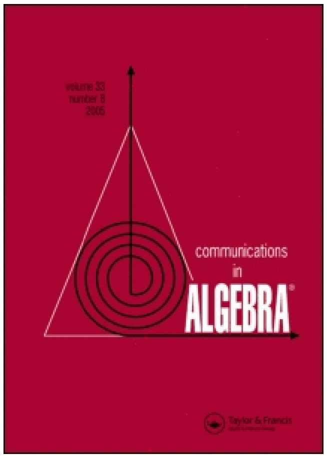

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MODULES WITH ANNIHILATOR CONDITIONS

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MODULES WITH ANNIHILATOR CONDITIONS

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ABSTRACT

Let S and R be rings. The objective is to study the bimodule ${}_SN_R$ satisfying the annihilator conditions $\mathbf{l}_N(\mathbf{r}_R(x)) = Sx$ for all $x \in N$. This approach will clearly show how the ring R or the module N_R is connected to the properties of the ring S through the annhilator condition. Specializing to the particular bimodule ${}_RR_R$ or ${}_{\mathrm{End}}(N)N_R$, we obtain some new results and known results as corollaries.

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INTRODUCTION

All rings are associative with identity and all modules are unitary. Let S and R be rings. One objective is to study the bimodule ${}_SN_R$ satisfying the annihilator condition (I) $\mathbf{l}_N(\mathbf{r}_R(x)) = Sx$ for all $x \in N$. Clearly, the ring R is a right principally injective ring (or P-injective ring) if and only if ${}_RR_R$ satisfies (I). For the detailed study of P-injective rings, we refer to[1–8]. For a right R-module N with $S = \mathrm{End}(N_R)$, ${}_SN_R$ satisfies (I) if and only if N_R is a principally quasi-injective module (due to[9]). Thus, the above annihilator condition naturally extends the P-injectivity of rings. An advantage of our approach using a bimodule setting is that one can see much better how the rings R, S and the modules ${}_SN, N_R$ are related to each other through the annihilator condition. For condition (I), our main results include a bijective correspondence between the set of simple submodules of ${}_SN$ and the set of maximal right ideals I of R, a characterization of right perfectness of S using a chain condition in N_R, and a determination of results on the endomorphism ring $\mathrm{End}(N_R)$. We also investigate how to characterize the Jacobson radical $J(S)$ using elements of S that are annihilated by essential submodules of N_R. Section 2 contains a characterization of V-modules using an annihilator condition, which extends a result of Faith and Menal on V-rings.

If M is a right R-module, we write $\mathbf{l}_M(r) = \{m \in M : mr = 0\}$ for all $r \in R$, $\mathbf{r}_R(m) = \{r \in R : mr = 0\}$ for all $m \in M$, $\mathbf{l}_M(A) = \cap_{a\in A}\mathbf{l}_M(a)$ for all $A \subseteq R$ and $\mathbf{r}_R(X) = \cap_{x\in X}\mathbf{r}_R(x)$ for all $X \subseteq M$. If M is a left R-module, $\mathbf{l}_R(X)$ and $\mathbf{r}_M(A)$ can be defined similarly. We use $K \leq_e N$ to indicate that K is an essential submodule of N. As usual, $J(N)$ and $\mathrm{Soc}(N)$ denote respectively the Jacobson radical and the socle of the module N. $J(R)$ stands for the Jacobson radical of the ring R.

1. ANNIHILATOR CONDITION (I)

Let ${}_SN_R$ be a bimodule. Then there is a canonical ring homomorphism $\lambda : S \to \mathrm{End}(N_R)$ given by $\lambda(s)(x) = sx$ for $x \in N$ and $s \in S$.

Lemma 1.1. *Let ${}_SN_R$ be a bimodule and $x \in N$. The following are equivalent:*

1. $\mathbf{l}_N(\mathbf{r}_R(x)) = Sx$.
2. *Every R-homomorphism $f : xR \to N_R$ extends to $\lambda(s) : N_R \to N_R$ for some $s \in S$.*
3. *If $\mathbf{r}_R(x) \subseteq \mathbf{r}_R(y)$ where $y \in N$*, then $Sy \subseteq Sx$.

Proof. The verification is straightforward. □

We say that a bimodule ${}_SN_R$ satisfies (I) if $\mathbf{l}_N(\mathbf{r}_R(x)) = Sx$ for all $x \in N$. Note that ${}_RR_R$ has (I) if and only if R is a right P-injective ring (see[3]) and ${}_{\mathrm{End}(N_R)}N_R$ has (I) if and only if N_R is a principally quasi-injective module (due to[9]).

