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NOTES ON COTORSION MODULES#

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Let R be a ring. A right R-module C is called cotorsion if $Ext_R^1(F, C) = 0$ for any flat right R-module F. In this paper, we first give some results on cotorsion envelopes that are analogous to those on injective envelopes. Then we characterize those rings R for which every cotorsion right R-module is $\mathcal{A}$-injective, where $\mathcal{A}$ is a nonemptry collection of right ideals of R. Finally, some new characterizations of right perfect rings and von Neumann regular rings are given.

Key Words: Cotorsion module; Cotorsion envelope; Flat cover; Perfect ring; von Neumann regular ring.

Mathematics Subject Classifications: Primary 16D50; Secondary 16E50.

1. INTRODUCTION

Let R be a ring. Recall that a right R-module C is called cotorsion (Enochs, 1984) if $Ext_R^1(F, C) = 0$ for any flat right R-module F. The class of cotorsion modules contains all pure-injective (hence injective) modules. Let M be an R-module. A homomorphism $\phi : M \to C$ with C cotorsion is called a cotorsion preenvelope of M (Xu, 1996) if for any homomorphism $f : M \to C'$, where C' is cotorsion, there is a homomorphism $g : C \to C'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of C when $C' = C$ and $f = \phi$, the cotorsion preenvelope ϕ is called a cotorsion envelope of M. Flat (pre)covers of M can be defined dually. Following Xu (1996, Propositions 2.1.3 and 2.1.4), a monomorphisim $\alpha : M \to C$ with C cotorsion is said to be a special cotorsion preenvelope of M if coker(α) is flat (dually we have the definition of a special flat precover). Special cotorsion preenvelopes (resp. special flat precovers) are obviously cotorsion preenvelopes (resp. flat precovers). An important feature of flat covers (resp. cotorsion envelopes) is that their kernels (resp. cokernels) are cotorsion (resp. flat) by Wakamatsu's Lemmas (Xu, 1996, Section 2.1).

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The existence of a flat cover and a cotorsion envelope for any module over any associative ring has been recently proved (Bican et al., 2001). Hence, while projective covers are rare, flat covers do exist for any module. Under this point of view the duality between flat and injective modules looks more accurate than between projective and injective modules. In turn, the right orthogonal class of the class of flat modules is the class of cotorsion modules, so it seems a natural question to study how is the analogy between very well known properties of injective envelopes and those of cotorsion envelopes, and this is the main goal of the first part of this paper. It is also studied that how are the rings satisfying that every (right) cotorsion module is injective with respect to a certain class of (right) ideals. Some new and known results are obtained as corollaries. For instance, we get that a commutative ring R is a PS ring if and only if every cotorsion R-module is mininjective.

On the other hand, cotorsion (and flat) modules have turned out to be a very useful in characterizing rings. In the second part of the paper new characterizations of perfect rings and von Neumann regular rings are given using, among other things, cotorsion envelopes and flat covers. For example, it is shown that a ring R is right perfect if and only if every flat right R-module has a cotorsion envelople with the unique mapping property if and only if R is a right cotorsion ring with $J(R)$ right T-nilpotent and R has no infinite set of nonzero orthogonal idempotents, and R is a von Neumann regular ring if and only if every cotorsion right R-module has a flat cover with the unique mapping property.

Throughout the paper, all rings are associative with identity and all modules are unitary. We write M_R to indicate a right R-module. For a right R-module M, $\varepsilon_M : F(M) \to M$ and $\sigma_M : M \to C(M)$ will denote a flat cover and a cotorsion envelope of M, respectively. We shall frequently identify M with its image in $C(M)$ and shall think of M as a submodule of $C(M)$. Sometimes we just call $C(M)$ a cotorsion envelope of M. As usual, $E(M)$ denotes the injective envelope of M, and $J(R)$ stands for the Jacobson radical of a ring R. We use $K \leq N$ and $K \leq_e N$ to mean that K is a submodule and an essential submodule of N, respectively. General background material can be found in Anderson and Fuller (1974), Enochs (1981), Enochs and Jenda (2000), Rotman (1979), and Xu (1996).

