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RELATIVE *FP*-PROJECTIVE MODULES#

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Let R be a ring and M a right R-module. M is called n-FP-projective if $\mathrm{Ext}^1(M, N) = 0$ for any right R-module N of FP-injective dimension $\leq n$, where n is a nonnegative integer or $n = \infty$. $v_R(M)$ is defined as $\sup\{n : M$ is n-FP-projective$\}$ and $v_R(M) = -1$ if $\mathrm{Ext}^1(M, N) \neq 0$ for some FP-injective right R-module N. The right v-dimension $r.v\text{-}\mathrm{dim}(R)$ of R is defined to be the least nonnegative integer n such that $v_R(M) \geq n$ implies $v_R(M) = \infty$ for any right R-module M. If no such n exists, set $r.v\text{-}\mathrm{dim}(R) = \infty$. The aim of this paper is to investigate n-FP-projective modules and the v-dimension of rings.

Key Words: Cotorsion theory; v-Dimension; *FP*-injective dimension; n-*FP*-Projective module.

Mathematics Subject Classification: Primary 16E10; Secondary 16D40.

1. NOTATION

In this section, we shall recall some known notions and definitions which we need in the later sections.

Throughout this paper, R is an associative ring with identity and all modules are unitary R-modules. We write M_R ($_RM$) to indicate a right (left) R-module. If $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution of an R-module M, let $K_0 = M$, $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$ for $i \geq 2$. The nth kernel K_n ($n \geq 0$) is called the nth syzygy of M. As usual, $wD(R)$ stands for the weak global dimension of R. $pd(M)$, $id(M)$ and $fd(M)$ denote the projective, injective and flat dimensions of M respectively. $\mathrm{Hom}(M, N)$ ($\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ ($\mathrm{Ext}^n_R(M, N)$) for an integer $n \geq 1$, and similarly $\mathrm{Tor}_n(M, N)$ denotes $\mathrm{Tor}^R_n(M, N)$ unless otherwise specified. For other concepts and notations, we refer the reader to Anderson and Fuller (1974), Enochs and Jenda (2000), Wisbauer (1991), and Xu (1996).

Let R be a ring and M a right R-module.

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M is called *FP*-injective (or absolutely pure) (Madox, 1967 and Stenström, 1970) if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented right R-modules N. Following Stenström (1970), the *FP*-injective dimension of M, denoted by $FP\text{-}id(M)$, is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for every finitely presented right R-module F (if no such n exists, set $FP\text{-}id(M) = \infty$), and $r.FP\text{-}\dim(R)$ is defined as $\sup\{FP\text{-}id(M) : M \text{ is a right } R\text{-module}\}$.

In Mao and Ding (2005), the *FP*-projective dimension $fpd(M)$ of M is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}(M, N) = 0$ for any *FP*-injective right R-module N. If no such n exists, set $fpd(M) = \infty$. The right *FP*-projective dimension $rfpD(R)$ of a ring R is defined as $\sup\{fpd(M) : M \text{ is a finitely generated right } R\text{-module}\}$. M is called *FP*-projective if $fpd(M) = 0$. Clearly, $fpd(M)$ measures how far away a right R-module M is from being *FP*-projective. Enochs (1976) proved that a finitely generated right R-module M is finitely presented if and only if $\mathrm{Ext}^1(M, N) = 0$ for any *FP*-injective right R-module N, and so a finitely generated *FP*-projective right R-module is finitely presented. It follows that $rfpD(R)$ measures how far away a ring R is from being right Noetherian.

A ring R is called right coherent if every finitely generated right ideal of R is finitely presented.

A pair $(\mathscr{F}, \mathscr{C})$ of classes of right R-modules is called a cotorsion theory (Enochs and Jenda, 2000) if $\mathscr{F}^{\perp} = \mathscr{C}$ and ${}^{\perp}\mathscr{C} = \mathscr{F}$, where $\mathscr{F}^{\perp} = \{C : \mathrm{Ext}^1(F, C) = 0 \text{ for all } F \in \mathscr{F}\}$, and ${}^{\perp}\mathscr{C} = \{F : \mathrm{Ext}^1(F, C) = 0 \text{ for all } C \in \mathscr{C}\}$.

