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$\mathcal{L}$-INJECTIVE HULLS OF MODULES

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Let R be a ring and $\mathcal{L}$ a class of R-modules. An R-module N is called $\mathcal{L}$-injective if $\mathrm{Ext}_R^1(L, N) = 0$ for all $L \in \mathcal{L}$. An $\mathcal{L}$-injective hull of an R-module M is defined to be a homomorphism $\phi : M \to F$ with F $\mathcal{L}$-injective such that for any monomorphism f: $M \to F^{'}$ with $F^{'}$ $\mathcal{L}$-injective, there is a monomorphism $g : F \to F^{'}$ satisfying $g\phi = f$. The aim of this paper is to study $\mathcal{L}$-injective hulls and their relations with $\mathcal{L}$-injective envelopes in Enochs' sense.

1. Introduction

Recall that an injective module E is called an injective hull of a module M if M essentially embeds in E. It is well known that the injective hull of M can be regarded simultaneously as the unique minimal injective extension and also the unique maximal essential extension of M (up to isomorphism). Eckmann and Schöpf [**3**] proved that every module has an injective hull. The result together with the Matlis' structure theorem [**11**] for injective modules has played an important role in homological algebra and commutative algebra.

Let R be a ring, $\mathcal{C}$ a class of R-modules and M an R-module. Enochs [**4**] introduced the concepts of $\mathcal{C}$-(pre)envelopes of M. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of M if for any homomorphism f: $M \to F^{'}$ with $F^{'} \in \mathcal{C}$, there is a homomorphism $g : F \to F^{'}$ such that $g\phi = f$. Moreover, if every endomorphism $g : F \to F$ such that $g\phi = \phi$ is an isomorphism, the $\mathcal{C}$-preenvelope ϕ is called a $\mathcal{C}$-envelope of M. $\mathcal{C}$-envelopes may not exist in general, but if they exist, they are unique up to isomorphism. In particular, let $\mathcal{C}$ be the class of all injective modules, then $\mathcal{C}$-envelopes in Enochs' sense agree with the injective hulls in Eckmann-Schöpf's sense by [**17**, Theorem 1.2.11].

Given a class $\mathcal{L}$ of R-modules. We let $\mathcal{L}^{\perp}$ be the class of R-modules M such that $\mathrm{Ext}_R^1(L, M) = 0$ for all $L \in \mathcal{L}$. Similarly, ${}^{\perp}\mathcal{L}$ denotes the class of R-modules N such that $\mathrm{Ext}_R^1(N, L) = 0$ for all $L \in \mathcal{L}$. An R-module M is called $\mathcal{L}$-injective (see [**7**]) if $M \in \mathcal{L}^{\perp}$, or equivalently, if M is injective with respect to every exact sequence $0 \to A$

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$\to B \to C \to 0$ with $C \in \mathcal{L}$. $\mathcal{L}$-injective modules stand for several known modules such as injective modules, FP-injective modules, divisible modules and cotorsion modules in case of different $\mathcal{L}$. $\mathcal{L}$-injective (pre)envelopes of modules for some special $\mathcal{L}$ have been studied by many authors (see, for example, [**6, 10, 16, 17**]).

In this short note, we introduce the concept of $\mathcal{L}$-injective hulls of modules which generalises that of injective hulls of modules from another point of view. An $\mathcal{L}$-injective hull of a module M is defined to be the "minimal" $\mathcal{L}$-injective extension of M. More precisely, an $\mathcal{L}$-injective hull of a module M is a homomorphism $\phi : M \to F$ with F $\mathcal{L}$-injective such that for any monomorphism f: $M \to F'$ with F' $\mathcal{L}$-injective, there is a monomorphism $g : F \to F'$ satisfying $g\phi = f$. It is shown that, if an R-module has an $\mathcal{L}$-injective hull, then it is unique up to isomorphism. It is also shown that, if $\mathcal{L}$ is closed under extensions, quotients and direct limits, then every R-module has an $\mathcal{L}$-injective hull. Some relations between $\mathcal{L}$-injective hulls and $\mathcal{L}$-injective envelopes are also studied.

Throughout this paper, R is an associative ring with identity and all modules are unitary right R-modules. $\mathcal{L}$ stands for a class of R-modules which is closed under isomorphisms and contains 0. For an R-module M, $E(M)$ denotes the injective hull of M. We use $N \leqslant_e M$ to indicate that N is an essential submodule of M. For other unexplained concepts and notations, we refer the reader to [**1, 6, 14, 17**].