Theorem 1.2. *Let ${}_SN_R$ be a bimodule satisfying (I) such that N_R is faithful. Then $J({}_SN) \subseteq \{x \in N : \mathbf{r}_R(x) \leq_e R_R\}$. The equality holds if in addition ${}_SN$ is cyclic.*

Proof. Let $x \in J({}_SN)$. If $\mathbf{r}_R(x)$ is not essential in R_R, then $\mathbf{r}_R(x) \cap aR = 0$ where $0 \neq a \in R$. It follows that $N = \mathbf{l}_N(0) = \mathbf{l}_N(\mathbf{r}_R(x) \cap aR) \supseteq \mathbf{l}_N(\mathbf{r}_R(x)) + \mathbf{l}_N(a) \supseteq Sx + \mathbf{l}_N(a)$. We show that $N = Sx + \mathbf{l}_N(a)$. To see this, let $y \in \mathbf{l}_N(\mathbf{r}_R(x) \cap aR)$. Then, $\mathbf{r}_R(xa) \subseteq \mathbf{r}_R(ya)$, and so $\mathbf{l}_N(\mathbf{r}_R(xa)) \supseteq \mathbf{l}_N(\mathbf{r}_R(ya))$. Since ${}_SN_R$ has (I), it follows that $Sxa \supseteq Sya$. Write $ya = txa$ where $t \in S$. Thus, $y - tx \in \mathbf{l}_N(a)$ and so $y = tx + (y - tx) \in Sx + \mathbf{l}_N(a)$. Therefore, $N = Sx + \mathbf{l}_N(a)$. Since $x \in J({}_SN)$, Sx is a small submodule of ${}_SN$. It follows that $N = \mathbf{l}_N(a)$, which gives $a = 0$ since N_R is faithful. This is a contradiction.

Suppose that ${}_SN$ is also cyclic. Let $x \in N$ such that $\mathbf{r}_R(x) \leq_e R_R$. To show $x \in J({}_SN)$, it suffices to prove that Sx is a small submodule of ${}_SN$. Let $N = Y + Sx$ where Y is a submodule of ${}_SN$. Because ${}_SN$ is cyclic, there exists a cyclic submodule Sy of Y such that $N = Sy + Sx$. Then $\mathbf{r}_R(N) = \mathbf{r}_R(y) \cap \mathbf{r}_R(x)$. Since N_R is faithful, $0 = \mathbf{r}_R(y) \cap \mathbf{r}_R(x)$. Because $\mathbf{r}_R(x)$ is essential in R_R, $\mathbf{r}_R(y) = 0$. Since ${}_SN_R$ has (I), $N = \mathbf{l}_N(\mathbf{r}_R(y)) = Sy \subseteq Y$. So $N = Y$. Thus we have proved that Sx is small in ${}_SN$, and so $x \in J({}_SN)$. □

Following Albu and Wisbauer,[10,2.6] a right R-module N_R is called a *Kasch module* if any simple module in $\sigma[N]$ embeds in N_R, where $\sigma[N]$ is the category consisting of all N-subgenerated right R-modules. For a right R-module N_R, we let $\mathcal{B}_N = \{I_R \subseteq R_R : I$ is a maximal right ideal of R and $R/I \in \sigma[N]\}$ and $J_N(R) = \cap\{I_R \subseteq R_R : I \in \mathcal{B}_N\}$. Note that $J_N(R)$ is a two-sided ideal of R. In fact, if $\mathcal{F}$ is the class of all simple right R-modules in $\sigma[N]$, then $J_N(R)$ is the reject of $\mathcal{F}$ in R_R (see[11, p. 109 and 8.23]).

The proof of the next lemma uses an idea of Gómez Pardo and Guil Asensio.[12]

Lemma 1.3. *Let ${}_SN_R$ be a bimodule such that N_R is a Kasch module and $\{M_i : i \in I\}$ is a family of maximal right ideals of R with all $R/M_i \in \sigma[N]$. Then there exists a subset K of I such that the family $\{\mathbf{l}_N(M_i) : i \in K\}$ of submodules of ${}_SN$ is independent and $\cap_{i\in I}M_i = \cap_{i\in K}M_i$. In particular, $R/J_N(R)$ is semisimple artinian if ${}_SN$ is also of finite uniform dimension.*

Proof. By Zorn's lemma, there exists a subset K of I such that $\{\mathbf{l}_N(M_i) : i \in K\}$ is a maximal independent subset of $\{\mathbf{l}_N(M_i) : i \in I\}$. Thus, for any $j \in I$, $\mathbf{l}_N(M_j) \cap [\Sigma_K \mathbf{l}_N(M_i)] \neq 0$. It follows that $\mathbf{l}_N(M_j + \cap_K M_i) = \mathbf{l}_N(M_j) \cap \mathbf{l}_N(\cap_K M_i) \neq 0$. So, $M_j + \cap_K M_i$ is a proper right ideal of R. Because M_j is a maximal right ideal of R, $\cap_K M_i \subseteq M_j$. Thus, we proved that $\cap_K M_i = \cap_I M_i$. If, in addition, ${}_SN$ has finite uniform dimension, K must be a finite set. Our proof implies that $J_N(R)$ must be an intersection of a finite number of maximal right ideals of R. Thus, $R/J_N(R)$ is semisimple artinian. □

Theorem 1.4. *Let ${}_SN_R$ be a bimodule satisfying (I) such that N_R is a Kasch module. Then*