Let $\mathscr{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathscr{C}$ is called a $\mathscr{C}$-preenvelope of M (Enochs and Jenda, 2000) if for any homomorphism $f\colon M \to F'$ with $F' \in \mathscr{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F' = F$ and $f = \phi$, the $\mathscr{C}$-preenvelope ϕ is called a $\mathscr{C}$-envelope of M. A $\mathscr{C}$-envelope $\phi : M \to F$ is said to have the unique mapping property (Ding, 1996) if for any homomorphism $f\colon M \to F'$ with $F' \in \mathscr{C}$, there is a unique homomorphism $g : F \to F'$ such that $g\phi = f$. Following Enochs and Jenda (2000), Definition 7.1.6, a monomorphism $\alpha : M \to C$ with $C \in \mathscr{C}$ is said to be a special $\mathscr{C}$-preenvelope of M if $\mathrm{coker}(\alpha) \in {}^{\perp}\mathscr{C}$. Dually we have the definitions of a (special) $\mathscr{C}$-precover and a $\mathscr{C}$-cover (with the unique mapping property). Special $\mathscr{C}$-preenvelopes (respectively special $\mathscr{C}$-precovers) are obviously $\mathscr{C}$-preenvelopes (resp. $\mathscr{C}$-precovers).

2. INTRODUCTION

The *FP*-projective dimension of modules was introduced in Mao and Ding (2005) to measure how far away a module is from being *FP*-projective. In this paper, we approach *FP*-projective modules from another point of view and introduce the concepts of *n-FP*-projective modules and v-dimensions of modules and rings. A right R-module M is called *n-FP*-projective if $\mathrm{Ext}^1(M, N) = 0$ for all right R-modules N of *FP*-injective dimension $\leq n$, where n is a nonnegative integer or $n = \infty$. We define $v_R(M) = \sup\{n : M \text{ is } n\text{-}FP\text{-projective}\}$ and $v_R(M) = -1$ if $\mathrm{Ext}^1(M, N) \neq 0$ for some *FP*-injective right R-module N. The right v-dimension $r \cdot v\text{-}\dim(R)$ of a ring R is defined to be the least nonnegative integer n such that $v_R(M) \geq n$ implies $v_R(M) = \infty$ for any right R-module M. If no such n exists, set $r \cdot v\text{-}\dim(R) = \infty$. The purpose of this paper is to investigate these new notions.

Let R be a right coherent ring and n a fixed nonnegative integer.

In Section 3, we prove that $(\mathcal{FP}_n, \mathcal{FI}_n)$ is a cotorsion theory, moreover, every right R-module has a special $\mathcal{FI}_n$-preenvelope, and every right R-module has a special $\mathcal{FP}_n$-precover, where $\mathcal{FI}_n$ $(\mathcal{FP}_n)$ denotes the class of all right modules of FP-injective dimension $\leq n$ (all n-FP-projective right R-modules) (see Theorem 3.8). Some characterizations of n-FP-projective right R-modules are given in Proposition 3.11.

Section 4 is devoted to rings whose every n-FP-projective module is projective. It is shown that $wD(R) \leq n$ if and only if every n-FP-projective right R-module is projective if and only if every 0-FP-projective right R-module is of projective dimension $\leq n$ if and only if every n-FP-projective right R-module has an $\mathcal{FI}_n$-envelope with the unique mapping property (see Theorem 4.1). In particular, R is a right semi-hereditary ring if and only if every 0-FP-projective right R-module has a monic $\mathcal{FI}_0$-cover (see Corollary 4.2), and $r.\nu\text{-dim}(R) = wD(R)$ if $wD(R) < \infty$ (see Corollary 4.3).

In Section 5, we characterize rings with the finite right ν-dimension. It is shown that $r \cdot \nu\text{-dim}(R) \leq n$ if and only if every n-FP-projective right R-module is $(n+1)$-FP-projective if and only if every right R-module with finite FP-injective dimension has FP-injective dimension $\leq n$ (see Theorem 5.1).

