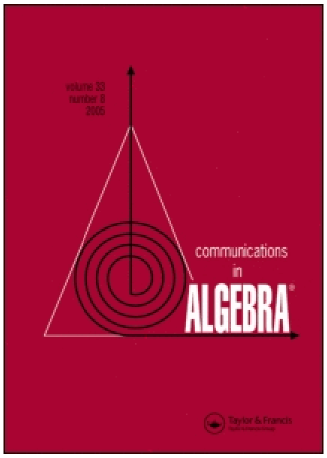

Communications in Algebra

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

ON (m, n)-INJECTIVITY OF MODULES

Jianlong Chen [a]; Nanqing Ding [b]; Yuanlin Li [c]; Yiqiang Zhou [c]
[a] Department of Mathematics, Harbin Institute of Technology. Harbin, 150001, 210096. P. R. China
[b] Department of Applied Mathematics, Southeast University. Nanjing. P. R. China
[c] Department of Mathematics, Nanjing University. Nanjing, 210093. P. R. China

First Published on: 31 December 2001
To link to this article: DOI: 10.1081/AGB-100107948
URL: http://dx.doi.org/10.1081/AGB-100107948

COMMUNICATIONS IN ALGEBRA, 29(12), 5589–5603 (2001)

ON (m, n)-INJECTIVITY OF MODULES

Jianlong Chen,[1] Nanqing Ding,[2] Yuanlin Li,[3] and Yiqiang Zhou[3]

[1]Department of Mathematics, Harbin Institute of Technology, Harbin 150001, P. R. China and Department of Applied Mathematics, Southeast University, Nanjing 210096, P. R. China
E-mail: jlchen@seu.edu.cn
[2]Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China
E-mail: nqding@netra.nju.edu.cn
[3]Memorial University of Newfoundland, St. John's, Newfoundland, Canada A1C 5S7
E-mail: yuanlin@math.mun.ca and zhou@math.mun.ca

ABSTRACT

Let R be a ring. For two fixed positive integers m and n, a right R-module M is called (m, n)-injective if every right R-homomorphism from an n-generated submodule of R^m to M extends to one from R^m to M. This definition unifies several definitions on generalizations of injectivity of modules. The aim of this paper is to investigate properties of the (m, n)-injective modules. Various results are developed, many extending known results.

1. INTRODUCTION

Throughout R is an associative ring with identity and all modules are unitary. We write M_R ($_RM$) to indicate a right (left) R-module, and we use the notation $R^{m\times n}$ for the set of all $m \times n$ matrices over R. For $A \in R^{m\times n}$, A^T will denote the transpose of A. In general, for an R-module N, we write $N^{m\times n}$ for the set of all formal $m \times n$ matrices whose entries are elements of N. Let M_R and $_RN$ be R-modules. For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in N^{n\times k}$, under the usual multiplication of matrices, xs (resp. sy) is a well-defined element in $M^{l\times n}$ (resp. $N^{m\times k}$). If $X \subseteq M^{l\times m}$, $S \subseteq R^{m\times n}$ and $Y \subseteq N^{n\times k}$, define

$$l_{M^{l\times m}}(S) = \{u \in M^{l\times m} : us = 0,\ \forall\, s \in S\}$$

$$r_{N^{n\times k}}(S) = \{v \in N^{n\times k} : sv = 0,\ \forall\, s \in S\}$$

$$r_{R^{m\times n}}(X) = \{s \in R^{m\times n} : xs = 0,\ \forall\, x \in X\}$$

$$l_{R^{m\times n}}(Y) = \{s \in R^{m\times n} : sy = 0,\ \forall\, y \in Y\}.$$

We will write $N^n = N^{1\times n}$, $N_n = N^{n\times 1}$, $R^n = R^{1\times n}$ and $R_n = R^{n\times 1}$. Multiplication maps $x \mapsto ax$ and $x \mapsto xa$ will be denoted $a\cdot$ and $\cdot a$, respectively.

Generalizations of injectivity have been discussed in many papers, for example, see [2, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15, 18]. In this paper, for two fixed positive integers m and n, (m,n)-injective modules are defined and studied. We prove that M_R is (m,n)-injective if and only if $l_{M^n}r_{R_n}\{\alpha_1,\alpha_2,\ldots,\alpha_m\} = M\alpha_1 + M\alpha_2 + \cdots + M\alpha_m$ for all $\alpha_i \in R^n$, $i = 1,2,\ldots,m$ [Theorem 2.4]. This fact is then used to prove that a left Kasch left $(n,m+1)$-injective ring R is right (m,n)-injective [Theorem 2.7]. The (m,n)-injective modules are also characterized as those $(m,1)$-injective modules M_R for which $l_{M^m}(I \cap K) = l_{M^m}(I) + l_{M^m}(K)$, where I and K are submodules of $(R_m)_R$ such that $I + K$ is n-generated [Theorem 2.9]. Any left Kasch, left P-injective and left IN-ring R is proved to be right f-injective and left dual (*i.e.,* every left ideal of R is a left annihilator) [Theorem 2.13]. Another characterization of (m,n)-injective modules is obtained as stated as follows: M_R is (m,n)-injective if and only if, for any $z = (m_1,m_2,\ldots,m_n) \in M^n$ and $A \in R^{m\times n}$ satisfying $r_{R_n}(A) \subseteq r_{R_n}(z)$, $z = yA$ for some $y \in M^m$ [Theorem 2.15]. We use this theorem to prove that R is right (m,n)-injective if and only if the exactness of $_RR^m \to {}_RR^n \to {}_RN \to 0$ implies the torsionlessness of N [Theorem 2.17] and that R is right (m,n)-injective and left (n,m)-injective if and only if R is right (m,n)-wlec and left (n,m)-wlec [Theorem 2.20]. Some known results appearing in [2, 6, 8, 10, 12, 14, 15] are obtained as corollaries of the main results of this paper.

