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ON RELATIVE INJECTIVE MODULES AND RELATIVE COHERENT RINGS

Lixin Mao
Department of Basic Courses, Nanjing Institute of Technology, Nanjing, China and Department of Mathematics, Nanjing University, Nanjing, China

Nanqing Ding
Department of Mathematics, Nanjing University, Nanjing, China

Let m and n be fixed positive integers and M a right R-module. Recall that M is said to be (m, n)-injective if $\mathrm{Ext}_R^1(P, M) = 0$ for any (m, n)-presented right R-module P; M is called (m, n)-flat if $\mathrm{Tor}_1^R(M, Q) = 0$ for any (m, n)-presented left R-module Q. In this article, M is defined to be (m, n)-projective (resp. (m, n)-cotorsion) if $\mathrm{Ext}_R^1(M, N) = 0$ (resp. $\mathrm{Ext}_R^1(N, M) = 0$) for any (m, n)-injective (resp. (m, n)-flat) right R-module N. These concepts are used to characterize von Neumann regular rings and (m, n)-coherent rings. Some known results are extended.

Key Words: Cotorsion theory; (m, n)-coherent ring; (m, n)-cotorsion module; (m, n)-flat module; (m, n)-injective module; (m, n)-projective module; Preenvelope.

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1. INTRODUCTION

Throughout this article, R is an associative ring with identity and all modules are unitary. For an R-module M, $E(M)$ denotes the injective envelope of M, the dual module $\mathrm{Hom}_R(M, R)$ is denoted by M^*, and the character module M^+ is defined by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. $\sigma_M : M \to M^{++}$ ($\delta_M : M \to M^{**}$) stands for the evaluation map. For an index set I, $M^{(I)}$ (M^I) is the direct sum (the direct product) of copies of M indexed by I. If n is a positive integer, M^n means the direct sum of n copies of M.

Given a class $\mathscr{C}$ of right R-modules, we will denote by $\mathscr{C}^{\perp} = \{X : \mathrm{Ext}_R^1(C, X) = 0 \text{ for all } C \in \mathscr{C}\}$ the right orthogonal class of $\mathscr{C}$, and by ${}^{\perp}\mathscr{C} = \{X : \mathrm{Ext}_R^1(X, C) = 0 \text{ for all } C \in \mathscr{C}\}$ the left orthogonal class of $\mathscr{C}$.

Let $\mathscr{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathscr{C}$ is called a *$\mathscr{C}$-preenvelope of M* (Enochs, 1981) if for any homomorphism $f : M \to F'$ with $F' \in \mathscr{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F

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Address correspondence to Nanqing Ding, Department of Mathematics, Nanjing University, Hankou Road, #22, Nanjing 210093 China; E-mail: nqding@nju.edu.cn

when $F' = F$ and $f = \phi$, the $\mathscr{C}$-preenvelope ϕ is called a *$\mathscr{C}$-envelope of M*. Following Enochs and Jenda (2000, Definition 7.1.6), a monomorphism $\alpha : M \to C$ with $C \in \mathscr{C}$ is said to be a *special $\mathscr{C}$-preenvelope of M* if $\mathrm{coker}(\alpha) \in {}^{\perp}\mathscr{C}$. Dually, we have the definitions of a *(special) $\mathscr{C}$-precover* and a *$\mathscr{C}$-cover*. Special $\mathscr{C}$-preenvelopes (resp. special $\mathscr{C}$-precovers) are obviously $\mathscr{C}$-preenvelopes (resp. $\mathscr{C}$-precovers). $\mathscr{C}$-envelopes ($\mathscr{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(\mathscr{F}, \mathscr{C})$ of classes of right R-modules is called a *cotorsion theory* (Enochs and Jenda, 2000) if $\mathscr{F}^{\perp} = \mathscr{C}$ and ${}^{\perp}\mathscr{C} = \mathscr{F}$. A cotorsion theory $(\mathscr{F}, \mathscr{C})$ is called *perfect* (*complete*) if every right R-module has a $\mathscr{C}$-envelope and an $\mathscr{F}$-cover (a special $\mathscr{C}$-preenvelope and a special $\mathscr{F}$-precover) (see Enochs et al., 2004; Trlifaj, 2000). A cotorsion theory $(\mathscr{F}, \mathscr{C})$ is said to be *hereditary* (Enochs et al., 2004) if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathscr{F}$, then L' is also in $\mathscr{F}$. By Enochs et al. (2004, Proposition 1.2), $(\mathscr{F}, \mathscr{C})$ is hereditary if and only if $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in \mathscr{C}$, then C'' is also in $\mathscr{C}$.

Let m and n be two fixed positive integers. Recently, (m, n)-injective modules, (m, n)-flat modules and (m, n)-coherent rings were introduced and studied in Chen et al. (2001), Shamsuddin (2001), and Zhang et al. (2005). In this article, the concepts of *(m, n)-projective modules* and *(m, n)-cotorsion modules* are introduced. A right R-module M is said to be *(m, n)-projective* (resp. *(m, n)-cotorsion*) if $\mathrm{Ext}^1_R(M, N) = 0$ (resp. $\mathrm{Ext}^1_R(N, M) = 0$) for any (m, n)-injective (resp. (m, n)-flat) right R-module N. Using the results of Eklof and Trlifaj (2001) and Trlifaj (2000), we prove that $(\mathscr{P}_{m,n}, \mathscr{I}_{m,n})$ $((\mathscr{F}_{m,n}, \mathscr{C}_{m,n}))$ is a complete (perfect) cotorsion theory, where $\mathscr{P}_{m,n}$ $(\mathscr{I}_{m,n})$ stands for the class of all (m, n)-projective ((m, n)-injective) left R-modules and $\mathscr{F}_{m,n}$ $(\mathscr{C}_{m,n})$ the class of all (m, n)-flat ((m, n)-cotorsion) right modules (Theorem 2.3). As a simple application, we get some new characterizations of von Neumann regular rings (Corollary 2.6). It is also shown that R is a left (m, n)-coherent ring if and only if every ((n, m)-presented) right R-module has an (m, n)-flat preenvelope (Theorem 3.1); R is a left (m, n)-coherent left (m, n)-injective ring if and only if every ((n, m)-presented) right R-module has a monic (m, n)-flat preenvelope (Theorem 4.1); R is a left (m, n)-coherent ring and submodules of (m, n)-flat right R-modules are (m, n)-flat if and only if every ((n, m)-presented) right R-module has an epic (m, n)-flat preenvelope (Theorem 5.1). In particular, some known results are obtained as corollaries.

