Publ. Math. Debrecen
Proof-sheets for paper Ref. no.: 3584 (2007), 1–16

On a new generalization of coherent rings

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Abstract. In this paper, we introduce a new generalization of coherent rings. Let m be a positive integer and d a positive integer or $d = \infty$. A ring R is called a left (m, d)-coherent ring in case every m-presented left R-module N with $pd(N) \leq d$ is $(m + 1)$-presented. It is shown that there are many similarities between coherent rings and (m, d)-coherent rings. Some applications are also given.

1. Notation

In this section we recall some known notions and definitions needed in the sequel.

Throughout this paper, R is an associative ring with identity and all modules are unitary. For an R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted M^+. Also $pd(M)$ and $fd(M)$ denote the projective and flat dimensions of M respectively.

Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (resp. $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (resp. $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}^R_n(M, N)$) for an integer $n \geq 1$.

Mathematics Subject Classification: 16P70, 16D40, 16D50.
Key words and phrases: (m, d)-coherent ring; (m, d)-injective module; (m, d)-flat module; preenvelope.
This research was partially supported by SRFDP (No. 20050284015), NSFC (No. 10331030), NSF of Jiangsu Province of China (No. BK 2005207), China Postdoctoral Science Foundation (No. 20060390926), Collegial Natural Science Research Program of Education, Department of Jiangsu Province (No. 06KJB110033), and Jiangsu Planned Projects for Postdoctoral Funds Research (No. 0601021B).
The authors would like to thank the referees for the very helpful comments and suggestions.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [10], we say that a homomorphism $\phi : M \to C$ is a $\mathcal{C}$-preenvelope if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a $\mathcal{C}$-envelope if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism.

Given a class $\mathcal{L}$ of R-modules, we will denote by $\mathcal{L}^{\perp} = \{C : \mathrm{Ext}^1(L, C) = 0$ for all $L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by ${}^{\perp}\mathcal{L} = \{C : \mathrm{Ext}^1(C, L) = 0$ for all $L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$. Following [11, Definition 7.1.6], a monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a special $\mathcal{C}$-preenvelope of M if $\mathrm{coker}(\alpha) \in {}^{\perp}\mathcal{C}$. Dually we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover. Special $\mathcal{C}$-preenvelopes (resp. special $\mathcal{C}$-precovers) are obviously $\mathcal{C}$-preenvelopes (resp. $\mathcal{C}$-precovers). $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair ($\mathcal{F}$, $\mathcal{C}$) of classes of R-modules is called a cotorsion theory [11] if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$. A cotorsion theory ($\mathcal{F}$, $\mathcal{C}$) is said to be perfect (complete) if every R-module has a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover (a special $\mathcal{C}$-preenvelope and a special $\mathcal{F}$-precover) (see [12], [20]). A cotorsion theory ($\mathcal{F}$, $\mathcal{C}$) is called hereditary [12] if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{F}$, then L' is also in $\mathcal{F}$.

Let M be a left R-module. M is called FP-injective [19] if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented left R-modules N. For a fixed nonnegative integer n, M is called n-presented (see [2], [7]) if it has a finite n-presentation, that is, there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is finitely generated free (or projective). Clearly, an R-module is 0-presented (resp. 1-presented) if and only if it is finitely generated (resp. finitely presented).

General background materials can be found in [1], [11], [18], [21], [22].

2. Introduction

A ring for which every finitely generated left ideal is finitely presented is called a left coherent ring. Coherent rings and their generalizations have been studied extensively by many authors (see, for example, [2], [3]–[7], [10], [11], [15]–[17], [19], [21], [22]).

Following Costa [7], a ring R is said to be left n-coherent for a fixed nonnegative integer n in case every n-presented left R-module is $(n+1)$-presented. It is easy to see that R is left 0-coherent (resp. 1-coherent) if and only if R is left noetherian (resp. coherent).

On the other hand, LEE [17] introduced another concept of n-coherent rings from a different point of view. Let n be a fixed positive integer or ∞. A ring R is called left n-coherent if every finitely generated submodule M of a free left R-module with $pd(M) \leq n-1$ is finitely presented. Accordingly, all rings are left 1-coherent, and the left coherent rings are exactly those which are d-coherent when d is the left global dimension of R, $0 < d \leq \infty$. Clearly the concept of n-coherent rings in [17] is different from that in [7].

In this paper, we introduce a new generalization of coherent rings, the so-called (m, d)-coherent rings, which unifies the above two definitions of n-coherent rings given in [7] and [17]. To characterize (m, d)-coherent rings, (m, d)-injective and (m, d)-flat modules are introduced. If $\mathcal{I}_{m,d}$ denotes the class of all (m, d)-injective left R-modules and $\mathcal{F}_{m,d}$ the class of all (m, d)-flat right modules, then it is shown that $({}^{\perp}\mathcal{I}_{m,d}, \mathcal{I}_{m,d})$ is a complete cotorsion theory and $(\mathcal{F}_{m,d}, \mathcal{F}_{m,d}^{\perp})$ is a perfect cotorsion theory. It is also shown that there are many similarities between coherent rings and (m, d)-coherent rings. For instance, we prove that a ring R is a left (m, d)-coherent ring if and only if any direct product of R as a right R-module is (m, d)-flat if and only if any direct product of (m, d)-flat right R-modules is (m, d)-flat if and only if any direct limit of (m, d)-injective left R-modules is (m, d)-injective if and only if every right R-module has an $\mathcal{F}_{m,d}$-preenvelope if and only if $\mathrm{Ext}^{m+1}(M, N) = 0$ for any m-presented left R-module M with $pd(M) \leq d$ and any (m, d)-injective left R-module N if and only if $({}^{\perp}\mathcal{I}_{m,d}, \mathcal{I}_{m,d})$ is a hereditary cotorsion theory. Finally, some applications are given.

