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On chief factors of finite groups

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Abstract

Let H and K be normal subgroups of a finite group G and let $K \leq H$. If A is a subgroup of G such that $AH = AK$ or $A \cap H = A \cap K$, we say that A covers or avoids H/K respectively. The purpose of this paper is to investigate factor groups of a finite group G using this concept. We get some characterizations of a finite group being solvable or supersolvable and generalize some known results.

MSC: 20D10; 20D20

1. Introduction

Covering and avoidance have proved to be very interesting and useful concepts when characterizing finite solvable groups and some of their subgroups. For example, Gaschütz [3] introduced a conjugacy class of subgroups of a finite solvable group, so-called pre-Frattini subgroups. Pre-Frattini subgroups avoid the complemented chief factors of a finite solvable group G but cover the rest of its chief factors. Chambers [1] got a sufficient condition for a subgroup of a finite solvable group to be an f-pre-Frattini subgroup. Tomkinson [7] gave a general method for constructing subgroups which either cover or avoid each chief factor of a finite solvable group. In these papers, the authors aimed to find subgroups having the covering and avoidance properties of a finite soluble group. In 1993, Ezquerro [2] considered problems converse to above approach and gave some characterization for a finite group G to be p-supersolvable and supersolvable under the assumption that all maximal subgroups of some Sylow subgroup of G have the covering and avoidance properties. In his paper, Guo [4] pushed this approach further and obtained some characterizations of a finite solvable group based on the assumption that some of its maximal subgroups or 2-maximal subgroups have the covering and avoidance properties.

We observe that the previous authors imposed strong conditions on a finite group, which confines authors to a very restricted area, to study a finite group. In this paper, we will extend the previous methods to chief factors of finite groups (not necessarily solvable). To be precise, we will investigate the solvability of chief factors of a finite group

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with the help of covering and avoidance of some subgroups. Fortunately, we get some meaning theorems. And thus we may obtain known results as corollaries of our theorems in many cases.

In this paper G always denotes a finite group, p a prime, $\pi(G)$ the set of prime divisors of the order of G. For the sake of convenience, we call H/K a factor group of a finite group G if H and K are normal subgroups with $K \leq H$.

2. Elementary results

Definition 1. Let A be a subgroup of a group G and H and K normal subgroups of G. Further suppose $H \geq K$. We will say that:

(1) A covers H/K if $HA = KA$.
(2) A avoids H/K if $H \cap A = K \cap A$.

We will say that A has the covering and avoidance properties in G if A either covers or avoids every chief factor of G. Also we say that A is a *CAP*-subgroup of G.

For convenience, further, we introduce, following [4], the following notation.

Let $\mathcal{F}$ be the set of maximal subgroups of G.

$\mathcal{F}_n = \{M \in \mathcal{F}$ and M is non-nilpotent$\}$.

$\mathcal{F}_c = \{M \in \mathcal{F} \mid |G : M|$ is composite$\}$.

$\mathcal{F}^p = \{M \in \mathcal{F}$ and $N_G(P) \leq M$ for a Sylow p-subgroup P of $G\}$.

$\mathcal{F}^{op} = \bigcup_{p \in \pi(G)-\{2\}} \mathcal{F}^p$.

$\mathcal{F}^{ocn} = \mathcal{F}^{op} \cap \mathcal{F}_c \cap \mathcal{F}_n$.

These are families of subgroups of G.

Definition 2. $S^{ocn} = \cap\{M \in \mathcal{F}^{ocn}\}$ if $\mathcal{F}^{ocn}$ is non-empty; otherwise $S^{ocn}(G) = G$.

We note that $S^{ocn}(G)$ is a characteristic subgroup of G, and that $\Phi(G) \leq S^{ocn}(G)$ always holds.

Lemma 1. (1) *Let A be a subgroup of G and H/K a factor group of G.*
 (a) *A covers H/K if and only if $K(H \cap A) = H$.*
 (b) *A avoids H/K if and only if $KA \cap H = K$.*

(2) *Let $A \leq G$ and H/K a chief factor of G. If $A \cap H \trianglelefteq G$, A covers or avoids H/K. In particular, normal subgroups cover or avoid any chief factor of G.*
(3) *G is a non-abelian simple group if and only if any non-trivial proper subgroup of G is not a CAP-subgroup.*
(4) *Let A be a subgroup of G and H/K a chief factor of G. Then A covers or avoids H/K if and only if there exists a normal subgroup N with $N \leq A \cap K$ and A/N covers or avoids $(H/N)/(K/N)$ respectively. Further, A is a CAP-subgroup of G if and only if there exists a normal subgroup L of G which is contained in A and such that A/L is a CAP-subgroup of G/L.*
(5) *Let A be a subgroup of G and H/K a factor group of G. If $(|A|, |H/K|) = 1$, A avoids H/K.*
(6) *Let H/K be a factor group of G and A a subgroup of G that covers or avoids H/K. Suppose that $B \geq A$ is a subgroup of G. Then $(H \cap B)/(K \cap B)$ is covered or avoided by A as is H/K.*

Proof. (1) Easy.