General background materials can be found in Anderson and Fuller (1974), Enochs and Jenda (2000), Lam (1999), and Rotman (1979).

2. DEFINITIONS AND GENERAL RESULTS

Let R be a ring, m and n two fixed positive integers, and M a right R-module.

M is called *(m, n)-presented* (Zhang et al., 2005) if there exists a right R-module exact sequence $0 \to K \to R^m \to M \to 0$, where K is n-generated. In particular, $(1, 1)$-presented modules are just cyclically presented modules in sense of Wisbauer (1991).

M is said to be *(m, n)-injective* (Chen et al., 2001) in case every right R-homomorphism from an n-generated submodule of R^m to M extends to one from R^m to M, or equivalently, $\mathrm{Ext}^1_R(P, M) = 0$ for all (m, n)-presented right R-modules P. This definition unifies several definitions on generalizations of injectivity of modules, such as f-injective modules (Gupta, 1969), divisible modules (Lam, 1999; Trlifaj, 2000) and *FP*-injective modules (Jain, 1973). (m, n)-*injective modules* are also called (R^n, R^m)-*injective modules* in the sense of Shamsuddin (2001). The ring R is a *right* (m, n)-*injective ring* if R is (m, n)-injective as a right R-module.

M is called (m, n)-*flat* (Zhang et al., 2005) in case $0 \to M \otimes_R I \to M \otimes_R R^m$ is exact for every n-generated submodule I of the left R-module R^m, or equivalently, $\mathrm{Tor}_1^R(M, N) = 0$ for any (m, n)-presented left R-module N. It is obvious that M is flat if and only if M is (m, n)-flat for all m and n if and only if M is $(1, n)$-flat for all n. M is $(1, n)$-flat if and only if M is n-flat in the sense of Shamsuddin (2001).

Definition 2.1. Let m and n be two fixed positive integers. A right R-module M is said to be (m, n)-*projective* (resp. (m, n)-cotorsion) if $\mathrm{Ext}_R^1(M, N) = 0$ (resp. $\mathrm{Ext}_R^1(N, M) = 0$) for any (m, n)-injective (resp. (m, n)-flat) right R-module N.

The left version can be defined similarly.

Remark 2.2. (1) Any (m, n)-presented R-module is (m, n)-projective.

(2) Note that $(1, 1)$-injective modules are exactly divisible modules. So $(1, 1)$-projective modules are just cyclically covered modules in the sense of Trlifaj (2000).

Since *FP*-injective modules $\Rightarrow (m, n)$-injective modules $\Rightarrow (1, 1)$-injective modules (see Chen et al., 2001, Remark 2.3), cyclically covered modules $\Rightarrow (m, n)$-projective modules $\Rightarrow$ finitely covered modules (see Trlifaj, 2000, Theorem 3.4).

(3) Every (m, n)-cotorsion module is cotorsion (Enochs and Jenda, 2000) (a right R-module C is called *cotorsion* provided that $\mathrm{Ext}_R^1(F, C) = 0$ for any flat right R-module F).

In what follows, we assume that m and n are fixed positive integers. $\mathscr{P}_{m,n}(\mathscr{I}_{m,n})$ stands for the class of all (m, n)-projective ((m, n)-injective) left R-modules, $\mathscr{F}_{m,n}$ ($\mathscr{C}_{m,n}$) denotes the class of all (m, n)-flat ((m, n)-cotorsion) right modules.

As is known to all, every R-module has a cotorsion envelope and a flat cover (Bican et al., 2001).

Now we have the following theorem.

Theorem 2.3. *Let R be a ring. Then*:

(1) $(\mathscr{P}_{m,n}, \mathscr{I}_{m,n})$ *is a complete cotorsion theory*;
(2) $(\mathscr{F}_{m,n}, \mathscr{C}_{m,n})$ *is a perfect cotorsion theory.*

Proof. (1) Let X be the set of representatives of (m, n)-presented left R-modules. Thus $\mathscr{I}_{m,n} = X^{\perp}$. Since $\mathscr{P}_{m,n} = {}^{\perp}(X^{\perp})$, the result follows from Eklof and Trlifaj (2001, Theorem 10) and Enochs and Jenda (2000, Definition 7.1.5).

(2) Note that a right R-module M is (m, n)-flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for all (m, n)-presented left R-modules N. So the result follows from Trlifaj (2000, Lemma 1.11 and Theorem 2.8). □

Remark 2.4. (1) By Theorem 2.3(1) and the proof of Trlifaj (2000, Theorem 3.4), a left R-module M is (m, n)-*projective* if and only if M is a direct summand of a left R-module N such that N is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is an (m, n)-presented left R-module for all $\alpha < \lambda$ (cf. Trlifaj, 2000, Definition 3.3).

(2) The statement of Theorem 2.3(1) is the best possible in the sense that $(\mathcal{P}_{m,n}, \mathcal{I}_{m,n})$ is not a perfect cotorsion theory because $\mathcal{I}_{m,n}$-envelopes may not exist in general (see Trlifaj, 2000, Proposition 4.8). However, if $\mathcal{P}_{m,n}$ is closed under direct limits, then $(\mathcal{P}_{m,n}, \mathcal{I}_{m,n})$ is a perfect cotorsion theory by Theorem 2.3(1) and Enochs and Jenda (2000, Theorem 7.2.6).

(3) Recall that a right R-module M is called *torsionfree* (Trlifaj, 2000, p. 4) if $\mathrm{Tor}_1^R(M, R/Rr) = 0$ for all $r \in R$ (if R is a (commutative integral) domain, an R-module M is torsionfree if and only if $rx = 0$ for $r \in R$ and $x \in M$ implies $r = 0$ or $x = 0$). Clearly, M is torsionfree if and only if M is $(1, 1)$-flat, and hence M is $(1, 1)$-cotorsion if and only if M is Warfield cotorsion (Trlifaj, 2000, p. 9). So Trlifaj (2000, Theorem 3.1.2) is a particular case of Theorem 2.3(2) where $m = n = 1$.