2. RESULTS

We start with a known result needed in the sequel.

Lemma 2.1 (Xu, 1996, Theorem 3.4.5). *Let M be a submodule of a cotorsion right R-module C. Then the following are equivalent*:

1. *The inclusion map $i : M \to C$ is a cotorsion envelope of M*;
2. *C is a flat essential extension of M (i.e., C/M is flat, and there are no nonzero submodules $S \leq C$ such that $S \cap M = 0$ and $C/(S \oplus M)$ is flat*).

As an immediate consequence of Lemma 2.1, we have

Corollary 2.2. *If M is a cotorsion right R-module, $N \leq_e M$, and M/N is flat, then M is a cotorsion envelope of N.*

We note that a cotorsion envelope $\sigma_M : M \to C(M)$ need not be an essential monomorphism in general as shown in the following example.

Example 2.3. Let $P = \{p : p \text{ is a prime}\}$, $\mathbb{Z}_{(p)} = \{a/b : b \notin \mathbb{Z}_p, (a, b) = 1\}$ where $p \in P$. Then

$$\varphi : \mathbb{Z} \to \prod_{p \in P} \mathbb{Z}_{(p)}$$
$$x \mapsto (x/1)$$

is a cotorsion envelope of $\mathbb{Z}$. However φ is not essential. In fact, it is easy to observe that $\Pi_{p \in P}(p/(p+1)) \neq 0$, but $\mathrm{im}(\varphi) \cap \Pi_{p \in P}(p/(p+1)) = 0$.

It is well known that the injective envelope of every simple R-module is indecomposable. Is it true for the cotorsion envelope? This is partially answered in the next proposition.

Recall that R is called a right SF ring if every simple right R-module is flat. It is easy to verify that R is a right SF ring if and only if the cotorsion envelope of every simple right R-module is flat.

Proposition 2.4. *Let R be a right SF ring such that the class of flat right R-modules is closed under cokernels of monomorphisms. Then the cotorsion envelope of every simple right R-module is indecomposable.*

Proof. Let M be a simple right R-module and $C(M) = M_1 \oplus M_2$, $M_i \neq 0$, $i = 1, 2$. Since M is simple, then $M \cap M_1 = 0$ or M. Assume $M \cap M_1 = 0$. There is an exact sequence

$$0 \to M \to C(M) \to C(M)/M \to 0$$

with $C(M)/M$ flat. Note that M_1 is flat, so the exactness of the sequence $0 \to M_1 \to C(M)/M \to C(M)/(M \oplus M_1) \to 0$ implies $C(M)/(M \oplus M_1)$ is flat by hypothesis. Thus $M_1 = 0$ by Lemma 2.1, a contradiction. It follows that $M \cap M_1 = M$, and hence $M \subseteq M_1$. Similarly $M \subseteq M_2$. So $M = 0$ a contradiction. The proof is complete. □

It it true that, if M is injective, then M has no proper essential extension (cf. Rotman, 1979, Theorem 3.29). Replacing “injective” with “cotorsion”, we have

Proposition 2.5. *If M is a cotorsion right R-module, and N is an essential extension of M with N/M flat, then $M = N$.*

Proof. This follows from the split exact sequence $0 \to M \to N \to N/M \to 0$. □

As is well known, for two right R-modules M and N, if $N \leq_e M$, then $E(N) = E(M)$ (see Anderson and Fuller, 1974, Proposition 18.12). Next, we focus our attention on the similar question when N and M share a common cotorsion (pre)envelope under the condition that $N \leq M$ (or $N \leq_e M$).

Proposition 2.6. *Let $\alpha : N \to M$ be a monomorphism.*

(1) *If coker(α) is flat, then $i\alpha : N \to H$ is a cotorsion preenvelop of N whenever $i : M \to H$ is a cotorsion preenvelope of M.*
(2) *coker(α) is flat if and only if $\sigma_M \alpha : N \to C(M)$ is a special cotorsion preenvelope of N.*

Proof. (1) Let $g : N \to L$ be any R-homomorphism with L a cotorsion right R-module. Since coker(α) is flat, $Hom_R(M, L) \to Hom_R(N, L) \to 0$ is epic. Therefore there exists $\beta : M \to L$ with $g = \beta\alpha$. The defining property of the cotorsion preenvelope now implies that there is $\gamma : H \to L$ such that $\beta = \gamma i$. Hence $\gamma(i\alpha) = (\gamma i)\alpha = \beta\alpha = g$, as required.

