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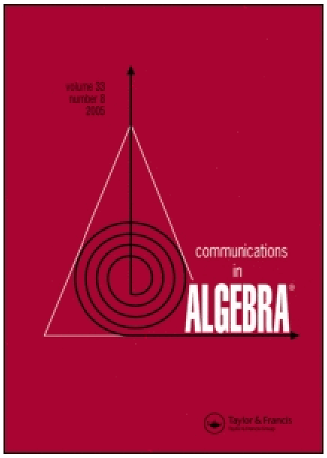

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On Generalizations of PF-Rings

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On Generalizations of PF-Rings[#]

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ABSTRACT

A ring R is called a right WPF-ring (weak PF-ring) if R is a semiperfect right simple-injective ring with essential right socle. The class of right WPF-rings is broader than that of right PF-rings. In this article, we study and provide several characterizations of this new class of rings. We also show that if R is a left perfect, left and right f-injective ring, then R is QF if and only if $Soc_2(R)$ is a finitely generated right R-module if and only if $R/Soc(R)$ is a finitely cogenerated left R-module. Some known results are obtained as corollaries.

Key Words: Simple-injective ring; PF-ring; WPF-ring; QF-ring.

1. INTRODUCTION

All rings are associative with identity and all modules are unitary. The socle of a module M is denoted by $Soc(M)$. If R is a ring, we denote by $Soc(R_R) = S_r$, $Soc({}_RR) = S_l$, $Z(R_R) = Z_r$ and $J(R) = J$ for the right socle, the left socle, the right singular ideal and the Jacobson radical of R, respectively. The left and right

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annihilators of a subset X of R are denoted by $\mathbf{l}(X)$ and $\mathbf{r}(X)$, respectively. We use $K \leq_e N$ to indicate that K is an essential submodule of N. General background material can be found in Anderson and Fuller (1974) and Faith (1976).

A ring R is called quasi-Frobenius, briefly QF, if it is right (or left) artinian and right (or left) self-injective. R is said to be a right PF-ring if the right R-module R_R is an injective cogenerator in the category of right R-modules, equivalently, R is semiperfect, right self-injective with essential right socle. Osofsky (1966) proved that a two-sided PF and one-sided perfect ring is QF. However, it is an open question whether a right PF and one-sided perfect (or two-sided perfect, or even semiprimary) ring is QF (see Faith, 1976, p. 218; 1990, Question 2.4). This question has been studied in many papers such as Armendariz and Park (1992), Clark and Huynh (1994a,b), Herbera and Shamsuddin (1996), Nicholson and Yousif (1997b, 2001a,b) and Xue (1996). In this article we consider a generalization of right PF-rings, namely the class of semiperfect right simple-injective rings with essential right socle (called right WPF-rings). We provide examples of right WPF-rings which are not right PF. We also show that many of the properties of right PF-rings are still valid for right WPF-rings. Several characterizations of right WPF-rings are provided. For instance, it is shown that R is a right WPF-ring if and only if R is semilocal, right Kasch and right simple-injective if and only if R is right continuous, right finitely cogenerated and right simple-injective. We prove that, for a right WPF-ring R, if R is left semi-dual or J^2 is a right annihilator of a finite subset of R, then $Soc_2(R)$ is a finitely generated left R-module. Finally, we show that if R is a left perfect, left and right f-injective ring, then R is QF if and only if $Soc_2(R)$ is a finitely generated right R-module if and only if $R/Soc(R)$ is a finitely cogenerated left R-module. Some known results appearing in Clark and Huynh (1994a), Nicholson and Yousif (2001a,b) and Xue (1996) are obtained as corollaries.

2. WEAK PF-RINGS

We say that a ring R is right simple-injective if every homomorphism from a right ideal of R to R with simple image can be given by left multiplication by an element of R. R is said to be right Kasch if every simple right R-module can be embedded in R_R. R is called right P-injective (mininjective) if every right R-homomorphism from a principal (simple) right ideal to R is given by left multiplication by an element of R.

Lemma 2.1. *Let R be a right Kasch right simple-injective ring. Then*

(1) $\mathbf{r}(\mathbf{l}(I)) = I$ *for every right ideal I of R. In particular, R is left P-injective.*
(2) $S_r = S_l$.
(3) $\mathbf{l}(J)$ *is an essential left ideal.*
(4) $J = \mathbf{r}(S) = \mathbf{r}(\mathbf{l}(J))$, *where* $S = S_r = S_l$.
(5) $Z_l = J$.
(6) *xR is minimal if and only if Rx is minimal for $x \in R$.*
(7) *Minimal left and right ideals are annihilators.*

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(8) *The map* $K \longmapsto \mathbf{r}(K)$ *gives a bijection from the set of all minimal left ideals of* R *onto the set of all maximal right ideals of* R*, whose inverse map is given by* $I \longmapsto \mathbf{l}(I)$.