(4) For every positive integer n, there exists a ring with a left ideal M which is $(1, n)$-flat but not $(1, n+1)$-flat (see Shamsuddin, 2001, Example 5.2). Thus M^+ is $(1, n)$-injective but not $(1, n+1)$-injective by Zhang et al. (2005, Theorem 4.3). This fact shows that there exists a module which is $(1, n+1)$-projective (resp. $(1, n+1)$-cotorsion) but not $(1, n)$-projective (resp. $(1, n)$-cotorsion).

If R is a domain, then R is a Dedekind ring if and only if every $(1, 1)$-injective R-module is injective if and only if every cyclic R-module is a direct summand of a direct sum of cyclically presented modules (Wisbauer, 1991, 40.5). Here we get the following corollary.

Corollary 2.5. *Let m and n be two fixed positive integers. The following are equivalent for a ring R:*

(1) *Every left R-module is (m, n)-projective;*
(2) *Every cyclic left R-module is (m, n)-projective;*
(3) *Every (m, n)-injective left R-module is injective;*
(4) *$(\mathcal{P}_{m,n}, \mathcal{I}_{m,n})$ is hereditary, and every (m, n)-injective left R-module is (m, n)-projective.*

In this case, R is left Noetherian.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear. (3) $\Rightarrow$ (1) holds by Theorem 2.3(1).

(2) $\Rightarrow$ (3) Let M be any (m, n)-injective left R-module and I any left ideal of R. Then $\mathrm{Ext}_R^1(R/I, M) = 0$ by (2). Thus M is injective, as desired.

(4) $\Rightarrow$ (1) By Theorem 2.3(1), for any left R-module M, there is a short exact sequence $0 \to M \to F \to L \to 0$, where F is (m, n)-injective and L is (m, n)-projective. So (1) follows from (4).

In this case, R is left Noetherian since every FP-injective left R-module is injective. $\square$

It is well known that R is a von Neumann regular ring if and only if every left R-module is $(1, 1)$-injective if and only if every cotorsion left R-module is injective if and only if every cotorsion right R-module is flat (Xu, 1996, Theorem 3.3.2). This fact can be generalized in the following corollary.

Corollary 2.6. *The following are equivalent for a ring R:*

(1) *R is a von Neumann regular ring;*
(2) *There exist m and n such that every left R-module is (m, n)-injective;*
(3) *There exist m and n such that every cotorsion left R-module is (m, n)-injective;*
(4) *There exist m and n such that every (m, n)-projective left R-module is projective;*
(5) *There exist m and n such that every right R-module is (m, n)-flat;*
(6) *There exist m and n such that every cotorsion right R-module is (m, n)-flat;*
(7) *There exist m and n such that every (m, n)-cotorsion right R-module is injective;*
(8) *There exist m and n such that $(\mathscr{F}_{m,n}, \mathscr{C}_{m,n})$ is hereditary and every (m, n)-cotorsion right R-module is (m, n)-flat;*
(9) *There exist m and n such that $(\mathscr{P}_{m,n}, \mathscr{I}_{m,n})$ is hereditary and every (m, n)-projective left R-module is (m, n)-injective.*

Proof. $(1) \Rightarrow (5) \Rightarrow (6)$ and $(2) \Rightarrow (3)$ are clear.

$(6) \Rightarrow (2)$ Let M be any left R-module. Note that M is a pure submodule of M^{++}. Since M^{+} is pure-injective and hence cotorsion, M^{+} is (m, n)-flat by (6). Thus M^{++} is (m, n)-injective by Zhang et al. (2005, Theorem 4.3), and so M is (m, n)-injective by Zhang et al. (2005, Corollary 2.7).

$(3) \Rightarrow (1)$ Let M be any left R-module. We get the exact sequence $0 \to M \to C \to L \to 0$, where $M \to C$ is a cotorsion envelope and L is flat. Since C is (m, n)-injective by (3) and M is a pure submodule of C, M is (m, n)-injective by Zhang et al. (2005, Corollary 2.7). Thus M is $(1, 1)$-injective, and so (1) follows.

$(2) \Leftrightarrow (4)$ and $(5) \Leftrightarrow (7)$ follow from Theorem 2.3.

$(2) \Rightarrow (9)$ and $(5) \Rightarrow (8)$ are obvious.

$(8) \Rightarrow (5)$ Let M be any right R-module. By Theorem 2.3(2), there exists an exact sequence $0 \to M \to C \to L \to 0$ with C (m, n)-cotorsion and L (m, n)-flat. So (5) follows from (8).

$(9) \Rightarrow (2)$ is similar to $(8) \Rightarrow (5)$ by Theorem 2.3(1). □

3. CHARACTERIZATIONS OF (m, n)-COHERENT RINGS

A ring R is called *left (m, n)-coherent* (Zhang et al., 2005) in case each n-generated submodule of the left R-module R^m is finitely presented. Clearly, R is left $(1, 1)$-coherent if and only if the left annihilator of any element of R is a finitely generated left ideal; R is left $(1, n)$-coherent if and only if R is left n-coherent in sense of Shamsuddin (2001); R is left coherent (i.e., every finitely generated left ideal of R is finitely presented) if and only if R is left (m, n)-coherent for all m and n if and only if R is left $(1, n)$-coherent for all n; R is left pseudo-coherent (Björk, 1970) (i.e., the left annihilator of every finite subset of R is finitely generated) if and only if R is left $(m, 1)$-coherent for all m.

Recall that, if M is a right R-module and U a subgroup of the Abelian group M, then U is said to be a *matrix subgroup of* M (Zimmermann, 1977) if there is a right R-module A and an element $x \in A$ such that U equals the set $H_{A,x}(M) = \{f(x) : f \in \mathrm{Hom}_R(A, M)\}$. Of course, every matrix subgroup of M is a left submodule of M over the endomorphism ring $\mathrm{End}_R M$.