2. RESULTS

In this section, m and n will be two fixed positive integers (unless specified otherwise). We start with the following.

Definition 2.1. *A right R-module M is called (m, n)-injective if every right R-homomorphism from an n-generated submodule of R^m (or R_m) to M extends to one from R^m (or R_m) to M. The ring R is a right (m, n)-injective ring if R_R is (m, n)-injective.*

It is easy to see that M_R is (m, n)-injective if and only if M_R is (m, k)-injective for all $1 \leq k \leq n$ if and only if M_R is (l, n)-injective for all $1 \leq l \leq m$ if and only if M_R is (l, k)-injective for all $1 \leq l \leq m$ and $1 \leq k \leq n$.

A module M_R is called n-injective if every right R-homomorphism from an n-generated right ideal to M extends to one from R_R to M, while M_R is f-injective [6] (= f.g.injective in [2] = Coflat in [5]) in case every right R-homomorphism from a finitely generated right ideal to M extends to one from R_R to M. We call M_R a P-injective module if every right R-homomorphism $aR \to M$, $a \in R$, extends to $R \to M$. A module M_R is FP-injective [8] in case, for every finitely generated submodule K of a free right R-module F, every homomorphism from K to M extends to one from F to M. The ring R is right n-injective (resp. f-injective, P-injective, FP-injective) in case R_R is n-injective (resp. f-injective, P-injective, FP-injective).

The next lemma is immediate.

Lemma 2.2. *Let M be a right R-module.*

1. *M is n-injective (resp. P-injective) if and only if M is (1, n)-injective (resp. (1, 1)-injective).*
2. *M is f-injective if and only if M is (1, n)-injective for all positive integers n.*
3. *M is FP-injective if and only if M is (m, n)-injective for all positive integers m and n if and only if M is (n, n)-injective for all positive integers n.*

Remark 2.3. The (m, n)-injective modules lie between P-injective modules and FP-injective modules. A right (m, n)-injective ring need not be left (m, n)-injective as shown by [3, Example 2]. Rutter ([17, Example 1]) has an example of right $(1, 1)$-injective which is not right $(1, 2)$-injective.

Let M be a right R-module and $\alpha = (r_1, r_2, \ldots, r_n) \in R^n$. In what follows, we write $M\alpha = \{x\alpha \mid x \in M\}$, where $x\alpha = (xr_1, xr_2, \ldots, xr_n) \in M^n$.

Theorem 2.4. *The following conditions are equivalent for a right R-module M:*

1. *M is (m, n)-injective.*
2. *$l_{M^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\} = M\alpha_1 + M\alpha_2 + \cdots + M\alpha_m$ for any m-element subset $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of R^n.*

Proof. (1) $\Rightarrow$ (2). Let $\alpha_i = (a_{1i}, a_{2i}, \ldots, a_{ni}) \in R^n$, $i = 1, 2, \ldots, m$. Suppose $x = (x_1, x_2, \ldots, x_n) \in l_{M^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Take $\beta_i = (a_{i1}, a_{i2}, \ldots, a_{im}) \in R^m$, $i = 1, 2, \ldots, n$, and define $g : \beta_1 R + \beta_2 R + \cdots + \beta_n R \to M$ such that

$$g\left(\sum_{i=1}^{n} \beta_i t_i\right) = \sum_{i=1}^{n} x_i t_i \quad \text{for} \quad t_i \in R,\ i = 1,\ 2,\ \ldots,\ n.$$

If $\sum_{i=1}^n \beta_i t_i = 0$, then $\sum_{i=1}^n a_{ij} t_i = 0$, $j = 1, 2, \ldots, m$. Let $\alpha = (t_1, t_2, \ldots, t_n) \in R^n$. Then $\alpha_j \alpha^T = 0$, $j = 1, 2, \ldots, m$, and so $\alpha^T \in r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Hence $\sum_{i=1}^n x_i t_i = 0$. This shows that g is well-defined. Since M is (m, n)-injective, g extends to a right R-homomorphism $\bar{g} : R^m \to M$. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^m$ (with 1 in the ith position and 0's in all other positions), $y_i = \bar{g}(e_i)$, $i = 1, 2, \ldots, m$, and $y = (y_1, y_2, \ldots, y_m) \in M^m$. Then, for any $u = (u_1, u_2, \ldots, u_m) \in R^m$, $\bar{g}(u) = y_1 u_1 + y_2 u_2 + y_m u_m = yu^T$. Thus $x_i = g(\beta_i) = \bar{g}(\beta_i) = y\beta_i^T = \sum_{j=1}^m y_j a_{ij}$, $i = 1, 2, \ldots, n$, and hence,

$$\begin{aligned} x = (x_1, x_2, \ldots, x_n) &= \left(\sum_{j=1}^{m} y_j a_{1j}, \sum_{j=1}^{m} y_j a_{2j}, \ldots, \sum_{j=1}^{m} y_j a_{nj}\right) \\ &= \sum_{j=1}^{m} y_j (a_{1j}, a_{2j}, \ldots, a_{nj}) = \sum_{j=1}^{m} y_j \alpha_j \in M\alpha_1 + M\alpha_2 + \cdots + M\alpha_m. \end{aligned}$$

So $l_{M^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subseteq M\alpha_1 + M\alpha_2 + \cdots + M\alpha_m$. The reverse inclusion is clear.