Section 6 studies how are the rings satisfying that every module is n-FP-projective. It is proven that every right R-module is n-FP-projective if and only if every right R-module with FP-injective dimension $\leq n$ has an $\mathcal{FP}_n$-cover with the unique mapping property (see Theorem 6.1). We conclude this paper by proving that $rfpD(R) \leq 1$ and $\mathcal{FP}_0$ is closed under direct products if and only if every right R-module has an epic $\mathcal{FP}_0$-envelope (see Theorem 6.3).

3. DEFINITION AND GENERAL RESULTS

We start with the following definition.

Definition 3.1. Let R be a ring and n a nonnegative integer or ∞. A right R-module M is called n-FP-projective provided that $\text{Ext}^1(M, N) = 0$ for any right R-module N with $FP\text{-}id(N) \leq n$.

For a right R-module M, let $\nu_R(M) = \sup\{n : M \text{ is } n\text{-}FP\text{-projective}\}$. We define $\nu_R(M) = -1$ if $\text{Ext}^1(M, N) \neq 0$ for some FP-injective right R-module N.

The right ν-dimension of a ring R, denoted by $r.\nu\text{-dim}(R)$, is defined to be the least nonnegative integer n such that $\nu_R(M) \geq n$ implies $\nu_R(M) = \infty$ for any right R-module M. If no such n exists, set $r.\nu\text{-dim}(R) = \infty$.

Remark 3.2. (1) 0-FP-projective modules were called FP-projective modules in Mao and Ding (2005) and finitely covered modules in Trlifaj (2000). Clearly, finitely presented R-modules are always 0-FP-projective, and projective modules are exactly ∞-FP-projective modules.

(2) It is clear that $\nu_R(M) \geq n$ if and only if M is n-FP-projective for an integer $n \geq 0$, and $\nu_R(M) = \infty$ if and only if M is m-FP-projective for any integer $m \geq 0$ if and only if $\text{Ext}^1(M, N) = 0$ for all right R-modules N with $FP\text{-}id(N) < \infty$.

(3) If $r.FP\text{-dim}(R) \leq n$, then the class of all n-FP-projective right R-modules and the class of all projective right R-modules are the same. Therefore it is always true that $r.\nu\text{-dim}(R) \leq r.FP\text{-dim}(R)$.

(4) Let R be an *n-FC* ring, i.e., R is a two-sided coherent ring with $FP\text{-}id({}_RR) \leq n$ and $FP\text{-}id(R_R) \leq n$ for some nonnegative integer n (see Ding and Chen, 1996). If M is an *n-FP*-projective right (or left) R-module, then $v_R(M) = \infty$. Indeed, this follows from the fact that $FP\text{-}id(N) \leq n$ if and only if $FP\text{-}id(N) < \infty$ for any right (or left) R-module N (see Ding and Chen, 1993, Proposition 3.16).

(5) If R is right coherent and $FP\text{-}id(M) = m$, then $\mathrm{Ext}^{m+k}(F, M) = 0$ for each finitely presented right R-module F and each $k \geq 1$.

Lemma 3.3. *Let R be a right coherent ring. If M is an n-FP-projective right R-module for some integer $n \geq 0$, then $\mathrm{Ext}^j(M, N) = 0$ for any integer $j \geq 2$ and any right R-module N with $FP\text{-}id(N) \leq n + 1$.*

Proof. For every right R-module N of *FP*-injective dimension $\leq n + 1$, there is a short exact sequence $0 \to N \to E \to L \to 0$ with E injective and $FP\text{-}id(L) \leq n$. Therefore $\mathrm{Ext}^2(M, N) \cong \mathrm{Ext}^1(M, L) = 0$ and the result follows by induction. □

Remark 3.4. Lemma 3.3 shows that, if R is a right coherent ring and M an *n-FP*-projective right R-module, then $\mathrm{Ext}^j(M, N) = 0$ for any integer $j \geq 1$ and any right R-module N with $FP\text{-}id(N) \leq n$.