Now we have the following characterizations of (m, n)-coherent rings.

Theorem 3.1. *The following are equivalent for a ring R:*

(1) *R is a left (m, n)-coherent ring;*
(2) *Any direct product of R as a right R-module is (m, n)-flat;*
(3) *Every direct product of (m, n)-flat right R-modules is (m, n)-flat;*
(4) *For any (n, m)-presented right R-module M, M^* is finitely generated;*
(5) *All matrix subgroups of the form $H_{M,x}(R)$ with M an (n, m)-presented right R-module and $x \in M$ are finitely generated left ideals of R;*
(6) *For any projective left R-module M, M^* is (m, n)-flat;*
(7) *Every right R-module has an (m, n)-flat preenvelope;*
(8) *Every (n, m)-presented right R-module has an (m, n)-flat preenvelope;*
(9) *Every (n, m)-presented right R-module has a projective preenvelope.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Zhang et al. (2005, Theorems 5.4 and 5.7).

(2) $\Rightarrow$ (6) For any projective left R-module P, there is a projective left R-module Q and an index set I such that $P \oplus Q \cong R^{(I)}$. So we have

$$P^* \oplus Q^* \cong (R^{(I)})^* \cong R^I.$$

Thus P^* is (m, n)-flat by (2).

(6) $\Rightarrow$ (2) is obvious by choosing P to be $R^{(I)}$ for any index set I.

(3) $\Rightarrow$ (7) Let N be any right R-module. By Enochs and Jenda (2000, Lemma 5.3.12), there is a cardinal number $\aleph_\alpha$ such that for any R-homomorphism $f : N \to L$ with L (m, n)-flat, there is a pure submodule Q of L such that $\mathrm{Card}(Q) \leq \aleph_\alpha$ and $f(N) \subseteq Q$. Note that Q is (m, n)-flat by Zhang et al. (2005, Proposition 4.5), and so N has an (m, n)-flat preenvelope by (3) and Enochs and Jenda (2000, Proposition 6.2.1).

(7) $\Rightarrow$ (8) is trivial.

(8) $\Rightarrow$ (9) Let $f : M \to N$ be an (m, n)-flat preenvelope with M (n, m)-presented. By Zhang et al. (2005, Theorem 4.3), there exist a finitely generated free right R-module R^k and homomorphisms $g : M \to R^k$, $h : R^k \to N$ such that $f = hg$. It is easy to check that g is a projective preenvelope of M.

(9) $\Rightarrow$ (5) Let M be an (n, m)-presented right R-module. Then, by (9), M has a projective preenvelope $f : M \to R^t$ with $t \in \mathbb{N}$. Let $\pi_k : R^t \to R$ be the canonical projection and $i_k : R \to R^t$ the canonical injection, $k = 1, 2, \ldots, t$. For any

$\alpha : M \to R$, there is $h : R^t \to R$ such that $hf = \alpha$. Set $g_k = \pi_k f$ and $s_k = hi_k$, $k = 1, 2, \ldots, t$. Then $\alpha(x) = \sum_{k=1}^{t} s_k g_k(x)$ for any $x \in M$. It follows that $H_{M,x}(R) = \sum_{k=1}^{t} R g_k(x)$, and hence (5) holds.

(5) $\Rightarrow$ (4) For any (n, m)-presented right R-module M, noting that M is finitely generated, we shall show that M^* is a finitely generated left R-module by induction on the number k of generating elements of M.

If $k = 1$ and $M = aR$. We define a left R-homomorphism $\beta : M^* \to H_{M,a}(R)$ via $f \mapsto f(a)$. It is clear that β is an isomorphism. Thus M^* is a finitely generated left R-module by (5).

Now suppose that the result is true for $k - 1$, and $M_R = a_1 R + a_2 R + \cdots + a_k R$. Consider the exact sequence $0 \to a_1 R \to M \to M/a_1 R \to 0$, which induces the exactness of the left R-module sequence

$$0 \to (M/a_1 R)^* \to M^* \to H_{M,a_1}(R) \to 0.$$

Since $(M/a_1 R)^*$ and $H_{M,a_1}(R)$ are finitely generated, so is M^*. □

Note that a ring R is left coherent if and only if R is left (m, n)-coherent for all m and n; R is left pseudo-coherent if and only if R is left $(m, 1)$-coherent for all m. So we have

Corollary 3.2. *Let R be a ring. Then*:

(1) (*Asensio Mayor and Martinez Hernandez,* 1988, *Proposition* 2) *R is left coherent if and only if every finitely presented right R-module has a projective preenvelope*;
(2) *R is left pseudo-coherent if and only if every finitely presented cyclic right R-module has a projective preenvelope*;
(3) *R is left* $(1, 1)$*-coherent if and only if every cyclically presented right R-module has a projective preenvelope.*

Corollary 3.3. *If R is a left (m, n)-coherent ring, then M^+ has an (m, n)-injective precover for any right R-module M.*

Proof. By Theorem 3.1, M has an (m, n)-flat preenvelope $f : M \to N$. We shall show that $f^+ : N^+ \to M^+$ is an (m, n)-injective precover of M^+. Indeed, let $\psi : H \to M^+$ be any homomorphism with H (m, n)-injective. Note that H^+ is (m, n)-flat by Zhang et al. (2005, Theorem 5.7(7)), so there exists $g : N \to H^+$ such that $gf = \psi^+ \sigma_M$. Thus $f^+ g^+ = \sigma_M^+ \psi^{++}$. Since $\psi^{++} \sigma_H = \sigma_{M^+} \psi$, $f^+(g^+ \sigma_H) = \sigma_M^+(\psi^{++} \sigma_H) = (\sigma_M^+ \sigma_{M^+})\psi = \psi$ by Anderson and Fuller (1974, Proposition 20.14), as required. □

Lemma 3.4. *Let R be left (m, n)-coherent and M an (n, m)-presented right R-module. Then M has a finitely generated projective envelope if and only if M^* has a projective cover.*

Proof. The proof is modeled on that of Asensio Mayor and Martinez Hernandez (1988, Proposition 1).