(2) $\Rightarrow$ (1). Let $N = \beta_1 R + \beta_2 R + \cdots + \beta_n R$ be an n-generated submodule of R^m and $f : N \to M$ a right R-homomorphism. Write $\beta_i = (a_{i1}, a_{i2}, \ldots, a_{im}) \in R^m$, $i = 1, 2, \ldots, n$, and $\alpha_j = (a_{1j}, a_{2j}, \ldots, a_{nj}) \in R^n$, $j = 1, 2, \ldots, m$. Let $u_i = f(\beta_i)$, $i = 1, 2, \ldots, n$, and $u = (u_1, u_2, \ldots, u_n)$. Then, for any $\xi = (t_1, t_2, \ldots, t_n)^T \in r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, we have $\alpha_j \xi = 0$, *i.e.*, $\sum_{i=1}^n a_{ij} t_i = 0$, $j = 1, 2, \ldots, m$. Thus $\sum_{i=1}^n (a_{i1}, a_{i2}, \ldots, a_{im}) t_i = 0$, *i.e.*, $\sum_{i=1}^n \beta_i t_i = 0$, and so $u\xi = \sum_{i=1}^n u_i t_i = \sum_{i=1}^n f(\beta_i) t_i = 0$, whence $u \in l_{M^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Therefore

$$u = (u_1, u_2, \ldots, u_n) \in M\alpha_1 + M\alpha_2 + \cdots + M\alpha_m$$

by (2). Let $(u_1, u_2, \ldots, u_n) = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_m\alpha_m$ for some $y_i \in M$, $i = 1, 2, \ldots, m$. Then

$$(u_1, u_2, \ldots, u_n) = \left(\sum_{j=1}^{m} y_j a_{1j}, \sum_{j=1}^{m} y_j a_{2j}, \ldots, \sum_{j=1}^{m} y_j a_{nj} \right),$$

and hence $u_i = \sum_{j=1}^{m} y_j a_{ij} = y\beta_i^T$, $i = 1, 2, \ldots, n$, where $y = (y_1, y_2, \ldots, y_m) \in M^m$. Now define $\bar{f}: R^m \to M$ such that $\bar{f}(x) = yx^T = \sum_{i=1}^{m} y_i x_i$ for each $x = (x_1, x_2, \ldots, x_m) \in R^m$. Then $\bar{f}(\beta_i) = y\beta_i^T = u_i = f(\beta_i)$, $i = 1, 2, \ldots, n$, and it follows that $\bar{f}$ is an extension of f. □

Corollary 2.5. *The following statements hold for a module M_R:*

1. *M_R is P-injective if and only if $l_M r_R(a) = Ma$ for all $a \in R$.*
2. *M_R is n-injective if and only if $l_{M^n} r_{R_n}(\alpha) = M\alpha$ for all $\alpha \in R^n$.*
3. *M_R is f-injective if and only if $l_{M^n} r_{R_n}(\alpha) = M\alpha$ for all $\alpha \in R^n$ and for all positive integers n.*
4. *M_R is $(m, 1)$-injective if and only if $l_M r_R(I) = MI$ for every m-generated left ideal I of R. In particular, R is right $(m, 1)$-injective if and only if every m-generated left ideal of R is a left annihilator.*

Remark 2.6. From Corollary 2.5 (4) we know that every finitely generated left ideal of R is a left annihilator if and only if R is right $(m, 1)$-injective for all positive integers m.

Recall that a ring R is left Kasch if every simple left R-module embeds in R.

Theorem 2.7. *Any left Kasch left $(n, m+1)$-injective ring R is right (m, n)-injective.*

Proof. By Theorem 2.4, it is sufficient to prove that $l_{R^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\} = R\alpha_1 + R\alpha_2 + \cdots + R\alpha_m$ for all $\alpha_i \in R^n$, $i = 1, 2, \ldots, m$. Clearly, $R\alpha_1 + R\alpha_2 + \cdots + R\alpha_m \subseteq l_{R^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Suppose $\beta \in l_{R^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, but $\beta \notin I = R\alpha_1 + R\alpha_2 + \cdots + R\alpha_m$. Since $(R\beta + I)/I$ is a nonzero finitely generated left R-module, it has a maximal submodule M/I. Hence $(R\beta + I)/M$ is a simple left R-module. Since R is left Kasch, let $\delta : (R\beta + I)/M \to_R R$ be an embedding, and define $f : R\beta + I \to_R R$ by $f(x) = \delta(x + M)$ for $x \in R\beta + I$. Clearly, $f(I) = 0$ and $f(\beta) \neq 0$. By hypothesis, f extends to a left R-homomorphism $\bar{f}: R^n \to_R R$. Thus there exists

$u = (u_1, u_2, \ldots, u_n) \in R^n$ such that $\bar{f}(x) = xu^T = x_1u_1 + x_2u_2 + \cdots + x_nu_n$ for any $x = (x_1, x_2, \ldots, x_n) \in R^n$. Therefore $0 = f(\alpha_i) = \bar{f}(\alpha_i) = \alpha_i u^T$, $i = 1, 2, \ldots, m$, and hence $u^T \in r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. But $\beta \in l_{R^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, and then $f(\beta) = \bar{f}(\beta) = \beta u^T = 0$. This is a contradiction, and the proof is complete. □

Corollary 2.8. *The following statements hold for a ring R:*

1. *([12, Theorem 3.1]). If R is left Kasch and left FP-injective, then R is right FP-injective.*
2. *([14, Lemma 2.2]). If R is left Kasch and left 2-injective, then R is right P-injective.*
3. *([2, Proposition 4.1]). Let R be left Kasch and left f-injective, then each finitely generated left ideal of R is a left annihilator.*
4. *If R is left Kasch and left (n, 2)-injective for all positive integers n, then R is right f-injective.*

Theorem 2.9. *The following conditions are equivalent for a module M_R:*

1. *M_R is (m, n)-injective.*
2. *M_R is (m,1)-injective and $l_{M^m}(I \cap K) = l_{M^m}(I) + l_{M^m}(K)$, where I and K are submodules of $(R_m)_R$ such that $I + K$ is n-generated.*
3. *M_R is $(m, 1)$-injective and $l_{M^m}(I \cap K) = l_{M^m}(I) + l_{M^m}(K)$, where I and K are submodules of $(R_m)_R$ such that I is cyclic and K is $(n-1)$-generated (if $n = 1$, $K = 0$).*