Proof. (1) and (2) follow from Nicholson and Yousif (1997a, Lemma 4.2).

(3) follows from Chen and Ding (2001, Lemma 2.1).

(4) Since R is right Kasch, $J = \mathbf{r}(S)$. So (4) holds.

(5) follows from (1) since R is left P-injective.

(6) and (7) follow since R is left and right mininjective.

(8) Let $K = Ra$ be a minimal left ideal. Then aR is a minimal right ideal, and so $\mathbf{r}(K) = \mathbf{r}(a)$ is a maximal right ideal. Clearly, $K = \mathbf{l}(\mathbf{r}(K))$ since K is an annihilator. Note that R is right Kasch and right simple-injective. Thus, for all maximal right ideals T, $\mathbf{l}(T)$ is simple and $T = \mathbf{r}(\mathbf{l}(T))$. So (8) follows.

Remark 1. The ring $\mathbb{Z}$ of integers is an example of a simple-injective ring which is not P-injective (and hence not self-injective). In Nicholson and Yousif (1997b, Example 4) and Björk (1970, Example, p. 70) examples are given of right P-injective rings which are not right simple-injective.

A ring R is called right finitely cogenerated if S_r is finitely generated and $S_r \leq_e R_R$. Recall that if M is a module, the submodules $Soc_1(M) \subseteq Soc_2(M) \subseteq \cdots$ are defined by setting $Soc_1(M) = Soc(M)$ and, if $Soc_n(M)$ has been specified, by $Soc_{n+1}(M)/Soc_n(M) = Soc(M/Soc_n(M))$. We also recall the following conditions:

C1: Every nonzero left ideal is essential in a direct summand of R.

C2: Every left ideal that is isomorphic to a direct summand of R is itself a direct summand.

C3: If $Re \cap Rf = 0$, where e and f are idempotents in R, then $Re \oplus Rf$ is a direct summand of R.

A ring R is called left continuous if it satisfies $C1$ and $C2$. If R satisfies only $C1$, it is called a left CS ring. R is called left min-CS if $C1$ is required only for minimal left ideals.

Lemma 2.2. *Let* R *be semilocal, right Kasch and right simple-injective. Then*

(1) *R is left GPF, i.e., R is semiperfect, left P-injective and $S_l \leq_e {}_RR$.*
(2) *R is left and right finitely cogenerated.*
(3) *R is left and right Kasch.*
(4) $Soc_n({}_RR) = Soc_n(R_R) = \mathbf{l}(J^n) = \mathbf{r}(J^n)$ *for* $n \geq 1$.
(5) *R is right continuous.*

Proof. (1)–(3) follows from Chen and Ding (2001, Theorem 2.3) and its proof.

(4) Since R is semilocal, $\mathbf{r}(J) = S_l = S_r = \mathbf{l}(J)$ by Lemma 2.1 (2). It is easy to see that $\mathbf{r}(J^n) = \mathbf{l}(J^n)$ for $n \geq 1$. By Goodearl and Warfield (1989, Proposition 3.14), $Soc_n({}_RR) = \mathbf{r}(J^n)$ and $Soc_n(R_R) = \mathbf{l}(J^n)$. So $Soc_n({}_RR) = Soc_n(R_R) = \mathbf{l}(J^n) = \mathbf{r}(J^n)$ for $n \geq 1$.

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(5) For every right ideal I of R, $I = \mathbf{r}(\mathbf{l}(I))$ by Lemma 2.1 (1). Since $S_l = S_r \leq_e R_R$, $I = \mathbf{r}(\mathbf{l}(I))$ is essential in a summand of R_R by Nicholson and Yousif (2001b, Lemma 3.11). So R is right CS. Since R is left Kasch, R is right $C2$ by Nicholson and Yousif (2001b, Proposition 4.1). So (5) holds.

Theorem 2.3. *The following are equivalent for a ring R.*

(1) *R is semilocal, right Kasch and right simple-injective.*
(2) *R is semiperfect, right Kasch and right simple-injective.*
(3) *R is semiperfect, right simple-injective and $S_r \leq_e R_R$.*
(4) *R is semiperfect, right simple-injective and $S_r \leq_e {}_RR$.*
(5) *R is left finitely cogenerated, right Kasch and right simple-injective.*
(6) *R is left finite dimensional, right Kasch and right simple-injective.*
(7) *R is left and right Kasch, and right simple-injective.*
(8) *R is left min-CS, right Kasch and right simple-injective.*
(9) *R is right continuous, right finitely cogenerated and right simple-injective.*
(10) *R is right Kasch and right simple-injective and S_r is a finitely generated left ideal.*

Proof. (1) ⇒ (2) by Lemma 2.2(1).

(2) ⇒ (3) and (4) by Lemma 2.2(2) and (4).