1. *The map $X \longmapsto \mathbf{r}_R(X)$ gives a bijection from the set of all simple submodules of ${}_SN$ onto the set $\mathcal{B}_N$, whose inverse map is given by $I \longmapsto \mathbf{l}_N(I)$.*
2. *For $x \in N$, ${}_S(Sx)$ is simple if and only if $(xR)_R$ is simple.*
3. $\mathrm{Soc}(N_R) = \mathrm{Soc}({}_SN) \leq_e {}_SN$.
4. *$J_N(R) = \mathbf{r}_R(W)$ where $W = \mathrm{Soc}(N_R) = \mathrm{Soc}({}_SN)$.*
5. *$R/J_N(R)$ is semisimple artinian if and only if ${}_SN$ is of finite uniform dimension.*

Proof. (1) Let $X = Sx$ be a simple submodule of ${}_SN$. Clearly, $\mathbf{r}_R(X) = \mathbf{r}_R(x) \neq R$. There exists a maximal right ideal K of R such that $\mathbf{r}_R(x) \subseteq K$. Then, R/K is a factor of $R/\mathbf{r}_R(x) \cong xR$. So, $K \in \mathcal{B}_N$. Since N_R is Kasch, $R/K \overset{\phi}{\hookrightarrow} N$. Let $x_0 = \phi(1+K) \in N$. Then $0 \neq x_0 \in \mathbf{l}_N(K) \subseteq \mathbf{l}_N(\mathbf{r}_R(x)) = Sx$. The last equality is because ${}_SN_R$ has (I). Since ${}_S(Sx)$ is simple, $Sx = \mathbf{l}_N(K)$. It follows that $K \subseteq \mathbf{r}_R(\mathbf{l}_N(K)) = \mathbf{r}_R(x)$. So, $\mathbf{r}_R(X) = K \in \mathcal{B}_N$.

Let $I \in \mathcal{B}_N$. Then R/I embeds in N_R, and, as above, $\mathbf{l}_N(I) \neq 0$. For any $0 \neq x \in \mathbf{l}_N(I)$, $\mathbf{r}_R(x) \neq R$ and $I \subseteq \mathbf{r}_R(\mathbf{l}_N(I)) \subseteq \mathbf{r}_R(x)$. So $I = \mathbf{r}_R(x)$ since I is a maximal right ideal of R. Then $Sx = \mathbf{l}_N(\mathbf{r}_R(x)) = \mathbf{l}_N(I)$. So $\mathbf{l}_N(I)$ is a simple submodule of ${}_SN$. Now (1) follows because $\mathbf{l}_N(\mathbf{r}_R(X)) = X$ for any simple submodule X of ${}_SN$ and $\mathbf{r}_R(\mathbf{l}_N(I)) = I$ for $I \in \mathcal{B}_N$.

(2) For $x \in N$, by (1), ${}_S(Sx)$ is simple if and only if $\mathbf{r}_R(x) \in \mathcal{B}_N$ if and only if $(xR)_R$ is simple.

(3) It follows from (2) that $\mathrm{Soc}(N_R) = \mathrm{Soc}({}_SN)$. Let $W = \mathrm{Soc}({}_SN)$. Suppose that $W \cap Sx = 0$ where $0 \neq x \in N$. Then ${}_S(Sx)$ is not simple. By (1), $\mathbf{r}_R(x)$ is not a maximal right ideal. There exists a maximal right ideal I of R such that $\mathbf{r}_R(x) \subseteq I$. Thus, $I \in \mathcal{B}_N$ and $\mathbf{l}_N(\mathbf{r}_R(x)) \supseteq \mathbf{l}_N(I) \neq 0$. Then, since ${}_SN_R$ has (I), $Sx = \mathbf{l}_N(\mathbf{r}_R(x))$, and $\mathbf{l}_N(I)$ is a simple submodule of ${}_SN$ by (1). So $\mathbf{l}_N(I) \subseteq W$, contradicting the assumption that $W \cap Sx = 0$.

(4) Clearly, $WJ_N(R) = \mathrm{Soc}(N_R)J_N(R) = 0$. So $J_N(R) \subseteq \mathbf{r}_R(W)$. Let $M \in \mathcal{B}_N$. Then, by (1), $\mathbf{l}_N(M) \subseteq W$. Thus, $\mathbf{r}_R(\mathbf{l}_N(M)) \supseteq \mathbf{r}_R(W)$. But, by (1), $M = \mathbf{r}_R(\mathbf{l}_N(M))$. So $M \supseteq \mathbf{r}_R(W)$. It follows that $J_N(R) \supseteq \mathbf{r}_R(W)$.