(2) Since $A \cap H \trianglelefteq G$, $(A \cap H)K/K \trianglelefteq G/K$. Noting that $(A \cap H)K/K \leq H/K$ and that H/K is a minimal normal subgroup of G/K, $(A \cap H)K/K = H/K$ or 1, that is, $(A \cap H)K = H$ or $(A \cap H)K = K$, which implies that A covers or avoids H/K.
(3) Easy.
(4) Let $N = (A \cap K)_G$. Then $AH = AK \iff (A/N)(H/N) = (A/N)(K/N)$ and $A \cap H = A \cap K \iff (A/N) \cap (H/N) = (A/N) \cap (K/N)$.
(5) We have the identity $|H/K| = |AH : AK||A \cap H : A \cap K|$; noting $|A \cap H : A \cap K| \mid |A|$, $|A \cap H : A \cap K| = 1$, this shows $A \cap H = A \cap K$, that is, A avoids H/K.
(6) Easy. $\square$

Lemma 2. *Let H/K be a factor group of G and A a subgroup which covers H/K of G. Further suppose that* **P** *is a group theoretical property which is inherited by subgroups and quotient groups. Then H/K has* **P** *if A has* **P**.

Proof. Suppose that A has the property **P** and that A covers H/K, that is, $AH = AK$. We have $H/K \leq AH/K = AK/K \cong A/A \cap K$. Noting the choice of **P**, the lemma is proved. □

Lemma 3. *Let H/K be a chief factor of G. Every cyclic subgroup of G covers or avoids $H/K \iff H/K$ is of prime order.*

Proof. $\Longrightarrow$ Suppose $p \in \pi(H/K)$ and let A/K be a subgroup with order p of H/K. We may assume $A/K = \langle x \rangle K/K$, where $x \in H$. Of course, x is not contained in K. By assumption, $H\langle x \rangle = K\langle x \rangle$, or $H \cap \langle x \rangle = K \cap \langle x \rangle$. If $H\langle x \rangle = K\langle x \rangle$, H/K is of order p by Lemma 2. Suppose $H \cap \langle x \rangle = K \cap \langle x \rangle$. But $H \cap \langle x \rangle = \langle x \rangle$ since $x \in H$. On the other hand, $K \cap \langle x \rangle < \langle x \rangle$ because x is not contained in K. This contradiction shows that H/K is of prime order.

$\Longleftarrow$ We need only to note the identity $|H/K| = |AH : AK||A \cap H : A \cap K|$ and $|H/K| = p$ for some prime. □

Proposition 1. *Let G be a finite group and let V be a subgroup of G. If V covers the factor group H/K with order a prime power of G, every Hall subgroup of V covers or avoids H/K.*

Proof. Let Q be a Hall subgroup of V. Set $\pi = \pi(Q)$. We prove that Q covers or avoids H/K.

By the assumption, we may suppose $|H/K| = p^\alpha$ for some prime $p \in \pi(G)$. If p is not contained in π, Q avoids H/K by Lemma 1(5).

Suppose $p \in \pi$. Observing $HV = KV$, $|HV|_\pi = |KV|_\pi$. Also $|HV|_\pi = |HQ|_\pi$, $|KV|_\pi = |KQ|_\pi$. Thus $|HQ|_\pi = |KQ|_\pi$. So

$$|HQ : KQ| = |HQ|/|KQ| = (|HQ|_\pi |HQ|_{\pi'})/(|KQ|_\pi |KQ|_{\pi'}) = |HQ|_{\pi'}/|KQ|_{\pi'},$$

which is a π'-number. On the other hand, $|HQ : KQ| \mid |H/K| = p^\alpha$. Hence $|HQ : KQ| = 1$, that is, $HQ = KQ$; in other words, Q covers H/K. □

Remark. Without the solvability of H/K, the above theorem is false. Let G, for example, be a non-abelian simple group. It is evident that G covers its only chief factor $G/1$. However, no proper non-trivial subgroup of G covers or avoids $G/1$. In particular, no Hall subgroup of G covers or avoids $G/1$.