3. (m, d)-Injective and (m, d)-flat modules

We begin with the following

Definition 3.1. Let R be a ring, m a positive integer, and d a positive integer or $d = \infty$.

A left R-module M is said to be (m, d)-injective if $\mathrm{Ext}^{m}(N, M) = 0$ for any m-presented left R-module N with $pd(N) \leq d$.

A right R-module F is said to be (m, d)-flat if $\mathrm{Tor}_{m}(F, N) = 0$ for any m-presented left R-module N with $pd(N) \leq d$.

Remark 3.2. The concept of (m, d)-injective modules unifies the concepts of n-FP-injective modules in [5] and n-absolutely pure modules in [17]. Similarly, the concept of (m, d)-flat modules unifies the concepts of n-flat modules in [5] and n-flat modules in [17]. In fact, we have the following implications for any integer $k \geq 1$:

FP-injective modules $\Rightarrow$ n-FP-injective modules in [5] $\Leftrightarrow$ (n, ∞)-injective modules $\Rightarrow$ (n, k)-injective modules, where n is a positive integer.

FP-injective modules $\Rightarrow$ n-absolutely pure modules in [17] $\Leftrightarrow$ $(1, n)$-injective modules $\Rightarrow$ (k, n)-injective modules, where n is a positive integer or $n = \infty$.

flat modules $\Rightarrow$ n-flat modules in [5] $\Leftrightarrow$ (n, ∞)-flat modules $\Rightarrow$ (n, k)-flat modules, where n is a positive integer.

flat modules $\Rightarrow$ n-flat modules in [17] $\Leftrightarrow$ $(1, n)$-flat modules $\Rightarrow$ (k, n)-flat modules, where n is a positive integer or $n = \infty$.

In what follows, m is a fixed positive integer and d a fixed positive integer or $d = \infty$. $\mathcal{P}_{m,d}$ stands for the class of all m-presented left R-modules N with $pd(N) \leq d$, $\mathcal{I}_{m,d}$ denotes the class of all (m, d)-injective left R-modules, $\mathcal{F}_{m,d}$ is the class of all (m, d)-flat right modules.

Theorem 3.3. *Let R be a ring. Then*

(1) $({}^{\perp}\mathcal{I}_{m,d}, \mathcal{I}_{m,d})$ *is a complete cotorsion theory.*

(2) $(\mathcal{F}_{m,d}, \mathcal{F}_{m,d}^{\perp})$ *is a perfect cotorsion theory.*

Proof. (1) Let M be any left R-module and $N \in \mathcal{P}_{m,d}$. Note that $\mathrm{Ext}^m(N, M) = 0$ if and only if $\mathrm{Ext}^1(K_{m-1}, M) = 0$, where K_{m-1} denotes the $(m-1)$th syzygy of N. Let X be the set of representatives of $(m-1)$th syzygy modules of all n-presented left R-modules N with $pd(N) \leq d$. Then $\mathcal{I}_{m,d} = X^{\perp}$, and so the result follows from [8, Theorem 10] and [11, Definition 7.1.5].

(2) Denote by $\mathcal{B}$ the class of all left R-modules B with $\mathrm{Tor}_1(N, B) = 0$ for all $N \in \mathcal{F}_{m,d}$. Then by dimension shifting one shows that $X \in \mathcal{F}_{m,d}$ if and only if $\mathrm{Tor}_1(X, B) = 0$ for all $B \in \mathcal{B}$. So (2) follows from [20, Lemma 1.11 and Theorem 2.8]. $\square$

Remark 3.4. (1) Note that $\mathcal{I}_{1,\infty}$ $(\mathcal{F}_{1,\infty})$ is just the class of all FP-injective left R-modules (all flat right R-modules). So [20, Theorems 3.1.1 and 3.4.2] are particular cases of Theorem 3.3 where $m = 1$ and $d = \infty$.

(2) Let M be an R-module over a commutative domain R. Then M is $(1, 1)$-flat if and only if M is torsionfree by [17, Lemma 1], so Theorem 3.3 (2) gives the well-known result that every R-module has a torsionfree cover. On the other hand, M is $(1, 1)$-injective if and only if M is divisible by [17, Lemma 3]. Since divisible envelopes may not exist (see [13] and [20, Proposition 4.8]), the statement of Theorem 3.3 (1) is the best possible in the sense that $({}^{\perp}\mathcal{I}_{m,d}, \mathcal{I}_{m,d})$ is not a perfect cotorsion theory. However, if ${}^{\perp}\mathcal{I}_{m,d}$ is closed under direct limits, then $({}^{\perp}\mathcal{I}_{m,d}, \mathcal{I}_{m,d})$ is a perfect cotorsion theory by Theorem 3.3 (1) and [11, Theorem 7.2.6].

