \leqslant 2$ are characterized.

Keywords: global dimension, projective resolution, envelope

MSC(2000): 16E10, 16E30, 16E05

1 Introduction

Throughout this paper, R always denotes a left coherent and right perfect ring and all modules are unitary. As usual, $J(R)$ and $rD(R)$ stand for the Jacobson radical and the right global dimension of R respectively. M_R $({}_RM)$ denotes a right (left) R-module. $\mathrm{pd}(M)$ stands for the projective dimension of an R-module M. We write $\mathcal{M_R}$ and $\mathcal{P}roj$ for the category of all right R-modules and the category of all projective right R-modules respectively. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}^n_R(M, N)$) for an integer $n \geqslant 1$. For other unexplained concepts and notations, we refer the reader to [1–3].

We first recall some known notions and facts needed in the sequel.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [4], we say that a homomorphism $\phi : M \to C$ is a $\mathcal{C}$-preenvelope if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a $\mathcal{C}$-envelope if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover. $\mathcal{C}$-envelopes

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($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let R be a ring. R is called left coherent[5] if every finitely generated left ideal of R is finitely presented. R is said to be right perfect[6] if every right R-module has a projective cover. The left coherent and right perfect rings are characterized by the condition that every right R-module has a projective (pre)envelope (see [7, Proposition 3.5]). So, for a left coherent and right perfect ring R, every right R-module M has a right $\mathcal{P}roj$-resolution, that is, there is a $\mathrm{Hom}(-,\mathcal{P}roj)$ exact complex $0 \to M \to P^0 \to P^1 \to \cdots$ (not necessarily exact) with each P^i projective. Let

$$L^0 = M, \quad L^1 = \mathrm{coker}(M \to P^0), \quad L^i = \mathrm{coker}(P^{i-2} \to P^{i-1}) \text{ for } i \geqslant 2.$$

The nth cokernel L^n ($n \geqslant 0$) is called the nth $\mathcal{P}roj$-cosyzygy of M.

Let $\cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M, which is clearly a left $\mathcal{P}roj$-resolution of M. Write

$$K_0 = M, \quad K_1 = \ker(P_0 \to M), \quad K_i = \ker(P_{i-1} \to P_{i-2}) \text{ for } i \geqslant 2.$$

The nth kernel K_n ($n \geqslant 0$) is called the nth ($\mathcal{P}roj$-) syzygy of M.

Note that $\mathrm{Hom}(-,-)$ is left balanced on $\mathcal{M}_\mathcal{R} \times \mathcal{M}_\mathcal{R}$ by $\mathcal{P}roj \times \mathcal{P}roj$ (see [1, Definition 8.2.13]). Thus the nth left derived functor, which is denoted by $\mathrm{Ext}_n(-,-)$, can be computed using a right $\mathcal{P}roj$-resolution of the first variable or a left $\mathcal{P}roj$-resolution of the second variable.

Following [1, Definition 8.4.1], the right $\mathcal{P}roj$-dimension of a right R-module M, denoted by the right $\mathcal{P}roj$-dim M, is defined as $\inf\{n$: there is a right $\mathcal{P}roj$-resolution of M of the form $0 \to M \to P^0 \to P^1 \to \cdots \to P^n \to 0\}$. If there is no such n, set the right $\mathcal{P}roj$-dim $M = \infty$. The global right $\mathcal{P}roj$-dimension of $\mathcal{M}_\mathcal{R}$, denoted by the gl right $\mathcal{P}roj$-dim $\mathcal{M}_\mathcal{R}$, is defined to be $\sup\{$right $\mathcal{P}roj$-dim M: $M \in \mathcal{M}_\mathcal{R}\}$ and is infinite otherwise.

It is well known that, for any ring R and a nonnegative integer n, $rD(R) \leqslant n$ if and only if $\mathrm{Ext}^{n+1}(N,M) = 0$ for all right R-modules N and M if and only if every nth syzygy of a right R-module is projective. In this paper, we characterize the right global dimension of a (left coherent and right perfect) ring R in terms of the left derived functors $\mathrm{Ext}_n(-,-)$. It is shown that $rD(R) \leqslant n$ ($n \geqslant 2$) if and only if the gl right $\mathcal{P}roj$-dim $\mathcal{M}_\mathcal{R} \leqslant n-2$ if and only if $\mathrm{Ext}_{n-1}(N,M) = 0$ for all right R-modules N and M. It is also proven that $rD(R) \leqslant n$ ($n \geqslant 1$) if and only if every $(n-1)$th $\mathcal{P}roj$-cosyzygy of a right R-module has an epic projective envelope if and only if every nth $\mathcal{P}roj$-cosyzygy of a right R-module is projective. As corollaries, the right hereditary rings and the rings R with $rD(R) \leqslant 2$ are characterized.