Proposition 3.5. *Let R be a right coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules.*

(1) *If $v_R(C) \geq 0$, then $v_R(A) \geq inf\{v_R(B), v_R(C) + 1\}$.*
(2) *$v_R(B) \geq inf\{v_R(A), v_R(C)\}$.*
(3) *If $B = A \oplus C$, then $v_R(A \oplus C) = inf\{v_R(A), v_R(C)\}$.*

Proof. The exact sequence $0 \to A \to B \to C \to 0$ gives rise to the exactness of the sequence $\mathrm{Ext}^1(C, N) \to \mathrm{Ext}^1(B, N) \to \mathrm{Ext}^1(A, N) \to \mathrm{Ext}^2(C, N)$ for any right R-module N. Now the result follows from Lemma 3.3 by a standard homological algebra argument. □

Corollary 3.6. *Let R be a right coherent ring.*

(1) *The nth syzygy K_n of every finitely presented right R-module is n-FP-projective.*
(2) *Every finitely generated submodule of any finitely generated 1-FP-projective right R-module is 1-FP-projective. In particular, each finitely generated right ideal of R is 1-FP-projective.*
(3) *For any right R-module homomorphism $\alpha: M \to N$ with M and N finitely generated 1-FP-projective, $\ker(\alpha)$ is 1-FP-projective. Furthermore, if M is 2-FP-projective, then $\ker(\alpha)$ is 2-FP-projective.*
(4) *The dual module $M^* = \mathrm{Hom}(M, R)$ of any finitely presented left R-module M is 2-FP-projective.*

Proof. (1) Let M be a finitely presented right R-module. There is an exact sequence

$$0 \to K_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with each P_i projective ($0 \leq i \leq n-1$). Let $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$, $2 \leq i \leq n$. Then $0 \to K_1 \to P_0 \to M \to 0$ is exact. Since $v_R(M) \geq 0$, $v_R(P_0) = \infty$ (for P_0 is projective), we have $v_R(K_1) \geq v_R(M) + 1 \geq 1$ by Proposition 3.5 (1). Thus (1) follows by induction.

(2) Let N be a finitely generated submodule of any finitely generated 1-*FP*-projective right R-module M. Then we have a short exact sequence $0 \to N \to M \to M/N \to 0$. Note that M/N is finitely presented (for every finitely generated 1-*FP*-projective module is *FP*-projective, and hence finitely presented), so $v_R(M/N) \geq 0$. Hence $v_R(N) \geq 1$ by Proposition 3.5 (1), i.e., N is 1-*FP*-projective. The rest is clear.

(3) Note that $\mathrm{im}(\alpha) \leq N$, and $\mathrm{im}(\alpha)$ is finitely generated, so $M/\ker(\alpha) \cong \mathrm{im}(\alpha)$ is 1-*FP*-projective by (2). Consider the exact sequence $0 \to \ker(\alpha) \to M \to M/\ker(\alpha) \to 0$. Then (3) follows from Proposition 3.5 (1).

(4) Let M be a finitely presented left R-module. Then there exists an exact sequence $F_1 \to F_0 \to M \to 0$ with F_1 and F_0 finitely generated free, which gives rise to the exactness of the sequence $0 \to M^* \to F_0^* \to F_1^*$. Therefore (4) holds by (3). $\square$

Recall that a right R-module M is called Gorenstein projective if there is an exact sequence

$$\widetilde{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective right R-modules such that $M = \ker(P^0 \to P^1)$, and $\mathrm{Hom}(\widetilde{P}, Q)$ is exact for any projective right R-module Q (see Enochs and Jenda, 2000, Definition 10.2.1). A ring R is called a Gorenstein ring if R is a left and right noetherian ring, $id(_RR) < \infty$ and $id(R_R) < \infty$. Furthermore, if $id(_RR) \leq n$ and $id(R_R) \leq n$ for an integer $n \geq 0$, then R is called an n-Gorenstein ring.