Lemma 3.5. *A right R-module M is (m,d)-flat if and only if M^+ is (m,d)-injective.*

Proof. The result follows from the standard isomorphism

$$\mathrm{Ext}^m(N, M^+) \cong \mathrm{Tor}_m(M, N)^+$$

for any left R-module N. □

Proposition 3.6. *Let R be a ring. Then*

(1) *$\mathcal{I}_{m,d}$ and $\mathcal{F}_{m,d}$ are closed under pure submodules.*

(2) *$\mathcal{I}_{m,d}$ is closed under direct products and $\mathcal{F}_{m,d}$ is closed under direct sums.*

(3) *$\mathcal{I}_{m,m}$ is closed under quotients and $\mathcal{F}_{m,m}$ is closed under submodules.*

(4) *$\mathcal{I}_{m,d}$ is closed under direct sums.*

Proof. (1) Let N be a pure submodule of an (m,d)-injective left R-module M. For any $P \in \mathcal{P}_{m,d}$ with a finite m-presentation $F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to P \to 0$, let $K = \ker(F_{m-2} \to F_{m-3})$, then K is finitely presented. Note that $\mathrm{Ext}^1(K, M) \cong \mathrm{Ext}^m(P, M) = 0$. Thus we have the exact sequence

$$\mathrm{Hom}(K, M) \to \mathrm{Hom}(K, M/N) \to \mathrm{Ext}^1(K, N) \to 0.$$

But the sequence $\mathrm{Hom}(K, M) \to \mathrm{Hom}(K, M/N) \to 0$ is exact since N is a pure submodule of M, so $\mathrm{Ext}^1(K, N) = 0$. Therefore $\mathrm{Ext}^m(P, N) \cong \mathrm{Ext}^1(K, N) = 0$, that is, N is (m,d)-injective.

Let N be a pure submodule of an (m,d)-flat right R-module M, then the exact sequence $0 \to N \to M \to M/N \to 0$ induces the split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. Thus N^+ is (m,d)-injective since M^+ is (m,d)-injective by Lemma 3.5. So N is (m,d)-flat by Lemma 3.5 again.

(2) and (3) are obvious.

(4) Let $(M_i)_{i\in I}$ be a family of (m,d)-injective left R-modules. For any $P \in \mathcal{P}_{m,d}$ with a finite m-presentation $F_m \overset{f_m}{\to} F_{m-1} \overset{f_{m-1}}{\to} \cdots \to F_1 \to F_0 \to P \to 0$, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}(F_{m-1}, \oplus M_i) \arrow[r] \arrow[d, "\alpha"'] & \mathrm{Hom}(\ker(f_{m-1}), \oplus M_i) \arrow[r] \arrow[d, "\beta"'] & \mathrm{Ext}^1(\ker(f_{m-2}), \oplus M_i) \arrow[r] \arrow[d, "\gamma"'] & 0 \\
\oplus \mathrm{Hom}(F_{m-1}, M_i) \arrow[r] & \oplus \mathrm{Hom}(\ker(f_{m-1}), M_i) \arrow[r] & \oplus \mathrm{Ext}^1(\ker(f_{m-2}), M_i) \arrow[r] & 0
\end{tikzcd}$$

Since α and β are isomorphisms by [1, Exercise 16.3, p. 189] (for $\ker(f_{m-1})$ is finitely generated), γ is an isomorphism by Five Lemma. Thus

$$\begin{aligned}\mathrm{Ext}^m(P, \oplus M_i) &\cong \mathrm{Ext}^1(\ker(f_{m-2}), \oplus M_i) \cong \oplus\, \mathrm{Ext}^1(\ker(f_{m-2}), M_i)\\ &\cong \oplus\, \mathrm{Ext}^m(P, M_i) = 0.\end{aligned}$$

So $\oplus M_i$ is (m, d)-injective. □

In the next section, we shall discuss when $\mathcal{F}_{m,d}$ is closed under direct products.

4. (m, d)-Coherent rings

We start with the following

Definition 4.1. Let m be a positive integer and d a positive integer or $d = \infty$. A ring R is called a left (m, d)-coherent ring in case every m-presented left R-module N with $pd(N) \leq d$ is $(m + 1)$-presented.

Remark 4.2. (1) Clearly, all rings are left (m, d)-coherent for any pair of positive integers m and d with $m \geq d$.

(2) R is a left coherent ring if and only if R is a left $(1, \infty)$-coherent ring if and only if R is a left $(1, d)$-coherent ring, where d denotes the left global dimension of R and $0 < d \leq \infty$.

(3) The concept of (m, d)-coherent rings unifies two different concepts of n-coherent rings appearing in [7], [17]. In fact, we have the following implications:

left coherent rings $\Rightarrow$ left n-coherent rings in [7] $\Leftrightarrow$ left (n, ∞)-coherent rings $\Rightarrow$ left (n, k)-coherent rings for every k positive or $k = \infty$, where n is a positive integer.

left coherent rings $\Rightarrow$ left n-coherent rings in [17] $\Leftrightarrow$ left $(1, n)$-coherent rings $\Rightarrow$ left (k, n)-coherent rings for every k positive, where n is a positive integer or $n = \infty$.

Next we shall characterize (m, d)-coherent rings in terms of, among others, (m, d)-flat and (m, d)-injective modules.

Theorem 4.3. *The following are equivalent for a ring R:*

(1) *R is a left (m, d)-coherent ring.*

(2) *Any direct product of R as a right R-module is (m, d)-flat.*

(3) *Any direct product of (m,d)-flat right R-modules is (m,d)-flat.*

(4) *Any direct limit of (m,d)-injective left R-modules is (m,d)-injective.*

(5) $\varinjlim \mathrm{Ext}^m(A, M_i) \to \mathrm{Ext}^m(A, \varinjlim M_i)$ *is an isomorphism for any* $A \in \mathcal{P}_{m,d}$ *and any direct system* $(M_i)_{i\in I}$ *of left R-modules.*

(6) $\mathrm{Tor}_m(\prod N_\alpha, A) \cong \prod \mathrm{Tor}_m(N_\alpha, A)$ *for any family* $\{N_\alpha\}$ *of right R-modules and any* $A \in \mathcal{P}_{m,d}$.