2. Definition and results

We start with the following

Definition 2.1: Let $\mathcal{L}$ be a class of R-modules and M an R-module. A homomorphism $\phi : M \to F$ with F $\mathcal{L}$-injective is called an $\mathcal{L}$-injective hull of M if for any monomorphism f: $M \to F'$ with F' $\mathcal{L}$-injective, there is a monomorphism $g : F \to F'$ such that $g\phi = f$.

Remark 2.2. (1) If we choose $\mathcal{L}$ to be the class of all R-modules, then $\mathcal{L}$-injective hulls agree with injective hulls by [**1**, Corollary 18.11]. However, if we choose $\mathcal{L}$ such that the class of injective modules is a proper subclass of $\mathcal{L}$-injective modules, then there exists an $\mathcal{L}$-injective M whose $\mathcal{L}$-injective hulls do not agree with its injective hulls.

(2) Note that the injective hull $E(M)$ of M is $\mathcal{L}$-injective and is an essential extension of M, so every $\mathcal{L}$-injective hull $\phi : M \to F$ is an essential monomorphism by [**1**, Exercise 5.14 (1), p. 77] (if it exists).

It is well known that $\mathcal{L}$-injective envelopes are unique up to isomorphism if they exist. Now we have the analogous result for $\mathcal{L}$-injective hulls.

Theorem 2.3. *If an R-module has an $\mathcal{L}$-injective hull, then it is unique up to isomorphism.*

Proof: Let M be an R-module and $\mathfrak{S} = \{N : M \leqslant N \leqslant E(M),\ N \text{ is } \mathcal{L}\text{-injective}\}$. Note that the set $\mathfrak{S}$ is nonempty since $E(M) \in \mathfrak{S}$. We shall show that $\mathfrak{S}$ has a minimal element. Let $\{N_\alpha \in \mathfrak{S} : \alpha \in I\}$ be a descending chain. It is enough to show that $\cap N_\alpha \in \mathfrak{S}$ by Zorn's Lemma. We shall prove that any exact sequence $0 \to \cap N_\alpha \xrightarrow{i} P \to C \to 0$ with $C \in \mathcal{L}$ is split (we may regard i as an inclusion). In fact, we have the following pushout diagram of the inclusions i and λ_α:

$$\begin{tikzcd}
0 \arrow[r] & \cap N_\alpha \arrow[r, "i"] \arrow[d, "\lambda_\alpha"'] & P \arrow[r] \arrow[d, "\mu_\alpha"'] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N_\alpha \arrow[r, "\nu_\alpha"] & A_\alpha \arrow[r, "t_\alpha"] & C \arrow[r] & 0,
\end{tikzcd}$$

where $A_\alpha = (P \oplus N_\alpha)/\{(a, -a) : a \in \cap N_\alpha\}$, $\mu_\alpha(p) = \overline{(p, 0)}$ for any $p \in P$, $\nu_\alpha(q) = \overline{(0, q)}$ for any $q \in N_\alpha$. Since N_α is $\mathcal{L}$-injective, the second row is split. Thus we get a split exact sequence $0 \to \cap N_\alpha \xrightarrow{\nu} \cap A_\alpha \xrightarrow{t} C \to 0$. We claim that $P \cong \cap A_\alpha$. Indeed, there exists $\beta : P \to \cap A_\alpha$ such that $\beta(p) = \mu_\alpha(p)$ for any $p \in P$ and $\alpha \in I$. Note that β is monic since μ_α is monic. Now we define $\gamma : \cap A_\alpha \to P$ via $\overline{(p_\alpha, n_\alpha)} \mapsto p_\alpha + n_\alpha$. Assume $\overline{(p_\alpha, n_\alpha)} \in \cap A_\alpha$, then for any $\beta \in I$, $\overline{(p_\alpha, n_\alpha)} \in A_\beta$, and so $\overline{(p_\alpha, n_\alpha)} = \overline{(p_\beta, n_\beta)}$ for some $p_\beta \in P$ and $n_\beta \in N_\beta$. Then $\overline{(p_\alpha - p_\beta, n_\alpha - n_\beta)} = 0$, and hence $n_\alpha - n_\beta = -a$ for some $a \in \cap N_\alpha$. Thus $n_\alpha = n_\beta - a \in N_\beta$, it follows that $n_\alpha \in \cap N_\alpha$. Therefore $p_\alpha + n_\alpha \in P$, and so γ is well-defined. Note that $\beta\gamma = 1$, and hence β is an isomorphism. Thus the first row in the pushout diagram above is split, and so $\cap N_\alpha$ is $\mathcal{L}$-injective. Consequently, $\mathfrak{S}$ has a minimal element N_0.