Proof. (1) ⇒ (2). Clearly, M_R is $(m, 1)$-injective and

$$l_{M^m}(I) + l_{M^m}(K) \subseteq l_{M^m}(I \cap K).$$

Conversely, let $x \in l_{M^m}(I \cap K)$, then $f : I + K \to M$ is well defined by $f(c + b) = xc$ for all $c \in I$ and $b \in K$, so $f = y\cdot$ for some $y = (y_1, y_2, \ldots, y_m) \in M^m$. Hence, for all $c \in I$ and $b \in K$, we have $yc = f(c) = xc$ and $yb = f(b) = 0$. Thus $x - y \in l_{M^m}(I)$ and $y \in l_{M^m}(K)$, so $x = (x - y) + y \in l_{M^m}(I) + l_{M^m}(K)$.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). We proceed by induction on n. Let $I = \alpha_1 R + \alpha_2 R + \cdots + \alpha_n R$ be an n-generated submodule of $(R_m)_R$, $I_1 = \alpha_1 R$ and $I_2 = \alpha_2 R + \cdots + \alpha_n R$. Suppose $f : I \to M$ is a right R-homomorphism. Then $f|_{I_1} = y_1\cdot$ by hypothesis and $f|_{I_2} = y_2\cdot$ by induction hypothesis for some $y_i \in M^m$, $i = 1, 2$. Thus $y_1 - y_2 \in l_{M^m}(I_1 \cap I_2) = l_{M^m}(I_1) + l_{M^m}(I_2)$, and so $y_1 - y_2 = z_1 + z_2$ for some $z_i \in l_{M^m}(I_i)$, $i = 1, 2$. Let $y = y_1 - z_1 = y_2 + z_2$. Then $f = y\cdot$. In fact, if $\alpha \in I = I_1 + I_2$, then $\alpha = \alpha_1 + \alpha_2$ with $\alpha_i \in I_i$, $i = 1, 2$,

and so $z_1\alpha_1 = 0$ and $z_2\alpha_2 = 0$. Hence $f(\alpha) = f(\alpha_1) + f(\alpha_2) = y_1\alpha_1 + y_2\alpha_2 = (y_1 - z_1)\alpha_1 + (y_2 + z_2)\alpha_2 = y\alpha_1 + y\alpha_2 = y(\alpha_1 + \alpha_2) = y\alpha$. So (1) follows. □

Corollary 2.10. *Let M be a right R-module.*

1. *The following conditions are equivalent:*
   (a) *M_R is n-injective.*
   (b) *M_R is P-injective and $l_M\ (I \cap K) = l_M(I) + l_M(K)$, where I and K are right ideals of R such that $I + K$ is n-generated.*
   (c) *M_R is P-injective and $l_M(I \cap K) = l_M(I) + l_M(K)$, where I is a principal right ideal of R and K is an $(n-1)$-generated right ideal of R.*
   In particular, M_R is 2-injective if and only if M_R is P-injective and $l_M(aR \cap bR) = l_M(a) + l_M(b)$ for all $a, b \in R$.
2. *([6, Theorem 2.1]). M_R is f-injective if and only if M_R is P-injective and $l_M(I \cap K) = l_M(I) + l_M(K)$ for each pair of finitely generated right ideals I and K of R.*
3. *M_R is $(m, 2)$-injective if and only if M_R is $(m, 1)$-injective and*

$$l_{M^m}(\alpha R \cap \beta R) = l_{M^m}(\alpha) + l_{M^m}(\beta)$$

   for $\alpha, \beta \in R_m$.
4. *M_R is FP-injective if and only if $l_M r_R(I) = MI$ for all finitely generated left ideals I of R and $l_{M^m}(H \cap K) = l_{M^m}(H) + l_{M^m}(K)$ for each pair of finitely generated submodules H and K of $(R_m)_R$ and for all positive integers m.*

In [8], Jain has shown that, if R is a right FP-injective ring, then every finitely generated left ideal is a left annihilator. This result can be improved as follows:

Corollary 2.11. *A ring R is right FP-injective if and only if every finitely generated left ideal is a left annihilator and $l_{R^m}(H \cap K) = l_{R^m}(H) + l_{R^m}(K)$ for each pair of finitely generated submodules H and K of $(R_m)_R$ and for all positive integers m.*

Recall that a ring R is called a left IN-ring [4] if $r_R(H \cap K) = r_R(H) + r_R(K)$ for all left ideals H and K of R. By [4, Example 16], an IN-ring need not be Kasch or P-injective. A ring R is called left simple-injective if every R-homomorphism with simple image from a left ideal of R to R is given by right multiplication by an element of R. We also recall the following conditions:

C1: Every nonzero left ideal is essential in a direct summand of R.

C2: Every left ideal that is isomorphic to a direct summand of R is itself a direct summand.

C3: If $Re \cap Rf = 0$, where e and f are idempotents in R, then $Re \oplus Rf$ is a direct summand of R.

A ring R is called left continuous if it satisfies $C1$ and $C2$, and R is called quasi-continuous if it satisfies $C1$ and $C3$.

By Corollary 2.10 (2), a left P-injective and left IN-ring is left f-injective. The proof of the next Lemma is essentially due to Hajarnavis and Norton [7, Proposition 5.2].

Lemma 2.12. *If R is a a left P-injective and left IN-ring, then R is left simple-injective and left continuous.*

Proof. Let I be a left ideal of R and $f: I \to_R R$ a homomorphism with simple image $f(I) = Ry$ for some $y \in R$. Choose $t \in I$ such that $f(t) = y$ and write $K = Kerf$. Then $I = Rt + K$. Since R is left P-injective, $f|_{Rt}$: $Rt \to_R R$ extends to ${}_R R$. Hence there exists $z \in R$ such that $f(x) = xz$ for all $x \in Rt$. Since $uz = f(u) = 0$ for all $u \in Rt \cap K$, $z \in r_R(Rt \cap K) = r_R(Rt) + r_R(K)$. Let $z = b + c$, where $b \in r_R(Rt)$ and $c \in r_R(K)$. For any $a \in I$, write $a = a_1 + a_2$, where $a_1 \in Rt$ and $a_2 \in K$. Then $a_1 b = 0 = a_2 c$, and so $f(a) = f(a_1) = a_1 z = a_1 c = ac$, i.e., $f = \cdot c$.