We observe that, if R is a Gorenstein ring, then each Gorenstein projective module is m-*FP*-projective for any integer m with $0 \leq m < \infty$, furthermore, if R is an n-Gorenstein ring, then a right (or left) R-module M is m-*FP*-projective ($n \leq m < \infty$) if and only if M is Gorenstein projective by Enochs and Jenda (2000, Theorem 9.1.10 and Corollary 11.5.3).

Remark 3.7. Obviously, if M is a projective right R-module, then $v_R(M) = \infty$. However, the converse is false in general because $v_R(M) = \infty$ for every Gorenstein projective module M over a Gorenstein ring as shown by the preceding observation.

It is well known that (the class of Gorenstein projective R-modules, the class of R-modules of finite projective dimension) is a cotorsion theory over any Gorenstein ring R (see Enochs and Jenda, 2000, Remark 11.5.10). Denote by $\mathcal{FI}_n$ ($\mathcal{FP}_n$) the class of all right modules of *FP*-injective dimension $\leq n$ (all n-*FP*-projective right R-modules). Then we have:

Theorem 3.8. *Let R be a right coherent ring and $n \geq 0$. Then $(\mathcal{FP}_n, \mathcal{FI}_n)$ is a cotorsion theory. Moreover, every right R-module has a special $\mathcal{FI}_n$-preenvelope, and every right R-module has a special $\mathcal{FP}_n$-precover.*

Proof. Let M be a right R-module. M admits an injective resolution

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to \cdots .$$

Write $L^n = \mathrm{im}(E^{n-1} \to E^n)$, $L^0 = M$. Then $M \in \mathcal{FI}_n$ if and only if L^n is FP-injective. Note that the latter is equivalent to $\mathrm{Ext}^1(R/I, L^n) = 0$ for all finitely generated right ideals I of R by Stenström (1970, Lemma 3.1). This means that $\mathrm{Ext}^{n+1}(R/I, M) = 0$ by dimension shifting. Denote by K_I the nth syzygy module of the cyclic finitely presented right R-module R/I. Then $\mathrm{Ext}^{n+1}(R/I, M) = 0$ if and only if $\mathrm{Ext}^1(K_I, M) = 0$. Thus $\mathcal{FI}_n = \left(\bigoplus K_I\right)^{\perp}$, where the sum is over all finitely generated right ideals I in R, and so the result follows from Eklof and Trlifaj (2001, Theorem 10) and Enochs and Jenda (2000, Definition 7.1.5). □

Remark 3.9. (1) Let m and n be nonnegative integers such that $m < n$. If M is n-FP-projective, then M is m-FP-projective. However, the converse is not true in general. In fact, take R to be a right coherent ring with $wD(R) = r.FP\text{-}\dim(R) = n$, for example, let $R = S[X_1, X_2, \ldots, X_n]$, the ring of polynomials in n indeterminates over a von Neumann regular ring S (see Glaz, 1989). Then the class of all right R-modules $= \mathcal{FI}_n \neq \mathcal{FI}_m$, so there exists an m-FP-projective right R-module which is not n-FP-projective by Theorem 3.8.

(2) It is known that $\mathcal{FI}_n$-envelopes may not exist in general (see Trlifaj, 2000, Theorem 4.9). However, if $\mathcal{FP}_n$ is closed under direct limits, then every right R-module has an $\mathcal{FI}_n$-envelope and every right R-module has an $\mathcal{FP}_n$-cover by Theorem 3.8 and Enochs and Jenda (2000, Theorem 7.2.6).

Corollary 3.10. *Let R be a right coherent ring. Suppose a right R-module M is the union of a continuous chain $(M_\alpha)_{\alpha<\lambda}$ of submodules. If $M_0 = 0$, $M_{\alpha+1}/M_\alpha$ is projective relative to each epimorphism $A \to B$, where A and B are FP-injective, whenever $\alpha + 1 < \lambda$, then M is 1-FP-projective.*

Proof. Let X be a right R-module with $FP\text{-id}(X) \leq 1$. Consider an exact sequence $0 \to X \to E \to Y \to 0$ with E injective. Then Y and E are FP-injective. Applying the functor $\mathrm{Hom}(M_{\alpha+1}/M_\alpha, -)$ to the sequence, one gets $\mathrm{Ext}^1(M_{\alpha+1}/M_\alpha, X) = 0$ whenever $\alpha + 1 < \lambda$. So the result follows from Theorem 3.8 and Enochs and Jenda (2000, Corollary 7.3.5). □

We end this section with the following characterizations of n-FP-projective R-modules.