2 Global dimension and left derived functors of Hom

As is mentioned in the introduction, if R is a left coherent and right perfect ring, then $\mathrm{Hom}(-,-)$ is left balanced on $\mathcal{M}_\mathcal{R} \times \mathcal{M}_\mathcal{R}$ by $\mathcal{P}roj \times \mathcal{P}roj$. Let $\mathrm{Ext}_n(-,-)$ denote the nth left derived functor of $\mathrm{Hom}(-,-)$. Then, for two right R-modules N and M, $\mathrm{Ext}_n(N,M)$ can be computed using a right $\mathcal{P}roj$-resolution of N or a left $\mathcal{P}roj$-resolution of M. Let $\cdots \to P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \to 0$ be a projective resolution of M. Applying $\mathrm{Hom}(N,-)$, we obtain the deleted complex

$$\cdots \to \mathrm{Hom}(N,P_1) \xrightarrow{f_*} \mathrm{Hom}(N,P_0) \to 0.$$

Then $\mathrm{Ext}_n(N,M)$ is exactly the nth homology of the complex above. There is a canonical map

$$\sigma : \mathrm{Ext}_0(N,M) = \mathrm{Hom}(N,P_0)/\mathrm{im}(f_*) \to \mathrm{Hom}(N,M)$$

defined by $\sigma(\alpha + \mathrm{im}(f_*)) = g\alpha$ for $\alpha \in \mathrm{Hom}(N, P_0)$.

We begin with the following

Proposition 2.1. *The following statements are equivalent for a right R-module M:*

(1) *M is projective*;

(2) *the canonical map $\sigma : \mathrm{Ext}_0(N, M) \to \mathrm{Hom}(N, M)$ is an epimorphism for any right R-module N*;

(3) *the canonical map $\sigma : \mathrm{Ext}_0(M, M) \to \mathrm{Hom}(M, M)$ is an epimorphism.*

Proof. (1) $\Rightarrow$ (2) is obvious by letting $P_0 = M$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). By (3), there exists $\alpha \in \mathrm{Hom}(M, P_0)$ such that $\sigma(\alpha + \mathrm{im}(f_*)) = g\alpha = 1_{\mathrm{Hom}(M,M)}$. Thus M is a direct summand of P_0, and hence it is projective.

Proposition 2.2. *The following statements are equivalent for a right R-module M:*

(1) *$pd(M) \leqslant 1$*;

(2) *the canonical map $\sigma : \mathrm{Ext}_0(N, M) \to \mathrm{Hom}(N, M)$ is a monomorphism for any right R-module N.*

Proof. (1) $\Rightarrow$ (2) By (1), M has a projective resolution $0 \to P_1 \to P_0 \to M \to 0$. Thus we get an exact sequence

$$0 \to \mathrm{Hom}(N, P_1) \to \mathrm{Hom}(N, P_0) \to \mathrm{Hom}(N, M)$$

for any right R-module N. Hence σ is a monomorphism.

(2) $\Rightarrow$ (1) Consider the exact sequence $0 \to K_1 \to P_0 \to M \to 0$ with P_0 projective. We only need to show that K_1 is projective. By [1, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & \mathrm{Ext}_0(K_1, K_1) & \longrightarrow & \mathrm{Ext}_0(K_1, P_0) & \longrightarrow & \mathrm{Ext}_0(K_1, M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_3} & & \\
0 & \longrightarrow & \mathrm{Hom}(K_1, K_1) & \longrightarrow & \mathrm{Hom}(K_1, P_0) & \longrightarrow & \mathrm{Hom}(K_1, M). & &
\end{array}$$

Note that σ_2 is an epimorphism by Proposition 2.1 and σ_3 is a monomorphism by (2). So σ_1 is an epimorphism by Snake Lemma (see [2, Theorem 6.5]). Thus K_1 is projective by Proposition 2.1, and so (1) follows.

Proposition 2.3. *The following statements are equivalent for a right R-module M and an integer $n \geqslant 2$:*

(1) *$pd(M) \leqslant n$*;

(2) $\mathrm{Ext}_{n+k}(N, M) = 0$ *for all right R-modules N and $k \geqslant -1$*;

(3) $\mathrm{Ext}_{n-1}(N, M) = 0$ *for all right R-modules N.*

Moreover, if R is right coherent and M is finitely presented, then the above conditions are also equivalent to:

(4) $\mathrm{Ext}_{n-1}(N, M) = 0$ *for all finitely presented right R-modules N.*

Proof. (1) $\Rightarrow$ (2) Let $0 \to P_n \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M, which induces an exact sequence $0 \to \mathrm{Hom}(N, P_n) \to \mathrm{Hom}(N, P_{n-1}) \to \mathrm{Hom}(N, P_{n-2})$ for any right

R-module N. Hence $\mathrm{Ext}_n(N,M)=\mathrm{Ext}_{n-1}(N,M)=0$. But it is clear that $\mathrm{Ext}_{n+k}(N,M)=0$ for all $k\geqslant 1$. Then (2) holds.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.

