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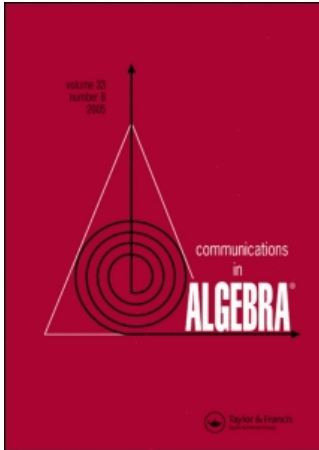
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Publisher: Taylor & Francis

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Communications in Algebra

Publication details, including instructions for authors and subscription information:

<http://www.informaworld.com/smpp/title-content=t713597239>

On a Generalization of Injective Rings

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First Published on: 09 February 2003

To link to this article: DOI: 10.1081/AGB-120023150

URL: <http://dx.doi.org/10.1081/AGB-120023150>

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ABSTRACT

A ring R is called left IP -injective if every homomorphism from a left ideal of R into R with principal image is given by right multiplication by an element of R . It is shown that R is left IP -injective if and only if R is left P -injective and left GIN (i.e., $\mathbf{r}(I \cap K) = \mathbf{r}(I) + \mathbf{r}(K)$ for each pair of left ideals I and K of R with I principal). We prove that R is QF if and only if R is right noetherian and left IP -injective if and only if R is left perfect, left GIN and right simple-injective. We

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DOI: 10.1081/AGB-120023150
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0092-7872 (Print); 1532-4125 (Online)
www.dekker.com



also show that, for a right *CF* left *GIN*-ring R , R is *QF* if and only if $Soc(R_R) \subseteq Soc({}_R R)$. Two examples are given to show that an *IP*-injective ring need not be self-injective and a right *IP*-injective ring is not necessarily left *IP*-injective respectively.

Key Words: *IP*-injective ring; *GIN* ring; *QF* ring.

2000 Mathematics Subject Classification: 16D50; 16P40.

1. INTRODUCTION

All rings are associative with identity and all modules are unitary. As usual, J , $Z(R_R)$, $Soc({}_R R)$ and $Soc(R_R)$ denote the Jacobson radical, the right singular ideal, the left and right socle of the ring R , respectively. The left and right annihilators of a subset X of R are denoted by $l(X)$ and $r(X)$, respectively. We use $K \leq_e N$ to indicate that K is an essential submodule of N . General background material can be found in Anderson and Fuller (1974).

Generalizations of injective rings have been studied in many papers such as Björk (1970), Camillo et al. (2000), Chen (1992), Chen et al. (2001), Hajarnavis and Norton (1985), Jain (1973), Nicholson and Yousif (1995, 1997a,b, 1998, 2001a,b) and Rutter (1975). Recall that a ring R is called left *FP*-injective (Jain (1973)) in case, for every finitely generated submodule K of a free left R -module F , every homomorphism from K to ${}_R R$ extends to one from F to ${}_R R$. Faith and Menal (1994) asked whether every right noetherian left *FP*-injective ring is *QF*. This question remains open, and some partial positive results have been obtained in Faith and Menal (1994), Nicholson and Yousif (1998, 2001a, 2001b). In this regard, we approach this question from another point of view. We say that a ring R is left *IP*-injective (images are principal) if every homomorphism from a left ideal of R into R with principal image is given by right multiplication by an element of R . The class of left *IP*-injective rings lies strictly between that of left self-injective rings and that of left *P*-injective rings. It is proven that a ring R is a left *IP*-injective ring if and only if R is left *P*-injective and left *GIN* (i.e., $r(I \cap K) = r(I) + r(K)$ for each pair of left ideals I and K with I principal). We show that if R is a right noetherian and left *IP*-injective ring, then R is *QF*. As a corollary, we prove that a right Johns left *GIN*-ring is *QF*. We also show that, for a right *CF* left *GIN*-ring R , R is *QF* if and only if $Soc(R_R) \subseteq Soc({}_R R)$. Two examples are given to show that an *IP*-injective ring need not be self-injective and a right *IP*-injective ring is not necessarily left *IP*-injective,

respectively. Finally, we prove that if R is left perfect, left GIN and right simple-injective then R is QF , which is a generalization of Osofsky's assertion that a left perfect, left and right self-injective ring is a QF ring (Osofsky, 1966).

