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ON DIVISIBLE AND TORSIONFREE MODULES

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A ring R is called left P-coherent in case each principal left ideal of R is finitely presented. A left R-module M (resp. right R-module N) is called D-injective (resp. D-flat) if $Ext^1(G, M) = 0$ (resp. $Tor_1(N, G) = 0$) for every divisible left R-module G. It is shown that every left R-module over a left P-coherent ring R has a divisible cover; a left R-module M is D-injective if and only if M is the kernel of a divisible precover $A \to B$ with A injective; a finitely presented right R-module L over a left P-coherent ring R is D-flat if and only if L is the cokernel of a torsionfree preenvelope $K \to F$ with F flat. We also study the divisible and torsionfree dimensions of modules and rings. As applications, some new characterizations of von Neumann regular rings and PP rings are given.

Key Words: *D*-flat module; *D*-injective module; Divisible module; *P*-coherent ring; (Pre)Cover; (Pre)Envelope; Torsionfree module; Warfield cotorsion module.

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1. INTRODUCTION

Let R be a ring. A left R-module M is said to be *divisible* (or *P-injective*) if $\mathrm{Ext}^1(R/Ra, M) = 0$ for all $a \in R$. A right R-module N is called *torsionfree* if $\mathrm{Tor}_1(N, R/Ra) = 0$ for all $a \in R$. The definitions of divisible and torsionfree modules coincide with the classical ones in case R is a commutative domain. It is clear that a right R-module N is torsionfree if and only if the character module N^+ is divisible by the standard isomorphism $\mathrm{Ext}^1(R/Ra, N^+) \cong \mathrm{Tor}_1(N, R/Ra)^+$ for every $a \in R$. These modules have been studied by many authors (see, for example, Couchot, 2006; Dauns and Fuchs, 2004; Enochs et al., 2001; Fuchs and Salce, 2001; Göbel and Trlifaj, 2006; Hattori, 1960; Lam, 1999; Lee, 2003; Mao and Ding, 2006; Nicholson and Yousif, 1995; Puninski et al., 1995; Shamsuddin, 2001; Xue, 1990; Yue Chi Ming, 1974; Zhang et al., 2005).

Let $\mathscr{D}$ ($\mathscr{TF}$) be the class of all divisible left R-modules (torsionfree right R-modules). In Section 2 of this article, we study the existence of $\mathscr{D}$-covers. To this aim, the concept of P-coherent rings is introduced as a generalization of

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coherent rings. Recall that R is called a *left coherent ring* if every finitely generated left ideal of R is finitely presented. We call R a *left P-coherent ring* if every principal left ideal of R is finitely presented. Some properties and characterizations of P-coherent rings are developed. It is shown that (1) R is a left coherent ring if and only if every $n \times n$ matrix ring $M_n(R)$ is a left P-coherent ring for every $n \geq 1$; (2) R is a left P-coherent ring if and only if any direct product of torsionfree right R-modules is torsionfree if and only if any direct limit of divisible left R-modules is divisible. Finally, we show that every left R-module over a left P-coherent ring R has a $\mathscr{D}$-cover.

In Section 3, divisible modules are used to define the concepts of D-injective and D-flat modules. It is shown that a left R-module M is D-injective if and only if M is the kernel of a $\mathscr{D}$-precover $A \to B$ with A injective. For a left P-coherent ring R, we prove that a left R-module M is D-injective if and only if M is a direct sum of an injective left R-module and a reduced D-injective left R-module; a finitely presented right R-module L is D-flat if and only if L is the cokernel of a $\mathscr{TF}$-preenvelope $K \to F$ with F flat.

Section 4 investigates the divisible and torsionfree dimensions of modules and rings. Suppose that R is a left P-coherent ring and n a nonnegative integer. It is shown that right $\mathscr{D}$-$\dim M \leq n$ for a left R-module M if and only if $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, \ker(F_{n-1} \to F_{n-2}))$ is an epimorphism for every left $\mathscr{D}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to N \to 0$ of every left R-module N; left $\mathscr{TF}$-$\dim G \leq n$ for a right R-module G if and only if $\mathrm{Hom}(F^n, G) \to \mathrm{Hom}(\mathrm{coker}(F^{n-2} \to F^{n-1}), G)$ is an epimorphism for every right $\mathscr{TF}$-resolution $0 \to H \to F^0 \to \cdots \to F^{n-1} \to F^n \to \cdots$ of every right R-module H. If R is a left strongly P-coherent ring, we get that gl right $\mathscr{D}$-$\dim {}_R\mathscr{M} =$ gl left $\mathscr{TF}$-$\dim \mathscr{M}_R =$ sup{projective (flat) dimensions of all cyclically presented left R-modules}.

Section 5 is devoted to some applications. It is shown that R is a von Neumann regular ring if and only if every Warfield cotorsion right R-module is injective (divisible) if and only if R is a left strongly P-coherent ring and every Warfield cotorsion right R-module is flat (torsionfree) if and only if every nonzero right R-module contains a nonzero torsionfree submodule. It is also shown that R is a left PP ring if and only if R is a left P-coherent ring and every D-injective left R-module is injective if and only if R is a left strongly P-coherent ring and gl right $\mathscr{D}$-$\dim {}_R\mathscr{M} \leq 1$ (gl left $\mathscr{TF}$-$\dim \mathscr{M}_R \leq 1$).

Let $\mathscr{C}$ be a class of R-modules and M an R-module. Recall that a homomorphism $\phi : C \to M$ is a *$\mathscr{C}$-precover of M* (Enochs, 1981) if $C \in \mathscr{C}$ and the abelian group homomorphism $\mathrm{Hom}(C', \phi) : \mathrm{Hom}(C', C) \to \mathrm{Hom}(C', M)$ is surjective for every $C' \in \mathscr{C}$. A $\mathscr{C}$-precover $\phi : C \to M$ is said to be a $\mathscr{C}$-cover of M if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. Dually we have the definitions of a $\mathscr{C}$-preenvelope and a $\mathscr{C}$-envelope. $\mathscr{C}$-covers ($\mathscr{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

Throughout this article, R is an associative ring with identity and all modules are unitary. $wD(R)$ stands for the weak global dimension of a ring R. ${}_R\mathscr{M}$ ($\mathscr{M}_R$) is the class of all left (right) R-modules. ${}_RM$ (M_R) denotes a left (right) R-module. For an R-module $M, E(M)$ stands for the injective envelope of $M, pd(M)$ denotes the projective dimension of M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+. For $a \in R$, the left annihilator of a in R is denoted by $l(a)$. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}_n^R(M, N)$) for an integer $n \geq 1$. For unexplained concepts and notations, we refer the reader to Enochs and Jenda (2000), Fuchs and Salce (2001), Göbel and Trlifaj (2006), Lam (1999), Rotman (1979), and Xu (1996).

2. *P*-COHERENT RINGS AND THE EXISTENCE OF DIVISIBLE COVERS

We begin with the following definition.

Definition 2.1. R is called a *left P-coherent ring* if every principal left ideal of R is finitely presented, or equivalently, $l(a)$ is finitely generated for every $a \in R$. The right version can be defined similarly.

The next example shows that the definition of P-coherent rings is not left-right symmetric.

Example 2.2. Let K be a field with a subfield L such that $\dim_L K = \infty$ and there exists a field isomorphism $\varphi : K \to L$ (for instance, $K = \mathbb{Q}(x_1, x_2, x_3, \ldots)$, $L = \mathbb{Q}(x_2, x_3, \ldots)$). Define a ring R by taking $R = K \times K$ with multiplication

$$(x, y)(x', y') = (xx', \varphi(x)y' + yx'), \qquad \text{where } x, y, x', y' \in K.$$

It is easy to see that R has exactly three right ideals: 0, R, and $(0, K) = (0, 1)R$. Thus R is a right P-coherent ring. On the other hand, let $a = (0, 1) \in R$, then $l(a)$ is not a finitely generated left ideal (see Lam, 1999, Example 4.46(e)). So R is not left P-coherent.

Coherent rings are obviously P-coherent. However, the converse is not true in general as shown by the following example.

Example 2.3. Let $x, y_1, y_2, \ldots$ be indeterminates over a field K, $S = K[x, y_i]$ and $R = K[x^2, x^3, y_i, xy_i]$. Then R is a subring of the commutative domain S. Hence R is also a commutative domain, and so R is a P-coherent ring. But R is not a coherent ring (see Glaz, 1989, p. 110).

Proposition 2.4. *A ring R is left coherent if and only if every $n \times n$ matrix ring $M_n(R)$ is left P-coherent for every $n \geq 1$.*

Proof. The necessity is clear since $M_n(R)$ is left coherent for every $n \geq 1$. Conversely, let $I = Ra_1 + Ra_2 + \cdots + Ra_n$ be a finitely generated left ideal of R. Put

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix} \in M_n(R).$$

Then

$$M_n(R)A = \begin{pmatrix} I & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ I & 0 & \cdots & 0 \end{pmatrix}$$

is a finitely presented left ideal of $M_n(R)$ by assumption. So there is a left $M_n(R)$-module exact sequence $0 \to K \to F \to M_n(R)A \to 0$, where F is a finitely generated free left $M_n(R)$-module and K is a finitely generated left $M_n(R)$-module. On the other hand, $M_n(R)$ is a free left R-module generated by the n^2 matrix units. Thus it is not difficult to verify that F is a finitely generated free left R-module and K is a finitely generated left R-module. Thus $M_n(R)A$ is a finitely presented left R-module, and hence I is a finitely presented left ideal of R since there exists a left R-isomorphism $M_n(R)A \cong I^n$. So R is a left coherent ring. □

Remark 2.5. It is well known that being left coherent is Morita invariant. But being left P-coherent is not Morita invariant by Proposition 2.4.