"$\Rightarrow$" Let $\phi : M \to F$ be a finitely generated projective envelope of M, and $\phi^* : F^* \to M^*$ the induced left R-module homomorphism. We shall show that ϕ^* is a projective cover of M^*. Note that F^* is a finitely generated projective left R-module and $\phi^* : F^* \to M^*$ is an epimorphism. So it is enough to show that if $g : F^* \to F^*$ satisfies $\phi^* g = \phi^*$, then g is an isomorphism. In fact,

$$\delta_F^{-1} g^* \delta_F \phi = \delta_F^{-1} g^* \phi^{**} \delta_M = \delta_F^{-1} (\phi^* g)^* \delta_M = \delta_F^{-1} \phi^{**} \delta_M = \phi.$$

Thus $\delta_F^{-1} g^* \delta_F$ is an isomorphism of F, and so $g = \delta_{F^*}^{-1} g^{**} \delta_{F^*}$ is an isomorphism.

"$\Leftarrow$" Let $h : P \to M^*$ be a projective cover of M^*. Since R is left (m, n)-coherent and M is (n, m)-presented, M^* is finitely generated by Theorem 3.1(4). It is easy to verify that P is finitely generated projective.

Let $f = h^* \delta_M : M \to P^*$. We claim that f is a finitely generated projective envelope of M. Indeed, let $\psi : M \to F$ be an R-homomorphism with F finitely generated projective. Since F^* is a finitely generated projective left R-module, there exists an R-homomorphism $\theta : F^* \to P$ such that $h\theta = \psi^*$.

Now let $\alpha = \delta_F^{-1} \theta^* : P^* \to F$, then

$$\alpha f = \delta_F^{-1} \theta^* h^* \delta_M = \delta_F^{-1} (h\theta)^* \delta_M = \delta_F^{-1} \psi^{**} \delta_M = \psi.$$

Therefore, f is a finitely generated projective preenvelope of M.

On the other hand, let $\gamma : P^* \to P^*$ satisfy $\gamma f = f$. Note that $\delta_{M^*} h = h^{**} \delta_P$ and $\delta_M^* \delta_{M^*} = 1_{M^*}$ by Anderson and Fuller (1974, Proposition 20.14), so $h = \delta_M^* h^{**} \delta_P$. Thus

$$h(\delta_P^{-1} \gamma^* \delta_P) = \delta_M^* h^{**} \gamma^* \delta_P = (\gamma h^* \delta_M)^* \delta_P = (\gamma f)^* \delta_P = f^* \delta_P = \delta_M^* h^{**} \delta_P = h,$$

and hence $\delta_P^{-1} \gamma^* \delta_P$ is an isomorphism. It follows that γ is an isomorphism. This completes the proof. □

Proposition 3.5. *Let R be a semiperfect ring. Then R is left (m, n)-coherent if and only if every (n, m)-presented right R-module has a finitely generated projective envelope.*

Proof. "$\Leftarrow$" holds by Theorem 3.1.

"$\Rightarrow$" It follows from Theorem 3.1(4), Lemma 3.4 and the fact R is a semiperfect ring if and only if every finitely generated R-module has a projective cover (see Anderson and Fuller, 1974, Theorem 27.6). □

If R is a domain, then R is a Prüfer ring if and only if every torsionfree R-module is flat if and only if every $(1, 1)$-injective R-module is FP-injective if and only if every finitely presented cyclic R-module is a direct summand of a direct sum of cyclically presented modules (Wisbauer, 1991, 40.4). We end this section with the following proposition.

Proposition 3.6. *The following are equivalent for a left (m, n)-coherent ring R:*

(1) *Every (m, n)-flat right R-module is flat;*
(2) *Every cotorsion right R-module is (m, n)-cotorsion;*

(3) *Every* (m, n)*-injective left* R*-module is* FP*-injective*;
(4) *Every finitely covered left* R*-module is* (m, n)*-projective.*

In this case, R *is a left coherent ring.*

Proof. (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) follow from Theorem 2.3.

(1) $\Rightarrow$ (3) Let M be any (m, n)-injective left R-module. Then M^+ is (m, n)-flat by Zhang et al. (2005, Theorem 5.7(7)), and so M^+ is flat by (1). On the other hand, by Chen and Ding (1996, Lemma 2.7(1)), for any finitely presented left R-module N, there is an exact sequence

$$\mathrm{Tor}_1^R(M^+, N) \to (\mathrm{Ext}_R^1(N, M))^+ \to 0.$$

Thus M is FP-injective.

(3) $\Rightarrow$ (1) Let M be any (m, n)-flat right R-module. Then M^+ is (m, n)-injective by Zhang et al. (2005, Theorem 4.3), and so M^+ is FP-injective by (3). Hence M is flat.

To prove the last statement, let M be an FP-injective left R-module with N a pure submodule, then M/N is (m, n)-injective by Zhang et al. (2005, Theorem 5.9(4)) since R is left (m, n)-coherent. Therefore M/N is FP-injective by (3), and hence R is a left coherent ring by Wisbauer (1991, 35.9). □

4. (m, n)-FLAT PREENVELOPES WHICH ARE MONOMORPHISMS

In Section 3, it is shown that R is a left (m, n)-coherent ring if and only if every right R-module has an (m, n)-flat preenvelope. In general, an (m, n)-flat preenvelope need not be a monomorphism. In this section, we investigate when every right R-module has an (m, n)-flat preenvelope which is a monomorphism.

We start with the following theorem.

Theorem 4.1. *Let* m *and* n *be two fixed positive integers. The following are equivalent for a ring* R:

(1) R *is a left* (m, n)*-coherent left* (m, n)*-injective ring*;
(2) R *is left* (m, n)*-coherent and every injective right* R*-module is* (m, n)*-flat*;
(3) R *is left* (m, n)*-coherent and every flat left* R*-module is* (m, n)*-injective*;
(4) R *is left* (m, n)*-coherent and the injective envelope of every simple right* R*-module is* (m, n)*-flat*;
(5) R *is left* (m, n)*-coherent and every* (n, m)*-presented right* R*-module is torsionless*;
(6) R *is left* (m, n)*-coherent and* R^+ *is* (m, n)*-flat as a right* R*-module*;
(7) *Every right* R*-module has a monic* (m, n)*-flat preenvelope*;
(8) *Every* (n, m)*-presented right* R*-module has a monic* (m, n)*-flat preenvelope*;
(9) *Every* (n, m)*-presented right* R*-module has a monic projective preenvelope.*

Proof. (3) $\Rightarrow$ (1), (2) $\Rightarrow$ (4), (2) $\Rightarrow$ (6), and (7) $\Rightarrow$ (8) are clear.