(2) Consider the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N \arrow[r, equal] \arrow[d, "\alpha"] & N \arrow[d, "\sigma_M\alpha"] & & \\
0 \arrow[r] & M \arrow[r, "\sigma_M"] \arrow[d] & C(M) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The necessity follows since L and Q are flat, and the sufficiency is true because of the flatness of H and Q. $\square$

Proposition 2.7. *Assume the class of flat right R-modules is closed under cokernels of monomorphisms. If $\alpha : N \to M$ is an essential monomorphism with coker(α) flat, then there exists $h : M \to C(N)$ such that h is a special cotorsion preenvelope of M.*

Proof. The exact sequence $0 \longrightarrow N \xrightarrow{\alpha} M \longrightarrow H \longrightarrow 0$ induces an exact sequence $Hom_R(M, C(N)) \to Hom_R(N, C(N)) \to 0$ since $H = \text{coker}(\alpha)$ is flat. So there exists $h : M \to C(N)$ such that $h\alpha = \sigma_N$. Since σ_N is a monomorphism and α is an essential monomorphism, then h is monic by Anderson and Fuller (1974, Corollary 5.13). Consider the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N \arrow[r, equal] \arrow[d, "\alpha"] & N \arrow[d, "\sigma_N"] & & \\
0 \arrow[r] & M \arrow[r, "h"] \arrow[d] & C(N) \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & L \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Note that H and K are flat, so L is flat by hypothesis, as desired. $\square$

Theorem 2.8. *If $N \leq_e M \leq_e C(M)$, then the following are equivalent*:

(1) *M/N is flat*;
(2) *$C(M)/N$ is flat*;
(3) *$C(M) = C(N)$ (up to isomorphism)*.

Proof. (1) $\Rightarrow$ (3) Let $i : N \to M$ be the inclusion map. By the flatness of M/N, there is a $\alpha : M \to C(N)$ such that $\alpha i = \sigma_N$. Note that α is monic since σ_N is monic and i is an essential monomorphism. By the defining property of a cotorsion envelope, if follows that α factors through $\sigma_M : M \to C(M)$, so there is $f : C(M) \to C(N)$ such that $f\sigma_M = \alpha$. Since α is monic and σ_M is an essential monomorphism, f is monic. Similarly, the map $\sigma_M i : N \to C(M)$ factors through $\sigma_N : N \to C(N)$, so there is $g : C(N) \to C(M)$ such that $\sigma_M i = g\sigma_N$. Thus $\sigma_N = \alpha i = f\sigma_M i = fg\sigma_N$, which implies fg is an automorphism of $C(N)$ by the defining property of a cotorsion envelope, and hence f is epic. It follows that f is an isomorphism.

(3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) There is an exact sequence

$$0 \to M/N \to C(M)/N \to C(M)/M \to 0.$$

The flatness of $C(M)/N$ and $C(M)/M$ implies that M/N is flat. □

It is well known that, if $N \leq M$, then $E(N) \leq E(M)$. It is straightforward to verify that if $N \leq M$, and $N \leq_e C(N)$, then $C(N) \leq C(M)$. In addition, we observe that, if $N \leq M \leq E(N)$, then $E(M) = E(N)$. Here we have the corresponding result for cotorsion envelopes.

Proposition 2.9. *If $N \leq M \leq C(N)$ and $M \leq_e C(M)$, then $C(M) = C(N)$ (up to isomorphism)*.