(5) One direction is by Lemma 1.3. Let $R/J_N(R)$ be semisimple artinian. As a right $R/J_N(R)$-module, $\mathbf{l}_N(J_N(R))$ is semisimple. Thus, $\mathbf{l}_N(J_N(R))$ is a semisimple right R-module. So $\mathbf{l}_N(J_N(R)) \subseteq \mathrm{Soc}(N_R)$. Clearly, $\mathbf{l}_N(J_N(R)) \supseteq \mathrm{Soc}(N_R)$, and so $\mathbf{l}_N(J_N(R)) = \mathrm{Soc}(N_R)$. Note that $R/J_N(R)$ is a finitely cogenerated right R-module and $\cap\{I/J_N(R) : I \in \mathcal{B}_N\} = \bar{0}$. So, there exists a finite subset $\mathcal{F}$ of $\mathcal{B}_N$ such that $\cap\{I/J_N(R) : I \in \mathcal{F}\} = \bar{0}$. Thus, there exists a finite subset $\{M_i : i = 1, \cdots, n\}$ of $\mathcal{B}_N$ such that $J_N(R) = \cap_{i=1}^n M_i$ and $J_N(R) \neq \cap_{i\neq j} M_i$ for any $1 \leq j \leq n$. Arguing as in the proof of[2, Lemma 2.7], we have that $\mathbf{l}_N(\cap_{i=1}^n M_i) = \Sigma_{i=1}^n \mathbf{l}_N(M_i)$. Thus, $W = \mathrm{Soc}(N_R) = \mathbf{l}_N(J_N(R)) = \Sigma_{i=1}^n \mathbf{l}_N(M_i)$. But, by (1), each $\mathbf{l}_N(M_i)$ is a simple left S-module. So $_SW$ is finitely generated. By (3), $_SN$ is of finite uniform dimension. □

The next corollary follows immediately.

Corollary 1.5.[9, Prop.1.4] *Let N_R be a principally quasi-injective, Kasch module with $S = \mathrm{End}(N_R)$. Then $\mathrm{Soc}(N_R) = \mathrm{Soc}(_SN) \subseteq \mathbf{l}_N(J(R))$ and $\mathrm{Soc}(_SN) \leq_e {_SN}$.*

For a right R-module N_R, it is easy to prove that the following conditions are equivalent:

1. $\mathbf{l}_N(I) \neq 0$ for every proper right ideal I of R.
2. $\mathbf{l}_N(I) \neq 0$ for every maximal right ideal I of R.
3. $I = \mathbf{r}_R(\mathbf{l}_N(I))$ for every maximal right ideal I of R.
4. Every simple right R-module embeds in N_R.

Note that the condition (4) of N_R above is strictly stronger than the one that N_R is a Kasch module. For instance, let $R = \mathbb{Z}$ and $N = \mathbb{Z}/p\mathbb{Z}$ (p is a prime number). Then N_R is a Kasch module, but, clearly, N_R does not satisfy the above condition (4).

Corollary 1.6. *Let $_SN_R$ be a bimodule satisfying (I) such that $\mathbf{l}_N(I) \neq 0$ for every maximal right ideal I of R. Then*

1. *The map $X \longmapsto \mathbf{r}_R(X)$ gives a bijection from the set of all simple submodules of $_SN$ onto the set of all maximal right ideals of R, whose inverse map is given by $I \longmapsto \mathbf{l}_N(I)$.*
2. *For $x \in N$, $_S(Sx)$ is simple if and only if $(xR)_R$ is simple.*
3. $\mathrm{Soc}(N_R) = \mathrm{Soc}(_SN) \leq_e {_SN}$.
4. *$J(R) = \mathbf{r}_R(W)$ where $W = \mathrm{Soc}(N_R) = \mathrm{Soc}(_SN)$.*
5. *$R/J(R)$ is semisimple artinian if and only if $_SN$ is of finite uniform dimension.*

Proof. Note that, if $\mathbf{l}_N(I) \neq 0$ for every maximal right ideal I of R, then $\mathcal{B}_N$ is the set of all maximal right ideals of R and $J_N(R) = J(R)$. $\square$

In Corollary 1.6, the condition that $\mathbf{l}_N(I) \neq 0$ for every maximal right ideal I of R cannot be replaced by the one that N_R is Kasch. To see this, let $R = \mathbb{Z}$ and $N = \mathbb{Z}/p\mathbb{Z}$ (p is a prime number). The bimodule ${}_RN_R$ satisfies (I), N_R is Kasch, $J(R) = 0$ and $\mathrm{Soc}(N) = N$. It is easy to see that none of the statements (1),(4) and (5) in Corollary 1.6 holds.

Note that Corollary 1.6 (1,5) extends[4, Theorem 1.2] and [4, Theorem 1.3] respectively.

For a bimodule ${}_SN_R$, let $W_N(S) = \{t \in S : \mathbf{r}_N(t) \leq_e N_R\}$. Then $W_N(S)$ is an ideal of S. To see this, let $t, s \in W_N(S)$ and $u \in S$. Since $\mathbf{r}_N(t) \cap \mathbf{r}_N(s) \subseteq \mathbf{r}_N(t+s)$ and $\mathbf{r}_N(t) \subseteq \mathbf{r}_N(ut)$, it follows that $t+s \in W_N(S)$ and $ut \in W_N(S)$. Since $\mathbf{r}_N(t) \leq_e N_R$, $\{x \in N : ux \in \mathbf{r}_N(t)\} \leq_e N_R$. Thus $tu \in W_N(S)$ since $\{x \in N : ux \in \mathbf{r}_N(t)\} \subseteq \mathbf{r}_N(tu)$. So $W_N(S)$ is an ideal of S. It is easy to see that $W_N(S) \subseteq \{t \in S : \mathbf{r}_N(1_S - st) = 0, \forall s \in S\}$.