2. IP-INJECTIVE RINGS

Definition 2.1. A ring R is called left IP -injective (images are principal) if every R -homomorphism from a left ideal of R into R with principal image is given by right multiplication by an element of R .

Recall that a ring R is called left f -injective (P -injective) if every left R -homomorphism from a finitely generated (principal) left ideal into R extends to an endomorphism of R . Clearly, left self-injective rings are left IP -injective, and left IP -injective rings are left P -injective, but no two of these rings are equivalent as shown below.

It is well known that a ring R is left f -injective if and only if R is left P -injective and $\mathbf{r}(I \cap K) = \mathbf{r}(I) + \mathbf{r}(K)$ for each pair of finitely generated left ideals I and K of R (see Chen et al., 2001, Corollary 2.10). We will characterize left IP -injective rings in a similar way. Following Camillo et al. (2000), a ring R is said to be a left IN -ring if $\mathbf{r}(I \cap K) = \mathbf{r}(I) + \mathbf{r}(K)$ for each pair of left ideals I and K of R . For convenience, we shall call a ring R a left GIN -ring if $\mathbf{r}(I \cap K) = \mathbf{r}(I) + \mathbf{r}(K)$ for each pair of left ideals I and K of R with I principal. Now we have the following.

Theorem 2.2. Let R be a ring. Then the following are equivalent:

1. R is left IP -injective.
2. R is left P -injective and left GIN .

Proof. (2) \Rightarrow (1). The proof was motivated by that of Hajarnavis and Norton (1985, Proposition 5.2). First we suppose that $f: I_1 + I_2 \rightarrow R$ is an R -homomorphism such that both $f|_{I_1}: I_1 \rightarrow R$ and $f|_{I_2}: I_2 \rightarrow R$ are given by right multiplication by elements z_1 and z_2 of R , respectively, where I_1 and I_2 are left ideals with I_1 principal. Now let $x \in I_1 \cap I_2$. Then $xz_1 = xz_2$, and so $z_1 - z_2 \in \mathbf{r}(I_1 \cap I_2) = \mathbf{r}(I_1) + \mathbf{r}(I_2)$. Hence $z_1 - z_2 = y_1 + y_2$ for some $y_i \in \mathbf{r}(I_i)$, $i = 1, 2$. Let $a_i \in I_i$, $i = 1, 2$. Then $a_i y_i = 0$, and so $f(a_1 + a_2) = a_1 z_1 + a_2 z_2 = a_1(z_1 - y_1) + a_2(z_2 + y_2)$. But $z_1 - y_1 = z_2 + y_2$. Therefore $f(a_1 + a_2) = (a_1 + a_2)(z_1 - y_1)$. Hence $f: I_1 + I_2 \rightarrow R$ is also given by right multiplication.



Now suppose that I is a left ideal of R and $f: I \rightarrow R$ is an R -homomorphism with principal image. Let $\text{Im}(f) = Rf(a)$ for some $a \in I$. Then $I = Ra + \text{Ker}(f)$. Since R is left P -injective, $f|_{Ra}$ is a right multiplication. Clearly $f|_{\text{Ker}(f)}$ is given by right multiplication by 0. Hence by earlier part of the proof it follows that f is given by right multiplication.

(1) \Rightarrow (2). Let $I = Rb$ for some $b \in R$, then every homomorphism from I to R is given by right multiplication by (1). So R is left P -injective.

Now let $A = R\sigma$ for some $\sigma \in R$ and B be a left ideal of R . Let $x \in \mathbf{r}(A \cap B)$ and define $f: A + B \rightarrow R$ by $f(a + b) = ax$ for all $a \in A$ and $b \in B$. It is easy to see that f is an R -homomorphism and $\text{Im}(f) = R\sigma x$. By (2), there exists $y \in R$ such that $ax = f(a + b) = (a + b)y$ for all $a \in A$ and $b \in B$. Let $b = 0$, then $x - y \in \mathbf{r}(A)$, and let $a = 0$, then $y \in \mathbf{r}(B)$. Thus $x = x - y + y \in \mathbf{r}(A) + \mathbf{r}(B)$, which implies $\mathbf{r}(A \cap B) \subseteq \mathbf{r}(A) + \mathbf{r}(B)$. The reverse inclusion is clear. So $\mathbf{r}(A \cap B) = \mathbf{r}(A) + \mathbf{r}(B)$. \square

Recall that a ring R is called a left HN -ring (simple-injective ring) if every homomorphism from a left ideal of R to R with finitely generated (simple) image is given by right multiplication by an element of R . It is obvious that a left HN -ring is left IP -injective, and a left IP -injective ring is left simple-injective. Note that a simple-injective ring need not be IP -injective. For instance, the ring \mathbb{Z} of integers is simple-injective, but it is not IP -injective.