Since R is left P-injective, R satisfies $C2$-condition by [14, Theorem 1.2]. On the other hand, R is left quasi-continuous by [4, Theorem 5]. So R is left continuous. □

Theorem 2.13. *Let R be a left Kasch, left P-injective and left IN-ring. Then every left ideal of R is a left annihilator, and R is right f-injective.*

Proof. By Lemma 2.12 and [13, Lemma 4.2], every left ideal of R is a left annihilator, and in particular, R is right P-injective. By Corollary 2.10 (2), it is sufficient to prove that $l_R(H \cap K) = l_R(H) + l_R(K)$ for each pair of finitely generated right ideals H and K of R. In fact, since R is a left P-injective and left IN-ring, $H = r_R l_R(H)$ and $K = r_R l_R(K)$ by [9, Lemma 5]. Clearly, $l_R(H) + l_R(K) \subseteq l_R(H \cap K)$. Suppose $l_R(H) + l_R(K) \neq l_R(H \cap K)$. Choose $b \in l_R(H \cap K)$ but $b \notin L = l_R(H) + l_R(K)$. Then $(Rb + L)/L$ has a maximal submodule M/L, and so $(Rb + L)/M$ is simple. Let $\sigma : (Rb + L)/M \to_R R$ be monic (for R is left Kasch) and $f: Rb + L \to_R R$ be defined by $f(x) = \sigma(x + M)$ for $x \in Rb + L$. Then Im(f) is simple. Thus $f = \cdot c$ for some $c \in R$ since R is left simple-injective by Lemma 2.12, and so $bc = f(b) \neq 0$. But $Mc = f(M) = 0$, and hence $Lc = 0$. Therefore $c \in r_R(L) = r_R(l_R(H) + l_R(K)) = r_R l_R(H) \cap r_R l_R(K) = H \cap K$, and so $bc = 0$, a contradiction. □

Remark 2.14. We already know that a left P-injective and left IN-ring is left f-injective, and a left Kasch and left FP-injective ring is right FP-injective. But we wonder whether a left Kasch and left f-injective ring is right f-injeceive.

Theorem 2.15. *The following conditions are equivalent for a right R-module M.*

1. *M_R is (m, n)-injective.*
2. *If $z = (m_1, m_2, \ldots, m_n) \in M^n$ and $A \in R^{m\times n}$ satisfy $r_{R_n}(A) \subseteq r_{R_n}(z)$, then $z = yA$ for some $y \in M^m$.*

Proof. (1) $\Rightarrow$ (2). Let $z = (m_1, m_2, \ldots, m_n) \in M^n$ and $A = (a_{ij}) \in R^{m\times n}$. Put $\alpha_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in R^n$, then $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix}$. Let $u \in r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Then $\alpha_i u = 0$, $i = 1, 2, \ldots, m$, and hence $Au = 0$. Thus $u \in r_{R_n}(A) \subseteq r_{R_n}(z)$, and so $zu = 0$. It follows that

$$z \in l_{M^n} r_{R_n}\{\alpha_1, \alpha_2, \ldots, \alpha_m\} = M\alpha_1 + M\alpha_2 + \cdots + M\alpha_m$$

by Theorem 2.4. Therefore there exists $y_i \in M$, $i = 1, 2, \ldots, m$, such that $z = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_m\alpha_m = (y_1, y_2, \ldots, y_m)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = yA$, where $y = (y_1, y_2, \ldots, y_m) \in M^m$.

(2) $\Rightarrow$ (1). Let $N = \alpha_1 R + \alpha_2 R + \cdots + \alpha_n R$ be an n-generated submodule of R_R^m and $f : N \to M$ a right R-homomorphism. Put $A = (\alpha_1^T, \alpha_2^T, \ldots, \alpha_n^T) \in R^{m\times n}$, $m_i = f(\alpha_i)$, $i = 1, 2, \ldots, n$, and $z = (m_1, m_2, \ldots, m_n) \in M^n$. Let $u = (u_1, u_2, \ldots, u_n)^T \in r_{R_n}(A)$. Then $Au = 0$, *i.e.,* $\alpha_1^T u_1 + \alpha_2^T u_2 + \cdots + \alpha_n^T u_n = 0$. Thus $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$, and hence $zu = m_1 u_1 + m_2 u_2 + \cdots + m_n u_n = f(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) = 0$, *i.e.,* $u \in r_{R_n}(z)$. By hypothesis, there exists $y = (y_1, y_2, \ldots, y_m) \in M^m$ such that $z = yA = y(\alpha_1^T, \alpha_2^T, \ldots, \alpha_n^T)$, and then $m_i = y\alpha_i^T$, $i = 1, 2, \ldots, n$. Define $\bar{f} : R^m \to M$ such that $\bar{f}(\xi) = y\xi^T$ for $\xi \in R^m$. Then $\bar{f}(\alpha_i) = y\alpha_i^T = m_i = f(\alpha_i)$, $i = 1, 2, \ldots, n$. So $\bar{f}$ is an extension of f. $\square$

Corollary 2.16. *The following statements hold:*

1. *The following conditions are equivalent:*
   - (a) *R is right (n, n)-injective.*
   - (b) *If $z = (m_1, m_2, \ldots, m_n) \in R^n$ and $A \in R^{n\times n}$ satisfy $r_{R_n}(A) \subseteq r_{R_n}(z)$, then $z = yA$ for some $y \in R^n$.*