(3) ⇒ (1) By (3), R is right minfull (i.e., R is semiperfect, right mininjective and $Soc(eR) \neq 0$ for each local idempotent $e \in R$), and so R is right Kasch by Nicholson and Yousif (1997a, Theorem 3.7).

(4) ⇒ (1) Since $S_r \leq_e {}_RR$, $S_r \cap Re \neq 0$ for every local idempotent $e \in R$. Let $0 \neq a \in S_r \cap Re$, then $a = ae \in S_re$. Thus $S_re \neq 0$, and so R is right Kasch by Nicholson and Yousif (1997a, Proposition 3.3).

(1) ⇒ (5) by Lemma 2.2(2).

(5) ⇒ (6) is clear.

(6) ⇒ (7) Since R is right Kasch, R is left $C2$. Hence R is semilocal by Nicholson and Yousif (2001b, Lemma 3.6). So R is left Kasch by Lemma 2.2(3).

(7) ⇒ (1) By Nicholson and Yousif (1997a, Lemma 4.2), every right ideal is a right annihilator. Since R is left Kasch, R is semilocal by Gómez Pardo and Guil Asensio (1998, Theorem 2.5).

(1) ⇒ (8) R is semiperfect, left finitely cogenerated and $S_r = S_l$ by Lemma 2.2. Thus $S_r \leq_e {}_RR$, and so $\mathbf{l}(\mathbf{r}(I))$ is essential in a summand of ${}_RR$ for every left ideal I of R by Nicholson and Yousif (2001b, Lemma 3.11). Since every minimal left ideal is a left annihilator by Lemma 2.1(7), R is left min-CS. So (8) holds.

(1) ⇒ (9) By Lemma 2.2(2) and (5).

(8) ⇒ (2) Let M be a maximal right ideal, then $\mathbf{l}(M)$ is a minimal left ideal since R is right Kasch and right mininjective. Hence $\mathbf{l}(M)$ is essential in a summand of ${}_RR$ by (8). Note that R is right Kasch, and so R is semiperfect by Nicholson and Yousif (2001b, Proposition 3.14).

(9) ⇒ (3) Since R is right finitely cogenerated, $S_r \leq_e R_R$. Next we'll show that R is semiperfect. By hypothesis, S_r is finitely generated. Let $S_r = K_1 \oplus K_2 \oplus \cdots \oplus K_n$,

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where each K_i is a simple right ideal, $i = 1, 2, \ldots, n$. Since R is right min-CS, there exist idempotents $e_i \in R$ such that $K_i \leq_e e_iR$, $i = 1, 2, \ldots, n$. Since $\{K_1, K_2, \ldots, K_n\}$ is an independent family, so is $\{e_1R, e_2R, \ldots, e_nR\}$ by Goodearl (1976, Proposition 1.1(d)).

Note that R is right $C2$, and so it is right $C3$. Hence $T = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$ is a direct summand. Since $S_r \subseteq T \subseteq R$ and $S_r \leq_e R_R$, $T \leq_e R_R$. So $T = R$, i.e., $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$.

Note that R_R is finite dimensional and $C2$, and hence monomorphisms $R_R \to R_R$ are epic by Nicholson and Yousif (2001b, Lemma 3.6). Let $0 \neq K$ be a submodule of e_iR. Since $K_i \leq_e e_iR$, $K \cap K_i \neq 0$. Note that K_i is simple, and so $K \cap K_i = K_i$, i.e., $K_i \subseteq K$. Similarly, for any nonzero submodule L of e_iR, we have $K_i \subseteq L$. Therefore $0 \neq K_i \subseteq K \cap L$, and hence e_iR is uniform for each $i = 1, 2, \ldots, n$. Consequently R is semiperfect by Nicholson and Yousif (2001b, Lemma 3.13).

(1) $\Rightarrow$ (10) By Lemma 2.1(2) and Lemma 2.2(2).

(10) $\Rightarrow$ (1) Let $S_r = Ra_1 + Ra_2 + \cdots + Ra_n$, where Ra_i is a simple left ideal, $i = 1, 2, \ldots, n$. Since R is right Kasch, $J = \mathbf{r}(S_r) = \bigcap_{i=1}^{n} \mathbf{r}(a_i)$. Note that each $\mathbf{r}(a_i) = \mathbf{r}(Ra_i)$ is a maximal right ideal by Lemma 2.1(8). So R is semilocal.

Definition 2.4. *A ring R is called a right WPF-ring (weak PF-ring) if it satisfies the equivalent conditions in Theorem 2.3.*

Remark 2. A ring R is called right *PF* if and only if R is semiperfect, right self-injective and $S_r \leq_e R_R$. Clearly every right *PF*-ring is right *WPF*, however the converse is not true in general. There is an example of a commutative *WPF*-ring with $J^2 = J$ which is not *PF* (see Nicholson and Yousif, 1997b, Example 3).

