On Almost Precovers and Almost Preenvelopes

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Abstract Let $\mathcal{C}$ be a class of R-modules closed under isomorphisms and finite direct sums. We first show that the finite direct sum of almost $\mathcal{C}$-precovers is an almost $\mathcal{C}$-precover and the direct sum of an almost $\mathcal{C}$-cover and a weak $\mathcal{C}$-cover is a weak $\mathcal{C}$-cover. Then the notion of almost $\mathcal{C}$-preenvelopes is introduced and studied.
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1. **Introduction**

Throughout this paper, R is an associative ring with identity and all modules are unitary right R-modules. $N \leq M$, $N \leq_e M$ and $N \ll M$ means that N is a submodule, an essential submodule and a superfluous submodule of M, respectively. The class $\mathcal{C}$ of R-modules are assumed to contain 0 and be closed under isomorphisms and finite direct sums. General background materials can be found in [1, 2].

Recall that an R-homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of a module M [3] if for any R-homomorphism $f : M \to F'$ where $F' \in \mathcal{C}$, there is an R-homomorphism $g : F \to F'$ such that $g\phi = f$. If, furthermore, when $F' = F$ and $f = \phi$, the only such g are automorphisms of F, then ϕ is called a $\mathcal{C}$-envelope of M. If $\mathcal{C}$ is the class of injective modules, then we get the usual injective envelopes. If envelopes exist, they are unique up to isomorphism. Dually we have the concepts of $\mathcal{C}$-precovers and $\mathcal{C}$-covers. Recently, the notions of almost $\mathcal{C}$-(pre)covers and weak $\mathcal{C}$-covers were introduced in [4] as generalizations of (pre)covers. In the present paper, we first show that the finite direct sum of weak $\mathcal{C}$-precovers is a weak $\mathcal{C}$-precover and the direct sum of an almost $\mathcal{C}$-cover and a weak $\mathcal{C}$-cover is a weak $\mathcal{C}$-cover. Then the notion of almost $\mathcal{C}$-preenvelopes is defined as the dual of almost $\mathcal{C}$-precovers and some properties are given.

2. **Properties of almost precovers**

Recall that an R-module homomorphism $f : M \to N$ is an essential monomorphism if f is monic and $\mathrm{im} f \leq_e N$; f is a superfluous epimorphism if f is epic and $\ker f \ll M$ (see [1]).

Following [4], an R-homomorphism $\varphi : X \to M$ with $X \in \mathcal{C}$ is called an almost $\mathcal{C}$-precover of M if for each $F \in \mathcal{C}$ and each R-homomorphism $f : F \to M$, there is an essential submodule $F^{'}$ of F with $F^{'} \in \mathcal{C}$, and an R-homomorphism $g : F^{'} \to X$ such that $\varphi g = f\iota$, where $\iota : F^{'} \to F$ is the inclusion map. It is easy to see that $\varphi : X \to M$ is an almost $\mathcal{C}$-precover if and only if for each $F \in \mathcal{C}$ and each R-homomorphism $f : F \to M$, there is an essential monomorphism $\psi : E \to F$ with $E \in \mathcal{C}$ and an R-homomorphism $g : E \to X$ such that $\varphi g = f\psi$.

Lemma 2.1. Consider the following pullback diagram:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & M \end{tikzcd}$$

where $A = \{(c, b)|f(c) = g(b), c \in C, b \in B\}$, $\alpha(c, b) = c$, $\beta(c, b) = b$.

(1) If g is an essential monomorphism and f a monomorphism, then α is an essential monomorphism.

(2) If f and g are both essential monomorphisms, then α and β are both essential monomorphisms.