Proof. Let $i : N \to M$ and $\alpha : M \to C(N)$ be the inclusion maps. Then $\sigma_N = \alpha i$, and there exist $f : C(M) \to C(N)$ and $g : C(N) \to C(M)$ such that $\alpha = f\sigma_M$ and $\sigma_M i = g\sigma_N$. The proof of (1) $\Rightarrow$ (3) in Theorem 2.8 shows that f is an isomorophism. This completes the proof. □

In what follows, a ring R is called right cotorsion if R_R is cotorsion. Let $\mathcal{A}$ be a nonempty collection of right ideals of a ring R. Following Smith (1981), a right R-module X is said to be $\mathcal{A}$-injective (or injective with respect to $\mathcal{A}$) provided each R-homomorphism $f : A \to X$ with $A \in \mathcal{A}$ extends to R. A right R-module X is called maxinjective (resp., mininjective) if X is $\mathcal{A}$-injective with $\mathcal{A} = \{$all maximal right ideals of $R\}$ (resp., $\{$all simple right ideals of $R\}$).

Proposition 2.10. *Let R be a ring and $\mathcal{A}$ a noempty collection of right ideals of R. Then the following are equivalent*:

(1) *Every cotorsion right R-module is $\mathcal{A}$-injective*;
(2) *Every pure-injective right R-module is $\mathcal{A}$-injective*;
(3) *R/A is a flat right R-module for any $A \in \mathcal{A}$.*

Moreover, if R is a right cotorsion ring, then the above equivalent conditions imply that $C(A)$ is a direct summand of R_R for any $A \in \mathscr{A}$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) For any $A \in \mathscr{A}$, let M be any pure-injective right R-module. The exactness of the sequence $0 \to A \to R \to R/A \to 0$ induces an exact sequence

$$Hom_R(R, M) \to Hom_R(A, M) \to Ext_R^1(R/A, M) \to 0.$$

Note that the homomorphism $Hom_R(R, M) \to Hom_R(A, M)$ is epic by (2), and so $Ext_R^1(R/A, M) = 0$. By the arbitrariness of the pure-injective M, it follows that R/A is flat by the proof of Xu (1996, Lemma 3.4.1).

(3) $\Rightarrow$ (1) Let M be any cotorsion right R-module. For any $A \in \mathscr{A}$, the exact sequence $0 \to A \to R \to R/A \to 0$ induces an exact sequence

$$Hom_R(R, M) \to Hom_R(A, M) \to Ext_R^1(R/A, M).$$

Note that $Ext_R^1(R/A, M) = 0$ by (3), so $Hom_R(R, M) \to Hom_R(A, M)$ is epic, and (1) follows.

If R is a right cotorsion ring, let R/A be a flat right R-module for any $A \in \mathscr{A}$, then the inclusion map $A \to R_R$ is a cotorsion preenvelope of A. Thus $C(A)$ is a direct summand of R_R by Enochs and Jenda (2000, Proposition 6.1.2) . □

As applications, we list some corollaries of Proposition 2.10 above.

Let $\mathscr{A}$ be the collection of all right ideals of R in Proposition 2.10, then we have the following result which is due to Xu (1996, Theorem 3.3.2).

Corollary 2.11. *Let R be a ring. Then the following are equivalent*:

(1) *Every cotorsion right R-module is injective*;
(2) *Every pure-injective right R-module is injective*;
(3) *R is a von Neumann regular ring.*

Specializing Proposition 2.10 to the case that $\mathscr{A}$ is the collections of all pure right ideals, all maximal right ideals, and all essential right ideals of R, respectively, we obtain

Corollary 2.12. *Let R be a ring. Then*

(1) *Every cotorsion right R-module is injective with respect to all pure right ideals.*
(2) *R is a right SF ring if and only if every cotorsion right R-module is maxinjective.*
(3) *R is a von Neumann regular ring if and only if every cyclic singular right R-module is flat.*

Proof. (1) and (2) are clear by Proposition 2.10.