(7) *A left R-module M is (m,d)-injective if and only if M^+ is (m,d)-flat.*

(8) *A left R-module M is (m,d)-injective if and only if M^{++} is (m,d)-injective.*

(9) *A right R-module M is (m,d)-flat if and only if M^{++} is (m,d)-flat.*

Proof. (6) $\Rightarrow$ (3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (4) are obvious.

(1) $\Rightarrow$ (6) follows from [5, Lemma 2.10 (2)].

(1) $\Rightarrow$ (5): Let $A \in \mathcal{P}_{m,d}$. Then A is an $(m+1)$-presented left R-module since R is left (m,d)-coherent, and so $\varinjlim \mathrm{Ext}^m(A, M_i) \cong \mathrm{Ext}^m(A, \varinjlim M_i)$ by [5, Lemma 2.9 (2)].

(2) $\Rightarrow$ (1): Let $P \in \mathcal{P}_{m,d}$ with a finite m-presentation $F_m \xrightarrow{f_m} F_{m-1} \xrightarrow{f_{m-1}} \cdots \to F_1 \to F_0 \to P \to 0$. Then we get an exact sequence $0 \to K \to F_{m-1} \to L \to 0$, where $K = \ker(f_{m-1}), L = \ker(f_{m-2})$. We shall show that K is finitely presented. Note that $\mathrm{Tor}_1(\prod R, L) \cong \mathrm{Tor}_m(\prod R, P) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (\prod R)\otimes K \arrow[r] \arrow[d, "\alpha"'] & (\prod R)\otimes F_{m-1} \arrow[r] \arrow[d, "\beta"'] & (\prod R)\otimes L \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \prod K \arrow[r] & \prod F_{m-1} \arrow[r] & \prod L \arrow[r] & 0
\end{tikzcd}$$

Since β and γ are isomorphisms by [11, Theorem 3.2.22], α is an isomorphism by Five Lemma. So K is finitely presented by [11, Theorem 3.2.22] again. Thus P is $(m+1)$-presented.

(4) $\Rightarrow$ (1): Let $P \in \mathcal{P}_{m,d}$ with a finite m-presentation $F_m \xrightarrow{f_m} F_{m-1} \xrightarrow{f_{m-1}} \cdots \to F_1 \to F_0 \to P \to 0$. Then we get an exact sequence $0 \to K \to F_{m-1} \to L \to 0$, where $K = \ker(f_{m-1}), L = \ker(f_{m-2})$. We claim that K is finitely presented. In fact, let $(M_i)_{i\in I}$ be a family of injective left R-modules, where I is a directed set. Then $\varinjlim M_i$ is (m,d)-injective by (4). Note that $\mathrm{Ext}^1(L, \varinjlim M_i) \cong$

$\mathrm{Ext}^m(P, \varinjlim M_i) = 0$. Thus we have a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}(L, \varinjlim M_i) \arrow[r] \arrow[d, "\alpha"'] & \mathrm{Hom}(F_{m-1}, \varinjlim M_i) \arrow[r] \arrow[d, "\beta"'] & \mathrm{Hom}(K, \varinjlim M_i) \arrow[r] \arrow[d, "\gamma"'] & 0 \\
\varinjlim \mathrm{Hom}(L, M_i) \arrow[r] & \varinjlim \mathrm{Hom}(F_{m-1}, M_i) \arrow[r] & \varinjlim \mathrm{Hom}(K, M_i) \arrow[r] & 0
\end{tikzcd}$$

Since α and β are isomorphisms by [16, Proposition 2.5], γ is an isomorphism by Five Lemma. So K is finitely presented by [16, Proposition 2.5] again. Therefore P is $(m+1)$-presented.

(1) $\Rightarrow$ (7): Let $A \in \mathcal{P}_{m,d}$. Since R is left (m,d)-coherent, A has a projective resolution $\cdots \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to A \to 0$ with each F_i finitely generated. Thus $\mathrm{Tor}_1(M^+, A) \cong \mathrm{Ext}^1(A, M)^+$ for any left R-module M by [18, Theorem 9.51] and the remark following it. So M is (m,d)-injective if and only if M^+ is (m,d)-flat.

(7) $\Rightarrow$ (8) is obvious since M^+ is (m,d)-flat if and only if M^{++} is (m,d)-injective by Lemma 3.5.

(8) $\Rightarrow$ (9): If M is an (m,d)-flat right R-module, then M^+ is (m,d)-injective by Lemma 3.5. So M^{+++} is (m,d)-injective by (8), and hence M^{++} is (m,d)-flat by Lemma 3.5. Conversely, if M^{++} is (m,d)-flat, then M is (m,d)-flat by Proposition 3.6 (1) since M is a pure submodule of M^{++}.