(3) $\Rightarrow$ (1) Let $\cdots\to P_n\to P_{n-1}\to\cdots\to P_0\to M\to 0$ be a projective resolution of M with $K_n=\ker(P_{n-1}\to P_{n-2})$. We only need to show that K_n is projective. In fact, we have the following exact commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[rr, "f"] \arrow[dr, "\pi"] & & P_{n-1} \arrow[r, "g"] & P_{n-2} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & M \arrow[r] & 0 \\
& & K_n \arrow[ur, "\lambda"] \arrow[dr] & & & & & & \\
& 0 \arrow[ur] & & 0 & & & & &
\end{tikzcd}$$

By (3), $\mathrm{Ext}_{n-1}(K_n,M)=0$. Thus the sequence

$$\mathrm{Hom}(K_n,P_n)\overset{f_*}{\to}\mathrm{Hom}(K_n,P_{n-1})\overset{g_*}{\to}\mathrm{Hom}(K_n,P_{n-2})$$

is exact. Since $g_*(\lambda)=g\lambda=0$, $\lambda\in\ker(g_*)=\mathrm{im}(f_*)$. Thus there exists $h\in\mathrm{Hom}(K_n,P_n)$ such that $\lambda=f_*(h)=fh=\lambda\pi h$, and hence $\pi h=1$ since λ is monic. Therefore K_n is projective.

The proof of (4) $\Rightarrow$ (1) is similar to that of (3) $\Rightarrow$ (1) by noting that K_n can be chosen to be finitely presented.

Corollary 2.4. *The following numbers are identical for a right R-module M with* $\mathrm{pd}(M)\geqslant 2$:

(1) $\mathrm{pd}(M)$;

(2) $\inf\{n:\mathrm{Ext}_{n-1}(N,M)=0$ *for all right R-modules N and* $n\geqslant 2\}$;

(3) *the integer n such that M admits a minimal projective resolution*:

$$0\to P_n\to\cdots\to P_0\to M\to 0,$$

where $P_i\to K_i$ is a projective cover of the ith syzygy K_i, $P_i\neq 0$, $i=0,1,\ldots,n$.

Proof. (1) = (2) holds by Proposition 2.3.

(1) $\leqslant$ (3) is trivial. Assume (1) $<$ (3). Let (1) $=m<\infty$. Then K_m is a projective right R-module. Since $P_m\to K_m$ is a projective cover of K_m, it follows that $K_{m+1}=0$, and hence $P_{m+1}=0$, a contradiction. Therefore (1) = (3).

Lemma 2.5. *The following statements are equivalent for a right R-module N and an integer* $n\geqslant 2$:

(1) *The right $\mathcal{P}roj$-dim $N\leqslant n-2$;*

(2) $\mathrm{Ext}_{n+k}(N,M)=0$ *for all right R-modules M and $k\geqslant -1$;*

(3) $\mathrm{Ext}_{n-1}(N,M)=0$ *for all right R-modules M.*

Proof. (1) $\Rightarrow$ (2) Let

$$0\to N\to P^0\to P^1\to\cdots\to P^{n-2}\to 0$$

be a right $\mathcal{P}roj$-resolution of N. Then we have the following complex

$$0\to\mathrm{Hom}(P^{n-2},M)\to\mathrm{Hom}(P^{n-3},M)\to\cdots\to\mathrm{Hom}(P^0,M)\to 0$$

for any right R-module M. Hence $\mathrm{Ext}_{n+k}(N,M)=0$ for all $k \geqslant -1$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) There exists a right $\mathcal{P}roj$-resolution of N:

$$0 \longrightarrow N \longrightarrow P^0 \longrightarrow \cdots P^{n-4} \xrightarrow{j} P^{n-3} \xrightarrow{h} P^{n-2} \xrightarrow{f} P^{n-1} \xrightarrow{g} P^n \cdots$$

with each P^i projective. Let $\pi : P^{n-1} \to L^n = P^{n-1}/\mathrm{im}(f)$ be the canonical projection, $i : L^n \to P^n$ the induced map, and let f and h factor through $\mathrm{im}(f)$ and $\ker(f)$ respectively in obvious ways, that is, $f = \lambda f'$ and $h = \iota h'$. Then we have the following commutative diagram:

$$\begin{tikzcd}
P^{n-4} \arrow[r, "j"] & P^{n-3} \arrow[rr, "h"] \arrow[dr, "h'"] & & P^{n-2} \arrow[rr, "f"] \arrow[dr, "f'"] & & P^{n-1} \arrow[rr, "g"] \arrow[dr, "\pi"] & & P^n \\
& & \ker(f) \arrow[ur, "\iota"] \arrow[dr] & & \mathrm{im}(f) \arrow[ur, "\lambda"] \arrow[dr] & & L^n \arrow[ur, "i"] \arrow[dr] & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