Lemma 1.7. *Let ${}_SN_R$ be a bimodule satisfying (I).*

1. $J(S) \subseteq W_N(S) = \{t \in S : \mathbf{r}_N(1_S - st) = 0, \forall s \in S\}$.
2. *If $s \notin W_N(S)$, then the inclusion $\mathbf{r}_N(s) \subset \mathbf{r}_N(s - sts)$ is proper for some $t \in S$.*

Proof. (1) Assume that $t \in S$ such that $\mathbf{r}_N(1_S - st) = 0$ for all $s \in S$. Let $\mathbf{r}_N(t) \cap xR = 0$ for some $x \in N$. Then $\mathbf{r}_R(tx) \subseteq \mathbf{r}_R(x)$, and so $x = stx$ for some $s \in S$ by Lemma 1.1. Hence $x \in \mathbf{r}_N(1_S - st) = 0$. This shows that $\mathbf{r}_N(t) \leq_e N_R$, i.e., $t \in W_N(S)$. Therefore $W_N(S) = \{t \in S : \mathbf{r}_N(1_S - st) = 0, \forall s \in S\}$, and hence $J(S) \subseteq W_N(S)$.

(2) If $s \notin W_N(S)$, then $\mathbf{r}_N(s) \cap xR = 0$ where $0 \neq x \in N$. Thus $\mathbf{r}_R(x) = \mathbf{r}_R(sx)$, and so $Sx = \mathbf{l}_N(\mathbf{r}_R(x)) = \mathbf{l}_N(\mathbf{r}_R(sx)) = S(sx)$. Write $x = tsx$ where $t \in S$. Then $(s - sts)x = 0$. Thus $\mathbf{r}_N(s) \subset \mathbf{r}_N(s - sts)$ is proper. $\square$

Proposition 1.8. *Let ${}_SN_R$ be a bimodule satisfying $\mathbf{l}_S(\mathbf{r}_N(t)) = St$ for all $t \in S$. Then $W_N(S) \subseteq J(S)$. If ${}_SN_R$ also satisfies (I), then $W_N(S) = J(S)$.*

Proof. We have $W_N(S) \subseteq \{t \in S : \mathbf{r}_N(1_S - st) = 0 \text{ for all } s \in S\} \subseteq J(S)$ since, if $\mathbf{r}_N(1_S - st) = 0$ for all $s \in S$, $S = \mathbf{l}_S(0) = \mathbf{l}_S(\mathbf{r}_N(1_S - st)) = S(1_S - st)$ for all $s \in S$. The second statement then follows from Lemma 1.7(1). $\square$

Lemma 1.9. *Let ${}_SN_R$ be a bimodule such that ${}_SN$ is faithful and, for any sequence $\{s_1, s_2, \ldots\} \subseteq S$, the chain $\mathbf{r}_N(s_1) \subseteq \mathbf{r}_N(s_2s_1) \subseteq \cdots$ terminates. Then*

1. *$W_N(S)$ is right T-nilpotent.*

2. *$S/W_N(S)$ contains no infinite set of nonzero pairwise orthogonal idempotents.*

Proof. (1) For $s_i \in W_N(S)$, $i = 1, 2, \ldots$, $\mathbf{r}_N(s_1) \subseteq \mathbf{r}_N(s_2 s_1) \subseteq \cdots$. Thus $\mathbf{r}_N(s_n \cdots s_1) = \mathbf{r}_N(s_{n+1} s_n \cdots s_1)$ for some $n > 0$. Hence $\mathbf{r}_N(s_{n+1}) \cap (s_n \cdots s_1) N = 0$. Since $s_{n+1} \in W_N(S)$, $\mathbf{r}_N(s_{n+1})$ is essential in N_R. It follows that $(s_n \cdots s_1)N = 0$. Thus, since ${}_SN$ is faithful, $s_n \cdots s_1 = 0$. So $W_N(S)$ is right T-nilpotent.

(2) Since $W_N(S)$ is right T-nilpotent, orthogonal sets of idempotents of $S/W_N(S)$ can be lifted to orthogonal sets of idempotents of S. Suppose (2) does not hold. Then, $S/W_N(S)$ contains an infinite set $\{\bar{t}_i\}$ of nonzero pairwise orthogonal idempotents, where $t_i^2 = t_i \in S$ and $t_i t_j = 0$ for $i \neq j$. Let $s_i = 1_S - (t_1 + \cdots + t_i)$, $i = 1, 2, \cdots$. Then, for all i, $s_{i+1} = s_i - s_i t_{i+1} s_i$, $s_{i+1} t_{i+1} = 0$, and $s_i t_{i+1} = t_{i+1} \neq 0$. It follows that $s_i(t_{i+1}N) = t_{i+1}N$ and $s_{i+1}(t_{i+1}N) = 0$. Since ${}_SN$ is faithful, $t_{i+1}N \neq 0$. Hence $\mathbf{r}_N(s_i) \subset \mathbf{r}_N(s_{i+1})$ is proper for all i. Let $b_i = 1_S - t_i$, then $s_i = b_i b_{i-1} \cdots b_1$, $i = 1, 2, \ldots$. Thus there is the following strictly ascending chain $\mathbf{r}_N(b_1) \subset \mathbf{r}_N(b_2 b_1) \subset \mathbf{r}_N(b_3 b_2 b_1) \subset \cdots$. This is a contradiction. □