Corollary. *Every Hall subgroup of the solvable group G is a CAP-group.*

Proof. Let $V = G$ in Proposition 1. Then we have the corollary. □

Lemma 4. *Let A and B be the subgroups with relatively prime indices of G. Suppose that H/K is a factor group of G. Then either A or B does not avoid H/K.*

Proof. We may assume $H/K \neq 1$. Let $p \mid |H/K|$. Then p does not divide either $|G : A|$ or $|G : B|$ since A and B are the subgroups with relatively prime indices of G. Thus we may choose a Sylow p-subgroup P of G such that $P \leq A$ or $P \leq B$. Without loss of generality, we may assume that $P \leq A$. Since $p \mid |H/K|$, we have $(PK/K) \cap (H/K) \neq 1$. So $K < PK \cap H \leq AK \cap H = (H \cap A)K$, which implies that $H \cap A > K \cap A$, that is, A does not avoid H/K. □

Lemma 5. *Let H/K be a factor group of G. Suppose that there exist two solvable subgroups A and B which have relatively prime indices. Then H/K is solvable if both A and B cover or avoid H/K.*

Proof. By Lemma 4, either A or B covers H/K. Without loss of generality, assume that A covers H/K. By Lemma 2, H/K is solvable. □

Proposition 2. *G is a solvable group if and only if there exist two subgroups A and B with relatively prime indices of G such that both A and B are solvable CAP-subgroups of G.*

Proof. Suppose that G is solvable. Let $A = 1$ and $B = G$. Then A and B are solvable *CAP*-subgroups with relatively prime indices of G.

Now assume that A and B are solvable *CAP*-subgroups with relatively prime indices of G. We need only to prove that H/K is solvable for any chief factor H/K of G. By Lemma 5, H/K is solvable. □

Corollary ([4, Theorem 3.2]). *Let H_1 and H_2 be two Hall subgroups of a group G such that $G = H_1H_2$. Then G is a solvable group if and only if H_1 and H_2 are both solvable CAP-subgroups of G.*

Proposition 3. *Let G be a finite group and H/K a chief factor of G. Then the following statements are equivalent pairwise:*

(1) *H/K is of prime order.*
(2) *Every subgroup of G covers or avoids H/K.*
(3) *Every cyclic subgroup of G covers or avoids H/K.*

Proof. (1) $\Longrightarrow$ (2) Assume $|H/K| = p$ for some prime p. Since $|H/K| = |PH : PK||P \cap H : P \cap K|$, $|PH : PK| = 1$, i.e., $PH = PK$, or $|P \cap H : P \cap K| = 1$, that is, $P \cap H = P \cap K$.

(2) $\Longrightarrow$ (3) Clear.

(3) $\Longrightarrow$ (1) By Lemma 3, H/K is of prime order. □

3. Main theorems

Theorem 1. *Let H/K be a chief factor of G. Then the following statements are equivalent in pairs.*

(1) *H/K is soluble.*
(2) *Every maximal subgroup of G covers or avoids H/K.*
(3) *Every maximal subgroup of G in $\mathcal{F}^{ocn}$ covers or avoids H/K.*
(4) *Every Hall subgroup of G covers or avoids H/K.*
(5) *There exists a prime $p \in \pi(H/K)$ and $P \in \mathrm{Syl}_p(G)$ such that P covers or avoids H/K.*

Proof. (1) $\Longrightarrow$ (2) Let M be a maximal subgroup of G. We may assume that H/K is an elementary abelian p-subgroup for some prime p in $\pi(G)$ since H/K is soluble.

Suppose that K is not contained in M. Then $G = MK$, noting the maximality of M. So $G = MK = MH$, which shows that M covers H/K.

Now assume $K \le M$. Then M/K is a maximal subgroup of G/K. If $H \le M$, $MH = M = MK$, i.e., M covers H/K clearly. Suppose that H is not contained in M. It is obvious that H/K is a minimal normal subgroup of G/K and is not contained in M/K. Therefore, by the maximality of M/K, $G/K = (M/K)(H/K)$ and $(M/K) \cap (H/K) = K/K$, which implies $K = M \cap H$. Hence $M \cap K = K = M \cap H$, that is, M avoids H/K. (2) is proved.

(2) $\Longrightarrow$ (3) Trivial.

(3) $\Longrightarrow$ (1) Assume $K \ne 1$. Then H/K is a minimal normal subgroup of G/K. We may assume that $S^{ocn}(G/K) < G/K$ by [4, Lemma 2.6]. On the other hand, by Lemma 1(4), G/K satisfies the assumption. Hence H/K is soluble by induction.