The following results can be proven in a similar way as in Theorem 2.2 above.

Theorem 2.3. *Let R be a ring. Then the following are equivalent:*

1. R is left HN -injective.
2. R is left P -injective and $\mathbf{r}(I \cap K) = \mathbf{r}(I) + \mathbf{r}(K)$ for any finitely generated left ideal I and left ideal K .

Theorem 2.4. *Let R be a ring. Then the following are equivalent:*

1. R is left simple-injective.
2. (a) $\mathbf{r}(I_1 \cap I_2) = \mathbf{r}(I_1) + \mathbf{r}(I_2)$, where I_1 and I_2 are left ideals of R with I_1 minimal.
(b) Every homomorphism from a principal left ideal of R to R with simple image is given by right multiplication by an element of R .

The following is an example of a commutative IP -injective ring which is not self-injective.

Example 1 (cf. Chen, 1992). Let $F = \mathbb{Z}_2 = \{0, 1\}$ be the field of two elements, \mathbb{N} the set of positive integers, $F_i = F$, $i \in \mathbb{N}$, $T = \prod_{i \in \mathbb{N}} F_i$ and $S = \bigoplus_{i \in \mathbb{N}} F_i$. Let R be the subring of T consisting of elements of the form

$$(a_1, a_2, \dots, a_n, a, a, a, \dots),$$

i.e., R is obtained by adjoining the identity of T to its ideal S . Then R is a commutative von Neumann regular ring, but it is not a self-injective ring by Goodearl (1976, Example 3.11 and Theorem 3.12).

Since R is von Neumann regular, it is P -injective. Next we'll show that R is GIN . In fact, we'll show that R is IN . Now let $p_i: T \rightarrow F_i$ be the i -th projection, and $e_i = (0, \dots, 0, 1, 0, \dots) \in T$ (with 1 in the i th position and 0's in all other positions), $i = 1, 2, \dots$.

First we claim that if K is an ideal of R , then either $K = \bigoplus_{i \in I} Re_i \subseteq S$ for some $I \subseteq \mathbb{N}$ or K is finitely generated.

If $K = 0$, we are done.

Suppose $0 \neq K \subseteq S$. Put $I = \{i \mid \exists a \in K \text{ such that } p_i(a) = 1\}$. Then it follows that $K \subseteq \bigoplus_{i \in I} Re_i$. Conversely, if $i \in I$, then there exists $a \in K$ such that $p_i(a) = 1$, and so $e_i = e_i a \in K$. Hence $K = \bigoplus_{i \in I} Re_i$.

Suppose $K \not\subseteq S$. Then there exist $a \in K$ and a least positive integer n such that $p_m(a) = 1$ for $m > n$. Note that $k - ka \in K \cap \bigoplus_{i=1}^n F_i$ for any $k \in K$. Thus $K = Ra + \sum_{u \in W} Ru$, where $W = K \cap \bigoplus_{i=1}^n F_i$. Since W is a finite set, K is finitely generated.

Next we show that R is IN , i.e.,

$$\mathbf{r}(K_1 \cap K_2) = \mathbf{r}(K_1) + \mathbf{r}(K_2) \quad (*)$$

for any two ideals K_1 and K_2 of R .