Recall that a ring R is a right *IN*-ring (Camillo et al., 2000) if $\mathbf{l}(A \cap B) = \mathbf{l}(A) + \mathbf{l}(B)$ for all right ideals A and B of R. Now we have the following result.

Theorem 2.5. *The following are equivalent for a ring R.*

(1) *R is a right Kasch, right P-injective and right IN-ring.*
(2) *R is a semiperfect, right P-injective, right IN-ring and $S_r \leq_e R_R$.*
(3) *R is a right finitely cogenerated, right P-injective and right IN-ring.*

Proof. (1) $\Rightarrow$ (3) By Camillo et al. (2000, Theorem 5), a right *IN*-ring is right *CS*. So R is right finitely cogenerated by Gómez Pardo and Guil Asensio (1998, Corollary 3.8).

(3) $\Rightarrow$ (2) Since R is right finitely cogenerated by (3), R is right finite dimensional and $S_r \leq_e R_R$. Since a right P-injective ring is right $C2$, R is semilocal by Nicholson and Yousif (2001b, Lemma 3.6). Note that R is right continuous by Chen et al. (2001, Lemma 2.12), and so it is semiregular. Thus R is semiperfect.

(2) $\Rightarrow$ (1) R is right *GPF* by (2), and so R is right Kasch by Nicholson and Yousif (1995, Corollary 2.3).

Remark 3. Every right P-injective and right *IN*-ring is right simple-injective by Chen et al. (2001, Lemma 2.12). Hence every right Kasch, right P-injective and right

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IN-ring is a right *WPF*-ring by Theorem 2.5, and in general need not be right *PF* (see Hajarnavis and Norton, 1985).

Lemma 2.6. *Let R be a semilocal ring. Then $J/J^2 = \bigoplus_{i\in I} \bar{j}_i R$, where $\bar{j}_i R$ is a simple right R-module and j_i can be chosen in J for each $i \in I$, $\bar{a} = a + J^2$ for any $a \in R$, $J = \sum_{i\in I} j_i R + J^2$ and $J^2 = \bigcap_{i\in I} A_i$ with $A_i = \sum_{i\neq k\in I} j_k R + J^2$, $i \in I$.*

Proof. Since R is semilocal, J/J^2 is a semisimple right R/J-module. Hence J/J^2 is a semisimple right R-module. Let $J/J^2 = \bigoplus_{i\in I} \bar{j}_i R$, where $\bar{j}_i R$ is a simple right R-module and j_i can be chosen in J for each $i \in I$. Clearly $J = \sum_{i\in I} j_i R + J^2$. Now let $A_i = \sum_{i\neq k\in I} j_k R + J^2$, then $J^2 \subseteq A_i$, $i \in I$. It is obvious that $J^2 \subseteq \bigcap_{i\in I} A_i$. Conversely, let $x \in \bigcap_{i\in I} A_i$. Then $x \in A_i$ for all $i \in I$. Write $x = \sum_{i\neq k\in I} j_k r_k + y_i$, where $y_i \in J^2$. Then $\bar{x} = \sum_{i\neq k\in I} \bar{j}_k r_k$. For any $k \neq i$, $\bar{x} = \sum_{k\neq t\in I} \bar{j}_t s_t$ since $x \in A_k$, and so $\sum_{i\neq k\in I} \bar{j}_k r_k = \sum_{k\neq t\in I} \bar{j}_t s_t$. Thus $\bar{j}_k r_k = 0$, i.e., $j_k r_k \in J^2$ for all $k \neq i$. Therefore $x = \sum_{i\neq k\in I} j_k r_k + y_i \in J^2$. So $J^2 = \bigcap_{i\in I} A_i$.

Lemma 2.7. *Let R be right simple-injective and S_r a finitely generated right ideal. If $K \subset I$ is a pair of right ideals such that I/K is semisimple, then*

$$\mathbf{l}(K)/\mathbf{l}(I) \cong Hom_R(I/K, R)$$

Proof. The proof is motivated by that of Herbera and Shamsuddin (1996, Lemma 2). Let

$$\phi : \mathbf{l}(K)/\mathbf{l}(I) \to Hom_R(I/K, R)$$

be the canonical map given by

$$\phi(r + \mathbf{l}(I))(x + K) = rx, \quad \text{for } r \in \mathbf{l}(K),\ x \in I.$$