Suppose $K = 1$. Then H is a minimal normal subgroup of G. We may assume $H < G$ by Lemma 1(3) and [4, Lemma 2.6]. Now applying Frattini's argument, $G = N_G(P)H$, where $p = \max \pi(H)$, $P \ne 1$, and is a Sylow p-subgroup of H. If $p = 2$, H is a 2-subgroup, and so soluble. Hence we may assume $p > 2$. If $N_G(P) = G$, $H = P$ since H is a minimal normal subgroup of G, as desired. Suppose $N_G(P) < G$. Let P_1 be a Sylow p-subgroup of G such that $P_1 \ge P$. Then $P = P_1 \cap H$. And so $N_G(P_1) \le N_G(P) \le N_G(Z(J(P)))$, where $J(P)$ is the Thompson subgroup of P. If $N_G(Z(J(P))) = G$, $H \le Z(J(P))$, which implies that H is soluble, since H is a minimal normal subgroup of G. Now suppose $N_G(Z(J(P))) < G$. Then there exists a maximal subgroup M of G such that $N_G(P_1) \le N_G(P) \le N_G(Z(J(P))) \le M$. Hence $G = N_G(P)H = MH$. Suppose that $|G : M| = q$ is a prime. By Sylow's theorem, we have $q = 1 + kp$ noting $|G : N_G(P_1)| = |G : M||M : N_G(P_1)| = |G : M||M : N_M(P_1)|$. On the other hand, $q = |G : M| = |HM : M| = |H : H \cap M|$, so $q \mid |H|$. This is a contradiction since p is the largest prime divisor of $|H|$ and $q = 1 + kp$. Hence $|G : M|$ is a composite number. If M is nilpotent, then so is $N_G(Z(J(P)))$ and therefore $N_H(Z(J(P)))$ is nilpotent. By the Glauberman–Thompson theorem, H is p-nilpotent. Thus H is a p-group and, in particular, soluble, noting the minimality of H. Thus we may assume that $|G : M|$ is a composite number and M is not nilpotent. Noting that $N_G(P_1) \le M$, we have $M \in \mathcal{F}^{ocn}$. By the assumption, M covers or avoids H. However, M does not avoid H, noting $1 < P \le M \cap H$. So M covers H, that is, $MH = M$, a contradiction.

(1) $\Longrightarrow$ (4) This is Proposition 1.

(4) $\Longrightarrow$ (5) Clear.

(5) $\Longrightarrow$ (1) Observe the identity $|H/K| = |PH : PK||P \cap H : P \cap K|$. If P covers H/K, H/K is a p-group and, in particular, solvable by Lemma 2. Now suppose that P avoids H/K, that is, $P \cap H = P \cap K$. Then $|H/K| = |PH : PK|$. Since $P \in \mathrm{Syl}_p(G)$, p does not divide $|PH : PK|$. Hence p does not divide $|H : K|$, a contradiction. □

Corollary 1 ([4, Theorem 3.1]). *A group G is solvable if and only if every maximal subgroup M of G in $\mathcal{F}^{ocn}$ is a CAP-subgroup of G.*

Proof. The assumption and Theorem 1(3) yield Corollary 1. □

Corollary 2 ([4, Theorem 3.9]). *G is p solvable if and only if there exists a Sylow p-subgroup P of G such that P covers or avoids H/K if H/K is a chief factor of G and p divides the order of H/K.*

Proof. The assumption and Theorem 1(5) yield Corollary 2.

Let T be a subgroup of G. We call T a 2-maximal subgroup of G if there exists a maximal subgroup S of G such that T is a maximal subgroup of S. □

Theorem 2. *Let H/K be a chief factor of G. H/K is solvable if one of the following statements is true:*

(1) *Every 2-maximal subgroup of G covers or avoids H/K.*
(2) *There exists a solvable subgroup M, whose index $|G : M|$ has at most two distinct prime divisors, such that M covers or avoids H/K.*
(3) *There exists a solvable maximal subgroup M such that M covers or avoids H/K.*
(4) *Every maximal subgroup of every Sylow p-subgroup of G covers or avoids H/K, where $p = \min \pi(H/K)$.*

Proof. (1) Suppose that (1) holds. Use induction on the order of G to prove H/K solvable. Suppose $K \neq 1$. Consider the quotient group G/K. The assumptions are satisfied by G/K by Lemma 1(4). So H/K is solvable.

Now assume $K = 1$. Then H is a minimal normal subgroup of G.

Suppose $H = G$. Let M be a 2-maximal subgroup of G. Then the assumption implies that $M = MH = G$ or $1 = \{1\} \cap M = G \cap M$. But $M = MH = G$ contradicts $M < G$. If $1 = \{1\} \cap M = G \cap M$, we have $M = 1$, which shows $|\pi(G)| \leq 2$, and so G is solvable. Hence H/K is solvable.

Suppose $H < G$. Set $p \in \pi(H)$ and $P \in \mathrm{Syl}_p(H)$. Applying Frattini's argument, $G = HN_G(P)$. If $N_G(P) = G$, $P = H$, which implies that H is solvable, because of the minimality of H.

Thus we may assume $N_G(P) < G$. And so $H \cap N_G(P) < N_G(P)$.