Proof. (1). Since g is a monomorphism, α is also a monomorphism by the property of a pullback. So it is enough to show that α is essential. Suppose there exists $0 \neq N \leq C$ such that $(\mathrm{im}\alpha) \cap N = 0$. Since f is a monomorphism, $f(N) \neq 0$. It follows that $(\mathrm{im} g) \cap f(N) \neq 0$ since g is an essential monomorphism. Hence there exist $0 \neq a \in N$, $0 \neq b \in B$ such that $f(a) = g(b) \neq 0$, and so $(a, b) \in A$. Since $\alpha(a, b) = a$, $a \in (\mathrm{im}\alpha) \cap N = 0$. So $a = 0$, a contradiction.

(2) follows from (1). $\square$

Recall that a class $\mathcal{C}$ of modules is called weakly hereditary [4] if for any $0 \neq M \in \mathcal{C}$, every non-zero submodule of M contains an essential submodule from $\mathcal{C}$.

It is well known that, if $\varphi_i : X_i \to M_i$, $i = 1, 2, \ldots, n$, are $\mathcal{C}$-precovers, then $\oplus\varphi_i : \oplus X_i \to \oplus M_i$ is a $\mathcal{C}$-precover [2]. Here we have the following

Theorem 2.2. Let $\mathcal{C}$ be weakly hereditary. If $\varphi_i : X_i \to M_i$, $i = 1, 2, \ldots, n$, are almost $\mathcal{C}$-precovers. Then $\oplus\varphi_i : \oplus X_i \to \oplus M_i$ is an almost $\mathcal{C}$-precover.

Proof. It is enough to show the case $n = 2$. Suppose $f : D \to M_1 \oplus M_2$ is any R-homomorphism with $D \in \mathcal{C}$. Let $\pi_i : M_1 \oplus M_2 \to M_i$ be the canonical projection, $i = 1, 2$. Since each $\varphi_i : X_i \to M_i$ is an almost $\mathcal{C}$-precover, there exist $B_i \leq_e D$ with $B_i \in \mathcal{C}$ and R-homomorphisms $\psi_i : B_i \to X_i$ such that the following diagram is commutative:

$$\begin{tikzcd}
B_i \arrow[r, "\iota_i"] \arrow[d, "\psi_i"'] & D \arrow[d, "\pi_i f"] \\
X_i \arrow[r, "\varphi_i"'] & M_i
\end{tikzcd}$$

where $\iota_i : B_i \to D$ is the inclusion map, $i = 1, 2$.

Consider the following pullback diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B_2 \arrow[d, "\iota_2"] \\
0 \arrow[r] & B_1 \arrow[r, "\iota_1"'] & D
\end{tikzcd}$$

By Lemma 2.1, α is essential. So $\iota_1\alpha = \iota_2\beta : A \to D$ is essential by [1, Exercise 5.14]. Let $a \in A$. Note that $(\varphi_1 \oplus \varphi_2)(\psi_1\alpha \oplus \psi_2\beta)(a) = (\varphi_1 \oplus \varphi_2)(\psi_1\alpha(a), \psi_2\beta(a)) = (\varphi_1\psi_1\alpha(a), \varphi_2\psi_2\beta(a)) = (\pi_1 f\iota_1\alpha(a), \pi_2 f\iota_2\beta(a)) = f\iota_1\alpha(a)$. Thus $(\varphi_1 \oplus \varphi_2)(\psi_1\alpha \oplus \psi_2\beta) = f\iota_1\alpha$. Since $\mathcal{C}$ is weakly hereditary, there exists $A' \leq_e A$ with $A' \in \mathcal{C}$. So we have the following commutative diagram:

$$\begin{tikzcd}
A' \arrow[r, "\iota_1\alpha\lambda"] \arrow[d, "(\psi_1\alpha\oplus\psi_2\beta)\lambda"'] & D \arrow[d, "f"] \\
X_1 \oplus X_2 \arrow[r, "\varphi_1\oplus\varphi_2"'] & M_1 \oplus M_2
\end{tikzcd}$$

where $\lambda : A' \to A$ is the inclusion map. Note that $\iota_1\alpha\lambda$ is an essential monomorphism, therefore $\varphi_1 \oplus \varphi_2 : X_1 \oplus X_2 \to M_1 \oplus M_2$ is an almost $\mathcal{C}$-precover, as desired. $\square$

Recall that an almost $\mathcal{C}$-precover $\varphi : G \to M$ of a module M is called a weak $\mathcal{C}$-cover [4] if each endomorphism f of G with $\varphi f = \varphi$ is an essential monomorphism, and φ is called an almost $\mathcal{C}$-cover [4] if each endomorphism f of G with $\varphi f = \varphi$ is an automorphism of G.