(9) $\Rightarrow$ (3): Let $(M_i)_{i\in I}$ be a family of (m,d)-flat right R-modules. Then $\oplus M_i$ is (m,d)-flat by Proposition 3.6 (2). So $(\oplus M_i)^{++} \cong (\Pi M_i^+)^+$ is (m,d)-flat by (9). But $\oplus M_i^+$ is a pure submodule of ΠM_i^+ by [3, Lemma 1 (1)]. Thus $(\Pi M_i^+)^+ \to (\oplus M_i^+)^+ \to 0$ is split. Hence $\Pi M_i^{++} \cong (\oplus M_i^+)^+$ is (m,d)-flat. Since ΠM_i is a pure submodule of ΠM_i^{++} by [3, Lemma 1 (2)], ΠM_i is (m,d)-flat by Proposition 3.6 (1). $\square$

It is well known that R is a left coherent ring if and only if every right R-module has a flat preenvelope (see [10, Proposition 5.1]) if and only if every factor module of an FP-injective left R-module by a pure submodule is FP-injective (see [21, 35.9]). Now we have similar characterizations of (m,d)-coherent rings as shown in the following theorem.

Theorem 4.4. *The following are equivalent for a ring R:*

(1) *R is a left (m,d)-coherent ring.*

(2) *Every right R-module has an $\mathcal{F}_{m,d}$-preenvelope.*

(3) $\mathrm{Ext}^{m+1}(M, N) = 0$ *for any $M \in \mathcal{P}_{m,d}$ and any $N \in \mathcal{I}_{m,d}$.*

(4) $\mathrm{Ext}^{m+1}(M,N)=0$ *for any* $M\in\mathcal{P}_{m,d}$ *and any* $N\in\mathcal{I}_{1,\infty}$.

(5) $\mathrm{Ext}^{m+j}(M,N)=0$ *for any* $j\geq 1$*, any* $M\in\mathcal{P}_{m,d}$ *and any* $N\in\mathcal{I}_{m,d}$.

(6) $\mathrm{Ext}^{m+j}(M,N)=0$ *for any* $j\geq 1$*, any* $M\in\mathcal{P}_{m,d}$ *and any* $N\in\mathcal{I}_{1,\infty}$.

(7) *If* $0\to A\to B\to C\to 0$ *is exact with* $B\in\mathcal{P}_{m,d}$ and $C\in\mathcal{P}_{m,d}$*, then* $A\in\mathcal{P}_{m,d}$.

(8) *If* $0\to N\to M\to L\to 0$ *is exact with* $N\in\mathcal{I}_{m,d}$ and $M\in\mathcal{I}_{m,d}$*, then* $L\in\mathcal{I}_{m,d}$.

(9) *If* $0\to N\to M\to L\to 0$ *is exact with* $N\in\mathcal{I}_{1,\infty}$ and $M\in\mathcal{I}_{m,d}$*, then* $L\in\mathcal{I}_{m,d}$.

(10) $({}^{\perp}\mathcal{I}_{m,d},\mathcal{I}_{m,d})$ *is a hereditary cotorsion theory.*

Proof. (1) $\Rightarrow$ (2): Let N be any right R-module. By [11, Lemma 5.3.12], there is an infinite cardinal number $\aleph_\alpha$ such that for any R-homomorphism $f: N\to L$ with L (m,d)-flat, there is a pure submodule Q of L such that $\mathrm{Card}(Q)\leq\aleph_\alpha$ and $f(N)\subseteq Q$. Note that Q is (m,d)-flat by Proposition 3.6 (1), and so N has an $\mathcal{F}_{m,d}$-preenvelope by Theorem 4.3 and [11, Proposition 6.2.1].

(2) $\Rightarrow$ (1): Note that $\mathcal{F}_{m,d}$ is closed under direct products by [4, Lemma 1] and so (1) follows from Theorem 4.3.

(1) $\Rightarrow$ (7): Since C is $(m+1)$-presented by (1), A is m-presented by [2, Exercise 6, p. 61]. In addition, $pd(A)\leq d$ is obvious.

(7) $\Rightarrow$ (3): Let $M\in\mathcal{P}_{m,d}$. There is an exact sequence $0\to K\to F\to M\to 0$ with F finitely generated projective. Then $K\in\mathcal{P}_{m,d}$ by (7). Thus for any $N\in\mathcal{I}_{m,d}$, we have the exact sequence

$$0=\mathrm{Ext}^m(K,N)\to\mathrm{Ext}^{m+1}(M,N)\to\mathrm{Ext}^{m+1}(F,N)=0.$$

So $\mathrm{Ext}^{m+1}(M,N)=0$.

(3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6) $\Rightarrow$ (4) and (8) $\Rightarrow$ (9) are trivial.

(3) $\Rightarrow$ (8): Let $P\in\mathcal{P}_{m,d}$. The exact sequence $0\to N\to M\to M/N\to 0$ induces the exactness of $0=\mathrm{Ext}^m(P,M)\to\mathrm{Ext}^m(P,M/N)\to\mathrm{Ext}^{m+1}(P,N)=0$. Thus $\mathrm{Ext}^m(P,M/N)=0$, that is, $M/N\in\mathcal{I}_{m,d}$.

(4) $\Rightarrow$ (9): The proof is similar to that of (3) $\Rightarrow$ (8).

(9) $\Rightarrow$ (1): Let $P\in\mathcal{P}_{m,d}$ with a finite m-presentation $F_m\overset{f_m}{\to}F_{m-1}\overset{f_{m-1}}{\to}\cdots\to F_1\to F_0\to P\to 0$. We claim that $K=\ker(f_{m-1})$ is finitely presented. In fact, for any FP-injective left R-module N, there is an exact sequence $0\to N\to E\to E/N\to 0$ with E injective. Note that E/N is (m,d)-injective by (9). Hence we get the exact sequence

$$0=\mathrm{Ext}^m(P,E/N)\to\mathrm{Ext}^{m+1}(P,N)\to\mathrm{Ext}^{m+1}(P,E)=0.$$

Thus $\mathrm{Ext}^{m+1}(P, N) = 0$, and so $\mathrm{Ext}^1(K, N) \cong \mathrm{Ext}^{m+1}(P, N) = 0$. It follows that K is finitely presented by [9] since K is finitely generated. Therefore R is left (m, d)-coherent.