Suppose $\phi : M \to F$ is any $\mathcal{L}$-injective hull of M. Then there exists a monomorphism ψ: $F \to N_0$ such that $\psi\phi = \iota$, where $\iota : M \to N_0$ is the inclusion. It is obvious that $\psi(F) \subseteq N_0$. In addition, $M = \iota(M) = \psi\phi(M) \subseteq \psi(F)$. Since $\psi(F) \cong F$ is $\mathcal{L}$-injective, $\psi(F) \in \mathfrak{S}$. So $\psi(F) = N_0$ by the minimality of N_0, and hence $F \cong N_0$.

This completes the proof. ∎

Remark 2.4. By Theorem 2.3, if an R-module M has an $\mathcal{L}$-injective hull, then we may choose the minimal $\mathcal{L}$-injective extension of M contained in $E(M)$ as its $\mathcal{L}$-injective hull.

Proposition 2.5. *Let $\phi : M \to F$ be a homomorphism.*

(1) *If ϕ is an $\mathcal{L}$-injective preenvelope, then ϕ is an $\mathcal{L}$-injective hull if and only if ϕ is an essential monomorphism.*

(2) *If M admits an $\mathcal{L}$-injective envelope, then ϕ is an $\mathcal{L}$-injective hull if and only if ϕ is an $\mathcal{L}$-injective envelope and ϕ is an essential monomorphism.*

Proof: (1) The necessity follows from Remark 2.2 (2). Conversely, assume that ϕ is essential. For any $\mathcal{L}$-injective module N and any monomorphism $f : M \to N$, there exists $g : F \to N$ such that $g\phi = f$ since ϕ is an $\mathcal{L}$-injective preenvelope. Thus g is a monomorphism by [**1**, Corollary 5.13], and so ϕ is an $\mathcal{L}$-injective hull.

(2) The sufficiency holds by (1). Conversely, suppose that ϕ is an $\mathcal{L}$-injective hull. Let $\lambda : M \to N$ be an $\mathcal{L}$-injective envelope of M, then there exists $f : N \to F$ such that $f\lambda = \phi$, and there exists a monomorphism $g : F \to N$ such that $g\phi = \lambda$. Thus $gf\lambda = \lambda$,

and hence gf is an isomorphism. Thus g is an isomorphism. It follows that $\phi : M \to F$ is an $\mathcal{L}$-injective envelope. ∎

Recall that an R-module M is called cotorsion [**5**] if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat R-modules F. It is well known that every R-module has a cotorsion envelope [**6**]. So, if $\phi : M \to F$ is a cotorsion hull of M, then ϕ is a cotorsion envelope of M by Proposition 2.5 (2). But the converse is not true in general as shown by the following example.

Example 2.6. Let $P = \{p : p \text{ is a prime}\}$, $\mathbb{Z}_{(p)} = \{a/b : b \notin \mathbb{Z}p,\ (a, b) = 1\}$, where $p \in P$. Then

$$\varphi : \mathbb{Z} \to \Pi_{p\in P}\mathbb{Z}_{(p)}$$
$$x \mapsto (x/1)$$

is a cotorsion envelope of $\mathbb{Z}$. However φ is not essential. In fact, it is easy to observe that $\Pi_{p\in P}\ \big(p/(p+1)\big) \neq 0$, but $\mathrm{im}(\varphi) \cap \Pi_{p\in P}\ \big(p/(p+1\big) = 0$. Thus φ is not a cotorsion hull of $\mathbb{Z}$ by Proposition 2.5 (1).

Proposition 2.7. *If $f : N \to M$ is a monomorphism with M $\mathcal{L}$-injective and* $\mathrm{coker}(f) \in \mathcal{L}$, *then the following are equivalent:*

(1) *f is an $\mathcal{L}$-injective hull of N.*

(2) *f is an essential monomorphism.*

Moreover, if $\mathcal{L}$ is closed under quotients, then the above conditions are also equivalent to:

(3) *f is an $\mathcal{L}$-injective envelope of N.*

Proof: We first note that $f : N \to M$ is an $\mathcal{L}$-injective preenvelope by assumption.