(3) Note that a ring R-module M is injective if and only if M is injective with respect to all essential right ideals. So (3) follows. □

Corollary 2.13 (Chen, 1991, Theorem 3). *If R is a right SF ring and every nonzero cyclic singular right R-module has a nonzero socle, then R is a von Neumann regular ring.*

Proof. Since R is a right SF ring, then every cotorsion right R-module is maxinjective by Corollary 2.12 (2). On the other hand, every maxinjective right R-module is injective by Smith (1981, Lemma 4). It follows that every cotorsion right R-module is injective, so R is von Neumann regular by Corollary 2.11. □

Lemma 2.14. *Let R be a commutative ring, then every simple R-module is cotorsion.*

Proof. Suppose $\{S_i\}_{i\in I}$ is an irredundant set of representatives of the simple R-modules. Let $E = E(\bigoplus_{i\in I} S_i)$, then E is an injective cogenerator by Anderson and Fuller (1974, Corollary 18.19). Let S be a simple R-module and F a flat R-module. Since E is injective, there is an isomorphism

$$Ext_R^1(F, Hom_R(S, E)) \cong Hom_R(Tor_1^R(F, S), E) = 0.$$

Note that $Hom_R(S, E) \cong S$ by the proof of Ware (1971, Lemma 2.6), so $Ext_R^1(F, S) = 0$, and hence S is cotorsion. □

Theorem 2.15. *Let R be a commutative ring, then the following are equivalent*:

(1) *R is a PS ring* (*i.e., every simple ideal is projective*);
(2) *Every cotorsion R-module is mininjective.*

Moreover, if R is a cotorsion ring with essential socle, then the above conditions are equivalent to

(3) *R is a nonsingular ring*;
(4) $J(R) = 0$;
(5) *R is a von Neumann regular ring.*

Proof. Let $Z(R)$ and $Soc(R)$ denote the singular ideal and the socle of the commutative ring R, respectively, and $ann_R(S) = \{x \in R : xs = 0 \text{ for all } s \in S\}$ for a nonempty subset S of R.

(1) $\Rightarrow$ (2) We shall show that $(Soc(R))^2 = Soc(R)$. In fact, $(Soc(R))^2 \subseteq Soc(R)$ is clear. We claim that $Soc(R)I \neq 0$ for any simple ideal I. If not, then there exists a simple ideal aR such that $Soc(R)aR = 0$, and so $(aR)^2 = 0$. Since R is a PS ring, we have $R = ann_R(a) \oplus K$. So $aR = aK \subseteq ann_R(a) \cap K = 0$, a contradiction. Thus $I = Soc(R)I$ for any simple ideal I. It follows that $Soc(R) \subseteq (Soc(R))^2$, and hence $(Soc(R))^2 = Soc(R)$. Consequently $R/Soc(R)$ is flat by Baccella (1980, Proposition 1.4).

Now let I be a simple ideal. The exact sequence

$$0 \to Soc(R)/I \to R/I \to R/Soc(R) \to 0$$

implies R/I is flat since $Soc(R)/I$ is projective. Thus (2) holds by Proposition 2.10.

(2) $\Rightarrow$ (1) Let I be a simple ideal. Then I is mininjective by (2) and Lemma 2.14, it follows that I is a direct summand of R, and hence that I is projective.

(1) $\Rightarrow$ (3) Note that $Z(Soc(R)) = Z(R) \cap Soc(R)$ and $Z(Soc(R)) = 0$ by Lam (1999, Exercise 12A(c), p. 269), so $Z(R) \cap Soc(R) = 0$, and hence $Z(R) = 0$ since $Soc(R) \leq_e R$.

(3) $\Rightarrow$ (1) follows from Nicholson and Watters (1988, Example 2.5(3)).

(3) $\Leftrightarrow$ (4) For a cotorsion ring with $Soc(R) \leq_e R$, we can prove a more general result: $Z(R) = J(R) = ann_R(Soc(R))$. In fact, it is clear that $J(R) \leq ann_R(Soc(R))$, we note that $Z(R) \leq J(R)$ by Nicholson and Yousif (2001, Thorem 1.2) or the remark just before Asensio and Herzog (2003, Theorem 6), and $ann_R(Soc(R)) \leq Z(R)$ by $Soc(R) \leq_e R$. Thus (3) $\Leftrightarrow$ (4) follows.

(5) $\Rightarrow$ (4) This is obvious.

(4) $\Rightarrow$ (5) Note that $R/J(R)$ is von Neumann regular by Asensio and Herzog (2003, Theorem 6). The proof is complete. $\square$

To prove the next main result, we need the following lemma which is of independent interest.