(3) $\Rightarrow$ (5) holds by induction and the equivalence of (3) and (7).

(8) $\Leftrightarrow$ (10) follows from Theorem 3.3 and [12, Proposition 1.2]. $\square$

5. Applications

Some applications are given in this section. We start by considering when every right R-module has a monic $\mathcal{F}_{m,d}$-preenvelope.

Proposition 5.1. *The following are equivalent for a ring R:*

(1) *Every right R-module has a monic $\mathcal{F}_{m,d}$-preenvelope.*

(2) *R is a left (m, d)-coherent ring and (m, d)-injective as a left R-module.*

(3) *R is left (m, d)-coherent and every injective right R-module is (m, d)-flat.*

(4) *R is left (m, d)-coherent and every flat left R-module is (m, d)-injective.*

Proof. (2) $\Rightarrow$ (1): Let M be any right R-module. Then M has an $\mathcal{F}_{m,d}$-preenvelope $f : M \to F$ by Theorem 4.4. Since $({}_RR)^+$ is a cogenerator in the category of right R-modules, there is an exact sequence $0 \to M \to \Pi({}_RR)^+$. Note that $({}_RR)^+$ is (m, d)-flat by Theorem 4.3 (7), and so $\Pi({}_RR)^+$ is (m, d)-flat by Theorem 4.3 (3). Thus f is monic, and hence (1) follows.

(1) $\Rightarrow$ (3) follows from Theorem 4.4.

(3) $\Rightarrow$ (4): Let M be a flat left R-module. Then M^+ is injective, and so M^+ is (m, d)-flat by (3). Thus M is (m, d)-injective by Theorem 4.3.

(4) $\Rightarrow$ (2) is clear. $\square$

Next we discuss when every right R-module has an epic $\mathcal{F}_{m,d}$-(pre)envelope.

Theorem 5.2. *The following are equivalent for a ring R:*

(1) *Every right R-module has an epic $\mathcal{F}_{m,d}$-envelope.*

(2) *Every left R-module has a monic $\mathcal{I}_{m,d}$-cover.*

(3) *R is a left (m, d)-coherent ring and submodules of (m, d)-flat right R-modules are (m, d)-flat.*

(4) *R is a left (m, d)-coherent ring and every left R-module in ${}^{\perp}\mathcal{I}_{m,d}$ has a monic $\mathcal{I}_{m,d}$-cover.*

(5) *Every quotient of any (m, d)-injective left R-module is (m, d)-injective.*

Proof. (1) $\Leftrightarrow$ (3) follows from [4, Theorem 2].

(3) $\Rightarrow$ (5): Let X be any (m,d)-injective left R-module and N any submodule of X. Then the exact sequence $0 \to N \to X \to X/N \to 0$ induces the exactness of $0 \to (X/N)^+ \to X^+ \to N^+ \to 0$. Since X^+ is (m,d)-flat by (3) and Theorem 4.3, so is $(X/N)^+$ by (3). Thus X/N is (m,d)-injective by Theorem 4.3 again.

(5) $\Rightarrow$ (3): R is a left (m,d)-coherent ring by Theorem 4.4.

Now let A be any submodule of an (m,d)-flat right R-module B. Then the exactness of $0 \to A \to B \to B/A \to 0$ induces an exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$. Note that B^+ is (m,d)-injective by Lemma 3.5, so A^+ is (m,d)-injective by (5), and hence A is (m,d)-flat by Lemma 3.5 again.

(2) $\Leftrightarrow$ (5) holds by [14, Proposition 4] since the class of (m,d)-injective left R-modules is closed under direct sums by Proposition 3.6 (4).

(4) $\Rightarrow$ (5): Let M be any (m,d)-injective left R-module and N any submodule of M. We have to prove that M/N is (m,d)-injective. In fact, note that N has a special $\mathcal{I}_{m,d}$-preenvelope by Theorem 3.3, that is, there exists an exact sequence $0 \to N \to E \xrightarrow{\beta} L \to 0$ with $E \in \mathcal{I}_{m,d}$ and $L \in {}^{\perp}\mathcal{I}_{m,d}$. Since L has a monic $\mathcal{I}_{m,d}$-cover $\phi : F \to L$ by (4), there is $\alpha : E \to F$ such that $\beta = \phi\alpha$. Thus ϕ is epic, and hence it is an isomorphism. So L is (m,d)-injective. For any $K \in \mathcal{P}_{m,d}$, we have the exact sequence

$$0 = \mathrm{Ext}^m(K,L) \to \mathrm{Ext}^{m+1}(K,N) \to \mathrm{Ext}^{m+1}(K,E).$$

Note that $\mathrm{Ext}^{m+1}(K,E) = 0$ by Theorem 4.4 since R is left (m,d)-coherent. So $\mathrm{Ext}^{m+1}(K,N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}^m(K,M) \to \mathrm{Ext}^m(K,M/N) \to \mathrm{Ext}^{m+1}(K,N) = 0.$$

Therefore $\mathrm{Ext}^m(K,M/N) = 0$, as desired.