Proposition 3.11. *Let R be a right coherent ring with FP-$id(R_R) \leq n$ for an integer $n \geq 0$. Then the following are equivalent for a right R-module M:*

1. *M is n-FP-projective;*
2. *M is projective with respect to every exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{FI}_n$;*
3. *For every exact sequence $0 \to K \to F \to M \to 0$ with $F \in \mathcal{FI}_n$, $K \to F$ is an $\mathcal{FI}_n$-preenvelope of K;*

(4) *M is a cokernel of an* $\mathscr{F}\mathscr{I}_n$*-preenvelope* $K \to F$ *with* F *projective*;
(5) *There exists a right R-module exact sequence*

$$\widetilde{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$$

with $M = \ker(E^0 \to E^1)$, $FP\text{-}id(E^i) \le n$, $FP\text{-}id(E_i) \le n$ *and* E_i *projective*, $i = 0, 1, 2, \ldots,$ *such that* $\mathrm{Hom}(\widetilde{E}, N)$ *is exact for all right R-modules N with FP-id*$(N) \le n$;
(6) *There exists a projective resolution* $\widetilde{E} = \cdots \to E_1 \to E_0 \to M \to 0$ *such that* $\mathrm{Hom}(\widetilde{E}, N)$ *is exact for all right R-modules N with FP-id*$(N) \le n$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) For every $N \in \mathscr{F}\mathscr{I}_n$, consider a short exact sequence $0 \to N \to E \to L \to 0$ with E injective.

(1) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) Let $0 \to K \to P \to M \to 0$ be an exact sequence with P projective. Note that $FP\text{-}id(P) \le n$ since $FP\text{-}id(R_R) \le n$, thus $K \to P$ is an $\mathscr{F}\mathscr{I}_n$-preenvelope.

(4) $\Rightarrow$ (1) By (4), there is an exact sequence $0 \to K \to P \to M \to 0$, where $K \to P$ is an $\mathscr{F}\mathscr{I}_n$-preenvelope with P projective. Hence there is an exact sequence $\mathrm{Hom}(P, N) \to \mathrm{Hom}(K, N) \to \mathrm{Ext}^1(M, N) \to 0$ for each $N \in \mathscr{F}\mathscr{I}_n$. Note that $\mathrm{Hom}(P, N) \to \mathrm{Hom}(K, N) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(M, N) = 0$, as desired.

(1) $\Rightarrow$ (5) Let $\cdots \to E_1 \to E_0 \to M \to 0$ be a projective resolution of M. By hypothesis, $FP\text{-}id(E_i) \le n$, $i = 0, 1, 2, \ldots$. Let N be any right R-module with $FP\text{-}id(N) \le n$. Since M is n-FP-projective, $\mathrm{Ext}^j(M, N) = 0$ for any integer $j \ge 1$ by Remark 3.4. Therefore the sequence

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(E_0, N) \to \mathrm{Hom}(E_1, N) \to \cdots$$

is exact. On the other hand, we can construct an exact sequence

$$0 \to M \to E^0 \to E^1 \to \cdots,$$

where $M \to E^0$, $\mathrm{coker}(M \to E^0) \to E^1$, $\mathrm{coker}(E^{n-1} \to E^n) \to E^{n+1}$ for $n \ge 1$ are $\mathscr{F}\mathscr{I}_n$-preenvelopes by Theorem 3.8. Thus we have the following exact sequence

$$\cdots \to \mathrm{Hom}(E^1, N) \to \mathrm{Hom}(E^0, N) \to \mathrm{Hom}(M, N) \to 0.$$

Let

$$\widetilde{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots.$$

Then $\mathrm{Hom}(\widetilde{E}, N)$ is exact.