It is easy to see ϕ is a monomorphism. To show that ϕ is an epimorphism, let $f \in (I/K)^* = Hom_R(I/K, R)$. Since I/K is semisimple, so is $\mathrm{Im}(f)$. Thus $\mathrm{Im}(f) \subseteq S_r$, and hence $\mathrm{Im}(f)$ is a direct summand of S_r. By hypothesis, we may assume that $\mathrm{Im}(f) = \bigoplus_{i=1}^n S_i$, where each S_i is a simple right R-module. Let $\pi : I \to I/K$ be the canonical map and $\pi_i : \mathrm{Im}(f) \to S_i$ be the ith projection, $i = 1, 2, \ldots, n$. Then $\mathrm{Im}(\pi_i f \pi) = S_i$ is simple, and so there exists $r_i \in R$ such that $\pi_i f \pi(x) = r_i x$ for all $x \in I$. Put $r = \sum_{i=1}^n r_i$, then

$$f(x + K) = f\pi(x) = \sum_{i=1}^n \pi_i f \pi(x) = \sum_{i=1}^n r_i x = rx$$

for all $x \in I$. Thus $r \in \mathbf{l}(K)$, and hence $f = \phi(r + \mathbf{l}(I))$. So ϕ is an isomorphism.

Recall that a ring is called left semi-dual (see Xue, 1996) if the sum of left annihilators is still a left annihilator.

Theorem 2.8. *Let R be a right WPF-ring. Assume either*

(1) *R is left semi-dual, or*
(2) *$J^2 = \mathbf{r}(A)$ for a finite subset A of R.*

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Then J/J^2 is a finitely generated right R-module and $Soc_2(R)$ is a finitely generated left R-module.

Proof. (1) By Lemma 2.6, $J/J^2 = \bigoplus_{i\in I} \bar{j}_i R$, where each $\bar{j}_i R$ is a simple right R-module and $J^2 = \bigcap_{i\in I} A_i$ with $A_i = \sum_{i\neq k\in I} j_k R + J^2$, $i \in I$. Since R is right Kasch, there exist monomorphisms $g_i : \bar{j}_i R \to R$, $i \in I$. Let $g = \bigoplus_{i\in I} g_i : J/J^2 \to R$ and $\pi : J \to J/J^2$ be the canonical map. Put $f = g\pi$, then $f(j_i) = g(\bar{j}_i) = g_i(\bar{j}_i) \neq 0$, $i \in I$, and Im(f)=Im(g) $\subseteq Soc(R_R) = S_r$. Since S_r is a finitely generated semisimple right R-module, so is Im(f). Note that R is right simple-injective. Hence f is given by a left multiplication by an element of R by the proof of Lemma 2.7, i.e., there exists $r \in R$ such that $f(j) = rj$ for all $j \in J$. Thus $rj_i = f(j_i) \neq 0$, $i \in I$, and $rJ^2 = f(J^2) = 0$, and so $r \in \mathbf{l}(J^2)$. Since R is left semi-dual, $\sum_{i\in I} \mathbf{l}(A_i) = \mathbf{l}(K)$ for some right ideal K of R. Therefore $K = \mathbf{r}(\mathbf{l}(K)) = \mathbf{r}(\sum_{i\in I} \mathbf{l}(A_i)) = \cap \mathbf{r}(\mathbf{l}(A_i))$ $= \cap A_i = J^2$, and hence $\mathbf{l}(J^2) = \mathbf{l}(K) = \sum_{i\in I} \mathbf{l}(A_i)$. Note that $r \in \mathbf{l}(J^2)$. Write $r = r_1 + r_2 + \cdots + r_n$, where $r_t \in \mathbf{l}(A_{i_t})$, $t = 1, 2, \ldots, n$. If $|I| = \infty$, then there exists $k \in I\backslash\{i_1, i_2, \ldots, i_n\}$. Hence $0 \neq rj_k = (r_1 + r_2 + \cdots + r_n)j_k$. But $k \neq i_t$, then $j_k \in A_{i_t}$, and so $r_t j_k = 0$, $t = 1, 2, \ldots, n$. This is a contradiction. Consequently $|I| < \infty$, i.e., J/J^2 is a finitely generated right R-module.

(2) Let $J^2 = \mathbf{r}(a_1, a_2, \ldots, a_n)$. Define $\phi : R/J^2 \to R_R^n$ via $\phi(a + J^2) = (a_1 a, a_2 a, \ldots, a_n a)$ for $a \in R$. Then ϕ is a monomorphism. Hence we may regard J/J^2 as a submodule of R_R^n. Note that R is left GPF by Lemma 2.2, and so $J/J^2 = Soc(J/J^2) \subseteq Soc(R_R^n) = (Soc(R_R))^n = S_r^n$. Since S_r is finitely generated by Chen and Ding (1999, Theorem 2.8), so is $(S_r)^n$. As a direct summand of $(S_r)^n$, J/J^2 is a finitely generated right R-module.