By (3), $\mathrm{Ext}_{n-1}(N,L^n)=0$. Thus the sequence

$$\mathrm{Hom}(P^n,L^n) \xrightarrow{g^*} \mathrm{Hom}(P^{n-1},L^n) \xrightarrow{f^*} \mathrm{Hom}(P^{n-2},L^n)$$

is exact. Since $f^*(\pi) = \pi f = 0$, $\pi \in \ker(f^*) = \mathrm{im}(g^*)$. So $\pi = g^*(l) = lg$ for some $l \in \mathrm{Hom}(P^n,L^n)$. But $g = i\pi$, and hence $\pi = li\pi$. Thus $li = 1$ since π is epic, and so L^n is projective. It follows that $\mathrm{im}(f)$ and $\ker(f)$ are projective. We claim that the complex

$$0 \to N \to P^0 \to P^1 \to \cdots \xrightarrow{j} P^{n-3} \to \ker(f) \to 0$$

is a right $\mathcal{P}roj$-resolution of N. In fact, it is enough to show that the complex

$$0 \longrightarrow \mathrm{Hom}(\ker(f),G) \xrightarrow{(h')^*} \mathrm{Hom}(P^{n-3},G) \xrightarrow{j^*} \mathrm{Hom}(P^{n-4},G)$$

is exact for any projective right R-module G. Note that we have the following exact commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(P^{n-1},G) \arrow[rr, "f^*"] \arrow[dr, "\lambda^*"] & & \mathrm{Hom}(P^{n-2},G) \arrow[rr, "h^*"] \arrow[dr, "\iota^*"] & & \mathrm{Hom}(P^{n-3},G) \\
& \mathrm{Hom}(\mathrm{im}(f),G) \arrow[ur, "(f')^*"] \arrow[dr] & & \mathrm{Hom}(\ker(f),G) \arrow[ur, "(h')^*"] \arrow[dr] & \\
0 \arrow[ur] & & 0 & & 0
\end{tikzcd}$$

So

$$\ker((h')^*\iota^*) = \ker(h^*) = \mathrm{im}(f^*) = \mathrm{im}((f')^*\lambda^*) = \mathrm{im}(f')^* = \ker(\iota^*).$$

Let $\alpha \in \ker(h')^*$. Since ι^* is epic, $\alpha = \iota^*(\beta)$ for some $\beta \in \mathrm{Hom}(P^{n-2},G)$. Thus $(h')^*\iota^*(\beta) = (h')^*(\alpha) = 0$, and hence $\alpha = \iota^*(\beta) = 0$. It follows that $(h')^*$ is monic. On the other hand,

$\ker(j^*) = \mathrm{im}(h^*) = \mathrm{im}((h')^*)$. So we obtain the desired exact sequence. This completes the proof.

Now we give the main result of this section.

Theorem 2.6. *The following statements are equivalent for a ring R and an integer $n \geqslant 2$:*

(1) *$rD(R) \leqslant n$;*

(2) *the gl right $\mathcal{P}roj$-dim $\mathcal{M}_{\mathcal{R}} \leqslant n-2$;*

(3) $\mathrm{Ext}_{n+k}(N, M) = 0$ *for all right R-modules N, M and $k \geqslant -1$;*

(4) $\mathrm{Ext}_{n-1}(N, M) = 0$ *for all right R-modules N and M;*

If R is right coherent, then the above conditions are also equivalent to:

(5) $\mathrm{Ext}_{n-1}(N, M) = 0$ *for all finitely presented right R-modules N and M.*

Proof. The result follows from Proposition 2.3, Lemma 2.5 and the fact that $rD(R)$ can be computed using just the projective dimensions of finitely presented right R-modules (see [8, Theorem 3.3]).

Next we shall see that the right global dimension of R can be computed more easily by using $\mathrm{Ext}_n(-,-)$ when R is right coherent. To this end, the following lemmas are required.

Lemma 2.7. *Let R be a right coherent ring, N a finitely presented right R-module and $\{M_i\}$ a family of right R-modules. The following statements are true for an integer $n \geqslant 1$:*

(1) $\mathrm{Ext}_n(N, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}_n(N, M_i)$;

(2) $\mathrm{Ext}_n(N, \coprod M_i) \cong \coprod \mathrm{Ext}_n(N, M_i)$.

Proof. (1) Note that N has a projective resolution with each term P_i finitely presented since R is right coherent. Since each $\mathrm{Hom}(P_i, -)$ commutes with direct limits by [9, Satz 3], $\mathrm{Ext}_n(N, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}_n(N, M_i)$ by [2, Exercise 6.4, p.170].

(2) follows from (1).

Lemma 2.8. *Let R be a right coherent ring. Then the following statements are equivalent for a finitely presented right R-module N and an integer $n \geqslant 2$:*

(1) *The right $\mathcal{P}roj$-dim $N \leqslant n-2$;*

(2) $\mathrm{Ext}_{n+k}(N, S) = 0$ *for all simple right R-modules S and $k \geqslant -1$;*

(3) $\mathrm{Ext}_{n-1}(N, S) = 0$ *for all simple right R-modules S;*

(4) $\mathrm{Ext}_{n-1}(N, R/J(R)) = 0$, *where $R/J(R)$ is regarded as a right R-module.*

Proof. (1) $\Rightarrow$ (2) holds by Lemma 2.5.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let M be any right R-module. We construct $\{M_\alpha\}$ by transfinite induction. Let $M_0 = 0, M_1 = \mathrm{Soc}(M)$. For any ordinal α, if α is not a limit ordinal, let M_α be a submodule of M such that $M_\alpha/M_{\alpha-1} = \mathrm{Soc}(M/M_{\alpha-1})$; if α is a limit ordinal, let $M_\alpha = \cup_{\beta<\alpha} M_\beta$. By the transfinite construction principle, $\{M_\alpha\}$ is well-defined. Note that R is semi-primary by [10, Proposition 3], and so there exists some ordinal α_0 such that $M = M_{\alpha_0}$ by [11, pp. 182–184].