If $N_G(P)$ is a maximal subgroup of G, let M be a 2-maximal subgroup of G such that $H \cap N_G(P) \leq M < N_G(P)$. By the assumption, $M = MH$ or $M \cap H = 1$. But $M \cap H = 1$ contradicts $1 \neq P \leq H \cap N_G(P) \leq M$. Supposing $M = MH$, $H \leq M$. So $G = HN_G(P) = MN_G(P) = N_G(P)$, a contradiction.

If $N_G(P)$ is not a maximal subgroup of G, let M be a 2-maximal subgroup of G such that $N_G(P) \leq M$. By the assumption, $M = MH$ or $M \cap H = 1$. But $M \cap H = 1$ contradicts $1 \neq P \leq H \cap N_G(P) \leq H \cap M$. Assuming $M = MH$, $H \leq M$. So $G = HN_G(P) = MN_G(P) = M$, again a contradiction.

(2) Assume that (2) is satisfied. If M covers H/K, H/K is solvable by Lemma 2. Assume that M avoids H/K, that is, $H \cap M = K \cap M$. Noting the identity $|H/K| = |MH : MK||M \cap H : M \cap K|$, we have $|H/K| = |MH : MK|$. So $|H/K| = |MH|/|MK| = |MH : M|/|MK : M|$. But both $|MH : M|$ and $|MK : M|$ divide $|G : M|$, whence $|H/K|$ has at most two distinct prime divisors, which implies that H/K is solvable by the well-known Burnside $p^\alpha q^\beta$-theorem.

(3) Suppose that (3) is true. Use induction on the order of G to prove H/K solvable. By Lemma 2, we may assume that M avoids H/K, that is, $M \cap H = M \cap K$.

We first assume $K \neq 1$. If K is not contained in M, $G = MK$ for the maximality of M. So $G = MK = MH$, a contradiction. Thus $K \leq M$. If $H \leq M$, $MH = MK$, a contradiction again. Hence $G = MH$ for the maximality of M. Consider the quotient group G/K. Then $G/K = (M/K)(H/K)$ and $(M/K) \cap (H/K) = K/K$, which imply that G/K satisfies the assumption of [8, Theorem 3.4]. Therefore G/K is solvable again. In particular, H/K is solvable.

Now we assume $K = 1$. Then H is a minimal normal subgroup of G and $1 = M \cap H$. Also $G = MH$. This shows that M is a c-normal maximal subgroup of G. Thus [8, 3.4] yields that G is solvable. In particular, H is solvable.

(4) Suppose that (4) is satisfied and that H/K is not solvable. By Lemma 2, every maximal subgroup of every Sylow subgroup of G avoids H/K. Let $p = \min \pi(H/K)$ and $P \in \mathrm{Syl}_p(H)$. Then P is not contained in K. So $P \cap K < P$. It is easy to see that p^2 divides $|H/K|$; otherwise H/K is p-nilpotent, which implies that H/K is a group of order p-power, a contradiction. Let $G_p \in \mathrm{Syl}_p(G)$ such that $P \leq G_p$.

Suppose $P = G_p$ and that P_1 is a maximal subgroup of P. Then P_1 is not contained in K since p^2 divides $|H/K|$ and $|P : P_1| = p$. Hence $P_1 \cap K < P_1$. But the assumption yields $P_1 \cap K = P_1 \cap H$. Thus $P_1 = P_1 \cap H = P_1 \cap K < P_1$, a contradiction.

Now assume $P < G_p$. Let P_2 be a maximal subgroup of G_p such that $P \leq P_2$. Thus $P \leq P_2 \cap H = P_2 \cap K$, which yields $P \leq K$, a contradiction again. $\square$

Remark. (i) The converse of (1) is not true. Let, for example, $G = A_4$ and H be the Sylow 2-subgroup of G. Then H is a chief factor of G. Suppose that P_1 is a maximal subgroup of H. P_1 neither avoids nor covers H clearly.
(ii) The converse of (4) is not true. Let, for example, G, H and P_1 be as above. P_1 neither avoids nor covers H clearly.

Corollary (*[4, Theorem 3.4]*). *G is solvable if every 2-maximal subgroup of a group G is a CAP-subgroup of G.*

Let G be a finite group. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasi-nilpotent subgroup of G. Now $F^*(G)$ is an important subgroup of G and is a natural generalization of $F(G)$. We can get a sufficient condition for a finite group to be solvable using $F^*(G)$ some of whose subgroups are required to avoid or cover some chief factors of G. To the end, we need the following lemma.