Let $J/J^2 = \bigoplus_{i=1}^n M_i$, where each M_i is a simple right R-module. Note that a right WPF-ring is right Kasch and right mininjective, and so each $(M_i)^* = Hom_R(M_i, R)$ is a simple left R-module. Let $S_2 = Soc_2({}_R R) = Soc_2(R_R)$ and $S = Soc({}_R R) = Soc(R_R)$, then $S_2/S = \mathbf{l}(J^2)/\mathbf{l}(J) \cong Hom(J/J^2, R) \cong \bigoplus_{i=1}^n (M_i)^*$ is a finitely generated left R-module by Lemmas 2.2 and 2.7. Since S is a finitely generated left R-module, so is S_2.

Lemma 2.9. *Let R be a left (resp. right) perfect ring. If J/J^2 is a finitely generated right (resp. left) R-module, then R is right (resp. left) artinian.*

Proof. See Rowen (1988, Exercise 8, p. 321) or Osofsky (1966, Lemma 11).

Corollary 2.10. *Let R be a left perfect and right simple-injective ring. Assume either*

(1) *R is left semi-dual, or*
(2) *$J^2 = \mathbf{r}(A)$ for a finite subset A of R.*

Then R is QF.

Proof. Clearly R is right WPF, and so J/J^2 is a finitely generated right R-module by Theorem 2.8. Hence R is right artinian by Lemma 2.9. Thus R is QF by Nicholson and Yousif (1997a, Corollary 4.8) since it is left and right mininjective.

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Remark 4. In general, if R is a left perfect left GPF ring with $J^2 = \mathbf{r}(A)$ for a finite subset A of R, R need not be QF. In fact, there is a two-sided perfect left P-injective ring and $J^2 = 0 = \mathbf{r}(1)$ which is not QF (see Rutter, 1975, Example 1).

3. ON F-INJECTIVE PERFECT RINGS

A ring R is called right f-injective (2-injective) if every right R-homomorphism from a finitely generated (2-generated) right ideal of R to R extends to an endomorphism of R.

Lemma 3.1. *Let R be a semilocal, right Kasch and right 2-injective ring. Then*

(1) *R is left and right Kasch.*
(2) *R is left and right P-injective.*
(3) $S_r = S_l$.
(4) $J = \mathbf{l}(S) = \mathbf{r}(S)$, *where* $S = S_r = S_l$.
(5) $S = \mathbf{l}(J) = \mathbf{r}(J)$, *where* $S = S_r = S_l$.
(6) *R is left and right finitely cogenerated.*
(7) $J = \mathbf{r}(k_1, k_2, \ldots, k_n) = \mathbf{l}(m_1, m_2, \ldots, m_s)$, *where* k_i, $m_j \in R$, $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, s$. *Moreover the elements k_i and m_j can be chosen so that Rk_i, k_iR, Rm_j and m_jR are simple.*

Proof. Since R is right Kasch and right 2-injective, R is left P-injective by Nicholson and Yousif (1995, Lemma 2.2). Thus R is left and right P-injective, and so $S_r = S_l$. It follows that R is left and right Kasch by Nicholson and Yousif (2001b, Lemma 3.3) (for R is two-sided mininjective). By Chen and Ding (1999, Theorem 2.8), R is left and right finitely cogenerated. Since R is semilocal, $\mathbf{l}(J) = Soc(R_R) = S = Soc({}_RR) = \mathbf{r}(J)$. Since R is left Kasch, $J = \mathbf{l}(S_l)$. Similarly, $J = \mathbf{r}(S_r)$. Since R is right finitely cogenerated, S_r is finitely generated. Let $S_r = m_1R + m_2R + \cdots + m_sR$, where each m_jR is a simple right ideal, $j = 1, 2, \ldots, s$. Since R is right P-injective, each Rm_j is a simple left ideal. Clearly, $J = \mathbf{l}(m_1, m_2, \ldots, m_s)$. Similarly, $J = \mathbf{r}(k_1, k_2, \ldots, k_n)$, where both Rk_i and k_iR are simple, $i = 1, 2, \ldots, n$.

Corollary 3.2. *The following are equivalent for a ring R.*

(1) *R is semilocal, right 2-injective and right Kasch.*
(2) *R is semilocal, right 2-injective and* $J = \mathbf{r}(k_1, k_2, \ldots, k_n)$, *where* $k_i \in R$, $i = 1, 2, \ldots, n$.
(3) *R is right finitely cogenerated, right 2-injective and right Kasch.*
(4) *R is left finitely cogenerated, right 2-injective and right Kasch.*
(5) *R is right finite dimensional, right 2-injective and right Kasch.*
(6) *R is left finite dimensional, right 2-injective and right Kasch.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 3.1(7).