Now we show that $\mathrm{Ext}_{n-1}(N, M_\alpha) = 0$ for any α.

If $\alpha = 0$, then $M_\alpha = 0$, and hence $\mathrm{Ext}_{n-1}(N, M_0) = 0$.

For each $\alpha > 0$, assume $\mathrm{Ext}_{n-1}(N, M_\beta) = 0$ for all $\beta < \alpha$.

If α is not a limit ordinal, then we have the exact sequence

$$0 \to M_{\alpha-1} \to M_\alpha \to M_\alpha/M_{\alpha-1} \to 0,$$

which induces an exact sequence

$$\mathrm{Ext}_{n-1}(N, M_{\alpha-1}) \to \mathrm{Ext}_{n-1}(N, M_\alpha) \to \mathrm{Ext}_{n-1}(N, M_\alpha/M_{\alpha-1})$$

by [1, Theorem 8.2.3]. Note that $\mathrm{Ext}_{n-1}(N, M_{\alpha-1}) = 0$ by induction hypothesis. In addition, since $M_\alpha/M_{\alpha-1}$ is semisimple, $M_\alpha/M_{\alpha-1} = \coprod S_i$, where each S_i is simple. Therefore $\mathrm{Ext}_{n-1}(N, M_\alpha/M_{\alpha-1}) = \mathrm{Ext}_{n-1}(N, \coprod S_i) \cong \coprod \mathrm{Ext}_{n-1}(N, S_i) = 0$ by (3) and Lemma 2.7. So $\mathrm{Ext}_{n-1}(N, M_\alpha) = 0$.

If α is a limit ordinal, then $M_\alpha = \cup_{\beta<\alpha} M_\beta = \varinjlim M_\beta$. Thus, by Lemma 2.7, we have

$$\mathrm{Ext}_{n-1}(N, M_\alpha) \cong \varinjlim \mathrm{Ext}_{n-1}(N, M_\beta) = 0.$$

Therefore, $\mathrm{Ext}_{n-1}(N, M_\alpha) = 0$ for all ordinals α. In particular, $\mathrm{Ext}_{n-1}(N, M) = 0$, and hence (1) follows from Lemma 2.5.

(3) $\Leftrightarrow$ (4) follows from Lemma 2.7 and the fact that every simple right R-module is the direct summand of $R/J(R)$ by [12, Theorem 9.3.4].

Theorem 2.9. *Let R be a right coherent ring. Then the following statements are equivalent for an integer $n \geqslant 2$:*

(1) $rD(R) \leqslant n$;

(2) $\mathrm{Ext}_{n-1}(N, S) = 0$ *for all simple right R-modules S and all finitely presented right R-modules N;*

(3) $\mathrm{Ext}_{n-1}(N, R/J(R)) = 0$ *for all finitely presented right R-modules N.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 2.6.

(2) $\Rightarrow$ (3) holds by Lemma 2.8.

(3) $\Rightarrow$ (1) Note that the right $\mathcal{P}roj$-dim $N \leqslant n-2$ for any finitely presented right R-module N by (3) and Lemma 2.8. Thus $\mathrm{Ext}_{n-1}(N, M) = 0$ for all finitely presented right R-modules M and N by Lemma 2.5, and so $rD(R) \leqslant n$ by Theorem 2.6.

Corollary 2.10. *The following statements are equivalent for a commutative ring R:*

(1) *R is a semisimple Artinian ring;*

(2) $\mathrm{Ext}_1(N, R/J(R)) = 0$ *for all finitely presented R-modules N.*

Proof. Note that R is an Artinian ring by [5, Theorem 3.4]. Thus the result holds by Theorem 2.9 and the fact that the global dimension of a commutative Artinian ring is either 0 or ∞ (see [13, Corollary 5.75]).

3 Global dimension and $\mathcal{P}roj$-cosyzygies

In this section, we will characterize the global dimension of R using, among others, the $\mathcal{P}roj$-cosyzygies of R-modules.

Following [8], an R-module M is called FP-injective if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented R-modules N. The FP-injective dimension of M, denoted by FP-id(M), is defined to be the smallest integer $n \geqslant 0$ such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for every finitely presented R-module F (if no such n exists, set FP-id$(M) = \infty$).

Recall that a right R-module M is called P-projective[14] if M is a cokernel of a projective preenvelope, or equivalently, if $\mathrm{Ext}^1(M, P) = 0$ for all projective right R-modules P. We will say that M is strongly P-projective if $\mathrm{Ext}^i(M, P) = 0$ for all projective right R-modules P and all $i \geqslant 1$.

Lemma 3.1. *Let $\mathcal{C}$ be a class of R-modules and M an R-module. If $M \to F$ and $M \to G$ are $\mathcal{C}$-preenvelopes with cokernels K and L respectively, then $K \oplus G \cong L \oplus F$.*

Proof. The proof is dual to that of [1, Lemma 8.6.3].