Lemma 2.16. *Let R and S be rings. If M_R is a cotorsion right R-module, and ${}_SF_R$ is an S-R-bimodule and flat as a right R-module, then $Hom_R({}_SF_R, M_R)$ is a cotorsion right S-module.*

Proof. Let N_S be a flat right S-module. There exists an exact sequence

$$0 \to K_S \to G_S \to N_S \to 0$$

with G_S projective, which yields the exactness of the sequence

$$0 \to K \otimes_S F_R \to G \otimes_S F_R \to N \otimes_S F_R \to 0.$$

Note that $N \otimes_S F_R$ is a flat right R-module. We have the exact sequence

$$Hom_R(G \otimes_S F_R, M_R) \to Hom_R(K \otimes_S F_R, M_R) \to Ext^1_R(N \otimes_S F_R, M_R) = 0,$$

which gives rise to the exactness of the sequence

$$Hom_S(G_S, Hom_R({}_SF_R, M_R)) \to Hom_S(K_S, Hom_R({}_SF_R, M_R)) \to 0.$$

On the other hand, the sequence

$$Hom_S(G_S, Hom_R({}_SF_R, M_R)) \to Hom_S(K_S, Hom_R({}_SF_R, M_R)) \to$$
$$Ext^1_S(N_S, Hom_R({}_SF_R, M_R)) \to Ext^1_S(G_S, Hom_R({}_SF_R, M_R)) = 0$$

is exact. Thus $Ext^1_S(N_S, Hom_R({}_SF_R, M_R)) = 0$, as desired. $\square$

Corollary 2.17. *If M is a flat submodule of a cotorsion right R-module N such that $f(M) \subseteq M$ for each R-homomorphism $f : M \to N$, then $End_R(M)$ is a right cotorsion ring.*

In particular, $End_R(M)$ is a right cotorsion ring for any flat cotorsion right R-module M.

Proof. Note that $Hom_R({}_{End_R(M)}M_R, N_R)$ is a cotorsion right $End_R(M)$-module by Lemma 2.16. Since $f(M) \subseteq M$ for each R-homorphism $f : M \to N$, we have $Hom_R({}_{End_R(M)}M_R, N_R) = End_R(M)$. Therefore $End_R(M)$ is a right cotorsion ring. $\square$

Recall that a cotorsion envelope $\sigma_M : M \to C(M)$ has the unique mapping property (Ding, 1996), if for any homomorphism $f : M \to N$ with N cotorsion, there exists a unique $g : C(M) \to N$ such that $g\sigma_M = f$. The concept of a flat cover with the unique mapping property can be defined similarly.

We are now in a position to prove the following

Theorem 2.18. *Let R be a ring. Then the following are equivalent.*

(1) *R is a right perfect ring*;
(2) *Every right R-module has a cotorsion envelope with the unique mapping property*;
(3) *Every flat right R-module has a cotorsion envelope with the unique mapping property*;
(4) *For any right R-homomorphism $f : M_1 \to M_2$ with M_1 and M_2 cotorsion, $ker(f)$ is cotorsion*;
(5) *For each (flat) right R-module M, the functor $Hom_R(-, M)$ is exact with respect to each pure exact sequence $0 \to K \to P \to L \to 0$ with P projective*;
(6) *$J(R)$ is right T-nilpotent and each right $R/J(R)$-module is cotorsion as a right R-module*;
(7) *R is a right cotorsion ring with $J(R)$ right T-nilpotent, and R has no infinite set of nonzero orthogonal idempotents.*

Proof. Note that R is a right perfect ring if and only if every (flat) right R-module is cotorsion by Xu (1996, Proposition 3.3.1). Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are trival.

(3) $\Rightarrow$ (1) Let M be any flat right R-module. There is the exact commutative diagram.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\sigma_M"] \arrow[drr, "0"'] & C(M) \arrow[r, "\gamma"] \arrow[dr, "\sigma_L\gamma"] & L \arrow[r] \arrow[d, "\sigma_L"] & 0 \\
 & & & C(L) &
\end{tikzcd}$$

Note that $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$, so $\sigma_L\gamma = 0$ by (3). Therefore $L = \mathrm{im}(\gamma) \subseteq \ker(\sigma_L) = 0$, and so M is cotorsion. Hence (1) follows.