(8) $\Rightarrow$ (9) comes from Theorem 3.1.

(1) $\Rightarrow$ (2) Let M be an injective right R-module and $f : N \to M$ any homomorphism with N (n, m)-presented. By Chen et al. (2001, Theorem 2.17), N is torsionless. Thus N embeds in an (m, n)-flat right R-module L by Theorem 3.1(2) since R is left (m, n)-coherent, and so f factors through L. It is easy to see that f factors through a finitely generated projective right R-module, and (2) holds by Zhang et al. (2005, Theorem 4.3).

(2) $\Rightarrow$ (3) Let M be a flat left R-module. Then M^+ is injective, and so M^+ is (m, n)-flat by (2). Thus M is (m, n)-injective by Zhang et al. (2005, Theorem 5.7(7)).

(2) $\Rightarrow$ (7) By Theorem 3.1, every right R-module M has an (m, n)-flat preenvelope $f : M \to F$. Since M embeds in its injective envelope, f must be monic.

(9) $\Rightarrow$ (1) R is obviously left (m, n)-coherent by Theorem 3.1. In addition, every (n, m)-presented right R-module is torsionless by (9), so R is a left (m, n)-injective ring by Chen et al. (2001, Theorem 2.17).

(4) $\Rightarrow$ (5) Let M be an (n, m)-presented right R-module. It is enough to show that for any $0 \neq x \in M$, there exists $f \in M^*$ such that $f(x) \neq 0$. In fact, there is a maximal submodule K of xR, and so xR/K is simple. By the injectivity of $E(xR/K)$, there exists $j : M \to E(xR/K)$ such that $j\iota = i\pi$, where $\iota : xR \to M$ and $i : xR/K \to E(xR/K)$ are the inclusions, and $\pi : xR \to xR/K$ is the natural map. Note that $E(xR/K)$ is (m, n)-flat by (4), and so there exist $t \in \mathbb{N}$, $g : M \to R^t$ and $h : R^t \to E(xR/K)$ such that $j = hg$. Since $j(x) = j\iota(x) = i\pi(x) \neq 0$, $g(x) = (r_1, r_2, \ldots, r_t) \neq 0$. Let $r_i \neq 0$, and $p_i : R^t \to R$ be the ith projection. Then $(p_i g)(x) \neq 0$, as desired.

(5) $\Rightarrow$ (2) Let E be an injective right R-module and $f : N \to E$ any homomorphism with N (n, m)-presented. Since N embeds in an (m, n)-flat right R-module L by (5), f factors through L. It is easy to see that f factors through a finitely generated projective right R-module, and (2) holds by Zhang et al. (2005, Theorem 4.3).

(6) $\Rightarrow$ (5) Let M be an (n, m)-presented right R-module. Then we have the commutative diagram

$$\begin{array}{ccc} R^+ \otimes M^* & \xrightarrow{\sigma} & \mathrm{Hom}_R(M, R^+) \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ (M^{**})^+ & \xrightarrow{\delta_M^+} & M^+ \end{array}$$

where α is a natural homomorphism and β is a natural isomorphism. Note that σ is epic by Zhang et al. (2005, Theorem 4.3(7)) since R^+ is (m, n)-flat, so δ_M^+ is an epimorphism. Thus δ_M is a monomorphism, and hence M is torsionless. $\square$

Corollary 4.2. *Let R be a ring. Then*:

(1) *R is left coherent and every finitely presented right R-module is torsionless if and only if every finitely presented right R-module has a monic projective preenvelope*;

(2) *R is left pseudo-coherent and every finitely presented cyclic right R-module is torsionless if and only if every finitely presented cyclic right R-module has a monic projective preenvelope*;
(3) *R is left $(1, 1)$-coherent and every cyclically presented right R-module is torsionless if and only if every cyclically presented right R-module has a monic projective preenvelope.*

Example 4.3. Let R be an algebra over a field F with basis $\{1\} \cup \{e_i : i = 0, 1, 2, \dots\} \cup \{x_i : i = 1, 2, \dots\}$ such that 1 is the unity of R and, for all i and j, $e_i e_j = \delta_{ij} e_j$, $x_i e_j = \delta_{i,j+1} x_i$, $e_i x_j = \delta_{ij} x_j$, and $x_i x_j = 0$. Then R is a right coherent ring which is not right FP-injective (see Colby, 1975, Example 2). Hence R is not right (m, n)-injective for some positive integers m and n. Thus there exists a left R-module that does not have a monic (m, n)-flat preenvelope by Theorem 4.1, while every left R-module has an (m, n)-flat preenvelope by Theorem 3.1.

Theorem 4.1 characterizes those rings such that every (n, m)-presented right R-module has a monic (m, n)-flat preenvelope. It is natural to ask when every (n, m)-presented right R-module has a monic (m, n)-flat envelope. We get the following theorem.

Theorem 4.4. *The following are equivalent for a right self-injective ring R:*

(1) *Every (n, m)-presented right R-module has a monic (m, n)-flat envelope*;
(2) *The injective envelope of every (n, m)-presented right R-module is finitely generated projective*;
(3) *Every injective right R-module is (m, n)-flat*;
(4) *For every (n, m)-presented right R-module, its (m, n)-flat envelope exists and coincides with its injective envelope.*

Proof. (1) $\Rightarrow$ (2) Let M be an (n, m)-presented right R-module. By (1), M has a monic (m, n)-flat envelope $\alpha : M \to F$. Since F is (m, n)-flat, α factors through a finitely generated projective module P, i.e., there exist $g : M \to P$ and $h : P \to F$ such that $\alpha = hg$. Note that g is monic and P is injective since R is right self-injective. Thus $E(M)$ is isomorphic to a direct summand of P, and hence $E(M)$ is finitely generated projective.