Lemma 6 (*[9, Lemma 2.3] and [5, X,13]*). *Let G be a finite group and N a subgroup of G.*

(1) *$F^*(N) \leq F^*(G)$ if N is normal in G.*
(2) *$F^*(G) > 1$ if $G > 1$.*
(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$. If $F^*(G)$ is solvable, $F^*(G) = F(G)$.*
(4) *Supposing that P is a normal p-subgroup of G for some prime p, then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$.*
(5) *If K is a subgroup of G contained in $Z(G)$, $F^*(G/K) = F^*(G)/K$.*

Theorem 3. *Let G be a group. Suppose that there exists a chief series $\mathcal{H}$ of G passing through $F^*(G)$ such that every maximal subgroup of every Sylow subgroup of $F^*(G)$ either covers or avoids each chief factor of G in $\mathcal{H}$. Then G is solvable.*

Proof. Assume the theorem is false and let G be a counterexample of minimal order. Clearly $G \neq 1$ and so $F^*(G) \neq 1$.

We prove first that $F^*(G)$ is solvable and so $F^*(G) = F(G)$. Assume $1 \leq K < H \leq F^*(G)$ and H/K is an insolvable chief factor of G in $\mathcal{H}$ if possible. Then $p^2 \mid |H/K|$, where $p = \min \pi(H/K)$. Let $H_p \in \mathrm{Syl}_p(H)$ and $P \in \mathrm{Syl}_p(F^*(G))$ such that $H_p \leq P$. H_p is not contained in K.

Suppose $H_p = P$. Let P_1 be a maximal subgroup of P. P_1 is not contained in K since $p^2 \mid |H/K|$. And so $P_1 \cap K < P_1$. On the other hand, $P_1 \cap H = P_1 \cap K$ by the assumption and Lemma 2. But $P_1 = P_1 \cap H$, which yields $P_1 \leq P_1 \cap H = P_1 \cap K$ contrary to $P_1 \cap K < P_1$.

Now assume $H_p < P$. Let P_2 be a maximal subgroup of P such that $H_p \leq P_2$. By the assumption and Lemma 2, $H_p \leq P_2 \cap H = P_2 \cap K$, which implies $H_p \leq K$, a contradiction. Thus $F^*(G)$ is solvable and so $F^*(G) = F(G)$ by Lemma 6(3).

Let N be the minimal normal subgroup of G which appears in $\mathcal{H}$. We have that N is an elementary abelian p-subgroup for some prime p divisor of the order of G. Denote by P the Sylow p-subgroup of $F(G)$. Obviously $N \leq P$.

If N is contained in every maximal subgroup of P, then $N \leq \Phi(P) \leq \Phi(G)$. Recall that $F^*(G/N) = F^*(G)/N$ by Lemma 6(4). Taking the quotient groups over N of all non-trivial normal subgroups of G appearing in $\mathcal{H}$, we obtain a chief series $\mathcal{H}'$ of G/N passing through $F^*(G/N)$. Let A/N be a maximal subgroup of some Sylow subgroup of $F^*(G/N)$. There exists a maximal subgroup A_1 of some Sylow subgroup of $F^*(G)$ such that $A = A_1 N$. It is easy to see that A either covers or avoids each chief factor of G in $\mathcal{H}$ and consequently A/N covers or avoids each chief factor of G/N in $\mathcal{H}'$. By the minimality of G, the quotient group G/N is solvable. Thus G is solvable, a contradiction. Let now P_1 be a maximal subgroup of P such that N is not contained in P_1. $P = P_1 N$ and $|P : P_1| = p$ clearly.

The assumption yields $P_1N = P_1$ or $P_1 \cap N = 1$. But $P_1N = P_1$ leads to $N \leq P_1$. This is a contradiction. Hence $P_1 \cap N = 1$ and $|N| = |P : P_1| = p$.

Assume that $N \leq Z(G)$. By Lemma 6(5) and Lemma 1(4), mimicking the preceding methods, G/N satisfies the assumption. Consequently G/N is solvable because of the minimality of G. Hence G is solvable, a contradiction. Therefore $C_G(N)$ is a proper normal subgroup of G. Moreover $F^*(G) = F(G) \leq C_G(N)$. Thus $F^*(G) = F^*(F^*(G)) \leq F^*(C_G(N)) \leq F^*(G)$. This implies that $F^*(C_G(N)) = F^*(G)$. Taking the intersection of all normal subgroups of G in $\mathcal{H}$ with $C_G(N)$, we construct a normal series of $C_G(N)$ passing $F^*(C_G(N))$. Considering the non-trivial factors of this normal series and refining we obtain a chief series $\mathcal{H}_0$ of $C_G(N)$. Let H/K be a chief factor of G in $\mathcal{H}$ below $F^*(G)$ and suppose that H_0/K_0 is a chief factor of $C_G(N)$ in $\mathcal{H}_0$ such that $K \leq K_0 < H_0 \leq H$. It is easy to check that if a subgroup A of $C_G(N)$ covers H/K, then A covers H_0/K_0, and if A avoids H/K, then A avoids H_0/K_0. Hence every maximal subgroup of every Sylow subgroup of $F^*(C_G(N))$ either covers or avoids the factors of a chief series $\mathcal{H}_0$ of passing through $F^*(C_G(N))$. By minimality of G, the group $C_G(N)$ is solvable. On the other hand, since N is of prime order, then $G/C_G(N)$ is isomorphic to a subgroup to a subgroup of $\mathrm{Aut}(N)$, which is cyclic. This shows that G is solvable. This is the final contradiction.