(c) $M_n(R)$ *is right P-injective.*
2. *([14, Theorem 4.2]). If $M_n(R)$ is right P-injective, then R is right n-injective.*

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from Theorem 2.15, and $(a) \Leftrightarrow (c)$ is by the remark following [15, Theorem 2.2]. (2) follows from (1) since the (n, n)-injectivity of M_R implies the n-injectivity of M_R. □

Theorem 2.17. *The following conditions are equivalent:*

1. *R is right (m, n)-injective.*
2. *If ${}_RR^m \to {}_RR^n \to {}_RN \to 0$ is exact, then N is torsionless.*

Proof. (1) $\Rightarrow$ (2). Let ${}_RR^m \xrightarrow{f} {}_RR^n \to {}_RN \to 0$ be exact. Then there exists $A \in M_{m\times n}(R)$ such that $f(z) = zA$ for $z \in {}_RR^m$, and so $\mathrm{Im}(f) = R^mA$, whence $N \cong R^n/(R^mA)$. We will show that $R^n/(R^mA)$ is torsionless. Let $0 \neq \bar{z} \in R^n/(R^mA)$, where $z = (z_1, z_2, \ldots, z_n) \in R^n \backslash (R^mA)$. By Theorem 2.15, $r_{R_n}(A) \nsubseteq r_{R_n}(z)$. Thus there exists $\alpha = (a_1, a_2, \ldots, a_n)^T \in R_n$ such that $A\alpha = 0$ but $z\alpha \neq 0$. Define $g : R^n/(R^mA) \to R$ such that $g(\bar{x}) = x\alpha$ for every $x \in R^n$. Clearly, g is well-defined, and $g(\bar{z}) = z\alpha \neq 0$. So $N \cong R^n/(R^mA)$ is torsionless.

(2) $\Rightarrow$ (1). Let $A \in R^{m\times n}$. Then $N = R^n/(R^mA)$ is torsionless by (2) because N is the cokernel of $f : {}_RR^m \to {}_RR^n$ defined by $f(x) = xA$. Let $z = (z_1, z_2, \ldots, z_n) \in R^n$. By Theorem 2.15, it is sufficient to show that, for $z \notin R^mA$, $r_{R_n}(A) \nsubseteq r_{R_n}(z)$. In fact, if $z \notin R^mA$, then $0 \neq \bar{z} \in R^n/(R^mA) = N$. Thus, there exists a left R-homomorphism $g : R^n/(R^mA) \to R$ such that $g(\bar{z}) \neq 0$ (for N is torsionless). Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$ (with 1 in the ith position and 0's in all other positions), $i = 1, 2, \ldots, n$, and $\alpha = (g(\bar{e}_1), g(\bar{e}_2), \ldots, g(\bar{e}_n)) \in R^n$. Then $0 \neq g(\bar{z}) = g(z_1\bar{e}_1 + z_2\bar{e}_2 + \cdots + z_n\bar{e}_n) = z\alpha^T$, *i.e.*, $\alpha^T \notin r_{R_n}(z)$.

On the other hand, let $\epsilon_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^m$ (with 1 in the jth position and 0's in all other positions), $j = 1, 2, \ldots, m$. Note that $g(\bar{x}) = x\alpha^T$ for $x \in R^n$. Thus $(\epsilon_jA)\alpha^T = g(\overline{\epsilon_jA}) = 0$ for $j = 1, 2, \ldots, m$, and hence $A\alpha^T = 0$, *i.e.*, $\alpha^T \in r_{R_n}(A)$. Therefore $r_{R_n}(A) \nsubseteq r_{R_n}(z)$, as required. □

Corollary 2.18. *The following statements hold for a ring R:*

1. *R is right n-injective if and only if the exactness of ${}_RR \to {}_RR^n \to {}_RN \to 0$ implies the torsionlessness of N.*
2. *The following conditions are equivalent:*
   (a) *R is right FP-injective.*
   (b) *Every finitely presented left R-module is torsionless.*
   (c) *For every positive integer n, the exactness of ${}_RR^n \to {}_RR^n \to {}_RN \to 0$ implies the torsionlessness of N.*

Remark 2.19. The equivalence of (a) and (b) in Corollary 2.18 (2) is due to S. Jain [8, Theorem 2.3].

Recall that a ring R is said to be right (m,n)-weakly linearly existentially closed (or (m,n)-wlec) [10] if every system of linear equations and a single linear inequation of the form

$$\begin{array}{ccccccccc} x_1a_{11} & + & x_2a_{12} & + & \cdots & + & x_ma_{1m} & = & b_1 \\ & \vdots & & & & & \vdots & & \\ x_1a_{n1} & + & x_2a_{n2} & + & \cdots & + & x_ma_{nm} & = & b_n \\ x_1a_{n+1,1} & + & x_2a_{n+1,2} & + & \cdots & + & x_ma_{n+1,m} & \neq & b_{n+1} \end{array}$$

which has a solution in some ring extension of R has a solution in R itself. A ring R is right weakly linearly existentially closed (or wlec) if R is right (m,n)-wlec for all positive integers m and n. Left (m,n)-wlec rings and left wlec rings can be defined similarly.

Let $X=(x_1,x_2,\ldots,x_m)$, $A=(a_{ij})^T\in R^{m\times n}$, $\gamma=(b_1,b_2,\ldots,b_n)\in R^n$ and $\alpha=(a_{n+1,1},a_{n+1,2},\ldots,a_{n+1,m})^T\in R_m$. The system above can be written in matrix form as

$$\begin{array}{ccc} XA & = & \gamma \\ X\alpha & \neq & b, \end{array}$$

where $b=b_{n+1}\in R$.

Theorem 2.20. *The ring R is right (m, n)-injective and left (n, m)-injective if and only if R is right (m, n)-wlec and left (n, m)-wlec.*

Proof. The proof is motivated by that of [10, Theorem 8].