(2) $\Rightarrow$ (3) Let M be any injective right R-module. For any (n, m)-presented right R-module N and any homomorphism $f : N \to M$, there exists $g : E(N) \to M$ such that $f = gi$, where $i : N \to E(N)$ is the inclusion. Since $E(N)$ is finitely generated projective by (2), M is (m, n)-flat.

(3) $\Rightarrow$ (4) Let M be an (n, m)-presented right R-module. By (3), $E(M)$ is (m, n)-flat. We claim that the inclusion $i : M \to E(M)$ is an (m, n)-flat envelope of M. In fact, for any (m, n)-flat right R-module F and any homomorphism $f : M \to F$, f factors through a finitely generated projective right R-module P, i.e., there exist $g : M \to P$ and $h : P \to F$ such that $f = hg$. Since R is right self-injective, P is injective. Therefore there is $j : E(M) \to P$ such that $g = ji$. Thus $f = h(ji) = (hj)i$, which means that i is an (m, n)-flat preenvelope, and hence i is an (m, n)-flat envelope of M since i is an injective envelope.

(4) $\Rightarrow$ (1) is clear. □

5. (m, n)-FLAT PREENVELOPES WHICH ARE EPIMORPHISMS

In this section we consider when every (n, m)-presented right R-module has an (m, n)-flat preenvelope which is an epimorphism.

Theorem 5.1. *Let m and n be two fixed positive integers. The following are equivalent for a ring R:*

(1) *R is a left (m, n)-coherent ring and submodules of (m, n)-flat right R-modules are (m, n)-flat;*
(2) *Every ((n, m)-presented) right R-module has an epic (m, n)-flat preenvelope;*
(3) *Every (n, m)-presented right R-module has an epic projective preenvelope;*
(4) *Every left R-module has a monic (m, n)-injective cover;*
(5) *$(\mathcal{P}_{m,n}, \mathcal{I}_{m,n})$ is hereditary, and every (m, n)-projective left R-module has a monic (m, n)-injective cover;*
(6) *Every quotient of any ((m, n)-)injective left R-module is (m, n)-injective;*
(7) *Every n-generated submodule of the left R-module R^m is projective;*
(8) *Every (m, n)-projective left R-module has projective dimension at most 1;*
(9) *M^* is finitely generated projective and $\delta_M : M \to M^{**}$ is an epimorphism for any (n, m)-presented right R-module M;*
(10) *All torsionless right R-modules are (m, n)-flat.*

Proof. (1) $\Rightarrow$ (2) Let N be any right R-module. Then N has an (m, n)-flat preenvelope $f : N \to F$ since R is left (m, n)-coherent. However $\mathrm{im}(f)$ is (m, n)-flat by (1), it follows that $f : N \to \mathrm{im}(f)$ is an epic (m, n)-flat preenvelope.

(2) $\Rightarrow$ (3) Let N be an (n, m)-presented right R-module. Then N has an epic (m, n)-flat preenvelope $f : N \to F$. By Zhang et al. (2005, Theorem 4.3), f factors through a finitely generated projective right R-module P, i.e., there exist $g : N \to P$ and $h : P \to F$ such that $f = hg$. On the other hand, since P is (m, n)-flat, there exists $\alpha : F \to P$ such that $g = \alpha f$. Thus $f = h\alpha f$, and so $h\alpha = 1$ since f is epic. Therefore F is projective, and so (3) follows.

(3) $\Rightarrow$ (1) R is clearly left (m, n)-coherent by Theorem 3.1. Now suppose that N is a submodule of an (m, n)-flat right R-module L and $\iota : N \to L$ the inclusion. For any (n, m)-presented right R-module K and $\alpha \in \mathrm{Hom}_R(K, N)$, $\iota\alpha$ factors through a finitely generated projective right R-module H, i.e., there exist $g : K \to H$ and $h : H \to L$ such that $\iota\alpha = hg$. By (3), K has an epic projective preenvelope $\beta : K \to Q$ with Q finitely generated projective. Thus there exists $\gamma : Q \to H$ such that $g = \gamma\beta$, which implies that $\ker(\beta) \subseteq \ker(\alpha)$ and so there exists $\varphi : Q \to N$ such that $\alpha = \varphi\beta$. Hence N is (m, n)-flat.

(1) $\Rightarrow$ (6) Let X be any (m, n)-injective left R-module and N any submodule of X. Then the exact sequence $0 \to N \to X \to X/N \to 0$ induces the exactness of $0 \to (X/N)^+ \to X^+ \to N^+ \to 0$. Since X^+ is (m, n)-flat by (1) and Zhang et al. (2005, Theorem 5.7(7)), so is $(X/N)^+$ by (1). Thus X/N is (m, n)-injective by Zhang et al. (2005, Theorem 5.7(7)) again.

(6) $\Rightarrow$ (1) R is left (m, n)-coherent by Zhang et al. (2005, Theorem 5.9(4)). Let A be any submodule of an (m, n)-flat right R-module B. Then the exactness of

$0 \to A \to B \to B/A \to 0$ induces an exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$. Note that B^+ is (m, n)-injective by Zhang et al. (2005, Theorem 4.3), so A^+ is (m, n)-injective by (6), and hence A is (m, n)-flat by Zhang et al. (2005, Theorem 4.3) again.

(4) $\Leftrightarrow$ (6) holds by García Rozas and Torrecillas (1994, Proposition 4) since the class of (m, n)-injective left R-modules is closed under direct sums by Zhang et al. (2005, Corollary 2.8).

(6) $\Rightarrow$ (5) is clear by the equivalence of (4) and (6).