In what follows, we write $\mathscr{D}$ and $\mathscr{TF}$ for the classes of all divisible left R-modules and all torsionfree right R-modules, respectively.

It is easy to see that the classes $\mathscr{D}$ and $\mathscr{TF}$ are closed under extensions, direct sums, and direct summands. Moreover, we have the following lemma.

Lemma 2.6. *The classes $\mathscr{D}$ and $\mathscr{TF}$ are closed under pure submodules.*

Proof. Let N be a pure submodule of a divisible left R-module M. For any principal left ideal I of R, we have the exact sequence

$$\mathrm{Hom}(R/I, M) \to \mathrm{Hom}(R/I, M/N) \to \mathrm{Ext}^1(R/I, N) \to \mathrm{Ext}^1(R/I, M) = 0.$$

But $\mathrm{Hom}(R/I, M) \to \mathrm{Hom}(R/I, M/N) \to 0$ is exact since R/I is finitely presented and N is a pure submodule of M, so $\mathrm{Ext}^1(R/I, N) = 0$. Thus N is divisible.

Now, let N be a pure submodule of a torsionfree right R-module M, then the pure exact sequence $0 \to N \to M \to M/N \to 0$ induces the split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. Since M^+ is divisible, so is N^+. Thus N is torsionfree. □

Now we give some characterizations of left P-coherent rings.

Theorem 2.7. *The following are equivalent for a ring R:*

(1) *R is a left P-coherent ring*;
(2) *Any direct product of copies of R_R is torsionfree*;
(3) *Any direct product of torsionfree right R-modules is torsionfree*;
(4) *Any direct limit of divisible left R-modules is divisible*;

(5) *A left R-module M is divisible if and only if M^+ is torsionfree*;
(6) *Every right R-module has a $\mathcal{TF}$-preenvelope.*

Proof. (1) $\Rightarrow$ (5) Note that $\mathrm{Tor}_1(M^+, R/I) \cong \mathrm{Ext}^1(R/I, M)^+$ for every left R-module M and every principal left ideal I of R by Chen and Ding (1996, Lemma 2.7(2)) since I is finitely presented. So M is divisible if and only if M^+ is torsionfree.

(5) $\Rightarrow$ (3) Let $(M_i)_{i\in J}$ be a family of torsionfree right R-modules. Then $\Pi M_i^{++} \cong (\bigoplus M_i^+)^+$ is torsionfree by (5). Since ΠM_i is a pure submodule of ΠM_i^{++} by Cheatham and Stone (1981, Lemma 1(2)), ΠM_i is torsionfree by Lemma 2.6.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let I be a principal left ideal of R. Then $\mathrm{Tor}_1(\prod R, R/I) = 0$ by (2). Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\prod R)\otimes I & \longrightarrow & (\prod R)\otimes R & \longrightarrow & (\prod R)\otimes R/I & \longrightarrow & 0\\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\
0 & \longrightarrow & \prod I & \longrightarrow & \prod R & \longrightarrow & \prod R/I & \longrightarrow & 0.
\end{array}$$

Note that β and γ are isomorphisms by Enochs and Jenda (2000, Theorem 3.2.22) since R/I is finitely presented. Thus α is an isomorphism by the Five Lemma, and so I is finitely presented by Enochs and Jenda (2000, Theorem 3.2.22) again. Hence R is a left P-coherent ring.

(1) $\Rightarrow$ (4) For any principal left ideal I of R and any direct system $(M_i)_{i\in J}$ of divisible left R-modules, we have $\mathrm{Ext}^1(R/I, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}^1(R/I, M_i) = 0$ by Chen and Ding (1996, Lemma 2.9(2)) because I is finitely presented. So $\varinjlim M_i$ is divisible.

(4) $\Rightarrow$ (1) Let I be a principal left ideal of R and $(M_i)_{i\in J}$ a family of injective left R-modules, where J is a directed set. Then $\varinjlim M_i$ is divisible by (4), and so $\mathrm{Ext}^1(R/I, \varinjlim M_i) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}(R/I, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}(R, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}(I, \varinjlim M_i) & \longrightarrow & 0\\
{\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\
\varinjlim \mathrm{Hom}(R/I, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}(R, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}(I, M_i) & \longrightarrow & 0.
\end{array}$$

Since α and β are isomorphisms by Jones (1982, Proposition 2.5), γ is an isomorphism by the Five Lemma. So I is finitely presented by Jones (1982, Proposition 2.5) again. Hence R is a left P-coherent ring.

(1) $\Leftrightarrow$ (6) follows from Mao and Ding (2006, Theorem 3.1). $\square$

The following lemmas are needed to prove the existence of $\mathcal{D}$-covers.

Lemma 2.8 (Enochs and Jenda, 2000, Proposition 5.2.2). *If $\mathcal{F}$ is a class of R-modules closed under direct sums, then an R-module M has an $\mathcal{F}$-precover if and only* *if there is a cardinal number* $\aleph_\alpha$ *such that any homomorphism* $D \to M$ *with* $D \in \mathcal{F}$ *has a factorization* $D \to C \to M$ *with* $C \in \mathcal{F}$ *and* $Card(C) \leq \aleph_\alpha$.

Lemma 2.9 (Bican et al., 2001, Theorem 5). *Let* R *be an arbitrary ring. Then for each cardinal* λ, *there is a cardinal* κ *such that for any* R*-module* M *and any* $L \leq M$ *satisfying* $Card(M) \geq \kappa$ *and* $Card(M/L) \leq \lambda$, *the submodule* L *contains a nonzero submodule that is pure in* M.

We are now in a position to prove the following theorem.

Theorem 2.10. *Let* R *be a left* P*-coherent ring. Then every left* R*-module has a* $\mathcal{D}$*-cover. In particular, if* R *is a left coherent ring or a domain, then every left* R*-module has a* $\mathcal{D}$*-cover.*

Proof. Assume that N is a left R-module with $\mathrm{Card}(N) = \lambda$. We first prove that N has a $\mathcal{D}$-precover. Let κ be a cardinal as in Lemma 2.9. By Lemma 2.8, it suffices to show that any homomorphism $f : D \to N$ with D divisible has a factorization $D \to C \to N$ with C divisible and $\mathrm{Card}(C) \leq \kappa$.

If $\mathrm{Card}(D) \leq \kappa$, then we are done. Hence we may assume $\mathrm{Card}(D) > \kappa$.

Let $K = \ker(f)$. Note that $\mathrm{Card}(D/K) \leq \lambda$ since D/K embeds in N. Thus K contains a nonzero submodule D_0 which is pure in D by Lemma 2.9. The pure exact sequence $0 \to D_0 \to D \to D/D_0 \to 0$ induces the split exact sequence $0 \to (D/D_0)^+ \to D^+ \to D_0^+ \to 0$. Thus $(D/D_0)^+$ is torsionfree since D^+ is torsionfree by Theorem 2.7. So D/D_0 is divisible by Theorem 2.7 again.

If $\mathrm{Card}(D/D_0) \leq \kappa$, then we are done by Lemma 2.8 since f factors through D/D_0.

Suppose $\mathrm{Card}(D/D_0) > \kappa$. Put

$$\mathcal{S} = \{X : D_0 \leq X \leq K \text{ and } D/X \text{ is divisible}\}.$$

Then $\mathcal{S}$ is a nonempty set since $D_0 \in \mathcal{S}$. Let $\{X_i \in \mathcal{S} : i \in I\}$ be an ascending chain. Note that $D_0 \leq \bigcup X_i \leq K$ and $D/\bigcup X_i = D/\varinjlim X_i = \varinjlim(D/X_i)$ is divisible by Theorem 2.7 since each D/X_i is divisible. Thus $\bigcup X_i \in \mathcal{S}$, and so $\mathcal{S}$ has a maximal element C by Zorn's Lemma.

We claim that $\mathrm{Card}(D/C) \leq \kappa$. Suppose $\mathrm{Card}(D/C) > \kappa$. Since $C \subseteq K$, there exists $g : D/C \to N$ with $\ker(g) = K/C$. Note that $\mathrm{Card}((D/C)/(K/C)) = \mathrm{Card}(D/K) \leq \lambda$, and so K/C contains a nonzero submodule C_1/C which is pure in D/C by Lemma 2.9. Therefore $D/C_1 \cong (D/C)/(C_1/C)$ is divisible by the proof above, and hence $C_1 \in \mathcal{S}$, which contradicts the maximality of C.

It is clear that D/C is divisible and f factors through D/C. So N has a $\mathcal{D}$-precover by Lemma 2.8.