"$\Rightarrow$". Let $A\in R^{m\times n}$, $X=(x_1,x_2,\ldots,x_m)$, $\alpha\in R_m$, $\gamma\in R^n$ and $b\in R$. If the system

$$\begin{array}{ccc} XA & = & \gamma \\ X\alpha & \neq & b, \end{array}$$

has a solution in the ring extension S of R, *i.e.,* there exists $X_0\in S^m$ such that $X_0A=\gamma$ and $X_0\alpha\neq b$. Since $X_0A=\gamma$, $r_{R_n}(A)\subseteq r_{R_n}(\gamma)$. By Theorem 2.15, there exists $\delta_0\in R^m$ such that $\gamma=\delta_0A$ (for R_R is (m,n)-injective). We claim that there exists $\sigma_1\in l_{R^m}(A)$ such that $(\delta_0+\sigma_1)\alpha\neq b$. Otherwise, $(\delta_0+\sigma)\alpha=b$ for all $\sigma\in l_{R^m}(A)$, and in particular, $\delta_0\alpha=b$. It follows that $\sigma\alpha=0$ for all $\sigma\in l_{R^m}(A)$, and hence $l_{R^m}(A)\subseteq l_{R^m}(\alpha)$. Therefore there exists

$\beta \in R_n$ such that $\alpha = A\beta$ by Theorem 2.15 (for ${}_RR$ is (n, m)-injective), and so $b = \delta_0\alpha = \delta_0(A\beta) = (\delta_0 A)\beta = \gamma\beta = (X_0 A)\beta = X_0(A\beta) = X_0\alpha$, a contradiction. Let $\delta_1 = \delta_0 + \sigma_1$. Then $\delta_1 \in R^m$ and $\delta_1 A = \delta_0 A = \gamma$ and $\delta_1\alpha \neq b$, *i.e.*, the system above has a solution in R. So R is right (m, n)-wlec. Similarly, R is left (n, m)-wlec.

"$\Leftarrow$". We shall show that R_R is (m, n)-injective. By Theorem 2.15, we have to show that if $\beta \in R^n$ and $A \in R^{m\times n}$ satisfy $r_{R_n}(A) \subseteq r_{R_n}(\beta)$, then $\beta = \xi A$ for some $\xi \in R^m$.

First, let E be an (R, R)-bimodule. Then we claim that $r_{E_n}(A) \subseteq r_{E_n}(\beta)$. Let

$$S = \left\{ \begin{pmatrix} a & 0 \\ x & a \end{pmatrix} \middle| a \in R,\ x \in E \right\}.$$

We now consider the map

$$a \to \hat{a} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

of R into S. It is clear that this is a monomorphism of the ring R into S. We shall identify R with its image in S, identifying a with $\hat{a}$. In this way we can regard S as a ring extension of R. Let $A = (a_{ij}) \in R^{m\times n} \subseteq S^{m\times n}$ and $\beta = (b_1, b_2, \ldots, b_n) \in R^n \subseteq S^n$. We write $\hat{A} = (\hat{a}_{ij}) \in S^{m\times n}$ and $\hat{\beta} = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n) \in S^n$. If $r_{S_n}(\hat{A}) \not\subseteq r_{S_n}(\hat{\beta})$, then there exists $u \in S_n$ such that $\hat{A}u = 0$ and $\hat{\beta}u \neq 0$. Note that A (resp. β) is identified with $\hat{A}$ (resp. $\hat{\beta}$). So the system

$$\begin{aligned} AX &= 0 \\ \beta X &\neq 0 \end{aligned}$$

has a solution in S. Since R is left (n, m)-wlec, the above system has a solution in R. Thus there exists $v \in R_n$ such that

$$\begin{aligned} Av &= 0 \\ \beta v &\neq 0, \end{aligned}$$

which contradicts $r_{R_n}(A) \subseteq r_{R_n}(\beta)$. So $r_{S_n}(\hat{A}) \subseteq r_{S_n}(\hat{\beta})$.

Now let $u = (u_1, u_2, \ldots, u_n)^T \in r_{E_n}(A)$, then $Au = 0$. Put $\bar{u}_i = \begin{pmatrix} 0 & 0 \\ u_i & 0 \end{pmatrix} \in S$, $i = 1, 2, \ldots, n$, and $\bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)^T$. It follows that $\hat{A}\bar{u} = 0$. Thus $\bar{u} \in r_{S_n}(\hat{A}) \subseteq r_{S_n}(\hat{\beta})$, and so $\hat{\beta}\bar{u} = 0$, whence $\beta u = 0$, *i.e.*, $u \in r_{E_n}(\beta)$. Therefore $r_{E_n}(A) \subseteq r_{E_n}(\beta)$.