(5) $\Rightarrow$ (6) Let M be any (m, n)-injective left R-module and N any submodule of M. We have to prove that M/N is (m, n)-injective. In fact, note that N has a special $\mathcal{I}_{m,n}$-preenvelope, i.e., there exists an exact sequence $0 \to N \to E \xrightarrow{\beta} L \to 0$ with $E \in \mathcal{I}_{m,n}$ and $L \in \mathcal{P}_{m,n}$. Since L has a monic $\mathcal{I}_{m,n}$-cover $\phi : F \to L$ by (5), there is $\alpha : E \to F$ such that $\beta = \phi\alpha$. Thus ϕ is epic, and hence it is an isomorphism. So L is (m, n)-injective. For any (m, n)-presented left R-module K, we have the exact sequence

$$0 = \mathrm{Ext}^1_R(K, L) \to \mathrm{Ext}^2_R(K, N) \to \mathrm{Ext}^2_R(K, E).$$

Note that $\mathrm{Ext}^2_R(K, E) = 0$ by Enochs et al. (2004, Proposition 1.2) since $(\mathcal{P}_{m,n}, \mathcal{I}_{m,n})$ is hereditary, and hence $\mathrm{Ext}^2_R(K, N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}^1_R(K, M) \to \mathrm{Ext}^1_R(K, M/N) \to \mathrm{Ext}^2_R(K, N) = 0.$$

Therefore $\mathrm{Ext}^1_R(K, M/N) = 0$, as desired.

(7) $\Rightarrow$ (6) Let X be any (m, n)-injective left R-module and N any submodule of X. We shall show that X/N is (m, n)-injective. To this end, let K be an n-generated submodule of left R-module R^m, $i : K \to R^m$ the inclusion and $\pi : X \to X/N$ the canonical map. For any $f : K \to X/N$, then there exists $g : K \to X$ such that $\pi g = f$ since K is projective by (7). Hence there is $h : R^m \to X$ such that $hi = g$ since X is (m, n)-injective. It follows that $(\pi h)i = f$, and so (6) holds.

(6) $\Rightarrow$ (8) Let M be an (m, n)-projective left R-module and N a left R-module, then there is a short exact sequence $0 \to N \to E \to L \to 0$ with E injective. Note that L is (m, n)-injective by (6). Thus $\mathrm{Ext}^2_R(M, N) = 0$, and hence M has projective dimension at most 1.

(8) $\Rightarrow$ (7) holds since every (m, n)-presented left R-module is (m, n)-projective.

(3) $\Rightarrow$ (9) Let M be an (n, m)-presented right R-module and $\phi : M \to F$ an epic projective preenvelope of M with F finitely generated projective. Then $\phi^* : F^* \to M^*$ is an isomorphism, and so M^* is finitely generated projective. On the other hand, δ_M is an epimorphism since $\delta_F\phi = \phi^{**}\delta_M$.

(9) $\Rightarrow$ (3) Let M be an (n, m)-presented right R-module, then M^* is a finitely generated projective left R-module by (9). Consequently, $\delta_M : M \to M^{**}$ is an epic projective preenvelope by the proof of Lemma 3.4 since δ_M is an epimorphism.

(1) $\Rightarrow$ (10) follows from Theorem 3.1.

(10) $\Rightarrow$ (2) By the foregoing proof, it suffices to prove that every (n, m)-presented right R-module M has an epic (m, n)-flat preenvelope. In fact, M has an (m, n)-flat preenvelope $f : M \to F$ since R is left (m, n)-coherent by Theorem 3.1 (2), and so there exist a finitely generated free right R-module P and homomorphisms $g : M \to P$, $h : P \to F$ such that $f = hg$. It is easily seen that $g : M \to P$ is an (m, n)-flat preenvelope of M. Note that $\mathrm{im}(g)$ is (m, n)-flat by (10), and hence $g : M \to \mathrm{im}(g)$ is an (m, n)-flat preenvelope which is an epimorphism. $\square$

Recall that a ring R is called *left semihereditary* (left *PP*) if every finitely generated (principal) left ideal of R is projective. It is well known that R is left semihereditary if and only if every finitely generated submodule of a free left R-module is projective if and only if R is left coherent and submodules of flat right R-modules are flat (see Lam, 1999; Rotman, 1979). Let m and n be arbitrary positive integers in Theorem 5.1, one can easily get some other characterizations of semihereditary rings. Specializing Theorem 5.1 to the case $m = n = 1$ gives some similar characterizations of left *PP* rings. To name only a few, we have the following corollary.

Corollary 5.2. *Let R be a ring. Then*:

(1) *R is left semihereditary if and only if every finitely presented right R-module has an epic projective preenvelope*;
(2) *Every cyclic submodule of a free left R-module is projective if and only if every finitely presented cyclic right R-module has an epic projective preenvelope*;
(3) *R is left PP if and only if every cyclically presented right R-module has an epic projective preenvelope.*

Example 5.3. Let $R = S[X_1, X_2]$, the ring of polynomials in 2 indeterminates over a commutative von Neumann regular ring S. Then R is a coherent ring with the weak global dimension $wD(R) = 2$ (see Glaz, 1989). By Theorem 5.1, there exists an R-module which does not have an epic (m, n)-flat preenvelope for some positive integers m and n. In fact, if for all positive integers m and n, every R-module has an epic (m, n)-flat preenvelope, then every submodule of a flat module is (m, n)-flat by Theorem 5.1. It follows that every submodule of a flat module is flat. So $wD(R) \leq 1$, this is a contradiction.

We conclude the article with the following remark.

Remark 5.4. (1) It is known that if R is a domain, then every cyclically covered (=(1, 1)-projective) R-module has projective dimension at most 1 (see Trlifaj, 2000, p. 21), and any torsionless R-module is torsionfree (= (1, 1)-flat) (see Lam, 1999, p. 144). These results are immediate consequences of Theorem 5.1 by letting $m = n = 1$.

(2) Although every right R-module over any ring R has a torsionfree cover by Trlifaj (2000, Theorem 3.1.2) or Theorem 2.3(2), torsionfree preenvelopes do not exist in general. In fact, R is left $(1, 1)$-coherent if and only if every right R-module has a torsionfree preenvelope by Theorem 3.1; R is left $(1, 1)$-coherent and left $(1, 1)$-injective if and only if every right R-module has a monic torsionfree preenvelope by Theorem 4.1; R is left *PP* if and only if every right R-module has an epic torsionfree preenvelope by Theorem 5.1.

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