Note that $\mathcal{D}$ is closed under direct limits by Theorem 2.7. Thus N has a $\mathcal{D}$-cover by Enochs and Jenda (2000, Corollary 5.2.7). □

3. *D*-INJECTIVE MODULES AND *D*-FLAT MODULES

Definition 3.1. A left R-module M is called *D-injective* if $\mathrm{Ext}^1(G, M) = 0$ for every divisible left R-module G.

A right R-module N is said to be *D-flat* if $\mathrm{Tor}_1(N, G) = 0$ for every divisible left R-module G.

Remark 3.2. (1) By Wakamutsu's Lemma (see Xu, 1996, Lemma 2.1.1), any kernel of a $\mathscr{D}$-cover is D-injective.

(2) A right R-module M is D-flat if and only if M^+ is D-injective by the standard isomorphism $\mathrm{Ext}^1(N, M^+) \cong \mathrm{Tor}_1(M, N)^+$ for every divisible left R-module N.

(3) Recall that a left R-module M is called *FP-injective* (Stenström, 1970) if $\mathrm{Ext}^1(N, M) = 0$ for every finitely presented left R-module N. M is called *FI-injective* (Mao and Ding, 2007) (resp. *copure injective*, Enochs and Jenda, 1993) if $\mathrm{Ext}^1(G, M) = 0$ for every FP-injective (resp. injective) left R-module G. A right R-module N is said to be *FI-flat* (Mao and Ding, 2007) (resp. *copure flat*, Enochs and Jenda, 1993) if $\mathrm{Tor}_1(N, G) = 0$ for every FP-injective (resp. injective) left R-module G. Obviously, we have the following implications:

D-injective modules $\Rightarrow$ FI-injective modules $\Rightarrow$ copure injective modules;

D-flat modules $\Rightarrow$ FI-flat modules $\Rightarrow$ copure flat modules.

By the way, we note that FI-flat modules are exactly copure flat modules. In fact, let M be a copure flat right module and N be an FP-injective left R-module. Then there exists a pure exact sequence $0 \to N \to E \to L \to 0$ with E injective. Thus we get a split exact sequence $0 \to L^+ \to E^+ \to N^+ \to 0$, and so N^+ is isomorphic to a direct summand of E^+. Note that $\mathrm{Tor}_1(M, E) = 0$ since M is copure flat, and so $\mathrm{Ext}^1(M, E^+) \cong \mathrm{Tor}_1(M, E)^+ = 0$. Thus $\mathrm{Tor}_1(M, N)^+ \cong \mathrm{Ext}^1(M, N^+) = 0$, and hence $\mathrm{Tor}_1(M, N) = 0$. So M is FI-flat.

Proposition 3.3. *The following are equivalent for a left R-module M:*

(1) *M is D-injective;*
(2) *For every exact sequence $0 \to M \to E \to L \to 0$ with E divisible, $E \to L$ is a $\mathscr{D}$-precover of L;*
(3) *M is the kernel of a $\mathscr{D}$-precover $f : A \to B$ with A injective;*
(4) *M is injective with respect to every exact sequence $0 \to A \to B \to C \to 0$ with C divisible.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear by definitions.

(2) $\Rightarrow$ (3) is obvious since there exists a short exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$.

(3) $\Rightarrow$ (1) Let M be the kernel of a $\mathscr{D}$-precover $f : A \to B$ with A injective. Then we have an exact sequence $0 \to M \to A \to A/M \to 0$. So, for any divisible left R-module N, the sequence $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to \mathrm{Ext}^1(N, M) \to 0$ is exact. It is easy to verify that $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to 0$ is exact by (3). Thus $\mathrm{Ext}^1(N, M) = 0$, and so (1) follows.

(4) $\Rightarrow$ (1) For every divisible left R-module N, there exists a short exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact

sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1(N, M) \to 0$. Note that $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(N, M) = 0$, as desired. □

Recall that a left R-module M is called *reduced* (Enochs and Jenda, 2000) if M has no nonzero injective submodules.

Proposition 3.4. *Let R be a left P-coherent ring. Then the following are equivalent for a left R-module M:*

(1) *M is a reduced D-injective left R-module*;
(2) *M is the kernel of a $\mathscr{D}$-cover $f : A \to B$ with A injective.*

Proof. (1) $\Rightarrow$ (2) By Proposition 3.3, the natural map $\pi : E(M) \to E(M)/M$ is a $\mathscr{D}$-precover. Note that $E(M)/M$ has a $\mathscr{D}$-cover, and $E(M)$ has no nonzero direct summand K contained in M since M is reduced. It follows that $\pi : E(M) \to E(M)/M$ is a $\mathscr{D}$-cover by Xu (1996, Corollary 1.2.8), and hence (2) follows.

(2) $\Rightarrow$ (1) Let M be the kernel of a $\mathscr{D}$-cover $\alpha : A \to B$ with A injective. By Remark 3.2(1), M is D-injective. Now let K be an injective submodule of M. Suppose $A = K \oplus L$, $p : A \to L$ is the projection and $i : L \to A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$ since $\alpha(K) = 0$. Therefore ip is an isomorphism since α is a cover. Thus i is epic, and hence $A = L$, $K = 0$. So M is reduced. □

Recall that a left R-module exact sequence $0 \to A \to B \to C \to 0$ is said to be *RD-exact* ("*RD*" for "relatively divisible") if, for every $a \in R$, the sequence $\mathrm{Hom}(R/Ra, B) \to \mathrm{Hom}(R/Ra, C) \to 0$ is exact, or equivalently, the sequence $0 \to (R/aR) \otimes A \to (R/aR) \otimes B$ is exact (see Warfield, 1969, Proposition 2). So an R-module A is divisible if and only if any exact sequence $0 \to A \to B \to C \to 0$ is RD-exact. On the other hand, an R-module L is torsionfree if and only if any exact sequence $0 \to M \to N \to L \to 0$ is RD-exact.

An R-module M is called *RD-injective* if for every RD-exact sequence $0 \to A \to B \to C \to 0$, the sequence $\mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to 0$ is exact. Clearly, a divisible RD-injective module is injective.

Corollary 3.5. *Let R be a left P-coherent ring. Then every RD-injective left R-module M has a $\mathscr{D}$-cover $f : F \to M$ with F injective. Moreover,* $\ker(f)$ *is a reduced D-injective left R-module.*

Proof. By Theorem 2.10, M has a $\mathscr{D}$-cover $f : F \to M$. There is an exact sequence $0 \to F \xrightarrow{i} E \to L \to 0$ with E injective. Since the exact sequence is RD-exact, there exists $g : E \to M$ such that $gi = f$. So there exists $\varphi : E \to F$ such that $f\varphi = g$ since f is a cover. Therefore $f\varphi i = f$ and hence φi is an isomorphism. It follows that F is isomorphic to a direct summand of E, and so F is injective.

By Proposition 3.4, $\ker(f)$ is a reduced D-injective left R-module. □

Theorem 3.6. *Let R be a left P-coherent ring. Then a left R-module M is D-injective if and only if M is a direct sum of an injective left R-module and a reduced D-injective left R-module.*

Proof. "$\Leftarrow$" is clear.

"$\Rightarrow$" Let M be a D-injective left R-module. Since $0 \to M \to E(M) \to E(M)/M \to 0$ is exact, $E(M) \to E(M)/M$ is a $\mathscr{D}$-precover of $E(M)/M$ by Proposition 3.3. Note that $E(M)/M$ has a $\mathscr{D}$-cover $L \to E(M)/M$ by Theorem 2.10, so we have the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "\phi"'] & L \arrow[r] \arrow[d, "\gamma"'] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"'] \arrow[d, "\sigma"'] & E(M) \arrow[r] \arrow[d, "\beta"'] & E(M)/M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, "f"] & L \arrow[r] & E(M)/M \arrow[r] & 0,
\end{tikzcd}$$

where K is a reduced D-injective left R-module by Proposition 3.4. Note that $\beta\gamma$ is an isomorphism, and so $E(M) \cong \ker(\beta) \oplus \operatorname{im}(\gamma)$. Thus L and $\ker(\beta)$ are injective (for $\operatorname{im}(\gamma) \cong L$). Since $\sigma\phi$ is an isomorphism by the Five Lemma, $M = \ker(\sigma) \oplus \operatorname{im}(\phi)$, where $\operatorname{im}(\phi) \cong K$. By the Snake Lemma (Rotman, 1979, Theorem 6.5), $\ker(\sigma) \cong \ker(\beta)$. This completes the proof. $\square$

Proposition 3.7. *Let R be a ring.*

(1) *If M is a finitely presented D-flat right R-module, then M is the cokernel of a $\mathscr{TF}$-preenvelope $f: K \to F$ with F flat.*
(2) *If R is a left P-coherent ring and L is the cokernel of a $\mathscr{TF}$-preenvelope $f: K \to F$ with F flat, then L is D-flat.*

Proof. (1) Let M be a finitely presented D-flat right R-module. There is an exact sequence $0 \to K \to F \to M \to 0$ with F projective and both F and K finitely generated. We claim that $K \to F$ is a $\mathscr{TF}$-preenvelope. In fact, for any torsionfree right R-module Q, we have $\operatorname{Tor}_1(M, Q^+) = 0$, and so we get the following commutative diagram with the first row exact:

$$\begin{tikzcd}
0 \arrow[r] & K \otimes Q^+ \arrow[r, "\alpha"] \arrow[d, "\tau_1"'] & F \otimes Q^+ \arrow[d, "\tau_2"'] \\
 & \operatorname{Hom}(K, Q)^+ \arrow[r, "\theta"] & \operatorname{Hom}(F, Q)^+.
\end{tikzcd}$$

By Colby (1975, Lemma 2), τ_1 is an epimorphism and τ_2 is an isomorphism. Thus θ is a monomorphism, and hence $\operatorname{Hom}(F, Q) \to \operatorname{Hom}(K, Q)$ is epic, as required.