(1) ⇔ (2) holds by Proposition 2.5 (1).

(3) ⇒ (2). Let X be a submodule of M such that $f(N) \cap X = 0$, and let $\pi : M \to M/X$ be the quotient map. Put $g = \pi f$, then we get an exact sequence $0 \to N \xrightarrow{g} M/X \to H \to 0$. So we have $H \cong M/X\big/g(N)$. Note that $g(N) = \big(f(N) + X\big)/X$, and hence

$$H \cong M/X\big/\big(f(N) + X\big)/X \cong M\big/\big(f(N) + X\big) \cong M/f(N)\big/\big(f(N) + X\big)/f(N).$$

Since $M/f(N) \in \mathcal{L}$ and $\mathcal{L}$ is closed under quotients, we have $H \in \mathcal{L}$. Thus there exists $h : M/X \to M$ such that $f = hg = h\pi f$, and hence $h\pi$ is an isomorphism by (3). Consequently $X \cong h\pi(X) = 0$. It follows that f is essential.

(2) ⇒ (3). Let α be an endomorphism of M such that $\alpha f = f$. Then α is an essential monomorphism by [**1**, Corollary 5.13 and Exercise 5.14 (1)] since f is essential. Note that the sequence $M/f(N) = M/\alpha f(N) \to M/\alpha(M) \to 0$ is exact. Therefore $M/\alpha(M) \in \mathcal{L}$ by assumption, and we obtain a split exact sequence $0 \to M \xrightarrow{\alpha} M \to M/\alpha(M) \to 0$. So $\alpha(M) = M$ since $\alpha(M) \leqslant_e M$. Thus α is an epimorphism, and hence an isomorphism, as desired. ∎

Remark 2.8. Let $\mathcal{S}$ be a set of R-modules, then for every R-module N, there is an exact sequence $0 \to N \xrightarrow{f} M \to C \to 0$ such that M is $\mathcal{S}$-injective and $C \in {}^{\perp}(\mathcal{S}^{\perp})$ by [**6**, Theorem 7.4.1]. Thus f is an $\mathcal{S}$-injective hull if and only if f is an essential monomorphism by Proposition 2.5 (1). In addition, if ${}^{\perp}(\mathcal{S}^{\perp})$ is closed under direct limits, then N has an $\mathcal{S}$-injective envelope by [**6**, Theorem 7.2.6], and so f is both an $\mathcal{S}$-injective hull and an $\mathcal{S}$-injective envelope by Proposition 2.5 (2) if f is essential.

As is well known, for two R-modules M and N, if $N \leqslant_e M$, then $E(N) = E(M)$ (see [**1**, Proposition 18.12]). Next we consider the similar question when N and M share a common $\mathcal{L}$-injective hull under the condition that $N \leqslant_e M$.

Proposition 2.9. *Let $\iota : N \to M$ be an essential extension of N with $M/N \in \mathcal{L}$.*

(1) *If $\mathcal{L}$ is closed under cokernels of monomorphisms, and N has an $\mathcal{L}$-injective hull $f : N \to K$ with* $\operatorname{coker}(f) \in \mathcal{L}$, *then M has an $\mathcal{L}$-injective hull $M \to K$.*

(2) *If $\mathcal{L}$ is closed under extensions, and M has an $\mathcal{L}$-injective hull $\lambda : M \to H$ with* $\operatorname{coker}(\lambda) \in \mathcal{L}$, *then N has an $\mathcal{L}$-injective hull $N \to H$.*

Proof: (1) Since $M/N \in \mathcal{L}$, there is $\alpha : M \to K$ such that $\alpha\iota = f$. Thus $K/\alpha(N) = K/f(N) = \operatorname{coker}(f) \in \mathcal{L}$. By the exactness of $0 \to M/N \xrightarrow{\overline{\alpha}} K/\alpha(N) \to K/\alpha(M) \to 0$, we have $K/\alpha(M) \in \mathcal{L}$ since $\mathcal{L}$ is closed under cokernels of monomorphisms. In addition, α is an essential monomorphism since f and ι are essential. So $\alpha : M \to K$ is an $\mathcal{L}$-injective hull by Proposition 2.7.