Theorem 3.2. *The following statements are equivalent for a ring R and an integer $n \geqslant 1$:*

(1) *$rD(R) \leqslant n$;*

(2) *FP-id$({}_RR) \leqslant n$ and an $(n-1)$th syzygy of a right R-module is P-projective if and only if it is projective;*

(3) *every $(n-1)$th $\mathcal{P}roj$-cosyzygy of a right R-module has an epic projective envelope;*

(4) *every nth $\mathcal{P}roj$-cosyzygy of a right R-module is projective.*

Proof. (1) $\Rightarrow$ (2) Since R is left coherent and right perfect, FP-id$({}_RR) = rD(R) \leqslant n$ by [8, Proposition 3.5]. Let K_{n-1} be an $(n-1)$th syzygy of a right R-module M. Then we get an exact sequence $0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ with P_{n-1} projective. Note that K_n is projective since $\mathrm{pd}(M) \leqslant n$. If K_{n-1} is P-projective, then $\mathrm{Ext}^1(K_{n-1}, K_n) = 0$. So the above exact sequence is split, and hence K_{n-1} is projective.

(2) $\Rightarrow$ (3) Let $0 \to M \to P^0 \to P^1 \to \cdots$ be a right $\mathcal{P}roj$-resolution of a right R-module M. By [1, Theorem 8.4.31], the sequence $0 \to L^n \to P^n \to \cdots \to P^{2n-2} \to L^{2n-1} \to 0$ is exact. Thus L^n is an $(n-1)$th syzygy of L^{2n-1}. So L^n is projective by (2) since L^n is P-projective. Let $K = \mathrm{im}(L^{n-1} \to P^{n-1})$. Then the sequence $0 \to K \to P^{n-1} \to L^n \to 0$ is split, and hence K is projective. Consequently $L^{n-1} \to K$ is an epic projective envelope.

(3) $\Rightarrow$ (4) Let $0 \to M \to P^0 \to P^1 \to \cdots$ be a right $\mathcal{P}roj$-resolution of a right R-module M. Then L^{n-1} has an epic projective envelope $L^{n-1} \to F^{n-1}$ by (3). Since $L^{n-1} \to P^{n-1}$ is a projective preenvelope, $L^n \oplus F^{n-1} \cong P^{n-1}$ by Lemma 3.1. Thus L^n is projective.

(4) $\Rightarrow$ (1) Note that the gl right $\mathcal{P}roj$-dim $\mathcal{M}_{\mathcal{R}} \leqslant n$ by (4), so $rD(R) \leqslant n+2 < \infty$ by Theorem 2.6. Thus $rD(R) = FP$-id$({}_RR)$ by [8, Proposition 3.5]. On the other hand, if L^n is an nth $\mathcal{P}roj$-cosyzygy of a right R-module, then L^n has a monic projective preenvelope (for L^n is projective). Hence FP-id$({}_RR) \leqslant n$ by [15, Lemma 3.6]. So $rD(R) \leqslant n$, as desired.

Corollary 3.3. *The following statements are equivalent for a ring R:*

(1) *R is a right hereditary ring;*

(2) *every P-projective right R-module is projective;*

(3) *every right R-module has an epic projective envelope;*

(4) *every finitely presented right R-module has an epic projective envelope.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold by letting $n = 1$ in Theorem 3.2.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) Let N be any submodule of a projective right R-module M. Suppose A is a finitely presented right R-module and $f : A \to N$ is a homomorphism. Then A has an epic projective envelope $g : A \to P$ by (4). So there exists $\alpha : P \to M$ such that $\alpha g = \iota f$, where $\iota : N \to M$ is the inclusion. Thus there is an induced map $\beta : P \to N$ such that $\beta g = f$, that is, f factors through a finitely generated projective right R-module P. Thus N is flat, and so projective (for R is right perfect). It follows that R is a right hereditary ring.

Remark 3.4. We note that the equivalence of (1) and (3) in Corollary 3.3 follows from [16, Theorem 4.5 and Corollary 4.7].

We state the following easy result for completeness.

Proposition 3.5. *The following statements are equivalent for a ring R*:

(1) *R is a QF ring*;

(2) *every right R-module has a monic projective envelope*;

(3) *every quotient of a P-projective right R-module is P-projective*;

(4) *every right R-module is* (*strongly*) *P-projective*;

(5) *every finitely presented right R-module is* (*strongly*) *P-projective.*

Proof. The equivalences of (1) through (4) follow from the fact that R is a QF ring if and only if every injective right R-module is projective if and only if every projective right R-module is injective.

(4) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (1) Note that R_R is FP-injective by (5). Thus R is a QF ring by [17, Theorem 4.3] since R is left coherent and right perfect.

The following result is well known in case $n = 0$.