(i) $K_i \subseteq S$, $i = 1, 2$. In this case, by the preceding proof, we may assume that $K_1 = \bigoplus_{i \in I} Re_i$, $K_2 = \bigoplus_{u \in U} Re_u$. It is easy to see that $K_1 \cap K_2 = \bigoplus_{i \in I \cap U} Re_i$, $\mathbf{r}(K_1 \cap K_2) = \bigoplus_{i \in I \cap U} (1 - e_i)R$, $\mathbf{r}(K_1) = \bigoplus_{i \in I} (1 - e_i)R$, $\mathbf{r}(K_2) = \bigoplus_{u \in U} (1 - e_u)R$. Let $x \in R$. It is straightforward to check that $x \in \bigoplus_{i \in \Lambda} (1 - e_i)R \Leftrightarrow p_i(x) = 0$ for all $i \in \Lambda$. Now let $x \in \bigoplus_{i \in I \cap U} (1 - e_i)R$. Choose $y \in \mathbf{r}(K_1) = \bigoplus_{i \in I} (1 - e_i)R$ such that $p_u(y) = p_u(x)$ for all $u \in U$. In fact, if $u \in I \cap U$, then $p_u(y) = 0 = p_u(x)$ for all $y \in \mathbf{r}(K_1)$. If $u \in U$ but $u \notin I$, take $y \in R$ such that $p_i(y) = 0$ for all $i \in I$ and $p_u(y) = p_u(x)$. Then $y \in \mathbf{r}(K_1)$ and $p_u(y) = p_u(x)$. Put $z = x - y$. Then $p_u(z) = 0$ for all $u \in U$, and so $z \in \mathbf{r}(K_2)$. Thus $x = y + z$ with $y \in \mathbf{r}(K_1)$ and $z \in \mathbf{r}(K_2)$. So (*) follows.



(ii) $K_1 \subseteq S$ and K_2 is finitely generated. In this case, we may assume $K_1 = \bigoplus_{i \in I} Re_i$ and $K_2 = Re$ with $e^2 = e \in R$ (for R is von Neumann regular).

(a) If $e \in S$, then $e = e_{i_1} + e_{i_2} + \cdots + e_{i_m}$ for some positive integer m . It is easy to see that $K_2 = \bigoplus_{v \in V} Re_v$ where $V = \{i_1, i_2, \dots, i_m\}$. Hence (*) follows from the proof of (i).

(b) If $e \notin S$, then $e = e_{i_1} + e_{i_2} + \cdots + e_{i_n} + c$, where n is a positive integer, $i_1 < i_2 < \cdots < i_n$ and $c = 1 - f$ with $f = e_1 + e_2 + \cdots + e_k$ for some $k > i_n$. Let $L = \{i_1, i_2, \dots, i_n\}$ and $X = L \cup \{k+1, k+2, \dots\}$. Then it is easily seen that $K_2 = (\bigoplus_{j \in L} Re_j) \oplus Rc$ and $K_1 \cap K_2 = \bigoplus_{i \in I \cap X} Re_i$. Thus, by the proof of (i), $\mathbf{r}(K_1 \cap K_2) = \bigoplus_{i \in I} (1 - e_i)R + \bigoplus_{j \in X} (1 - e_j)R = \mathbf{r}(K_1) + \bigoplus_{j \in X} (1 - e_j)R$. But $\mathbf{r}(K_2) = (\bigoplus_{i \in L} (1 - e_i)R) \cap fR$, and it is easy to see that $x \in fR \Leftrightarrow p_t(x) = 0$ for all $t \in \{k+1, k+2, \dots\}$. Thus $x \in \mathbf{r}(K_2) \Leftrightarrow p_i(x) = 0$ for all $i \in X \Leftrightarrow x \in \bigoplus_{j \in X} (1 - e_j)R$. Therefore $\mathbf{r}(K_1 \cap K_2) = \mathbf{r}(K_1) + \mathbf{r}(K_2)$.

(iii) If K_1 and K_2 are finitely generated, then (*) holds (for every von Neumann regular ring is f -injective).

Consequently, R is P -injective and IN , and so it is an IP -injective ring.

It is well known that a right noetherian left self-injective ring is QF . However, in general, a right noetherian left P -injective ring need not be QF (see Rutter, 1975). In the next theorem we show that a right noetherian left IP -injective ring is QF .