For the sake of convenience, we need to introduce a new term. A subgroup L is called a pure 2-maximal subgroup of G if L is a maximal element in $\mathcal{M}$, where $\mathcal{M}$ is the set of 2-maximal subgroups of G. $\square$

Remark. It is clear that pure 2-maximal subgroups are 2-maximal subgroups. But 2-maximal subgroups need not to be pure 2-maximal subgroups. We have the following.

Example. Let $G = A_5$, $B = \langle (123), (12)(45) \rangle$ and $B_2 \in \mathrm{Syl}_2(B)$. $|B| = 6$ and B is a maximal subgroup of G clearly. So $B_2 = \langle (12)(45) \rangle$ is a 2-maximal subgroup of G. On the other hand, $A_4 = \langle (123), (124) \rangle$ is a maximal subgroup of G evidently. Suppose $D \in \mathrm{Syl}_2(A_4)$ and $D_1 = \langle (12)(34) \rangle$. It follows that D_1 is a maximal subgroup of D with $|D_1| = 2$ and D is a 2-maximal subgroup of G. Let $x = (345)$. Then ${D_1}^x = B_2$. So $B_2 = {D_1}^x < D^x < {A_4}^x < G$, which implies that B_2 is a 2-maximal subgroup but not a pure 2-maximal subgroup of G.

Proposition 4. *Let G be a group. If the trivial subgroup is a 2-maximal subgroup of G, then G is solvable.*

Proof. Because 1 is a 2-maximal subgroup of G, there is a maximal subgroup M of G such that 1 is maximal in M. We have easily that M is of order a prime, say p. If G is a p-group, G is solvable, and so we are done. Hence we can assume that G is not a p-group. It is easy to see that M is a Sylow p-subgroup of G. If $M < N_G(M)$, then $N_G(M) = G$, that is, $M \trianglelefteq G$. Thus $|G : M|$ is a prime, say q (which is distinct from p as G is not a p-group). This means G is of order pq and so solvable, as claimed. Therefore $M = N_G(M)$, and $N_G(M) = C_G(M)$ obviously. The well-known Burnside theorem implies that G is p-nilpotent. Let L be the normal p-complement of G. Pick $r \in \pi(L)$. Assume $|\pi(L)| > 1$. Then L has a subgroup $R \in \mathrm{Syl}_r(L)$ such that $R^M = \langle R^g | g \in M \rangle = R$ [6, 8.2.3]. In particular MR is a proper subgroup of G. This contradicts the maximality of M. Hence L is an r-group and G is a group of order $p^\alpha r^\beta$. Now G is solvable by Burnside's $p^a q^b$-theorem, as desired. $\square$

Theorem 4. *Let G be a group and consider a chief factor H/K of G. Assume that there exists a solvable pure 2-maximal subgroup L of G such that L avoids H/K. Then H/K is solvable.*

Proof. Assume the theorem is false and let G be a counterexample of minimal order. In the group G there exists a solvable pure 2-maximal subgroup L of G such that L avoids a non-abelian chief factor H/K of G. By Proposition 4, we can assume that $L \neq 1$. $\square$

Suppose $L_G \neq 1$. By Lemma 2, L_G avoids H/K. Then $(HL_G/L_G)/(KL_G/L_G) \cong H/K$ is a chief factor of G/L_G. Also L/L_G is a pure 2-maximal subgroup of G/L_G clearly. Now the choice of G implies that H/K is solvable. This is a contradiction. Thus $L_G = 1$.

Assume that $K \neq 1$. Consider the quotient group G/K. Then $LK > L$. If $LK = G$, $G = LK = LH$, that is, L covers H/K, which contradicts the assumption that L avoids H/K. Hence LK and LK/K are maximal subgroups of G and G/K respectively since L is a pure 2-maximal subgroup of G. By Lemma 1(4), LK/K avoids the chief factor H/K of G/K. Now Theorem 2(3) implies that H/K is solvable; this is a contradiction.

Hence $K = 1$. Then H is a minimal normal subgroup of G and $L \cap H = 1$. Set $S = LH$. Then $L < LH \leq G$. Noting that L is a pure 2-maximal subgroup of G, either $S = G$ or S is a maximal subgroup of G and L is a maximal subgroup of S.