(4) $\Rightarrow$ (1) Let M be any flat right R-module. The above commutative diagram implies $M = \ker(\gamma) = \ker(\sigma_L\gamma)$ is cotorsion by (4), as desired.

(1) $\Rightarrow$ (5) This is clear since L is flat.

(5) $\Rightarrow$ (1) Let M and N be flat right R-modules. There exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact sequence

$$Hom_R(P, M) \to Hom_R(K, M) \to Ext_R^1(N, M) \to 0.$$

Since $Hom_R(P, M) \to Hom_R(K, M)$ is epic by (5), $Ext_R^1(N, M) = 0$. Thus M is cotorsion, and so (1) follows.

(1) $\Rightarrow$ (6) and (7) follows from Anderson and Fuller (1974, Theorem 28.4).

(6) $\Rightarrow$ (1) Let F be a flat right R-module. Then F/FJ ($J = J(R)$) is projective as an R/J-module by Sandomierski (1973, Lemma 5.1), and so F is projective by Sandomierski (1973, Theorem 5.2), as required.

(7) $\Rightarrow$ (1) Since R has no infinite set of nonzero orthogonal idempotents, we have $R_R = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ such that I_i is indecomposable and $I_i = e_iR$, $e_i^2 = e_i$, $i = 1, 2, \ldots, n$, by Lam (1999, Proposition 6.60). Let $S_i = End_R(I_i)$, then S_i is a right cotorsion ring by Corollary 2.17 since each I_i is a flat cotorsion right R-module, $i = 1, 2, \ldots, n$. In addition, 0 and 1 are the only idempotents in S_i since I_i is indecomposable. It follows that S_i is local by Asensio and Herzog (2003, Corollary 7). Note that each $e_iRe_i \cong S_i$, $i = 1, 2, \ldots, n$. Consequently, R is a semiperfect ring by Anderson and Fuller (1974, Theorem 27.6), and hence (1) follows. $\square$

We conclude this paper with the following easy result which may be viewed as the dual of Theorem 2.18.

Proposition 2.19. *Let R be a ring. Then the following are equivalent*:

(1) *R is a von Neumann regular ring*;
(2) *Every cotorsion right R-module is absolutely pure*;
(3) *Every right R-module has a flat cover with the unique mapping property*;
(4) *Every cotorsion right R-module has a flat cover with the unique mapping property*;
(5) *For any right R-homorphism $f: M_1 \to M_2$ with M_1 and M_2 flat, coker(f) is flat.*

Proof. (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (1) Let M be a right R-module. There is an exact sequence

$$0 \longrightarrow K \longrightarrow F(M) \xrightarrow{\varepsilon_M} M \longrightarrow 0.$$

Since K is cotorsion, K is absolutely pure by (2). Thus the sequence is pure, and so M is flat.

(1) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) are trivial.

(4) $\Rightarrow$ (1) Let M be any cotorsion right R-module. There is the exact commutative diagram.

$$\begin{tikzcd} & F(K) \arrow[d, "\varepsilon_K"'] \arrow[dr, "\alpha\varepsilon_K"] \arrow[drr, "0"] & & & \\ 0 \arrow[r] & K \arrow[r, "\alpha"'] \arrow[d] & F(M) \arrow[r, "\varepsilon_M"'] & M \arrow[r] & 0 \\ & 0 & & & \end{tikzcd}$$

Note that $\varepsilon_M \alpha \varepsilon_K = 0 = \varepsilon_M 0$, so $\alpha\varepsilon_K = 0$ by (4). Therefore $K = \mathrm{im}(\varepsilon_K) \subseteq \ker(\alpha) = 0$, and so M is flat. Hence (1) follows from Xu (1996, Theorem 3.3.2).

(5) $\Rightarrow$ (1) Let M be any cotorsion right R-module. The above commutative diagram implies that $M = \mathrm{coker}(\alpha) = \mathrm{coker}(\alpha\varepsilon_K)$ is flat by (5), as required. $\square$

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