Lemma 2.5. *Let R be left P -injective and the ascending chain $\mathbf{r}(s_1) \subseteq \mathbf{r}(s_2s_1) \subseteq \cdots$ terminates for any sequence $\{s_1, s_2, \dots\} \subseteq R$. Then $\text{Soc}(R_R)$ and $\mathbf{l}(J)$ are essential left ideals of R .*

Proof. First, we claim that, for any $0 \neq x_1 \in R$, there exists $y \notin \mathbf{l}(x_1)$ such that yx_1R is a minimal right ideal. If not, for any $y \notin \mathbf{l}(x_1)$, yx_1R is not a minimal right ideal. In particular, x_1R is not minimal. Thus there exists $t \in R$ such that $0 \neq x_1tR \subsetneq x_1R$, and so $\mathbf{l}(x_1) \subsetneq \mathbf{l}(x_1t)$ (if $\mathbf{l}(x_1) = \mathbf{l}(x_1t)$, then $x_1R = \mathbf{r}(\mathbf{l}(x_1)) = \mathbf{r}(\mathbf{l}(x_1t)) = x_1tR$ by the P -injectivity of R , a contradiction). Hence there is $x_2 \in \mathbf{l}(x_1t)$ but $x_2 \notin \mathbf{l}(x_1)$, i.e., $x_2x_1t = 0$, $x_2x_1 \neq 0$. Therefore $t \in \mathbf{r}(x_2x_1)$, but $t \notin \mathbf{r}(x_1)$, i.e., $\mathbf{r}(x_1) \subsetneq \mathbf{r}(x_2x_1)$. Since $x_2 \notin \mathbf{l}(x_1)$, x_2x_1R is not minimal by assumption. Hence there exists $x_3 \in R$ such that $\mathbf{r}(x_2x_1) \subsetneq \mathbf{r}(x_3x_2x_1)$ by the preceding proof. Repeating the above-mentioned process, we get a strictly ascending chain $\mathbf{r}(x_1) \subsetneq \mathbf{r}(x_2x_1) \subsetneq \mathbf{r}(x_3x_2x_1) \cdots$. This is a contradiction. So our claim follows.

Now let $0 \neq x \in R$, then there exists $y \in R$ such that yxR is minimal by the above claim. So $0 \neq yx \in \text{Soc}(R_R) \cap Rx$, which shows that $\text{Soc}(R_R) \leq_e R_R$. Note that $\text{Soc}(R_R) \subseteq \mathbf{I}(J)$ is always true, and hence $\mathbf{I}(J) \leq_e R_R$. \square

Lemma 2.6. *If R is right noetherian and left P -injective. Then*

1. J is nilpotent.
2. $\mathbf{I}(J) \leq_e R_R$.
3. $\mathbf{I}(J) \leq_e R_R$.
4. $\text{Soc}(R_R) \leq_e R_R$.

Proof. (1)–(3) follows from Gómez Pardo and Guil Asensio (1998, Theorem 2.7).

(4). By Lemma 2.5. \square

Theorem 2.7. *Let R be a right noetherian and left IP -injective ring, then R is QF .*

Proof. First, we claim that R is left finite dimensional. In fact, let $\bigoplus_{i=1}^{\infty} Ra_i \subseteq R$, where each $a_i \in R$, $i = 1, 2, \dots$, and $I_n = \mathbf{r}(a_n, a_{n+1}, \dots)$ for $n \geq 1$. Then we have an ascending chain $I_1 \subseteq I_2 \subseteq I_3 \dots$, and so there exists an integer $m \geq 1$ such that $I_k = I_m$ for all $k \geq m$ (for R is right noetherian). Now let $k \geq m$, then $\mathbf{r}(a_{k+1}, a_{k+2}, \dots) = I_{k+1} = I_k \subseteq \mathbf{r}(a_k)$. Since $Ra_k \cap \bigoplus_{i=k+1}^{\infty} Ra_i = 0$, $R = \mathbf{r}(Ra_k \cap \bigoplus_{i=k+1}^{\infty} Ra_i) = \mathbf{r}(Ra_k) + \mathbf{r}(\bigoplus_{i=k+1}^{\infty} Ra_i) = \mathbf{r}(a_k) + \mathbf{r}(a_{k+1}, a_{k+2}, \dots) = \mathbf{r}(a_k)$. Thus $a_k = 0$ for $k \geq m$, and so R is left finite dimensional. Hence R is semilocal by Nicholson and Yousif (1995, Theorem 3.3). Since J is nilpotent by Lemma 2.6, R is semi-primary. Thus R is right artinian and so R has ascending chain condition on left annihilators. By hypothesis, we have $\mathbf{r}(\mathbf{l}(a)) = aR$ and $\mathbf{r}(Ra \cap Rb) = \mathbf{r}(a) + \mathbf{r}(b)$. Hence R is left 2-injective by Chen et al. (2001, Corollary 2.10). Therefore R is QF by Rutter (1975, Corollary 3). \square

As a consequence of the preceding result, we obtain a very short proof of Faith and Menal (1992, Theorem 3.4) or Gómez Pardo and Guil Asensio (1998, Corollary 2.12).