(2) There is an exact sequence $0 \to \operatorname{im}(f) \xrightarrow{i} F \to L \to 0$. It is clear that $i: \operatorname{im}(f) \to F$ is a $\mathscr{TF}$-preenvelope. For any divisible left R-module N, N^+ is torsionfree by Theorem 2.7. Thus we obtain an exact sequence $\operatorname{Hom}(F, N^+) \to \operatorname{Hom}(\operatorname{im}(f), N^+) \to 0$, which yields the exactness of $(F \otimes N)^+ \to (\operatorname{im}(f) \otimes N)^+ \to 0$.

So the sequence $0 \to \operatorname{im}(f) \otimes N \to F \otimes N$ is exact. Thus the exactness of $0 \to \operatorname{Tor}_1(L, N) \to \operatorname{im}(f) \otimes N \to F \otimes N$ implies $\operatorname{Tor}_1(L, N) = 0$. □

4. DIVISIBLE AND TORSIONFREE DIMENSIONS

Since every left R-module over a left P-coherent ring R has a $\mathscr{D}$-cover by Theorem 2.10, every left R-module M has a *left $\mathscr{D}$-resolution*, that is, there is a $\operatorname{Hom}(\mathscr{D}, -)$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ (not necessarily exact) with each F_i divisible. On the other hand, every left R-module M over any ring R has a $\mathscr{D}$-preenvelope (see Göbel and Trlifaj, 2006). So M has a *right $\mathscr{D}$-resolution*, that is, there is a $\operatorname{Hom}(-, \mathscr{D})$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ with each F^i divisible. Obviously, this complex is exact.

Following Enochs and Jenda (2000, Definition 8.4.1), the *right* $\mathscr{D}$-dimension of a left R-module M, denoted by right $\mathscr{D}$-dim M, is defined as $\inf\{n$: there is a right $\mathscr{D}$-resolution of M of the form $0 \to M \to F^0 \to F^1 \to \cdots \to F^n \to 0\}$. If there is no such n, set right $\mathscr{D}$-dim $M = \infty$. The *global right* $\mathscr{D}$-dimension of ${}_R\mathscr{M}$, denoted by gl right $\mathscr{D}$-dim ${}_R\mathscr{M}$, is defined to be sup{right $\mathscr{D}$-dim M: $M \in {}_R\mathscr{M}$} and is infinite otherwise. The left versions can be defined similarly.

If R is a left P-coherent ring, then $\operatorname{Hom}(-, -)$ is left balanced on ${}_R\mathscr{M} \times {}_R\mathscr{M}$ by $\mathscr{D} \times \mathscr{D}$ (see Enochs and Jenda, 2000, Definition 8.2.13). Let $\operatorname{Ext}_n(-, -)$ denote the nth left derived functor of $\operatorname{Hom}(-, -)$ with respect to $\mathscr{D} \times \mathscr{D}$. Then, for two left R-modules M and N, $\operatorname{Ext}_n(M, N)$ can be computed using a right $\mathscr{D}$-resolution of M or a left $\mathscr{D}$-resolution of N.

Let $0 \to M \xrightarrow{g} F^0 \xrightarrow{f} F^1 \to \cdots$ be a right $\mathscr{D}$-resolution of M. Applying $\operatorname{Hom}(-, N)$, we obtain the deleted complex $\cdots \to \operatorname{Hom}(F^1, N) \xrightarrow{f^*} \operatorname{Hom}(F^0, N) \to 0$. Then $\operatorname{Ext}_n(M, N)$ is exactly the nth homology of the complex above. There is a canonical map

$$\sigma : \operatorname{Ext}_0(M, N) = \operatorname{Hom}(F^0, N)/\operatorname{im}(f^*) \to \operatorname{Hom}(M, N)$$

defined by $\sigma(\alpha + \operatorname{im}(f^*)) = \alpha g$ for $\alpha \in \operatorname{Hom}(F^0, N)$.

Proposition 4.1. *Let R be a left P-coherent ring. The following are equivalent for a left R-module M:*

(1) *M is divisible*;
(2) *The canonical map $\sigma : \operatorname{Ext}_0(M, N) \to \operatorname{Hom}(M, N)$ is an isomorphism* (*epimorphism*) *for every left R-module N*;
(3) *The canonical map $\sigma : \operatorname{Ext}_0(M, M) \to \operatorname{Hom}(M, M)$ is an isomorphism* (*epimorphism*).

Proof. (1) ⇒ (2) is obvious by letting $F^0 = M$.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) By (3), there exists $\alpha \in \operatorname{Hom}(F^0, M)$ such that $\sigma(\alpha + \operatorname{im}(f^*)) = \alpha g = 1_M$. Thus M is isomorphic to a direct summand of F^0, and hence it is divisible. □

Recall that a right R-module M is *cyclically presented* if $M \cong R/rR$ for some $r \in R$. In what follows, we will call R a *left divisible ring* if ${}_RR$ is divisible.

Proposition 4.2. *The following are equivalent for a left P-coherent ring R:*

(1) *R is a left divisible ring;*
(2) *The canonical map $\sigma : \mathrm{Ext}_0({}_RR, N) \to \mathrm{Hom}({}_RR, N)$ is an isomorphism (epimorphism) for every left R-module N;*
(3) *The canonical map $\sigma : \mathrm{Ext}_0({}_RR, {}_RR) \to \mathrm{Hom}({}_RR, {}_RR)$ is an isomorphism (epimorphism);*
(4) *Every left R-module has an epic $\mathscr{D}$-cover;*
(5) *Every right R-module is a submodule of a torsionfree right R-module;*
(6) *Every injective right R-module is torsionfree;*
(7) *Every cyclically presented right R-module embeds in a free right R-module.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Proposition 4.1.

(1) $\Rightarrow$ (4) Let M be a left R-module, then M has a $\mathscr{D}$-cover g. On the other hand, there is an exact sequence $F \to M \to 0$ with F free. Since F is divisible by (1), g is an epimorphism.

(4) $\Rightarrow$ (1) Let $f : N \to {}_RR$ be an epic $\mathscr{D}$-cover. Then ${}_RR$ is isomorphic to a direct summand of N, and so ${}_RR$ is divisible.

(1) $\Rightarrow$ (5) Let N be a right R-module. Then N embeds in $\Pi({}_RR)^+$. Note that $\Pi({}_RR)^+$ is torsionfree by (1) and Theorem 2.7, and so (5) follows.

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (7) Let N be a cyclically presented right R-module. Then N embeds in a torsionfree right R-module since $E(N)$ is torsionfree by (6). So N embeds in a free right R-module by Zhang et al. (2005, Theorem 4.3).

(7) $\Rightarrow$ (1) Let N be a cyclically presented right R-module. Then there is a monomorphism $i : N \to F$ with F a free right R-module by (7). For any $f : N \to ({}_RR)^+$, since $({}_RR)^+$ is injective, there exists $g : F \to ({}_RR)^+$ such that $f = gi$. Thus $({}_RR)^+$ is torsionfree by Zhang et al. (2005, Theorem 4.3), and so ${}_RR$ is divisible by Theorem 2.7. □

Proposition 4.3. *Let R be a left P-coherent ring. Then the following are equivalent for a left R-module M:*

(1) *right $\mathscr{D}$-dim $M \leq 1$;*
(2) *The canonical map $\sigma : \mathrm{Ext}_0(M, N) \to \mathrm{Hom}(M, N)$ is a monomorphism for every left R-module N.*

Proof. (1) $\Rightarrow$ (2) By (1), M has a right $\mathscr{D}$-resolution $0 \to M \to F^0 \to F^1 \to 0$. Thus we get an exact sequence $0 \to \mathrm{Hom}(F^1, N) \to \mathrm{Hom}(F^0, N) \to \mathrm{Hom}(M, N)$ for every left R-module N. Hence σ is a monomorphism.