(2) $\Rightarrow$ (1). Let K be a simple right R-module. Then K is a simple right R/J-module. Since R/J is semisimple, there is a monic R/J-homomorphism

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$\phi : K \to R/J$. Clearly, ϕ is a monic R-homomorphism. By hypothesis, $J = \mathbf{r}(k_1, k_2, \ldots, k_n)$, and so there is a monomorphism $\psi : R/J \to R^n$. Hence $f = \psi\phi$ is monic. Let $\pi_i : R^n \to R$ be the ith projection, $i = 1, 2, \ldots, n$. Then it is easy to see $\pi_i f$ is monic for some i. So K embeds in R_R.

(1) ⇒ (3) and (4). By Lemma 3.1(6).

(3) ⇒ (5) Clear.

(5) ⇒ (1) follows since a right P-injective and right finite dimensinal ring is semilocal by Nicholson and Yousif (1995, Theorem 3.3).

(4) ⇒ (6) Obvious.

(6) ⇒ (1) A right Kasch left finite dimensinal ring is semilocal by Gómez Pardo and Guil Asensio (1998, Proposition 2.3).

Remark 5. Since right FP-injective rings are right 2-injective, the above corollary extends the work in Nicholson and Yousif (2001b, Theorem 3.7 (1)(3)(5)(7)).

Theorem 3.3. *The following are equivalent for a ring R.*

(1) *R is semiperfect, right 2-injective and right Kasch.*
(2) *R is semiperfect, right 2-injective and $S_r \leq_e R_R$.*
(3) *R is semiperfect, right 2-injective and $S_r \leq_e {}_RR$.*
(4) *R is left min-CS, right 2-injective and right Kasch.*
(5) *R is right min-CS, right 2-injective and right finitely cogenerated.*

Proof. (1) ⇒ (2) and (3) R is left and right finitely cogenerated and $S_r = S_l$ by Lemma 3.1. So (2) and (3) follow.

(2) ⇒ (1) R is right GPF by hypothesis, and so R is left and right Kasch by Nicholson and Yousif (1995, Corollary 2.3).

(3) ⇒ (1) By the proof of (4) ⇒ (1) in Theorem 2.3.

(1) ⇒ (4) and (5) R is left and right finitely cogenerated and $S_r = S_l$ by Lemma 3.1. Thus $S_l \leq_e R_R$ and $S_r \leq_e {}_RR$, and so $\mathbf{r}(\mathbf{l}(K))$ is essential in a summand of R_R and $\mathbf{l}(\mathbf{r}(I))$ is essential in a summand of ${}_RR$ for every right ideal K and every left ideal I of R by Nicholson and Yousif (2001b, Lemma 3.11). Note that R is left and right P-injective by Lemma 3.1. Hence $aR = \mathbf{r}(\mathbf{l}(a))$ is essential in a summand of R_R and $Ra = \mathbf{l}(\mathbf{r}(a))$ is essential in a summand of ${}_RR$ for any $a \in R$. In particular, R is left and right min-CS.

(4) ⇒ (1) Since R is right Kasch and right mininjective, $\mathbf{l}(M)$ is a minimal left ideal for every maximal right ideal M. Thus $\mathbf{l}(M)$ is essential in a summand of ${}_RR$, and so R is semiperfect by Nicholson and Yousif (2001b, Proposition 3.14).

(5) ⇒ (2) This can be proven in the same way as in the proof of (9) ⇒ (3) in Theorem 2.3 by noting that a right 2-injective ring is a right $C2$-ring.

Lemma 3.4. *Let R be a semiperfect, right f-injective ring with $S_r \leq_e R_R$. Then*

(1) *R is left and right Kasch.*

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(2) *R is left and right finitely cogenerated and $S_r = S_l$.*
(3) *Every finitely generated right ideal of R is a right annihilator.*

Proof. R is a right *GPF* ring by hypothesis, and so it is right Kasch by Nicholson and Yousif (1995, Corollary 2.3).

(1) and (2) follow from Lemma 3.1.

(3) holds by Björk (1970, Proposition 4.2) or Chen et al. (2001, Corollary 2.8(3)).

The next theorem extends the work in Clark and Huynh (1994a, Corollary) and Xue (1996, Proposition 9).

Theorem 3.5. *Let R be a one-sided perfect and left f-injective ring with $S_l \leq_e {}_RR$. If $\mathbf{r}(J) = \mathbf{r}(A)$ for some finitely generated left ideal A, then R is QF.*

Proof. Since R is semilocal, $\mathbf{r}(J) = S_l$. Hence $J = \mathbf{l}(S_l) = \mathbf{l}(\mathbf{r}(J)) = \mathbf{l}(\mathbf{r}(A)) = A$ is a finitely generated left ideal by the left version of Lemma 3.4. Thus J is nilpotent by Lam (1995, Exercise 23.1, p. 259) or Kasch (1982, Exercise 9, p. 305), and hence R is semiprimary. So R is left artinian by Lemma 2.9. Note that R is left f-injective. Therefore R is QF.