(2) Consider the exact sequence $0 \to M/N \xrightarrow{\overline{\lambda}} H/\lambda(N) \to H/\lambda(M) \to 0$. Then $H/\lambda(N) \in \mathcal{L}$ since $\mathcal{L}$ is closed under extensions. Note that $\lambda\iota$ is essential, and hence $\lambda\iota : N \to H$ is an $\mathcal{L}$-injective hull by Proposition 2.7. ∎

Now we give a sufficient condition for the existence of $\mathcal{L}$-injective hulls.

Theorem 2.10. *If $\mathcal{L}$ is closed under extensions, quotients and direct limits, then every R-module has an $\mathcal{L}$-injective hull.*

Proof: Let M be an R-module. Put $\mathfrak{T} = \{N : M \leqslant N \leqslant E(M), \text{ and } N/M \in \mathcal{L}\}$. Then $\mathfrak{T}$ is a nonempty set since $M \in \mathfrak{T}$. Let $\{N_i \in \mathfrak{T} : i \in I\}$ be an ascending chain. Note that $M \leqslant \cup N_i \leqslant E(M)$ and $(\cup N_i)/M = \cup(N_i/M) = \varinjlim(N_i/M) \in \mathfrak{T}$ since $\mathcal{L}$ is closed under direct limits. Thus $\cup N_i \in \mathfrak{T}$, and so $\mathfrak{T}$ has a maximal element N' by Zorn's Lemma. We shall prove that N' is $\mathcal{L}$-injective. It is enough to show that any exact sequence $0 \to N' \xrightarrow{f} B \to C \to 0$ with $C \in \mathcal{L}$ is split. Let $\iota : N' \to E(N')$ be the inclusion and $\pi : E(N') \to E(N')/N'$ the quotient map. Then there exist $\alpha : B \to E(N')$ and $\beta : C \to E(N')/N'$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, "\alpha"] & C \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\iota"] & E(N') \arrow[r, "\pi"] & E(N')/N' \arrow[r] & 0.
\end{tikzcd}$$

Since $\beta(C) \leqslant E(N')/N'$, there exists H such that $N' \leqslant H \leqslant E(N')$ and $\beta(C) = H/N'$. So $H/N' \in \mathcal{L}$ since $C \in \mathcal{L}$ and $\mathcal{L}$ is closed under quotients. Thus the exactness of $0 \to N'/M \to H/M \to H/N' \to 0$ implies that $H/M \in \mathcal{L}$ by hypothesis. But the maximality of N' forces that $N' = H$, and hence $\beta(C) = 0$. So $\alpha(B) \subseteq N'$. It follows that the first row is split, and hence N' is $\mathcal{L}$-injective.

On the other hand, M is an essential submodule of N' since $M \leqslant N' \leqslant E(M)$. Therefore the inclusion $M \to N'$ is an $\mathcal{L}$-injective hull by Proposition 2.7. ∎

Recall that an R-module M is called FP-injective (or absolutely pure) [**12, 15**] if $\mathrm{Ext}^1_R(N, M) = 0$ for any finitely presented R-module N. M is called divisible (or P-injective) [**13, 16**] if $\mathrm{Ext}^1_R(R/aR, M) = 0$ for all $a \in R$. If R is a commutative domain, then M is divisible if and only if $Mr = M$ for any $0 \neq r \in R$. A ring R is called right semihereditary (right PP) if every finitely generated (principal) right ideal of R is projective.

Corollary 2.11. *The following are true:*

(1) *Every R-module over a right semihereditary ring R has an $\mathcal{FI}$-injective hull, where $\mathcal{FI}$ denotes the class of all FP-injective R-modules.*

(2) *Every R-module over a right PP ring R has a $\mathcal{DI}$-injective hull, where $\mathcal{DI}$ denotes the class of all divisible R-modules.*

Proof: (1) Note that $\mathcal{FI}$ is closed under extensions, direct limits by [**15**, Theorem 3.2] and quotients by [**12**, Theorem 2] since R is a right semihereditary ring. Thus (1) follows from Theorem 2.10.