(2) $\Rightarrow$ (1) Consider the exact sequence $0 \to M \to F^0 \to L^1 \to 0$, where $M \to F^0$ is a $\mathscr{D}$-preenvelope. We only need to show that L^1 is divisible. By Enochs

and Jenda (2000, Theorem 8.2.3), we have the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & \mathrm{Ext}_0(L^1, L^1) & \longrightarrow & \mathrm{Ext}_0(F^0, L^1) & \longrightarrow & \mathrm{Ext}_0(M, L^1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_2} & & \downarrow{\scriptstyle \sigma_3} & & \\ 0 & \longrightarrow & \mathrm{Hom}(L^1, L^1) & \longrightarrow & \mathrm{Hom}(F^0, L^1) & \longrightarrow & \mathrm{Hom}(M, L^1). & & \end{array}$$

Note that σ_2 is an epimorphism by Proposition 4.1 and σ_3 is a monomorphism by (2). Hence σ_1 is an epimorphism by the Snake Lemma. Thus L^1 is divisible by Proposition 4.1, and so (1) follows. □

Proposition 4.4. *Let R be a left P-coherent ring. Then the following are equivalent for an integer $n \geq 2$:*

(1) *gl right $\mathscr{D}$-dim$_R\mathscr{M} \leq n$;*
(2) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M and N, and all $k \geq -1$;*
(3) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M and N.*

Proof. (1) $\Rightarrow$ (2) Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^n \to 0$ be a right $\mathscr{D}$-resolution of a left R-module M, which induces an exact sequence

$$0 \to \mathrm{Hom}(F^n, N) \to \mathrm{Hom}(F^{n-1}, N) \to \mathrm{Hom}(F^{n-2}, N)$$

for every left R-module N. Hence $\mathrm{Ext}_n(M, N) = \mathrm{Ext}_{n-1}(M, N) = 0$. It is clear that $\mathrm{Ext}_{n+k}(M, N) = 0$ for all $k \geq 1$. So (2) holds.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Suppose that $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-2} \xrightarrow{f} F^{n-1}$ is a partial right $\mathscr{D}$-resolution of M with $L^n = \mathrm{coker}(F^{n-2} \to F^{n-1})$. We only need to show that L^n is divisible. Let $\pi : F^{n-1} \to L^n$ be the canonical projection, $\lambda : L^n \to F^n$ be a $\mathscr{D}$-preenvelope and $g = \lambda\pi$. By (3), $\mathrm{Ext}_{n-1}(M, L^n) = 0$. Thus the sequence

$$\mathrm{Hom}(F^n, L^n) \xrightarrow{g^*} \mathrm{Hom}(F^{n-1}, L^n) \xrightarrow{f^*} \mathrm{Hom}(F^{n-2}, L^n)$$

is exact. Since $f^*(\pi) = \pi f = 0$, $\pi \in \ker(f^*) = \mathrm{im}(g^*)$. Thus there exists $h \in \mathrm{Hom}(F^n, L^n)$ such that $\pi = g^*(h) = hg = h\lambda\pi$, and hence $h\lambda = 1$ since π is epic. Therefore L^n is divisible. □

Theorem 4.5. *Let R be a left P-coherent ring, M a left R-module, and $n \geq 0$ an integer. Then right $\mathscr{D}$-dim$M \leq n$ if and only if for every left $\mathscr{D}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of every left R-module N, $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism, where $K_n = \ker(F_{n-1} \to F_{n-2})$.*

Proof. We proceed by induction on n. Let $n = 0$. If M is divisible, it is clear that $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, K_0)$ is an epimorphism. Conversely, put $N = M$. Then $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, M)$ is an epimorphism, and so M is divisible.

Let $n \geq 1$. By Göbel and Trlifaj (2006, Theorem 4.1.6(a)), there is an exact sequence $0 \to M \to E \to L \to 0$ with E divisible and $\mathrm{Ext}^1(L, G) = 0$ for all divisible left R-modules G. Then we have the following exact commutative diagrams:

$$\begin{array}{ccccc}
\mathrm{Hom}(E, F_n) & \longrightarrow & \mathrm{Hom}(E, K_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(M, F_n) & \longrightarrow & \mathrm{Hom}(M, K_n) & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(L, K_n) & \longrightarrow & \mathrm{Hom}(L, F_{n-1}) & \longrightarrow & \mathrm{Hom}(L, K_{n-1}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(E, K_n) & \longrightarrow & \mathrm{Hom}(E, F_{n-1}) & \longrightarrow & \mathrm{Hom}(E, K_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(M, K_n) & \longrightarrow & \mathrm{Hom}(M, F_{n-1}) & \longrightarrow & \mathrm{Hom}(M, K_{n-1}) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where $K_{n-1} = \ker(F_{n-2} \to F_{n-3})$. Then right $\mathscr{D}$-dim $M \leq n$ if and only if right $\mathscr{D}$-dim $L \leq n-1$ if and only if $\mathrm{Hom}(L, F_{n-1}) \to \mathrm{Hom}(L, K_{n-1})$ is epic by induction if and only if $\mathrm{Hom}(E, K_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the first diagram. □

Corollary 4.6. *The following are equivalent for a left P-coherent ring R and an integer $n \geq 0$:*

(1) *right $\mathscr{D}$-dim ${}_RR \leq n$;*
(2) *Every left $\mathscr{D}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of every left R-module N is exact at F_i for every $i \geq n-1$.*

Proof. (1) ⇒ (2) By Theorem 4.5, $\mathrm{Hom}({}_RR, F_n) \to \mathrm{Hom}({}_RR, K_n)$ is an epimorphism. So $F_n \to K_n$ is an epimorphism. It follows that $F_n \to F_{n-1} \to F_{n-2}$ is exact. In addition, right $\mathscr{D}$-dim ${}_RR \leq k$ for every $k \geq n+1$ by (1). So $F_k \to F_{k-1} \to F_{k-2}$ is exact, and hence (2) holds.

(2) ⇒ (1) holds by Theorem 4.5. □

Note that every right R-module over a left P-coherent ring R has a $\mathcal{TF}$-preenvelope by Theorem 2.7, so every right R-module M has a *right* $\mathcal{TF}$*-resolution*, that is, there is a $\mathrm{Hom}(-, \mathcal{TF})$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ (not necessarily exact) with each F^i torsionfree. On the other hand, every right R-module M over any ring R has a torsionfree cover (see Göbel and Trlifaj, 2006). So M has a *left* $\mathcal{TF}$*-resolution*, that is, there is a $\mathrm{Hom}(\mathcal{TF}, -)$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ with each F_i torsionfree. Obviously, this complex is exact. The *left* $\mathcal{TF}$*-dimension of a right R-module M*, denoted by left $\mathcal{TF}$-dim M, is defined as $\inf\{n$: there is a left $\mathcal{TF}$-resolution of M of the form $0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0\}$. If no such n exists, set left $\mathcal{TF}$-dim $M = \infty$. The *global left* $\mathcal{TF}$*-dimension of* $\mathcal{M}_R$, denoted by gl left $\mathcal{TF}$-dim $\mathcal{M}_R$, is defined to be $\sup\{$left $\mathcal{TF}$-dim M: $M \in \mathcal{M}_R\}$ and is infinite otherwise.

Recall that a right R-module C is called *Warfield cotorsion* (Fuchs and Salce, 2001; Göbel and Trlifaj, 2006) provided that $\mathrm{Ext}^1(F, C) = 0$ for every torsionfree right R-module F. Clearly, any RD-injective module is Warfield cotorsion.

Theorem 4.7. *Let R be a left P-coherent ring, G a right R-module, and $n \geq 0$ an integer. Then left $\mathcal{TF}$-dim $G \leq n$ if and only if for every right $\mathcal{TF}$-resolution $0 \to H \to F^0 \to F^1 \to \cdots F^{n-1} \to F^n \to \cdots$ of every right R-module H,* $\mathrm{Hom}(F^n, G) \to \mathrm{Hom}(L^n, G)$ *is an epimorphism, where* $L^n = \mathrm{coker}(F^{n-2} \to F^{n-1})$.

Proof. By Göbel and Trlifaj (2006, Theorem 4.1.1(b)) and Wakamutsu's Lemma, for any right R-module G, there is an exact sequence $0 \to K \to T \to G \to 0$ with T torsionfree and K Warfield cotorsion. So the result holds by the proof dual to that of Theorem 4.5. □

Corollary 4.8. *The following are equivalent for a left P-coherent ring R and an integer $n \geq 0$:*

(1) *left $\mathcal{TF}$-dim $({}_RR)^+ \leq n$;*
(2) *Every right $\mathcal{TF}$-resolution $0 \to N \to F^0 \to F^1 \to \cdots F^{n-1} \to F^n \to \cdots$ of every right R-module N is exact at F^i for every $i \geq n-1$.*

Proof. The proof is dual to that of Corollary 4.6 by Theorem 4.7. □

Lemma 4.9. *The following are equivalent for a ring R:*

(1) *For every left R-module exact sequence $0 \to A \to B \to C \to 0$ with A and B divisible, C is divisible;*
(2) *R is left P-coherent and if $0 \to N \to M \to Q \to 0$ is an exact sequence of right R-modules with M and Q torsionfree, N is torsionfree;*
(3) $\mathrm{Ext}^i(R/Ra, M) = 0$ *for every divisible left R-module M, every $a \in R$ and every $i \geq 1$;*
(4) *R is left P-coherent and* $\mathrm{Tor}_i(N, R/Ra) = 0$ *for every torsionfree right R-module N, every $a \in R$ and every $i \geq 1$.*

Proof. (3) $\Rightarrow$ (1) is easy.