Theorem 1.10. *Let ${}_SN_R$ be a bimodule satisfying (I) such that ${}_SN$ is faithful. The following are equivalent:*

1. *S is a right perfect ring.*
2. *For any sequence $\{s_1, s_2, \ldots\} \subseteq S$, the chain $\mathbf{r}_N(s_1) \subseteq \mathbf{r}_N(s_2 s_1) \subseteq \cdots$ terminates.*

Proof. (1) $\Rightarrow$ (2). Let $s_i \in S$, $i = 1, 2, \ldots$. Since S is right perfect, R satisfies DCC on principal left ideals. So the chain $Ss_1 \supseteq Ss_2 s_1 \supseteq \cdots$ terminates. Thus there exists $n > 0$ such that $S(s_n \cdots s_1) = S(s_{n+1} s_n \cdots s_1) = \cdots$. It follows that $\mathbf{r}_N(s_n \cdots s_1) = \mathbf{r}_N(s_{n+1} s_n \cdots s_1) = \cdots$.

(2) $\Rightarrow$ (1). Note that, for any $s \in S$ and $t \in S$, if $\overline{s - sts}$ is a regular element of $S/W_N(S)$, then so is $\bar{s}$. So, by (2) and Lemma 1.7(2), $S/W_N(S)$ is von Neumann regular by an argument similar to that in the proof of.[2, Theorem 3.4] By Lemmas 1.7 and 1.9, $J(S) = W_N(S)$ is right T-nilpotent. Thus, $S/J(S)$ is semisimple artinian because of Lemma 1.9(2). Therefore S is right perfect. □

Lemma 1.11. *Let ${}_SN_R$ be a bimodule such that ${}_SN$ is faithful and N_R satisfies ACC on $\{\mathbf{r}_N(A) : A \subseteq S\}$. Then $W_N(S)$ is nilpotent.*

Proof. By Lemma 1.9(1), $W_N(S)$ is right T-nilpotent. Then it is easy to show that $W_N(S)$ is nilpotent by a standard argument. □

The next corollary follows from Theorem 1.10 and Lemma 1.11.

Corollary 1.12. *Let ${}_SN_R$ be a bimodule satisfying (I) such that ${}_SN$ is faithful and N_R satisfies ACC on $\{\mathbf{r}_N(A) : A \subseteq S\}$. Then S is semiprimary.*

For a module N_R, a submodule X of N_R is called a *kernel submodule* if $X = \ker(f)$ for some $f \in \mathrm{End}(N_R)$, and X is called an *annihilator submodule* if $X = \cap_{f\in A}\ker(f)$ for some $A \subseteq \mathrm{End}(N_R)$. Part 2 of the next corollary extends a result of Fisher and Harada-Ishii that the endomorphism ring of a noetherian QI-module is semiprimary (see[13, Theorem 1.1] and [14, Theorem 1]).

Corollary 1.13. *Let N_R be a principally quasi-injective module and $S = \mathrm{End}(N_R)$.*

1. *If N_R satisfies ACC on kernel submodules, then S is right perfect.*
2. *If N_R satisfies ACC on annihilator submodules, then S is semiprimary.*

It was proved in[7, Theorem] that, if R is right P-injective and has ACC on annihilator right ideals, then R is left artinian. But we do not know if the ring S in Corollary 1.13(2) is left artinian.

2. V-MODULES AND ANNIHILATOR CONDITIONS

In this section, the V-modules are characterized using an annihilator condition, extending a result of Faith and Menal. All modules in this section are right R-modules.

Given two R-modules M and N, consider $\mathrm{Hom}_R(N, M)$, a left $\mathrm{End}(M)$-module. For a subset K of N and a subset X of $\mathrm{Hom}_R(N, M)$, put $An(K) = \{f : f \in \mathrm{Hom}_R(N, M) \text{ and } f(K) = 0\}$ and $\mathrm{Ke}(X) = \cap\{\ker(g) : g \in X\}$.

Definition 2.1. *A module N is said to be M-annular if, for every submodule K of N, $K = Ke(An(K))$.*

It can easily be proved that, for a submodule K of N, $K = Ke(An(K))$ if and only if N/K is cogenerated by M. Therefore, N is M-annular if and only if every factor of N is cogenerated by M.

Example 2.2.

1. R_R is R_R-annular if and only if every right ideal of R is a right annihilator. In this case, the ring R is called right dual.
2. R_R is M-annular if and only if $I = r_R(l_M(I))$ for every right ideal I. This condition was termed by Faith-Menal[15] as saying that M satisfies the double annihilator condition with respect to right ideals.

A module M_R is called a V-module if every submodule of M is an intersection of maximal submodules, or equivalently, every simple R-module is M-injective. When R_R is a V-module, we call R a right V-ring. It was proved in[15] that R is a right V-ring if and only if R is M-annular for some semisimple module M, and that in this case M is a cogenerator in Mod-R. They further show that if R is right noetherian and right dual (i.e., R is a right Johns ring by[15]) then $R/J(R)$ is a right V-ring. These results can be extended as follows.