Recall that a ring R is called an *FP*-ring (Nicholson and Yousif, 2001a) if R is semiperfect, right *FP*-injective and $S_r \leq_e R_R$, or equivalently, if R is semiperfect, left *FP*-injective and $S_l \leq_e {}_RR$.

Corollary 3.6. *Let R be a one-sided perfect FP-ring. If $\mathbf{r}(J) = \mathbf{r}(A)$ for some finitely generated left ideal A, then R is QF.*

Corollary 3.7. *Let R be a left perfect and right FP-injective (or right P-injective and right IN) ring. If $\mathbf{r}(J) = \mathbf{r}(A)$ for some finitely generated left ideal A, then R is QF.*

Proof. If R is a left perfect and right *FP*-injective ring, then R is an *FP*-ring.

If R is a left perfect, right P-injective and right *IN*, then R is right *WPF*. Thus R is left f-injective by Chen et al. (2001, Theorem 2.13) and left finitely cogenerated by Theorem 2.3. So the result follows from Theorem 3.5.

Lemma 3.8. *Let R be a semiperfect, left and right f-injective ring with $S_r \leq_e R_R$. Then J is finitely generated as a left ideal if and only if R/S is finitely cogenerated as a right R-module, where $S = S_r = S_l$.*

Proof. "$\Leftarrow$". By Lemma 3.4(2), S is a finitely generated right ideal. Thus S is a right annihilator by Lemma 3.4(3), and so R/S is a torsionless right R-module. Since R/S is a finitely cogenerated right R-module, there is a monomorphism $\phi : R/S \to R^n$ for some positive integer n. Let $\phi(1 + S) = (a_1, a_2, \ldots, a_n)$, then $S = \mathbf{r}(a_1, a_2, \ldots, a_n)$. Since R is left Kasch and left f-injective, $J = \mathbf{l}(S_l) = \mathbf{l}(S) = \mathbf{l}(\mathbf{r}(a_1, a_2, \ldots, a_n)) = Ra_1 + Ra_2 + \cdots + Ra_n$ is a finitely generated left ideal.

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"$\Rightarrow$". Let $J = Rx_1 + Rx_2 + \cdots + Rx_n$. Since R is semilocal, $S = \mathbf{r}(J) = \mathbf{r}(x_1, x_2, \ldots, x_n)$. Hence R/S embeds in the right R-module R^n. Since R is a right finitely cogenerated right R-module, so is R^n. Therefore the result follows.

The following theorem extends the work in Clark and Huynh (1994a, Theorem), Nicholson and Yousif (2001a, Theorem 7) and Xue (1996, Theorem 7).

Theorem 3.9. *Let R be a left perfect, left and right f-injective ring. Then*

(1) *R is QF if and only if $Soc_2(R)$ is a finitely generated right R-module.*
(2) *R is QF if and only if R/S is a finitely cogenerated left R-module, where $S = S_r = S_l$.*
(3) *If R is also right perfect, then R is QF if and only if $Soc_2(R)$ is a finitely generated left R-module.*

Proof. (1) Since R is left perfect, it is right semiartinian by Stenström (1975). Hence R/S has an essential right socle. Note that $Soc(R/S) = Soc_2(R)/S$. If $Soc_2(R)$ is a finitely generated right R-module, so is $Soc(R/S)$. Thus R/S is a finitely cogenerated right R-module, and so J is a finitely generated left ideal by Lemma 3.8. Therefore J is nilpotent by Lam (1995, Exercise 23.1, p. 259) or Kasch (1982, Exercise 9, p. 305), and hence R is semiprimary. So R is left artinian by Lemma 2.9. Since R is left f-injective, it is injective. Thus R is QF.

(2) If R/S is a finitely cogenerated left R-module, then J is a finitely generated right ideal by Lemmas 3.4 and 3.8. Note that R is left perfect, and so R is right artinian by Lemma 2.9. Clearly R is QF.

(3) If R is also right perfect, it is left semiartinian. Thus R/S has an essential left socle. If $Soc_2(R)$ is a finitely generated left R-module, so is $Soc(R/S)$. Thus R/S is a finitely cogenerated left R-module. Therefore R is QF by (2).

Corollary 3.10. *Let R be a left perfect, right P-injective and right IN-ring. Then*

(1) *R is QF if and only if $Soc_2(R)$ is a finitely generated right R-module.*
(2) *R is QF if and only if R/S is a finitely cogenerated left R-module.*
(3) *If R is also right perfect, then R is QF if and only if $Soc_2(R)$ is a finitely generated left R-module.*

Proof. It is clear that R is right f-injective. R is left f-injective by the proof of Corollary 3.7.

Corollary 3.11. *Let R be a left perfect, right P-injective and right IN-ring. If J/J^2 is countably generated as a left R-module, then R is QF.*

Proof. The result follows from Corollary 3.10 and Nicholson and Yousif (1997b, Remark (1), p. 983).

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