(5) $\Rightarrow$ (4) follows from Theorem 4.4 (8) and the equivalence of (2) and (5). $\square$

Note that all rings are left (m,m)-coherent by Remark 4.2 (1). So we have the following

Corollary 5.3. *The following are true for any ring R:*

(1) *$\mathcal{F}_{m,m}$ is closed under direct products and $\mathcal{I}_{m,m}$ is closed under direct limits.*

(2) *Every right R-module has an epic $\mathcal{F}_{m,m}$-envelope and every left R-module has a monic $\mathcal{I}_{m,m}$-cover.*

(3) *R is (m,m)-injective as a left R-module if and only if every (injective) right R-module is (m,m)-flat if and only if every (flat) left R-module is (m,m)-injective.*

Proof. (1) follows from Theorem 4.3.

(2) holds by Proposition 3.6 (3) and Theorem 5.2.

(3) comes from (2) and Proposition 5.1. □

Corollary 5.4. *The following are equivalent for a ring R:*

(1) *Every m-presented left R-module has projective dimension at most m.*

(2) *Every quotient of any (m,∞)-injective left R-module is (m,∞)-injective.*

(3) *R is a left (m,∞)-coherent ring and submodules of (m,∞)-flat right R-modules are (m,∞)-flat.*

(4) *R is a left (m,∞)-coherent ring and every m-presented left R-module has flat dimension at most m.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear.

(2) $\Rightarrow$ (1): Let M be an m-presented left R-module and N any left R-module, then there is a short exact sequence $0 \to N \to E \to L \to 0$ with E injective. Note that L is (m,∞)-injective by (2). Thus we have an exact sequence $0 = \mathrm{Ext}^m(M,L) \to \mathrm{Ext}^{m+1}(M,N) \to \mathrm{Ext}^{m+1}(M,E) = 0$, and so $\mathrm{Ext}^{m+1}(M,N) = 0$. It follows that M has projective dimension at most m.

(2) $\Leftrightarrow$ (3) follows from Theorem 5.2.

(4) $\Rightarrow$ (1): Let M be any m-presented left R-module. Then M is $(m+1)$-presented since R is left (m,∞)-coherent. So there is an exact sequence $F_{m+1} \xrightarrow{f_{m+1}} F_m \xrightarrow{f_m} \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i finitely generated projective. Note that $\ker(f_{m-1})$ is finitely presented. Since $fd(M) \leq m$ by (4), $\ker(f_{m-1})$ is flat. Thus $\ker(f_{m-1})$ is projective, and hence $pd(M) \leq m$. □

Proposition 5.5. *The following are equivalent for a ring R:*

(1) *All finitely presented left R-modules are of projective dimension $\leq d$.*

(2) *Every $(1,d)$-injective left R-module is FP-injective.*
Moreover, if R is a left $(1,d)$-coherent ring, then the above conditions are equivalent to:

(3) *Every $(1,d)$-flat right R-module is flat.*

In this case, R is a left coherent ring.

Proof. (1) $\Rightarrow$ (2) is obvious by definition.

(2) $\Rightarrow$ (1): Let M be a finitely presented left R-module. Then $M \in {}^{\perp}\mathcal{I}_{1,d}$ by (2). So by Theorem 3.3 (1) and the proof of [20, Theorem 3.4], M is a direct summand in a left R-module N such that N is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is a finitely presented left

R-module of projective dimension $\leq d$ for all $\alpha < \lambda$. It follows that $pd(M) \leq d$ by [11, Exercise 7.3.2, p. 162].

(2) $\Rightarrow$ (3): Let M be any $(1, d)$-flat right R-module. Then M^+ is $(1, d)$-injective by Lemma 3.5, and so M^+ is FP-injective by (2). Hence M is flat.

(3) $\Rightarrow$ (2): Let M be any $(1, d)$-injective left R-module. Then M^+ is $(1, d)$-flat by Theorem 4.3 since R is left $(1, d)$-coherent, and so M^+ is flat by (3). On the other hand, by [5, Lemma 2.7 (1)], for any finitely presented left R-module N, there is an exact sequence

$$\mathrm{Tor}_1(M^+, N) \to (\mathrm{Ext}^1(N, M))^+ \to 0.$$

Thus M is FP-injective.

In this case, note that every direct product ΠM_i of any family $\{M_i\}$ of flat right R-modules is $(1, d)$-flat by Theorem 4.3 since R is left $(1, d)$-coherent, and hence ΠM_i is flat by (3). Thus R is left coherent. □

Let d_1 and d_2 be positive integers such that $d_1 < d_2$. If M is (m, d_2)-injective (resp. (m, d_2)-flat), then M is (m, d_1)-injective (resp. (m, d_1)-flat). However, the converse is not true in general as shown by the following example.

Example 5.6. Take R to be a commutative coherent ring with $wD(R) = d_2$, for example, let $R = S[X_1, X_2, \ldots, X_{d_2}]$, the ring of polynomials in d_2 indeterminates over a commutative von Neumann regular ring S (see [15, Theorem 1.3.17]). Then all finitely presented R-modules are of projective dimension $\leq d_2$ by [19, Theorem 3.3]. Thus there exists a $(1, d_1)$-injective R-module (resp. $(1, d_1)$-flat R-module) which is not $(1, d_2)$-injective (resp. $(1, d_2)$-flat) by Proposition 5.5.