It is known that if $\varphi_i : X_i \to M_i$, $i = 1, 2$, are $\mathcal{C}$-covers, then $\varphi_1 \oplus \varphi_2 : X_1 \oplus X_2 \to M_1 \oplus M_2$ is a $\mathcal{C}$-cover [2]. Here we have

Theorem 2.3. Let $\mathcal{C}$ be weakly hereditary. If $\varphi_1 : X_1 \to M_1$ is an almost $\mathcal{C}$-cover, $\varphi_2 : X_2 \to M_2$ is a weak $\mathcal{C}$-cover, then $\varphi_1 \oplus \varphi_2 : X_1 \oplus X_2 \to M_1 \oplus M_2$ is a weak $\mathcal{C}$-cover.

Proof. By Theorem 2.2, $\varphi_1 \oplus \varphi_2 : X_1 \oplus X_2 \to M_1 \oplus M_2$ is an almost $\mathcal{C}$-precover. Now suppose that f is an endomorphism of $X_1 \oplus X_2$ such that $\varphi_1 \oplus \varphi_2 = (\varphi_1 \oplus \varphi_2)f$. We shall show that f is an essential monomorphism. Let $\pi_i : X_1 \oplus X_2 \to X_i$ be the canonical projection and $\iota_i : X_i \to X_1 \oplus X_2$ the canonical injection, $i = 1, 2$. For convenience we express the elements in $X_1 \oplus X_2$ as columns $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ for $x_1 \in X_1, x_2 \in X_2$. Then $\varphi_1 \oplus \varphi_2 = \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}$, $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$, where $f_{11} = \pi_1 f \iota_1$, $f_{12} = \pi_1 f \iota_2, f_{21} = \pi_2 f \iota_1$, $f_{22} = \pi_2 f \iota_2$. Note that $\varphi_1 \oplus \varphi_2 = (\varphi_1 \oplus \varphi_2)f$ means that $\varphi_1 f_{11} = \varphi_1, \varphi_1 f_{12} = 0$, $\varphi_2 f_{21} = 0$, $\varphi_2 f_{22} = \varphi_2$. By hypothesis, f_{11} is an automorphism of X_1. Consider the matrix equation

$$\begin{pmatrix} 1 & 0 \\ -f_{21}f_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ 0 & -f_{21}f_{11}^{-1}f_{12} + f_{22} \end{pmatrix}$$

Since $\varphi_2 f_{21} = 0$, $\varphi_2 f_{22} = \varphi_2$, we get $\varphi_2(-f_{21}f_{11}^{-1}f_{12} + f_{22}) = \varphi_2$. Hence $-f_{21}f_{11}^{-1}f_{12} + f_{22}$ is an essential monomorphism by hypothesis. Now by a standard matrix argument we see that f is monic. So the proof is complete if we show that $\mathrm{im} f \leq_e X_1 \oplus X_2$. Let $g = \begin{pmatrix} f_{11} & f_{12} \\ 0 & -f_{21}f_{11}^{-1}f_{12} + f_{22} \end{pmatrix}$. We claim that $\mathrm{im} g \leq_e X_1 \oplus X_2$. In fact, $\mathrm{im}(f_{11} \oplus (-f_{21}f_{11}^{-1}f_{12} + f_{22})) = \mathrm{im} \begin{pmatrix} f_{11} & 0 \\ 0 & -f_{21}f_{11}^{-1}f_{12} + f_{22} \end{pmatrix} \leq \mathrm{im} g \leq X_1 \oplus X_2$. Note that $\mathrm{im}(f_{11} \oplus (-f_{21}f_{11}^{-1} f_{12} + f_{22})) \leq_e X_1 \oplus X_2$ by [1, Proposition 5.20], and so $\mathrm{im} g \leq_e X_1 \oplus X_2$. Thus $\mathrm{im} f \leq_e X_1 \oplus X_2$. □

3. **Almost preenvelopes**

In this section, the concept of almost preenvelopes is introduced as the dual of almost precovers.