Proposition 3.6. *The following statements are equivalent for a ring R and an integer $n \geqslant 0$*:

(1) *$rD(R) \leqslant n$*;

(2) *$rD(R) < \infty$ and every nth syzygy of a right R-module is P-projective*;

If R is right coherent, then the above conditions are also equivalent to:

(3) *$rD(R) < \infty$ and every nth syzygy of a finitely presented right R-module is P-projective.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1) Let M be any right R-module. Then $\mathrm{Ext}^{n+1}(M,P) \cong \mathrm{Ext}^1(K_n,P) = 0$ for all projective right R-modules P by (2). It is easy to verify that $\mathrm{Ext}^{j+1}(M,P) = 0$ for all $j \geqslant n$ and all projective right R-modules P by induction. On the other hand, we assume $rD(R) = m < \infty$. Thus $\mathrm{pd}(N) \leqslant m$ for any right R-module N, and hence there exists an exact sequence $0 \to P_m \to \cdots \to P_0 \to N \to 0$ with each P_i projective. Then $\mathrm{Ext}^{n+1}(M,N) \cong \mathrm{Ext}^{m+n+1}(M,P_m) = 0$. So $\mathrm{pd}(M) \leqslant n$, and hence (1) follows.

(3) $\Rightarrow$ (1) follows from the proof of (2) $\Rightarrow$ (1) and [8, Theorem 3.3].

Proposition 3.7. *The following statements are equivalent for a ring R with $rD(R) < \infty$*:

(1) *R is a right hereditary ring*;

(2) *every right R-module has an epic P-projective envelope*;

(3) *every submodule of a P-projective right R-module is P-projective*;

(4) *every right ideal of R is P-projective*;

(5) *every P-projective right R-module is strongly P-projective*;

(6) *R is right coherent, and every finitely presented right R-module has an epic P-projective envelope*;

(7) *R is right coherent, and every finitely generated right ideal of R is P-projective*;

(8) *R is right coherent, and every finitely generated submodule of a projective right R-module is P-projective.*

Proof. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6) follow from Corollary 3.3.

(2) $\Rightarrow$ (3) Let N be a submodule of a P-projective right R-module M. Then N has an epic P-projective envelope by (2), and hence N is P-projective (for N embeds in a P-projective

right R-module).

(3) $\Rightarrow$ (1) and (8) $\Rightarrow$ (1) hold by Proposition 3.6.

(4) $\Rightarrow$ (3) Let I be a right ideal of R and P a projective right R-module. The exact sequence $0 \to I \to R \to R/I \to 0$ induces the exactness of $\mathrm{Ext}^1(I,P) \to \mathrm{Ext}^2(R/I,P) \to 0$. Thus $\mathrm{Ext}^2(R/I,P)=0$ since I is P-projective by (4), and hence P has injective dimension at most 1.

Now let N be a submodule of a P-projective right R-module M. Then the exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of $0 \to \mathrm{Ext}^1(N,P) \to \mathrm{Ext}^2(M/N,P)=0$ for any projective right R-module P. Thus $\mathrm{Ext}^1(N,P)=0$, and so N is P-projective.

(5) $\Rightarrow$ (4) Let I be a right ideal of R and $I \to P^0$ the projective preenvelope of I. Then $I \to P^0$ is a monomorphism, and so we get the short exact sequence $0 \to I \to P^0 \to L \to 0$. By definition, L is P-projective, and hence L is strongly P-projective by (5). Thus $\mathrm{Ext}^1(I,P) \cong \mathrm{Ext}^2(L,P)=0$ for any projective right R-module P, that is, I is P-projective.

The proof of (6) $\Rightarrow$ (7) is similar to that of (2) $\Rightarrow$ (3).

(7) $\Rightarrow$ (8) follows from the proof of (4) $\Rightarrow$ (3) noting that every projective right R-module has FP-injective dimension at most 1 by [8, Lemma 3.1].

Remark 3.8. We note that Proposition 3.7 does not hold for a ring R with $rD(R)=\infty$. For instance, let $R=\mathbb{Z}/4\mathbb{Z}$, then R is a QF ring. It is easily seen that R satisfies the conditions (2) through (8) in Proposition 3.7, but R is not hereditary.

Recall that a $\mathcal{C}$-envelope $\phi: M \to C$ is said to have the unique mapping property[15] if for any homomorphism f: $M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g: C \to C'$ such that $g\phi = f$. Dually we have the definition of a $\mathcal{C}$-cover with the unique mapping property.

Theorem 3.9. *Consider the following conditions for a ring R and an integer $n \geqslant 2$:*

(1) $rD(R) \leqslant n$;

(2) *every $(n-2)$th $\mathcal{P}roj$-cosyzygy of a right R-module has a projective envelope with the unique mapping property*;

(3) $rD(R) < \infty$ *and every nth syzygy of a right R-module has a P-projective envelope with the unique mapping property*;

(4) $rD(R) < \infty$ *and every nth syzygy of a right R-module has a P-projective cover with the unique mapping property.*