Recall that a ring R is called left semihereditary if every finitely generated left ideal of R is projective. Specializing Proposition 5.5 to the case $d = 1$, we have

Corollary 5.7 ([17, Corollary 1])**.** *The following are equivalent for a ring R:*

(1) *R is a left semihereditary ring.*

(2) *Every $(1, 1)$-injective left R-module is FP-injective.*

(3) *Every $(1, 1)$-flat right R-module is flat.*

Theorem 5.8. *The following are equivalent for a ring R:*

(1) *Every left R-module in $\mathcal{P}_{m,d}$ is of projective dimension $\leq m - 1$.*

(2) *Every left R-module is (m, d)-injective.*

(3) *Every right R-module is (m, d)-flat.*

(4) *Every left R-module in $^{\perp}\mathcal{I}_{m,d}$ is projective.*

(5) *Every right R-module in $\mathcal{F}_{m,d}^{\perp}$ is injective.*

(6) *R is a left (m,d)-coherent ring, and every left R-module in $^{\perp}\mathcal{I}_{m,d}$ is (m,d)-injective.*

(7) *R is a left (m,d)-coherent ring, and every right R-module in $\mathcal{F}_{m,d}^{\perp}$ is (m,d)-flat.*

Proof. (1) $\Rightarrow$ (2): Let M be any left R-module and $N \in \mathcal{P}_{m,d}$. Then $pd(N) \leq m-1$ by (1), and so $\mathrm{Ext}^m(N,M) = 0$. Thus M is (m,d)-injective.

(2) $\Rightarrow$ (3) follows from Lemma 3.5.

(3) $\Rightarrow$ (1): Let $P \in \mathcal{P}_{m,d}$. Then there is an exact sequence $F_m \stackrel{f_m}{\to} F_{m-1} \stackrel{f_{m-1}}{\to} \cdots \to F_1 \to F_0 \to P \to 0$ with each F_i finitely generated projective. Since $\mathrm{Tor}_m(Q,P) = 0$ for any right R-module Q by (3), $fd(P) \leq m-1$. Hence $K = \ker(f_{m-2})$ is flat. Note that K is finitely presented, and so K is projective. Thus $pd(P) \leq m-1$.

(2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) follow from Theorem 3.3.

(2) $\Rightarrow$ (6) and (3) $\Rightarrow$ (7) are obvious by Theorem 4.3.

(6) $\Rightarrow$ (2): Let M be a left R-module. By Theorem 3.3, M has a special $^{\perp}\mathcal{I}_{m,d}$-precover, that is, there is a short exact sequence $0 \to K \to N \to M \to 0$, where $K \in \mathcal{I}_{m,d}$ and $N \in {}^{\perp}\mathcal{I}_{m,d}$. Since $N \in \mathcal{I}_{m,d}$ by (6), $M \in \mathcal{I}_{m,d}$ by Theorem 4.4.

(7) $\Rightarrow$ (3): Let M be a right R-module. By Theorem 3.3 and Wakamatsu's Lemma (see [22, Section 2.1]), there is a short exact sequence $0 \to M \to N \to L \to 0$, where $N \in \mathcal{F}_{m,d}^{\perp}$ and $L \in \mathcal{F}_{m,d}$. Then $0 \to L^+ \to N^+ \to M^+ \to 0$ is exact. Note that $N \in \mathcal{F}_{m,d}$ by (7). So $L^+ \in \mathcal{I}_{m,d}$ and $N^+ \in \mathcal{I}_{m,d}$ by Lemma 3.5. Thus $M^+ \in \mathcal{I}_{m,d}$ by Theorem 4.4, and hence $M \in \mathcal{F}_{m,d}$, as required. □

Corollary 5.9. *The following are equivalent for a ring R:*

(1) *For any exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, if A and B are finitely generated and projective, so is C.*

(2) *For any exact sequence $0 \to A_n \to A_{n-1} \to \cdots \to A_1 \to A_0 \to 0$ with $n \geq 2$, if each A_i $(1 \leq i \leq n)$ is finitely generated and projective, then so is A_0.*

(3) *Every left R-module is $(1,1)$-injective.*

(4) *Every right R-module is $(1,1)$-flat.*

(5) *R is $(1,1)$-injective as a left R-module.*

Proof. The equivalence of (1) through (4) follows from Theorem 5.8 by letting $m = d = 1$. (4) $\Leftrightarrow$ (5) comes from Corollary 5.3 (3). □

We end this paper with the following

Remark 5.10. (1) Any von Neumann regular ring satisfies the equivalent conditions in Corollary 5.9, but the converse is not true. For example, let R be an algebra over a field F with basis $\{1\} \cup \{e_i : i = 0, 1, 2, \dots\} \cup \{x_i : i = 1, 2, \dots\}$ such that 1 is the unity of R and, for all i and j, $e_i e_j = \delta_{ij} e_j$, $x_i e_j = \delta_{i,j+1} x_i$, $e_i x_j = \delta_{ij} x_j$, and $x_i x_j = 0$. Then R is FP-injective as a left R-module but not FP-injective as a right R-module (see [6, Example 2]). So R satisfies the equivalent conditions in Corollary 5.9 but it is not von Neumann regular.

(2) Let $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules with A and B finitely generated projective. In general, C is not projective. For example, let $R = \mathbb{Z}$, the ring of integers. In the exact sequence $0 \to 2R \to R \to R/2R \to 0$, $2R$ and R are finitely generated projective, but $R/2R$ is not projective.

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(Received October 17, 2005; revised April 6, 2006)