Corollary 2.8. *Let R be a ring with ACC on right annihilators. If R is left P -injective and left IN , then R is QF .*

Proof. By Johns (1977, Lemma 5), every finitely generated right ideal is a right annihilator. Thus R is right noetherian by hypothesis, and so R is QF by Theorem 2.7. \square



We say that a ring R is strongly right Johns if $M_n(R)$ is right Johns for every $n \geq 1$. Strongly right Johns rings have been characterized by Faith and Menal (1994) as the right noetherian left FP -injective rings, and Faith and Menal asked whether these rings are QF . The following corollary is a partial positive answer to this question.

Corollary 2.9. *If R is right Johns and left GIN , then R is QF .*

Remark 1. A right Johns ring need not be left GIN as shown by Example 1 in Rutter (1975).

The following example shows that a left FP -injective ring need not be a left IP -injective ring. However we do not know whether a left IP -injective ring is left FP -injective.

Example 2. Let V be an infinite-dimensional right vector space over a division ring D and $R = \text{End}_D V$, then R is a von Neumann regular right self-injective ring, but R is not left self-injective (see Goodearl, 1976, Proposition 2.23). Since R is von Neumann regular, it is left FP -injective. We shall show that R is not left GIN , whence it is not left IP -injective.

The following argument is similar to that in the proof of Goodearl (1976, Proposition 2.23). Since V is an infinite-dimensional right vector space over D , we can find an infinite linearly independent sequence $\{v_1, v_2, \dots\}$ in V . Write $V = V_0 \oplus (\oplus v_i D)$ for some V_0 , and define idempotents e_0, e_1, \dots in R as follows: e_0 is the projection on V_0 , while for $i > 0$, e_i is the projection on $v_i D$. It follows that $\{e_i\}_{i=0}^\infty$ is a set of orthogonal idempotents and $V = \bigoplus_{i=0}^\infty e_i V$. Put $K = \bigoplus_{i=0}^\infty R e_i$, then it is easily seen that $\mathbf{r}(K) = 0$.

Now define $p \in R$ by setting $pV_0 = 0$ and $p v_i = v_1$ for all $i > 0$. We claim that $Rp \cap K = 0$. Given any $f \in R$ such that $fp \in K$, we must have $fp = r_0 e_0 + r_1 e_1 + \dots + r_n e_n$ for some $n \geq 0$ and some $r_i \in R$, $i = 1, 2, \dots, n$. For $i = 1, 2, \dots, n$, we observe that $r_i e_i v_i = f p v_i = f v_1 = f p v_{n+1} = 0$, whence $r_i e_i = 0$. Then $fp = r_0 e_0$, and as a result, $fp = f p e_0 = 0$. Thus $Rp \cap K = 0$.

If R is left GIN , then $R = \mathbf{r}(Rp \cap K) = \mathbf{r}(p) + \mathbf{r}(K) = \mathbf{r}(p)$, which implies $p = 0$. This is a contradiction.

Remark 2. (i) We note that Example 2 above also shows that a right IP -injective ring (GIN -ring) need not be a left IP -injective ring (GIN -ring).

(ii) Every left *IP*-injective ring is a left *f*-injective ring by Chen et al. (2001, Corollary 2.10. 1(c)), but the converse is not true in general as shown by Example 2 above again.

Lemma 2.10. *If R is a left GIN ring, then every closed left ideal is a left annihilator.*

Proof. First we claim that $L \leq_e \mathbf{l}(\mathbf{r}(L))$ for any left ideal L of R . In fact, if $L \cap Rx = 0$ with $x \in \mathbf{l}(\mathbf{r}(L))$, then $\mathbf{r}(x) \supseteq \mathbf{r}(\mathbf{l}(\mathbf{r}(L))) = \mathbf{r}(L)$, and so $R = \mathbf{r}(L \cap Rx) = \mathbf{r}(x) + \mathbf{r}(L) = \mathbf{r}(x)$. Thus $x = 0$, it follows that $L \leq_e \mathbf{l}(\mathbf{r}(L))$. If L is a closed ideal, then $L = \mathbf{l}(\mathbf{r}(L))$, as required. \square