Lemma 2.3. *Let K be a submodule of N_R. Then K is an intersection of maximal submodules of N_R if and only if $Ke(An(K)) = K$ for some semisimple module M.*

Proof. '$\Leftarrow$'. By assumption, $K = Ke(An(K)) = \cap \ker(g) : g \in An(K)\}$. For $g \in An(K)$, $K \subseteq \ker(g)$ and $N/\ker(g) \hookrightarrow M$. Thus, $N/\ker(g)$ is semisimple. Then $\ker(g)$ is an intersection of maximal submodules of N. Because $K = Ke(An(K)) = \cap\{\ker(g) : g \in An(K)\}$, it follows that K is an intersection of maximal submodules of N.

'$\Rightarrow$'. Let $\{M_i\}$ be a complete set of non-isomorphic simple modules in $\sigma[N]$ and $M = \oplus M_i$. Since K is an intersection of maximal submodules of N, it follows that $N/K \hookrightarrow \Pi X_j$ with each $X_j \in \sigma[N]$ a simple module. Therefore, there exists an embedding $l : N/K \hookrightarrow M^I$ for an index set I. Let $p : N \longrightarrow N/K$ be the natural homomorphism and $\pi_\alpha : M^I \longrightarrow M$ be the canonical projection onto the αth-component. Then $\{f_\alpha = \{pi_\alpha \circ l \circ p : \alpha \in I\} \subseteq An(K)$. Thus, $K \subseteq Ke(An(K)) \subseteq Ke(\{f_\alpha\})$. But, it is clear that $Ke(\{f_\alpha\}) \subseteq K$. So, $K = Ke(An(K))$. □

Theorem 2.4. *1. A module N is a V-module if and only if N is M-annular for some semisimple module M.*

2. If N is M-annular for a semisimple module M then M is a cogenerator in $\sigma[N]$.

3. If N is M-annular and $\mathbf{l}_M(J(R)) = Soc(M)$, then $N/NJ(R)$ is a V-module.

Proof. (1) By Lemma 2.3.

(2). By (1), N is a V-module and thus every simple module is N-injective. Hence the N-injective hull of any simple module $X \in \sigma[N]$ is itself. By Wisbauer,[16,17.12, p.143] we only need to show that M contains a copy of X for each simple module $X \in \sigma[N]$. For a simple module $X \in \sigma[N]$, $X \hookrightarrow N/A$ for some $A \subseteq N$ by[17, 2.3]. Since N is M-annular, $N/A \hookrightarrow M^I$ for some index set I. It follows that $X \hookrightarrow M^I$, so $X \hookrightarrow M$.

(3). We first show that, if X is a submodule of N such that, for any $g \in \mathrm{Hom}(N, M)$ with $g(X) = 0$, $g(N) \subseteq \mathrm{Soc}(M)$, then N/X is a V-module. To see this, let Y be a submodule of N containing X. By the assumptions, we have $\cap\{\ker(g) : g \in \mathrm{Hom}(N, \mathrm{Soc}(M)), g(Y) = 0\} = \cap\{\ker(g) : g \in \mathrm{Hom}(N, M), g(Y) = 0\} = Y$. Therefore, $\cap\{\ker(g) : g \in \mathrm{Hom}(N/X, \mathrm{Soc}(M)), g(Y/X) = 0\} = [\cap\{\ker(f) : f \in \mathrm{Hom}(N, \mathrm{Soc}(M)), f(Y) = 0\}]/X = Y/X$. This shows that N/X is $\mathrm{Soc}(M)$-annular. By (1), N/X is a V-module.

Now let $g \in \mathrm{Hom}(N, M)$ with $g(NJ(R)) = 0$. Thus, $g(N)J(R) = 0$, implying $g(N) \subseteq \mathbf{l}_M(J(R)) = \mathrm{Soc}(M)$. As seen above, $N/NJ(R)$ is a V-module.

From Anderson-Fuller,[11,15.17 and 15.18] $R/J(R)$ is semisimple if and only if $\mathrm{Soc}(M) = \mathbf{l}_M(J(R))$ for every right R-module M, and in this case, $J(M) = MJ(R)$ for every right R-module M. The following is immediate.

Corollary 2.5. *Suppose R is semilocal. If N is M-annular, then $N/J(N)$ is a V-module.*

Part 2 of the next Corollary extends a result in[7].

Corollary 2.6. *Let R be a right dual ring satisfying ACC on essential right ideals.*

1. $\mathrm{Soc}(R_R) = \mathbf{l}_R(J(R)) = \mathbf{r}_R(J(R))$ *is an essential right ideal of R.*
2. *$R/J(R)$ is a right V-ring.*