We start with the following

Definition 3.1. Let $\mathcal{C}$ be a class of modules.

A homomorphism $\phi : M \to G$ with $G \in \mathcal{C}$ is called an almost $\mathcal{C}$-preenvelope of M if for each $F \in \mathcal{C}$ and each homomorphism $f : M \to F$, there are superfluous epimorphism $\pi : F \to F'$ with $F' \in \mathcal{C}$ and a homomorphism $g : G \to F'$ such that $g\phi = \pi f$.

An almost $\mathcal{C}$-preenvelope $\varphi : M \to G$ with $G \in \mathcal{C}$ is called a weak $\mathcal{C}$-envelope if each endomorphism f of G with $f\varphi = \varphi$ is a superfluous epimorphism and φ is called an almost $\mathcal{C}$-envelope if each endomorphism f of G with $f\varphi = \varphi$ is an automorphisms of G.

$\mathcal{C}$ is said to be weakly homomorphically closed if for any $A \in \mathcal{C}$ and any epimorphism $A \to B$, there exists $C \ll B$ such that $B/C \in \mathcal{C}$.

Theorem 3.2. Let $\mathcal{C}$ be a weakly homomorphically closed class of modules and the following diagram

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "f"'] & F' \arrow[d, "g"] \\ M \arrow[r, "\psi"] & G \end{tikzcd}$$

a pushout diagram. If φ is an almost $\mathcal{C}$-preenvelope and $\pi : G \to N$ a superfluous epimorphism with $N \in \mathcal{C}$, then $\pi\psi$ is an almost $\mathcal{C}$-preenvelope.

Proof. Let $H \in \mathcal{C}$ and $\alpha : M \to H$ be an R-homomorphism. Since φ is an almost $\mathcal{C}$-preenvelope, there exist a superfluous epimorphism $\beta : H \to L$ with $L \in \mathcal{C}$ and an R-homomorphism $\gamma : F' \to L$ such that $\gamma\varphi = \beta(\alpha f) = (\beta\alpha)f$. So by the property of a pushout, there exists $\phi : G \to L$ such that the following diagram is commutative:

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "f"'] & F' \arrow[d, "g"] \arrow[ddr, "\gamma"] & \\ M \arrow[r, "\psi"] \arrow[dr, "\alpha"'] & G \arrow[dr, "\phi"] & \\ & H \arrow[r, "\beta"'] & L \end{tikzcd}$$

Thus $\phi\psi = \beta\alpha$.

Let $K = L/\phi(\ker \pi)$, $p : L \to K$ be the canonical map. Since π is a superfluous epimorphism, $\phi(\ker \pi) \ll L$. Hence p is a superfluous epimorphism. Let $\delta : N \to K$

be the induced homomorphism. Then we have the following commutative diagram:

$$\begin{tikzcd} L \arrow[r, "p"] & K \\ G \arrow[u, "\phi"] \arrow[r, "\pi"] & N \arrow[u, "\delta"'] \end{tikzcd}$$

Since $\mathcal{C}$ is weakly homomorphically closed, there exists a superfluous epimorphism $\theta : K \to E$ with $E \in \mathcal{C}$. Thus we get the following commutative diagram:

$$\begin{tikzcd} H \arrow[r, "\theta p \beta"] & E \\ M \arrow[u, "\alpha"] \arrow[r, "\pi\psi"'] & N \arrow[u, "\theta\delta"'] \end{tikzcd}$$

Note that $\theta p \beta$ is a superfluous epimorphism by [1, Exercises 5.14]. So $\pi\psi$ is an almost $\mathcal{C}$-preenvelope. $\square$

Corollary 3.3. Let $\mathcal{C}$ be a weakly homomorphically closed class of modules. Then every module has an almost $\mathcal{C}$-preenvelope if and only if every flat module has an almost $\mathcal{C}$-preenvelope.