Lemma 2.11. *If R is a right dual ring (i.e., every right ideal of R is a right annihilator), then $\mathbf{l}(J) \leq_e R$.*

Proof. Let $\mathbf{l}(J) \cap Ra = 0$, where $a \in R$. Then $R = \mathbf{r}(\mathbf{l}(J) \cap Ra) = J + \mathbf{r}(a)$ by Nicholson and Yousif (1998, Lemma 2.1). Thus $R = \mathbf{r}(a)$, and so $a = 0$. \square

Corollary 2.12. *If R is left GIN and right dual, then R is semiperfect and $\mathbf{l}(J) = \text{Soc}(R_R)$ is an essential left ideal.*

Proof. The result follows from Lemma 2.10, Lemma 2.11 and Gómez Pardo and Guil Asensio (1998, Proposition 2.11). \square

Recall that a ring R is said to be right *CF* if every cyclic right R -module embeds in a free right R -module.

Theorem 2.13. *If R is right CF and left GIN, then the following are equivalent:*

1. R is *QF*.
2. $J \subseteq Z(R_R)$.
3. $\text{Soc}(R_R) \subseteq \text{Soc}({}_R R)$.
4. R is right mininjective.

Proof. (1) \Rightarrow (2) and (1) \Rightarrow (4) are obvious. (4) \Rightarrow (3) by Nicholson and Yousif (1997a, Theorem 1.14(4)).

(3) \Rightarrow (1). Since R is right *CF* and left *GIN*, R is semiperfect and $\mathbf{l}(J) = \text{Soc}(R_R) \leq_e R$ by Corollary 2.12. But $\text{Soc}(R_R) \subseteq \text{Soc}({}_R R)$ by (3),



and so $\text{Soc}({}_R R) \leq_e R$. Therefore R is left *GPF* (i.e., R is semiperfect, left P -injective and $\text{Soc}({}_R R) \leq_e R$), and hence R_R is finitely cogenerated by Nicholson and Yousif (1995, Theorem 2.3(5)). It follows that R is right artinian (for R is right *CF*). By hypotheses, we have $\mathbf{r}(Ra \cap Rb) = \mathbf{r}(a) + \mathbf{r}(b)$, $\mathbf{r}(\mathbf{l}(a)) = aR$, and so R is left 2-injective by Chen et al. (2001, Corollary 2.10). Since R has *DCC* on right annihilators, it has *ACC* on left annihilators. Consequently, R is *QF* by Rutter (1975, Corollary 3).

(2) \Rightarrow (3). By the proof of (3) \Rightarrow (1), R is semiperfect. Hence $\text{Soc}({}_R R) = \mathbf{r}(J) \supseteq \mathbf{r}(Z(R_R)) \supseteq \text{Soc}(R_R)$. \square

A well known result of Osofsky (1966) asserts that a left perfect, left and right self-injective ring is a *QF* ring. This result can be improved as the following.

Theorem 2.14. *If R is left perfect, left *GIN* and right simple-injective, then R is *QF*.*

Proof. By hypothesis, R is right minfull, i.e., R is semiperfect, right mininjective and $\text{Soc}(eR) \neq 0$ for each local idempotent $e \in R$, and so R is right Kach by Nicholson and Yousif (1997a, Theorem 3.7). Thus $\mathbf{r}(\mathbf{l}(I)) = I$ for every right ideal I of R by Nicholson and Yousif (1997a, Lemma 4.2), and hence R is left P -injective. Therefore R is left *IP*-injective by Theorem 2.2, and then it is left simple-injective. So R is *QF* by Nicholson and Yousif (1997b, Proposition 3). \square

Remark 3. A left perfect and left *GIN*-ring need not be right simple-injective in general. For instance, Björk (1970, Example, P. 70) is a left perfect and left (*G*)*IN*-ring but it is not right simple-injective.

ACKNOWLEDGMENTS

Part of this work was carried out during a visit by the second author to the Ohio State University at Lima. He is grateful to the members of the Mathematics Department for their kind hospitality. This research was supported in part by NNSF of China (No. 10171011 and 10071035), NSF of Jiangsu Province (No. BK 2001001) and by the Ohio State University.

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Received June 2002