(1) $\Rightarrow$ (3) Let M be a divisible left R-module. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective, and so L is divisible by (1). Thus $\mathrm{Ext}^2(R/Ra, M) \cong \mathrm{Ext}^1(R/Ra, L) = 0$ for every $a \in R$, and hence (3) holds by induction.

(2) $\Leftrightarrow$ (4) The proof is dual to that of (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) Let I be a principal left ideal of R and N an FP-injective left R-module. Then $\mathrm{Ext}^2(R/I, N) = 0$ since (1) $\Leftrightarrow$ (3). It follows that $\mathrm{Ext}^1(I, N) \cong \mathrm{Ext}^2(R/I, N) = 0$. Thus I is finitely presented by Enochs (1976). So R is left P-coherent.

Now let $0 \to N \to M \to Q \to 0$ be an exact sequence of right R-modules with M and Q torsionfree. Then we get an exact sequence $0 \to Q^+ \to M^+ \to N^+ \to 0$. Since Q^+ and M^+ are divisible, so is N^+ by (1). Thus N is torsionfree.

(2) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules with A and B divisible. Then we get an exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Note that A^+ and B^+ are torsionfree by Theorem 2.7. Thus C^+ is torsionfree by (2), and so C is divisible, as desired. □

We will call R a *left strongly P-coherent ring* if every principal left ideal of R is cyclically presented. Examples of such rings include not only von Neumann regular rings, but also left morphic rings (a ring R is called *left morphic* by Nicholson and Sánchez Campos, 2004, if $l(a) \cong R/Ra$ for every $a \in R$) as well as left generalized morphic rings (a ring R is called *left generalized morphic* by Zhu and Ding, 2007, if, for every $a \in R$, there is $b \in R$ with $l(a) \cong R/Rb$).

Lemma 4.10. *Let R be a left strongly P-coherent ring. Then R satisfies the equivalent conditions of Lemma* 4.9.

Proof. Let M be a divisible left R-module and $a \in R$, then Ra is cyclically presented. Thus $\mathrm{Ext}^1(Ra, M) = 0$, and hence $\mathrm{Ext}^2(R/Ra, M) = 0$. So $\mathrm{Ext}^i(R/Ra, M) = 0$ for every $i \geq 2$ by induction. □

Lemma 4.11. *Let R be a left strongly P-coherent ring. If M is an RD-injective left R-module, then M has a left $\mathcal{D}$-resolution $\cdots \to F_{n-2} \to F_{n-3} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i injective.*

Proof. By Corollary 3.5, M has a $\mathcal{D}$-cover $f : F_0 \to M$ with F_0 injective and $\ker(f)$ D-injective. Let $g : F_1 \to \ker(f)$ be a $\mathcal{D}$-cover of $\ker(f)$. Consider the short exact sequence $0 \to F_1 \xrightarrow{i} E \to L \to 0$ with E injective. Note that L is divisible by Lemma 4.9(1), and so there exists $j : E \to \ker(f)$ such that $ji = g$ since $\ker(f)$ is D-injective. Thus there exists $h : E \to F_1$ such that $gh = j$ since g is a cover. Therefore $ghi = g$, and hence hi is an isomorphism. It follows that F_1 is injective. Note that $\ker(g)$ is also D-injective. So we can continue the above process to get the desired left $\mathcal{D}$-resolution of M. □

Theorem 4.12. *Consider the following conditions for a left strongly P-coherent ring R and an integer* $n \geq 2$:

(1) *gl left* $\mathscr{D}$*-dim* ${}_R\mathscr{M} \leq n-2$;
(2) *gl right* $\mathscr{D}$*-dim* ${}_R\mathscr{M} \leq n$;
(3) *left* $\mathscr{D}$*-dim* $N \leq n-2$ *for all RD-injective left R-modules N.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Let N be any left R-module. By (1), N has a left $\mathscr{D}$-resolution

$$0 \to F_{n-2} \cdots \to F_1 \to F_0 \to N \to 0.$$

Then we have the complex

$$0 \to \mathrm{Hom}(M, F_{n-2}) \to \mathrm{Hom}(M, F_{n-3}) \to \cdots \to \mathrm{Hom}(M, F_0) \to 0$$

for every left R-module M. Hence $\mathrm{Ext}_{n+k}(M, N) = 0$ for all $k \geq -1$. So gl right $\mathscr{D}$-dim ${}_R\mathscr{M} \leq n$ by Proposition 4.4.

(2) $\Rightarrow$ (3) Let N be an RD-injective left R-module. Then N has a left $\mathscr{D}$-resolution:

$$\cdots \to F_n \xrightarrow{f} F_{n-1} \xrightarrow{g} F_{n-2} \xrightarrow{h} F_{n-3} \xrightarrow{j} F_{n-4} \to \cdots \to F_1 \to F_0 \to N \to 0$$

with each F_i injective by Lemma 4.11. Put $K = \ker(g)$, $H = F_{n-1}/K$. Let $\lambda : K \to F_{n-1}$ be the inclusion and $\pi : F_{n-1} \to H$ the canonical projection. Then there exists $p : F_n \to K$ such that $f = \lambda p$ and there exists a monomorphism $\alpha : H \to F_{n-2}$ such that $g = \alpha\pi$. Put $L = F_{n-2}/\mathrm{im}(\alpha)$ and let $\beta : F_{n-2} \to L$ be the canonical projection. Then there exists a homomorphism $i : L \to F_{n-3}$ such that $h = i\beta$. So we have the following commutative diagram:

$$\begin{tikzcd}
F_n \arrow[rr, "f"] \arrow[dr, "p"] & & F_{n-1} \arrow[rr, "g"] \arrow[dr, "\pi"] & & F_{n-2} \arrow[rr, "h"] \arrow[dr, "\beta"] & & F_{n-3} \\
& K \arrow[ur, "\lambda"] \arrow[dr] & & H \arrow[ur, "\alpha"] \arrow[dr] & & L \arrow[ur, "i"] \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0.
\end{tikzcd}$$

By (2) and Proposition 4.4, $\mathrm{Ext}_{n-1}(K, N) = 0$. Thus the sequence

$$\mathrm{Hom}(K, F_n) \xrightarrow{f_*} \mathrm{Hom}(K, F_{n-1}) \xrightarrow{g_*} \mathrm{Hom}(K, F_{n-2})$$

is exact. Since $g_*(\lambda) = g\lambda = 0$, $\lambda \in \ker(g_*) = \mathrm{im}(f_*)$. So $\lambda = f_*(l) = fl$ for some $l \in \mathrm{Hom}(K, F_n)$. But $f = \lambda p$, and hence $\lambda = \lambda pl$. Thus $pl = 1$ since λ is monic, and so

K is injective. It follows that H and L are injective. We claim that the complex

$$0 \to L \xrightarrow{i} F_{n-3} \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a left $\mathscr{D}$-resolution of N. In fact, it is enough to show that the complex

$$0 \longrightarrow \mathrm{Hom}(G, L) \xrightarrow{i_*} \mathrm{Hom}(G, F_{n-3}) \xrightarrow{j_*} \mathrm{Hom}(G, F_{n-4})$$

is exact for every divisible left R-module G. Note that we have the following exact commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(G, F_{n-1}) \arrow[rr, "g_*"] \arrow[dr, "\pi_*"] & & \mathrm{Hom}(G, F_{n-2}) \arrow[rr, "h_*"] \arrow[dr, "\beta_*"] & & \mathrm{Hom}(G, F_{n-3}) \\
& \mathrm{Hom}(G, H) \arrow[ur, "\alpha_*"] \arrow[dr] & & \mathrm{Hom}(G, L) \arrow[ur, "i_*"] \arrow[dr] & \\
0 \arrow[ur] & & 0 & & 0.
\end{tikzcd}$$

So $\ker(i_*\beta_*) = \ker(h_*) = \mathrm{im}(g_*) = \mathrm{im}(\alpha_*\pi_*) = \mathrm{im}(\alpha_*) = \ker(\beta_*)$. Let $\theta \in \ker(i_*)$. Since β_* is epic, $\theta = \beta_*(\gamma)$ for some $\gamma \in \mathrm{Hom}(G, F_{n-2})$. Thus $i_*\beta_*(\gamma) = 0$, and hence $\theta = \beta_*(\gamma) = 0$. It follows that i_* is monic. On the other hand, $\ker(j_*) = \mathrm{im}(h_*) = \mathrm{im}(i_*)$. So we obtain the desired exact sequence. This completes the proof. $\square$

Proposition 4.13. *Let R be a left strongly P-coherent ring and n a fixed non-negative integer. The following are equivalent for a left R-module M:*

(1) *right $\mathscr{D}$-dim $M \leq n$;*
(2) $\mathrm{Ext}^{n+k}(R/Ra, M) = 0$ *for every $a \in R$ and every $k \geq 1$;*
(3) $\mathrm{Ext}^{n+1}(R/Ra, M) = 0$ *for every $a \in R$;*
(4) *If $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to L \to 0$ is exact and each F^i is divisible, then L is divisible.*

Proof. (1) $\Rightarrow$ (2) Since right $\mathscr{D}$-dim $M \leq n$, there is a right $\mathscr{D}$-resolution of the form $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to F^n \to 0$. So $\mathrm{Ext}^{n+k}(R/Ra, M) \cong \mathrm{Ext}^k(R/Ra, F^n) = 0$ for every $a \in R$ and every $k \geq 1$ by Lemma 4.9(3).