Then (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (1), (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Let $0 \to M \to P^0 \to P^1 \to \cdots \to P^{n-3}$ be a partial right $\mathcal{P}roj$-resolution of M and $f: L^{n-2} \to P^{n-2}$ the projective envelope of L^{n-2}. Thus $P^{n-2}/\mathrm{im}(f)$ is an $(n-1)$th $\mathcal{P}roj$-cosyzygy of M, and so $P^{n-2}/\mathrm{im}(f)$ has an epic projective envelope $g: P^{n-2}/\mathrm{im}(f) \to P^{n-1}$ by Theorem 3.2 since $rD(R) \leqslant n$. Note that $P^{n-1}=0$ by [14, Proposition 3.2], so we get the commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{Hom}(P^{n-2},P) & \longrightarrow & \mathrm{Hom}(P^{n-3},P) \\ & & \Big\downarrow{\scriptstyle f^*} & & \Big\| \\ 0 & \longrightarrow & \mathrm{Hom}(L^{n-2},P) & \longrightarrow & \mathrm{Hom}(P^{n-3},P), \end{array}$$

where P is projective. Therefore f^* is monic, and hence f is a projective envelope with the unique mapping property.

(2) $\Rightarrow$ (1) Let M be any right R-module and let $0 \to M \to P^0 \to \cdots \to P^{n-3} \xrightarrow{\alpha} P^{n-2} \xrightarrow{\beta} P^{n-1} \to \cdots$ be a right $\mathcal{P}roj$-resolution of M such that $g : L^{n-2} \to P^{n-2}$ is a projective envelope of L^{n-2} with the unique mapping property by (2). So we get the exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
\mathrm{Hom}(P^{n-1}, P^{n-1}) \arrow[r, "\beta^*"] & \mathrm{Hom}(P^{n-2}, P^{n-1}) \arrow[r, "\alpha^*"] \arrow[d, "g^*"'] & \mathrm{Hom}(P^{n-3}, P^{n-1}) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Hom}(L^{n-2}, P^{n-1}) \arrow[r] & \mathrm{Hom}(P^{n-3}, P^{n-1}).
\end{tikzcd}$$

Then α^* is monic. Thus $\beta^* = 0$, and hence $\beta = 0$. It follows that L^{n-1} has a projective preenvelope $L^{n-1} \to 0$. Therefore

$$0 \to M \to P^0 \to \cdots \to P^{n-3} \xrightarrow{\alpha} P^{n-2} \to 0$$

is a right $\mathcal{P}roj$-resolution of M, and so right $\mathcal{P}roj$-dim $M \leqslant n-2$. Thus $rD(R) \leqslant n$ by Theorem 2.6.

(3) $\Rightarrow$ (1) Let $P_{n-1} \xrightarrow{\varphi} P_{n-2} \to \cdots \to P_0 \to M \to 0$ be a partial projective resolution of a right R-module M. Then K_n has a P-projective envelope $\theta : K_n \to H$ with the unique mapping property by (3). Thus there exists $\delta : H \to P_{n-1}$ such that $i = \delta\theta$, where $i : K_n \to P_{n-1}$ is the inclusion. Thus $\varphi\delta\theta = \varphi i = 0$, and hence $\varphi\delta = 0$, which implies that $\mathrm{im}(\delta) \subseteq \ker(\varphi) = \mathrm{im}(i)$. So there exists $\gamma : H \to K_n$ such that $i\gamma = \delta$. Note that $i\gamma\theta = i$, and so $\gamma\theta = 1_{K_n}$ since i is monic. Thus K_n is isomorphic to a direct summand of H, and hence K_n is P-projective. So (1) holds by Proposition 3.6.

(4) $\Rightarrow$ (1) Let $P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$ be a partial projective resolution of a right R-module M. Then K_n has a P-projective cover $\theta : G \to K_n$ with the unique mapping property by (4). It is clear that θ is epic. Suppose $i : \ker(\theta) \to G$ is the inclusion. Note that there exists an exact sequence $P \xrightarrow{\gamma} \ker(\theta) \to 0$ with P projective. Since $\theta i\gamma = 0$, $i\gamma = 0$. Therefore $\ker(\theta) = 0$, and so K_n is P-projective. Thus (1) holds by Proposition 3.6.

Corollary 3.10. *The following statements are equivalent for a ring R:*

(1) $rD(R) \leqslant 2$;

(2) $\mathrm{Ext}_1(N, M) = 0$ *for all right R-modules N and M*;

(3) *every P-projective right R-module has an epic projective envelope*;

(4) *the second $\mathcal{P}roj$-cosyzygy of a right R-module is projective*;

(5) $rD(R) < \infty$ *and the second syzygy of a right R-module is P-projective*;

(6) *every right R-modules has a projective envelope with the unique mapping property.*

Proof. The result follows from Theorems 2.6, 3.2, 3.9 and Proposition 3.6 by letting $n = 2$.

Remark 3.11. The equivalence of (1) and (6) in Corollary 3.10 has been proven in [15, 18] using different methods (see [15, Corollary 3.9] and [18, Proposition 4.1]).

Finally, combining Theorem 2.6 with Corollary 3.10, we have

Corollary 3.12. *Let R be a ring. Then the gl right $\mathcal{P}roj$-dim $\mathcal{M}_{\mathcal{R}} = rD(R)-2$ if $rD(R) \geqslant 2$, or 0 if $rD(R)$ is 0 or 1.*

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