Next let G be the $\mathbb{Z}$-injective envelope of the additive group of R and put $E = Hom_{\mathbb{Z}}(R, G)$. It is easy to see that E is an (R, R)-bimodule, E_R is injective and ${}_RE$ is faithful. Let $S = End_R(E_R)$, then E is a left S-module by defining $sx = s(x)$ for $s \in S$ and $x \in E$. For any $r \in R$, we define $\hat{r} \in S$ such that $\hat{r}(x) = rx$ for $x \in E_R$. It is easy to see that the map $r \to \hat{r}$ of R into S is a monomorphism. We shall now identify r with $\hat{r}$. Then R is identified with a subring of S. By the first part of the proof, $r_{E_n}(A) \subseteq r_{E_n}(\beta)$. Write $AE_n = \{A\gamma | \gamma \in E_n\} \subseteq E_m$ and define $f : AE_n \to E_R$ such that $f(A\gamma) = \beta\gamma$, then f is a right R-homomorphism. Since E_R is injective, f extends to $g : E_m \to E_R$. Let $\lambda_i : E_R \to E_m$ be the ith injection and $f_i = g\lambda_i$, then $f_i \in S$, $i = 1, 2, \ldots, m$. For any $\alpha = (a_1, a_2, \ldots, a_n)^T \in E_m$, $g(\alpha) = g(\lambda_1(a_1) + \lambda_2(a_2) + \cdots + \lambda_m(a_m)) = f_1(a_1) + f_2(a_2) + \cdots + f_m(a_m) = (f_1, f_2, \ldots, f_m)\alpha$. Since $g|_{AE_n} = f$, for any $\gamma \in E_n$, we have $\beta\gamma = f(A\gamma) = g(A\gamma) = (f_1, f_2, \ldots, f_m)A\gamma$. In particular, for any $x \in E$, let $\gamma_i = (0, \ldots, 0, x, 0 \ldots, 0)^T \in E_n$ (with x in the ith position and 0's in all other positions), $i = 1, 2, \ldots, n$. From $(f_1, f_2, \ldots, f_m)A\gamma_i = \beta\gamma_i$ we have $\sum_{j=1}^m f_j(a_{ji}x) = b_ix$, *i.e.*, $\sum_{j=1}^m f_j\hat{a}_{ji}(x) = \hat{b}_i(x)$ for all $x \in E$, and so $\sum_{j=1}^m f_j\hat{a}_{ji} = \hat{b}_i$, $i = 1, 2, \ldots, n$. Therefore $(f_1, f_2, \ldots, f_m)\hat{A} = \hat{\beta}$. Identifying A (resp. β) with $\hat{A}$ (resp. $\hat{\beta}$), we have that the system $YA = \beta$ has a solution in S. Choose $\alpha \in R_m$ and $b \in R$ such that

$$(f_1, f_2, \ldots, f_m)\hat{\alpha} \neq \hat{b}.$$

For example, take $\alpha = (1, 0, \ldots, 0)^T$, and

$$b = \begin{cases} 1, & \text{if } f_1 = 0 \\ 0, & \text{if } f_1 \neq 0. \end{cases}$$

Thus the system

$$\begin{array}{rcl} YA & = & \beta \\ Y\alpha & \neq & b \end{array}$$

has a solution in S, and hence it has a solution in R (for R is right (m, n)-wlec). Therefore there exists $\xi \in R^m$ such that $\beta = \xi A$, as required. So R_R is (m, n)-injective. Similarly, ${}_RR$ is (n, m)-injective. □

Corollary 2.21. *The following statements hold for a ring R:*

1. *R is left and right P-injective if and only if R is left and right (1, 1)-wlec.*

2. *R is right f-injective and every finitely generated right ideal of R is a right annihilator if and only if R is right (1, n)-wlec and left (n, 1)-wlec for all positive integers n.*
3. *The following conditions are equivalent:*
   (a) *R is left and right FP-injective.*
   (b) *R is left and right wlec.*
   (c) *R is left and right (n, n)-wlec for all positive integers n.*

Remark 2.22. The equivalence of (a) and (b) in Corollary 2.21 (3) is due to P. Menal and P. Vamos [10, Theorem 8].

ACKNOWLEDGMENTS

This work was carried out during a visit of the second author to Memorial University of Newfoundland, St. John's, Canada. It is the second author's pleasure to thank the Department of Mathematics and Statistics of Memorial University for its kind hospitality. The research was supported in part by the National Natural Science Foundation of China and the Natural Sciences and Engineering Research Council of Canada.

REFERENCES

1. Anderson, F.W.; Fuller, K.R. *Rings and Categories of Modules*; Springer-Verlag New York Inc.: New York, USA, 1974; 339 pp.
2. Björk, J.E. Rings Satisfying Certain Chain Conditions. J. Reine Angew. Math. **1970**, *245*, 63–73.
3. Colby, R.R. Rings Which Have Flat Injective Modules. J. Algebra **1975**, *35*, 239–252.
4. Camillo, V.; Nicholson, W.K.; Yousif, M.F. Ikeda-Nakayama Rings. J. Algebra **2000**, *226* (2), 1001–1010.
5. Damiano, R.F. Coflat Rings and Modules. Pacific J. Math. **1979**, *81* (1), 349–369.
6. Gupta, R.N. On f-Injective Modules and Semihereditary Rings. Proc. Nat. Inst. Sci. India Part A **1969**, *35*, 323–328.
7. Hajarnavis, C.R.; Norton, N.C. On Dual Rings and Their Modules. J. Algebra **1985**, *93*, 253–266.
8. Jain, S. Flat and FP-Injectivity. Proc. Amer. Math. Soc. **1973**, *41*, 437–442.
9. Johns, B. Annihilator Conditions in Noetherian Rings. J. Algebra **1977**, *49*, 222–224.

10. Menal, P.; Vamos, P. Pure Ring Extensions and Self FP-Injective Rings. Math. Proc. Camb. Phil. Soc. **1989**, *105*, 447–458.
11. Mohamed, S.H.; Müller, B.J. *Continuous and Discrete Modules*; Cambridge University Press: Cambridge, England, 1990; 126 pp.
12. Nicholson, W.K.; Yousif, M.F. On Quasi-Frobenius Rings. in *Proceedings of the Third Korea-China-Japan International Symposium on Ring Theory*; to appear.
13. Nicholson, W.K.; Yousif, M.F. Mininjective Rings. J. Algebra **1997**, *187* (2), 548–578.
14. Nicholson, W.K.; Yousif, M.F. Principally Injective Rings. J. Algebra **1995**, *174* (1), 77–93.
15. Nicholson, W.K.; Yousif, M.F. On a Theorem of Camillo. Comm. Algebra **1995**, *23* (14), 5309–5314.
16. Rotman, J.J. *An Introduction to Homological Algebra*; Academic Press: New York, USA, 1979; 376 pp.
17. Rutter, E.A. Rings with the Principal Extension Property. Comm. Algebra **1975**, *3* (3), 203–212.
18. Yousif, M.F. On continuous Rings. J. Algebra **1997**, *191* (2), 495–509.

Received May 2000
Revised November 2000