Proof. One direction is obvious. Now suppose every flat module has an almost $\mathcal{C}$-preenvelope. Let M be any R-module. By [5], M has a flat cover $\alpha : F(M) \to M$. It follows that $F(M)$ has an almost $\mathcal{C}$-preenvelope $\phi : F(M) \to L$ by hypothesis. Consider the following pushout diagram:

$$\begin{tikzcd} F(M) \arrow[r, "\phi"] \arrow[d, "\alpha"'] & L \arrow[d, "\beta"] \\ M \arrow[r, "\psi"] & N \end{tikzcd}$$

Note that α is epic, so is β. Since $\mathcal{C}$ is weakly homomorphically closed, there is a superfluous epimorphism $\pi : N \to H$ with $H \in \mathcal{C}$. Thus $\pi\psi$ is an almost $\mathcal{C}$-preenvelope by Theorem 3.2. $\square$

Lemma 3.4. Consider the following pushout diagram:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d, "g"] \\ C \arrow[r, "f"'] & M \end{tikzcd}$$

where $M = (C \oplus B)/\{(\alpha(a), -\beta(a)) |\ a \in A\}$, $f(c) = \overline{(c,0)}$, $g(b) = \overline{(0,b)}$.

(1) If α is a superfluous epimorphism and β an epimorphism, then g is a superfluous epimorphism.

(2) If α and β are both superfluous epimorphisms, then f and g are both superfluous epimorphisms.

Proof. (1). Since α is an epimorphism, g is also an epimorphism by the property of a pushout. So it is enough to show that g is superfluous. Let $\ker g + N = B$ with $N \le B$. We first claim that $\ker\alpha + \beta^{-1}(N) = A$. In fact, let $a \in A$. Then there exist $x \in \ker g$, $y \in N$ such that $\beta(a) = x + y$. Since β is epic, there exists $s \in A$ such that $\beta(s) = y$. Note that $g\beta(a) = g(x) + g(y) = g\beta(s)$, and so $g\beta(a - s) = 0$. Thus $f\alpha(a-s) = g\beta(a-s) = 0$, that is, $\overline{(\alpha(a - s), 0)} = 0$. Hence there exists $t \in A$ such that $\alpha(a-s) = \alpha(t)$ and $0 = -\beta(t)$. Note that $a-s-t \in \ker\alpha$, $s+t \in \beta^{-1}(N)$ and so $a = (a - s - t) + (s + t) \in \ker\alpha + \beta^{-1}(N)$, therefore $\ker\alpha + \beta^{-1}(N) = A$. It follows that $\beta^{-1}(N) = A$ since α is superfluous. Thus $N = B$, as required.

(2) follows from (1). □

We omit the proofs of the following two results which are dual to those of Theorems 2.2 and 2.3 using Lemma 3.4 in place of Lemma 2.1.

Theorem 3.5. Let $\mathcal{C}$ be weakly homomorphically closed. If $\varphi_i : M_i \to G_i$, $i = 1, 2, \ldots, n$, are almost $\mathcal{C}$-preenvelopes, then $\oplus\varphi_i : \oplus M_i \to \oplus G_i$ is an almost $\mathcal{C}$-preenvelope.

Theorem 3.6. Let $\mathcal{C}$ be weakly homomorphically closed. If $\varphi_1 : M_1 \to G_1$ is an almost $\mathcal{C}$-envelope, $\varphi_2 : M_2 \to G_2$ is a weak $\mathcal{C}$-envelope, then $\varphi_1 \oplus \varphi_2 : M_1 \oplus M_2 \to G_1 \oplus G_2$ is a weak $\mathcal{C}$-envelope.

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