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to L \to 0$ be exact with each F^i divisible. Then $\mathrm{Ext}^1(R/Ra, L) \cong \mathrm{Ext}^{n+1}(R/Ra, M) = 0$ for every $a \in R$ by Lemma 4.9(3). So L is divisible.

(4) $\Rightarrow$ (1) Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1}$ be a partial right $\mathscr{D}$-resolution of M. Then we get an exact sequence $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to L \to 0$. By (4), L is divisible. Thus right $\mathscr{D}$-dim $M \leq n$. $\square$

Proposition 4.14. *Let R be a left strongly P-coherent ring and n a nonnegative integer. The following are equivalent for a right R-module N:*

(1) *left* $\mathcal{TF}$*-dim* $N \leq n$;
(2) $\mathrm{Tor}_{n+k}(N, R/Ra) = 0$ *for every* $a \in R$ *and every* $k \geq 1$;
(3) $\mathrm{Tor}_{n+1}(N, R/Ra) = 0$ *for every* $a \in R$;
(4) *If* $0 \to K \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ *is exact with each* F_i *torsionfree, then* K *is torsionfree.*

Proof. The proof is analogous to that of Proposition 4.13 by Lemma 4.9(4). □

Theorem 4.15. *The following are equivalent for a left strongly P-coherent ring R and an integer* $n \geq 0$:

(1) *gl right* $\mathcal{D}$*-dim* ${}_R\mathcal{M} \leq n$;
(2) *gl left* $\mathcal{TF}$*-dim* $\mathcal{M}_R \leq n$;
(3) *left* $\mathcal{TF}$*-dim* $N \leq n$ *for every Warfield cotorsion right R-module N*;
(4) $\mathrm{Ext}^{n+1}(R/Ra, M) = 0$ *for every* $a \in R$ *and every left R-module M*;
(5) $\mathrm{Tor}_{n+1}(N, R/Ra) = 0$ *for every* $a \in R$ *and every right R-module N*;
(6) *Every cyclically presented left R-module has projective dimension* $\leq n$;
(7) *Every cyclically presented left R-module has flat dimension* $\leq n$.

In this case, every Warfield cotorsion right R-module has injective dimension $\leq n$.

Proof. (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (5) follow from Propositions 4.13 and 4.14, respectively.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2) Let M be any right R-module. Then, by Göbel and Trlifaj (2006, Theorem 4.1.1(b)), there is an exact sequence $0 \to M \to N \to L \to 0$, where N is Warfield cotorsion and L is torsionfree. Thus we get an induced exact sequence $0 = \mathrm{Tor}_{n+2}(L, R/Ra) \to \mathrm{Tor}_{n+1}(M, R/Ra) \to \mathrm{Tor}_{n+1}(N, R/Ra) = 0$ for every $a \in R$ by (3) and Proposition 4.14. So left $\mathcal{TF}$-dim $M \leq n$ and (2) follows.

(4) $\Rightarrow$ (5) holds because $\mathrm{Tor}_{n+1}(N, R/Ra)^+ \cong \mathrm{Ext}^{n+1}(R/Ra, N^+)$ for every $a \in R$ and every right R-module N.

(5) $\Rightarrow$ (4) holds because $\mathrm{Ext}^{n+1}(R/Ra, M)^+ \cong \mathrm{Tor}_{n+1}(M^+, R/Ra)$ for every $a \in R$ and every left R-module M by Rotman (1979, Theorem 9.51) and the remark following it.

(4) $\Leftrightarrow$ (6) and (5) $\Leftrightarrow$ (7) are obvious.

Next we prove the last statement. Let M be a Warfield cotorsion right R-module and N any right R-module. Then, by (5), there is an exact sequence $0 \to F_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$ with F_n torsionfree and each P_i projective, and so $\mathrm{Ext}^{n+1}(N, M) \cong \mathrm{Ext}^1(F_n, M) = 0$. Thus M has injective dimension $\leq n$. □

Let $\mathcal{C}$ be a class of left R-modules and M a left R-module. Recall that a $\mathcal{C}$-cover $\phi : C \to M$ is said to *have the unique mapping property* (Ding, 1996) if for any homomorphism $f: C' \to M$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g : C' \to C$ such that $\phi g = f$.

Corollary 4.16. *The following are equivalent for a left strongly P-coherent ring R*:

(1) *gl right* $\mathscr{D}$-*dim* ${}_R\mathscr{M} \leq 2$;
(2) *gl left* $\mathscr{TF}$-*dim* $\mathscr{M}_R \leq 2$;
(3) *Every left R-module has a* $\mathscr{D}$-*cover with the unique mapping property.*

Proof. (1) $\Leftrightarrow$ (2) holds by Theorem 4.15.

(1) $\Rightarrow$ (3) Let M be a left R-module. Then M has a $\mathscr{D}$-cover $f: F \to M$ by Theorem 2.10. It is enough to show that, for any divisible left R-module G and any homomorphism $g: G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists β : $F/\mathrm{im}(g) \to M$ such that $\beta\pi = f$ since $\mathrm{im}(g) \subseteq \ker(f)$, where $\pi: F \to F/\mathrm{im}(g)$ is the natural map. Consider the exact sequence $0 \to \ker(g) \to G \to F \to F/\mathrm{im}(g) \to 0$. Note that $F/\mathrm{im}(g)$ is divisible by (1) and Proposition 4.13. Thus there exists α : $F/\mathrm{im}(g) \to F$ such that $\beta = f\alpha$, and so $f\alpha\pi = \beta\pi = f$. Hence $\alpha\pi$ is an isomorphism since f is a cover. Therefore π is monic, and so $g = 0$.

(3) $\Rightarrow$ (1) follows from Theorem 4.12 by letting $n = 2$. □

Theorem 4.17. *The following are equivalent for a left strongly P-coherent ring R and an integer* $n \geq 0$:

(1) *right* $\mathscr{D}$-*dim* ${}_RR \leq n$;
(2) *left* $\mathscr{TF}$-*dim* $({}_RR)^+ \leq n$;
(3) *right* $\mathscr{D}$-*dim* $F \leq n$ *for every flat left R-module F*;
(4) *left* $\mathscr{TF}$-*dim* $E \leq n$ *for every injective right R-module E*;
(5) *Every right* $\mathscr{TF}$-*resolution* $0 \to N \to F^0 \to F^1 \to \cdots F^{n-1} \to F^n \to \cdots$ *of every right R-module N is exact at* F^i *for every* $i \geq n-1$;
(6) *Every left* $\mathscr{D}$-*resolution* $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ *of every left R-module N is exact at* F_i *for every* $i \geq n-1$.

Proof. (1) $\Rightarrow$ (2) follows from Propositions 4.13 and 4.14 and the isomorphism

$$\mathrm{Ext}^{n+1}(R/Ra, {}_RR)^+ \cong \mathrm{Tor}_{n+1}(({}_RR)^+, R/Ra)$$

for every $a \in R$.

(2) $\Rightarrow$ (4). Let E be an injective right R-module. Then E is isomorphic to a direct summand of $\Pi({}_RR)^+$. Note that $\mathrm{Tor}_{n+1}(\Pi({}_RR)^+, R/Ra) \cong \Pi\mathrm{Tor}_{n+1}(({}_RR)^+, R/Ra)$ for every $a \in R$ by Chen and Ding (1996, Lemma 2.10). So left $\mathscr{TF}$-dim $E \leq n$ by (2) and Proposition 4.14.

(4) $\Rightarrow$ (3) Let F be a flat left R-module. Then F^+ is injective and so left $\mathscr{TF}$-dim $F^+ \leq n$ by (4). Since $\mathrm{Ext}^{n+1}(R/Ra, F)^+ \cong \mathrm{Tor}_{n+1}(F^+, R/Ra)$ for every $a \in R$, we have right $\mathscr{D}$-dim $F \leq n$ by Propositions 4.13 and 4.14.

(3) $\Rightarrow$ (1) is trivial.

(1) $\Leftrightarrow$ (6) holds by Corollary 4.6.

(2) $\Leftrightarrow$ (5) comes from Corollary 4.8. □

Proposition 4.18. *Let R be a left strongly P-coherent ring with $wD(R) < \infty$. Then right $\mathcal{D}$-dim$_R R$ = gl right $\mathcal{D}$-dim$_R\mathcal{M}$ = gl left $\mathcal{TF}$-dim$\mathcal{M}_R \leq wD(R)$.*

Proof. It suffices to show that gl right $\mathcal{D}$-dim$_R\mathcal{M} \leq$ right $\mathcal{D}$-dim$_R R$. We may assume that right $\mathcal{D}$-dim$_R R = m < \infty$. For any left R-module M, there exist an integer $n \geq 0$ and an exact sequence $0 \to F_n \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to M \to 0$ with each F_i flat by hypothesis. Since right $\mathcal{D}$-dim$F_i \leq m$ by Theorem 4.17, $i = 0, 1, \ldots, n$, we have right $\mathcal{D}$-dim$M \leq m$ by Proposition 4.13. It follows that right $\mathcal{D}$-dim$_R R$ = gl right $\mathcal{D}$-dim$_R\mathcal{M}$. □

5. WARFIELD COTORSION MODULES

It is easy to see that every right R-module is Warfield cotorsion if and only if every torsionfree right R-module is projective by Göbel and Trlifaj (2006, Theorem 4.1.1(b)). In addition, we have the following result, which has been proven for commutative domains (see Lee, 2003, Theorem 3.3).