Proof. (1) Let $Z_r = Z(R_R)$ be the right singular ideal of R, $J = J(R)$ and $S_r = \mathrm{Soc}(R_R)$. For convenience, we shall abbreviate $\mathbf{l}_R(X)$ and $\mathbf{r}_R(X)$ to $\mathbf{l}(X)$ and $\mathbf{r}(X)$ respectively for a subset X of R. By[18, Theorem 2.9], J is nilpotent. Then $\mathbf{l}(J)$ is essential in R_R as argued in Johns.[19] Since R has ACC on essential right ideals, R/S_r is a right noetherian ring by.[20, Cor.2.9] By[17, Lemma 18.3], Z_r is nilpotent. So $Z_r \subseteq J$. Define $\mathbf{l}^{n+1}(J) = \mathbf{l}(\mathbf{l}^n(J))$ for $n \geq 1$. Following Johns' arguments,[19] we have $\mathbf{l}^2(J) \subseteq Z_r \subseteq J$ and this implies that $\mathbf{l}(J) \subseteq \mathbf{l}^3(J) \subseteq \mathbf{l}^5(J) \subseteq \cdots$. Note that this is a chain of essential right ideals. Since R satisfies ACC on essential right ideals, there exists an $m > 0$ such that $\mathbf{l}^m(J) = \mathbf{l}^{m+2}(J)$. Taking right annihilators $(m+1)$-times, we have $\mathbf{r}(J) = \mathbf{l}(J)$. Finally, arguing as the proof of[19, Lemma 4], we have that $\mathbf{r}(J) \subseteq S_r \subseteq \mathbf{l}(J)$.

(2) By (1) and Theorem 2.4(3), $(R/J)_R$ is a V-module. Thus, R/J is a right V-ring. □

CONCLUSION

Let S and R be rings. The paper studies the bimodule ${}_SN_R$ satisfying the annihilator condition $\mathbf{l}_N(\mathbf{r}_R(x)) = Sx$ for all $x \in N$. This approach clearly shows how the ring R or the module N_R is connected to the properties of the ring S through the annihilator condition. Specializing to the particular bimodule ${}_RR_R$ or $\mathrm{End}(N_R)N_R$, we obtain some new results and known results as corollaries.

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REFERENCES

1. Camillo, V.P. Commutative Rings Whose Principal Ideals are Annihilators. Portugal. Math. **1989**, *46* (1), 33–37.
2. Chen, J.; Ding, N. On General Principally Injective Rings. Comm. Algebra **1999**, *27* (5), 2097–2116.
3. Nicholson, W.K.; Yousif, M.F. Principally Injective Rings. J. Algebra **1995**, *174*, 77–93.
4. Nicholson, W.K.; Yousif, M.F. On a Theorem of Camillo. Comm. Algebra **1995**, *23* (14), 5309–5314.
5. Nicholson, W.K.; Yousif, M.F. On Quasi-Frobenius Rings. In *International Symposium on Ring Theory*; South Korea, June 28-July 3, 1999; Birkenmeier, G.F., Park, J.K., Park, Y.S., Eds.; Birkhäuser: Boston, 2001; 245–277.
6. Puninski, G.; Wisbauer, R.; Yousif, M.F. On *p*-Injective Rings, Glasgow Math. J. **1995**, *37* (3), 373–378.
7. Rutter, E.A. Jr. Rings With the Principal Extension Property. Comm. Algebra **1975**, *3* (3), 203–212.
8. Yue Chi Ming, R. On Injectivity and *p*-Injectivity. J. Math. Kyoto Univ. **1987**, *27*, 439–452.
9. Nicholson, W.K.; Park, J.K.; Yousif, M.F. Principally Quasi-Injective Modules. Comm. Algebra **1999**, *27* (4), 1683–1693.
10. Albu, T.; Wisbauer, R. Kasch Modules. In *Advances in Ring Theory*; Granville, May 1996; Jain, S.K., Rizvi, S.T., Eds.; Birkhäuser: Boston, 1997; 1–16.
11. Anderson, F.W.; Fuller, K.R. Rings and Categories of Modules; Springer-Verlag: New York, 1974.

12. Gómez Pardo, J.L.; Guil Asensio, P.A. Torsionless Modules and Rings with Finite Essential socle. In *Abelian Groups, Module Theory, and Topology*; Padua, 1997; Lecture Notes in Pure and Appl. Math.; Marcel Dekker, Inc.: New York, 1998; Vol. 201, 261–278.
13. Fisher, J.W. Finiteness Conditions for Projective and Injective Modules. Proc. Amer. Math. Soc. **1973**, *40*, 389–394.
14. Harada, M.; Ishii, T. On Endomorphism Rings of Noetherian Quasi-Injective Modules. Osaka J. Math. **1972**, *9*, 217–223.
15. Faith, C.; Menal, P. A New Duality Theorem for Semisimple Modules and Characterization of Villamayor Rings. Proc. Amer. Math. Soc. **1995**, *123* (6), 1635–1637.
16. Wisbauer, R. *Foundation of Module and Ring Theory*; Gordon and Breach: New York, 1991.
17. Dung, N.V.; Huynh, D.V.; Smith, P.F.; Wisbauer, R. *Extending Modules; Pitman Res. Notes Math. Ser.*; Longman: Harlow, 1994; Vol. 313.
18. Page, S.S.; Zhou, Y. Quasi-Dual Rings. Comm. Algebra **2000**, *28* (1), 489–504.
19. Johns, B. Annihilator Conditions in Noetherian Rings. J. Algebra **1977**, *49*, 222–224.
20. Page, S.S.; Yousif, M.F. Relative Injectivity and Chain Conditions. Comm. Algebra **1989**, *17* (4), 899–924.
21. Faith, C.; Menal, P. The Structure of Johns Rings. Proc. Amer. Math. Soc. **1994**, *120* (4), 1071–1081.

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