Suppose first that S is a proper subgroup of G. Since L is a maximal subgroup of S and $H \cap L = 1$, we have that H is a minimal normal subgroup of S. Now Theorem 2(3) implies that H is solvable and this is a contradiction. Hence $G = S = LH$.

Let T be a minimal normal subgroup of L. Then T is a p-group for some prime $p \in \pi(L)$. It follows that $L \leq N_G(T) < G$. Since L is a pure 2-maximal subgroup of G, either $L = N_G(T)$ or $N_G(T)$ is a maximal subgroup of G and L is a maximal subgroup of $N_G(T)$.

Assume that $L = N_G(T)$. We consider the subgroup TH. If $p \in \pi(H)$, there exists a $H_p \in \mathrm{Syl}_p(H)$ such that H_p is T-invariant, that is, T normalizes H_p. Then $C_{H_p}(T) > 1$. This contradicts that $N_G(T) = L$, noting $C_{H_p}(T) \leq C_H(T) \leq N_H(T) \leq N_G(T) = L$. Therefore H is a p'-group and $C_H(T) = 1$. Let $r \in \pi(H)$. Then H has the unique $R \in \mathrm{Syl}_r(H)$ such that $R^T = \langle R^g | g \in T\rangle = R$ [6, 8.2.3]. Let $g \in L$. We have $(R^g)^T = R^{gT} = R^{Tg} = (R^T)^g = R^g$. So $R^g = R$ and $R^L = R$. Consider LR. Then LR is solvable. If $LR = G$, G is solvable, and we are done. So we may assume $LR < G$. Because L is a pure 2-maximal subgroup of G and $L < LR$, L is a maximal subgroup of LR and LR is a maximal subgroup of G. We have that R is a minimal normal subgroup of LR. Hence R is an elementary abelian r-group and $R \leq C_H(R)$. If $N_H(R) > R$, $R \trianglelefteq G$ since LR is a maximal subgroup of G and L normalizes R. But this shows $R = H$ since H is a minimal normal subgroup of G. This is impossible, noting the assumption that $LR < G$.

As a result, $N_H(R) = R$. So $R \leq C_H(R) \leq N_H(R) = R$, that is, $N_H(R) = C_H(R) = R$. Now, by the well-known Burnside theorem, H is r-nilpotent. Since H is a minimal normal subgroup of G, $H = R$. This implies that H is solvable and this is a contradiction.

So $N_G(T)$ is a maximal subgroup of G and L is a maximal subgroup of $N_G(T)$. We have $N_G(T) = N_G(T) \cap G = N_G(T) \cap LH = L(N_G(T) \cap H)$. Now [8, Theorem 3.4] yields that $N_G(T)$ is solvable. It follows easily that $N_G(T) \cap H$ is a minimal normal subgroup of $N_G(T)$. And so $N_G(T) \cap H$ is an elementary abelian q-group for some prime $q \in \pi(N_G(T))$. Write $Q = N_G(T) \cap H$. If Q is normal in G, then $Q = H$. In this case $G = N_G(T)$ and this is impossible. Then $N_G(Q) = N_G(T)$. If Q is not a Sylow q-subgroup of H, then Q is a proper normal subgroup of a q-subgroup Q_0 of H. But then $Q_0 \leq N_G(Q) = N_G(T)$ and then $Q_0 \leq Q$. This is not possible and hence $Q \in \mathrm{Syl}_q(H)$. Clearly $N_H(Q) \leq N_G(Q) \cap H = Q$ and then $N_G(Q) = C_G(Q)$. Thus the well-known Burnside theorem implies that H is q-nilpotent. Further $H = Q$, and this is the final contradiction.

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References

[1] G.A. Chambers, On f-preFrattini subgroups, Canad. Math. Bull. 15 (3) (1972).
[2] L.M. Ezquerro, A contribution to the theory of finite supersolvable groups, Rend. Sem. Mat. Univ. Padova 89 (1993) 161–170.
[3] W. Gaschütz, Praefrattini gruppen, Arch. Math. 13 (1962) 418–426.
[4] X. Guo, Cover-avoidance properties and the structure of finite groups, J. Pure Appl. Algebra 181 (2003) 297–308.
[5] B. Huppert, N. Blackburn, Finite Groups III, Springer-Verlag, New York, Berlin, 1982.
[6] H. Kurzweil, B. Stellmacher, The Theory of Finite Groups, an Introduction, Springer, 2004.
[7] M.J. Tomkinson, Cover-avoidance properties in finite solvable groups, Canad. Math. Bull. 19 (2) (1976) 213–216.
[8] Y. Wang, c-Normality of groups and its properties, J. Algebra 180 (1996) 954–965.
[9] H. Wei, Y. Wang, Y. Li, On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups. II, Commun. Algebra 31 (10) (2003) 4807–4816.