Theorem 5.1. *The following are equivalent for a ring R:*

(1) *Every quotient of a Warfield cotorsion right R-module is Warfield cotorsion;*
(2) *All torsionfree right R-modules are of projective dimension ≤ 1;*
(3) *For any RD-exact sequence $0 \to A \to B \to C \to 0$ of right R-modules with B projective, A is projective.*

Proof. (1) ⇒ (3) Let $0 \to A \to B \to C \to 0$ be an RD-exact sequence of right R-modules with B projective. Then C is torsionfree. Let M be any right R-module. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Note that L is Warfield cotorsion by (1), and hence $\mathrm{Ext}^2(C, M) \cong \mathrm{Ext}^1(C, L) = 0$. Thus $pd(C) \leq 1$, and so A is projective.

(3) ⇒ (2) Let M be any torsionfree right R-module. There exists an exact sequence $0 \to N \to P \to M \to 0$ with P projective. Note that the sequence is RD-exact, so N is projective by (3). It follows that $pd(M) \leq 1$.

(2) ⇒ (1) Let E be any Warfield cotorsion right R-module and K a submodule of E. For any torsionfree right R-module F, the exactness of the sequence $0 \to K \to E \to E/K \to 0$ induces the exact sequence $0 = \mathrm{Ext}^1(F, E) \to \mathrm{Ext}^1(F, E/K) \to \mathrm{Ext}^2(F, K)$. Note that $\mathrm{Ext}^2(F, K) = 0$ by (2), so $\mathrm{Ext}^1(F, E/K) = 0$, as required. □

Next we characterize von Neumann regular rings.

Theorem 5.2. *The following are equivalent for a ring R:*

(1) *R is a von Neumann regular ring;*
(2) *Every Warfield cotorsion right R-module is injective;*
(3) *Every Warfield cotorsion right R-module is divisible;*
(4) *R is a left strongly P-coherent ring and every Warfield cotorsion right R-module is flat (torsionfree);*

(5) *Every nonzero right R-module contains a nonzero torsionfree submodule*;
(6) *R is a left strongly P-coherent and left divisible ring with* $wD(R) < \infty$;
(7) *gl right* $\mathcal{D}$*-dim* $_R\mathcal{M} = 0$;
(8) *gl left* $\mathcal{TF}$*-dim* $\mathcal{M}_R = 0$.

Proof. (1) $\Rightarrow$ (6), (2) $\Rightarrow$ (3), and (8) $\Rightarrow$ (5) are clear.

(1) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) follow from Mao and Ding (2006, Corollary 2.6) or Dauns and Fuchs (2004, Theorem 2.2).

(2) $\Leftrightarrow$ (8) holds by Göbel and Trlifaj (2006, Theorem 4.1.1(b)).

(3) $\Rightarrow$ (1) Let M be any Warfield cotorsion right R-module. For any $a \in R$, $\mathrm{Ext}^1(R/aR, M) = 0$ by (3). Thus R/aR is torsionfree by Göbel and Trlifaj (2006, Theorem 4.1.1(b)) and so it is projective by Dauns and Fuchs (2004, Proposition 1.2). It follows that aR is a direct summand of R, which implies that R is von Neumann regular.

(4) $\Leftrightarrow$ (8) comes from Theorem 4.15.

(5) $\Rightarrow$ (2) Assume that $0 \to A \to B \to C \to 0$ is any right R-module exact sequence. To simplify the notation, we think of A as a submodule of B. Let M be a Warfield cotorsion right R-module and $f : A \to M$ be any homomorphism. By a simple application of Zorn's Lemma, we can find a $g : D \to M$ where $A \subseteq D \subseteq B$, $g|_A = f$, such that g cannot be extended to any submodule of B properly containing D. We claim that $D = B$. Indeed, if $D \neq B$, then $B/D \neq 0$. By (5), there exists a nonzero submodule N/D of B/D such that N/D is torsionfree. Since M is Warfield cotorsion, there is $h : N \to M$ such that $h|_D = g$. It is obvious that h extends g, this yields the desired contradiction, and so M is injective.

(6) $\Rightarrow$ (7) holds by Proposition 4.18. □

Finally, we give some new characterizations of *PP* rings. Recall that a ring R is called *left PP* if every principal left ideal of R is projective. Obviously, any left *PP* ring is left strongly P-coherent. But the converse is false in general. For example, $\mathbb{Z}_4$ is a strongly P-coherent ring, but it is not a *PP* ring.

Theorem 5.3. *The following are equivalent for a ring R*:

(1) *R is a left PP ring*;
(2) *Every quotient of every divisible left R-module is divisible*;
(3) *Every left R-module has a monic divisible cover*;
(4) *R is a left P-coherent ring and every D-injective left R-module is injective*;
(5) *R is a left strongly P-coherent ring and every D-injective left R-module is divisible*;
(6) *R is a left strongly P-coherent ring and gl right* $\mathcal{D}$*-dim* $_R\mathcal{M} \leq 1$;
(7) *R is a left strongly P-coherent ring and gl left* $\mathcal{TF}$*-dim* $\mathcal{M}_R \leq 1$;
(8) *R is a left strongly P-coherent ring and left* $\mathcal{TF}$*-dim* $N \leq 1$ *for every Warfield cotorsion right R-module N*.

Proof. (1) $\Leftrightarrow$ (2) follows from Xue (1990, Theorem 2).

(2) $\Rightarrow$ (3) R is left P-coherent since (1) and (2) are equivalent. Let M be a left R-module, then M has a divisible cover $f: E \to M$ by Theorem 2.10. Note that $\mathrm{im}(f)$ is divisible by (2), so $\mathrm{im}(f) \to M$ is a monic divisible cover of M.

(3) $\Rightarrow$ (2) Let $B \to C \to 0$ be an exact sequence of left R-modules with B divisible. Since C has a monic divisible cover $E \to C$ by (3), we have $C \cong E$ is divisible.

(2) $\Rightarrow$ (4) Let M be a D-injective left R-module. Then there is an exact sequence $0 \to M \to E(M) \to L \to 0$. Note that L is divisible by (2). So the exact sequence is split, and hence M is injective.

(4) $\Rightarrow$ (2) Let M be a quotient of a divisible left R-module. Suppose $f: F \to M$ is a $\mathscr{D}$-cover of M. Then f is an epimorphism. By Remark 3.2(1), $\ker(f)$ is D-injective, and hence it is injective by (3). So M is divisible.

(4) $\Rightarrow$ (5) is clear since (4) is equivalent to (1).

(5) $\Rightarrow$ (4) Let M be a D-injective left R-module. Then there is an exact sequence $0 \to M \to E(M) \to L \to 0$, and so L is divisible by (5) and Lemma 4.9(1). Thus the sequence is split, and hence M is injective.

(2) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (2) Let B be any divisible left R-module and A a submodule of B. By (6), there is a right $\mathscr{D}$-resolution $0 \to A \to M \to N \to 0$. Consider the following pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & B/A \arrow[r] & 0 \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Thus H is divisible, so is B/A by Lemma 4.9(1), as desired.

(6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) follow from Theorem 4.15. $\square$

We conclude this article with the following corollary.

Corollary 5.4. *The following are equivalent for a ring R:*

(1) *R is a two-sided PP ring*;

(2) *R is a two-sided strongly P-coherent ring and every D-flat right R-module is flat* (*torsionfree*);
(3) *R is a two-sided strongly P-coherent ring and every finitely presented D-flat right R-module is flat* (*torsionfree*).

Proof. (1) $\Rightarrow$ (2) Let M be a D-flat right R-module. Then M^+ is D-injective by Remark 3.2(2), and hence M^+ is injective by Theorem 5.3. So M is flat.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let $a \in R$. Then aR has a monic $\mathcal{TF}$-preenvelope $f : aR \to T$ by Theorem 2.7. There is an exact sequence $P \xrightarrow{g} T \to 0$ with P projective. Since aR is cyclically presented, there is $h : aR \to P$ such that $f = gh$. It is easy to check that h is a monic $\mathcal{TF}$-preenvelope of aR. We may choose P to be finitely generated. So coker(h) is finitely presented D-flat by Proposition 3.7(2). Thus coker(h) is torsionfree by (3), and hence aR is torsionfree by Lemma 4.9(2). Therefore aR is flat by Shamsuddin (2001, p. 2047, 5(a)). Note that Ra is also flat by Jøndrup (1971). It follows that R is a two-sided PP ring. □

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