(2) $\mathcal{DI}$ is clearly closed under extensions and direct sums. Since R is right PP, $\mathcal{DI}$ is closed under quotients by [**18**, Theorem 2]. Note that the sequence $\oplus M_i \to \varinjlim M_i \to 0$ is exact, and so $\mathcal{DI}$ is closed under direct limits. Therefore (2) holds by Theorem 2.10. ∎

It is known that every finite direct sum of $\mathcal{L}$-injective envelopes is still an $\mathcal{L}$-injective envelope. But $\mathcal{L}$-injective envelopes are not closed under arbitrary direct sums in general (even if the class of $\mathcal{L}$-injective modules is closed under arbitrary direct sums) (see [**17**]). The next proposition shows that $\mathcal{L}$-injective hulls are preserved under arbitrary direct sums.

Proposition 2.12. *The following are true:*

(1) *If $\phi_i : M_i \to F_i$ is an $\mathcal{L}$-injective hull for $i = 1, 2$, then $\phi_1 \oplus \phi_2 : M_1 \oplus M_2 \to F_1 \oplus F_2$ is an $\mathcal{L}$-injective hull.*

(2) *If the class of $\mathcal{L}$-injective modules is closed under direct sums, and $\phi_i : M_i \to F_i$ is an $\mathcal{L}$-injective hull for any $i \in I$, then $\oplus\phi_i : \oplus M_i \to \oplus F_i$ is an $\mathcal{L}$-injective hull.*

Proof: (1) Let $f : M_1 \oplus M_2 \to N$ with N $\mathcal{L}$-injective be any monomorphism. Suppose $\iota_i : M_i \to M_1 \oplus M_2$ is the canonical injection and $\pi_i : F_1 \oplus F_2 \to F_i$ the canonical projection, $i = 1, 2$. Then there exist monomorphisms $g_i : F_i \to N$ such that $g_i \phi_i = f \iota_i$. Define $g : F_1 \oplus F_2 \to N$ by $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$. It is easy to verify that $g(\phi_1 \oplus \phi_2) = f$. Note that $\phi_1 \oplus \phi_2$ is an essential monomorphism by [**1**, Proposition 5.20] since ϕ_i are essential monomorphisms by Remark 2.2 (2). So g is a monomorphism by [**1**, Corollary 5.13], as desired.

(2) Note that $\oplus \phi_i$ is an essential monomorphism by [**9**, Proposition 1.1 (d)]. Thus (2) holds by the proof of (1). ∎

We should point out that, although the class of $\mathcal{L}$-injective modules is closed under direct products, $\mathcal{L}$-injective hulls are not preserved under direct products in general (see [**17**, Example, p. 15]).

Finally, as an application of the results above, we consider the special case that R is a commutative domain.

Proposition 2.13. *The following are equivalent for a commutative domain R:*

(1) *Every free R-module has a divisible hull which is a divisible preenvelope.*

(2) *R has a divisible hull which is a divisible preenvelope.*

(3) *R has a divisible envelope.*

Proof: (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) follows from Proposition 2.12 and [**17**, Proposition 1.2.4] since the class of divisible modules is closed under direct sums.

(2) $\Rightarrow$ (3). Let $f : R \to N$ be a divisible hull of R. We may assume that f is an inclusion. For any $0 \neq r \in R$, there exists $t_r \in N$ such that $rt_r = 1$ since N is divisible. Define $p_r : R \to N$ via $s \mapsto st_r$. If $st_r = 0$, then $s = srt_r = rst_r = 0$, so p_r is a monomorphism. Thus there exists a monomorphism $g_r : N \to N$ such that $t_r = p_r(1) = g_r f(1) = g_r(1)$. Define $h_r : N \to N$ via $x \mapsto rx$, then $f = g_r h_r f$. Thus $g_r h_r$ is a monomorphism since f is essential by Remark 2.2 (2), and hence h_r is a monomorphism. It follows that N is torsionfree. So N is injective by [**2**, Proposition VII. 1.3] or [**8**, Theorem VI. 4.1]. Therefore f is an injective hull (envelope) since f is essential. Hence every endomorphism $g : N \to N$ such that $gf = f$ is an isomorphism. Thus f is a divisible envelope since f is a divisible preenvelope.

(3) $\Rightarrow$ (2). Let $f : R \to N$ be a divisible envelope of R. We may assume that f is an inclusion. It is easy to show that N is injective using an argument similar to that in the proof of (2) $\Rightarrow$ (3). Therefore f is an injective envelope (hull) since f is a divisible envelope. Hence f is a divisible hull by Proposition 2.5 (1) since f is essential. ∎

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