(5) $\Rightarrow$ (6) is obvious.

$(6) \Rightarrow (1)$ It follows since

$$\cdots \to E_1 \to E_0 \to M \to 0$$

is a projective resolution of M and $\mathrm{Hom}(E_0, N) \to \mathrm{Hom}(E_1, N) \to \mathrm{Hom}(E_2, N)$ is exact for all right R-modules N with $FP\text{-}id(N) \leq n$. □

4. RINGS WHOSE EVERY *n-FP*-PROJECTIVE MODULE IS PROJECTIVE

It is well–known that a ring R is von Neumann regular if and only if every right R-module is FP-injective if and only if every finitely presented right R-module is projective (flat). So R is von Neumann regular if and only if every 0-FP-projective right R-module is projective (flat). In addition, if R is a right coherent ring, then R is von Neumann regular if and only if every 0-FP-projective right R-module is FP-injective (see Mao and Ding, 2005, Corollary 4.3).

In what follows, let $\sigma_M : M \to \mathcal{FI}_n(M)$ ($\epsilon_M : \mathcal{FP}_n(M) \to M$) denote the $\mathcal{FI}_n$-envelope ($\mathcal{FP}_n$-cover) of a right R-module M. Now we have:

Theorem 4.1. *Let R be a right coherent ring and n a fixed nonnegative integer. Then the following are equivalent*:

(1) $r.FP\text{-}\dim(R) \leq n$;
(2) $wD(R) \leq n$;
(3) *Every n-FP-projective right R-module is projective*;
(4) *Every n-FP-projective right R-module is flat*;
(5) $pd(M) \leq n$ *for every* 0-*FP-projective right R-module M*;
(6) $FP\text{-}id(M) \leq n$ *for every n-FP-projective right R-module M*;
(7) *Every* (*n-FP-projective*) *right R-module has an* $\mathcal{FI}_n$*-envelope with the unique mapping property.*

Moreover if $n \geq 1$, *then the above conditions are also equivalent to*

(8) $pd(M) \leq 1$ ($fd(M) \leq 1$) *for every* $(n-1)$-*FP-projective right R-module M*;
(9) *Every* ($(n-1)$-*FP-projective*) *right R-module M has a monic* $\mathcal{FI}_{n-1}$*-cover* $\phi : F \to M$.

Proof. $(1) \Leftrightarrow (2) \Leftarrow (5)$ hold by Stenström (1970, Theorem 3.3), $(1) \Rightarrow (3) \Rightarrow (4)$ and $(1) \Rightarrow (7)$ are trivial.

$(3) \Rightarrow (1)$ Since $r \cdot FP\text{-}\dim(R) \leq n$ is equivalent to $FP\text{-}id(M) \leq n$ for every right R-module M, the result follows from Theorem 3.8.

$(4) \Rightarrow (2)$ Let M be any finitely presented right R-module. By Corollary 3.6 (1), the nth syzygy of M is n-FP-projective, and so it is flat by (4). Thus $fd(M) \leq n$, which implies that $wD(R) \leq n$ by Enochs and Jenda (2000, Theorem 8.4.20).

$(6) \Rightarrow (1)$ Let M be a right R-module. By Theorem 3.8, M has a special $\mathcal{FP}_n$-precover, and hence there is a short exact sequence $0 \to K \to N \to M \to 0$, where $FP\text{-}id(K) \leq n$ and N is n-FP-projective. Since $FP\text{-}id(N) \leq n$ by (6), $FP\text{-}id(M) \leq n$. So (1) follows.

(7) ⇒ (6) Let M be an n-FP-projective right R-module. There is the following exact commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\sigma_M"] \arrow[drr, "0"'] & \mathcal{F}\mathcal{I}_n(M) \arrow[r, "\gamma"] \arrow[dr, "\sigma_L\gamma"] & L \arrow[r] \arrow[d, "\sigma_L"] & 0 \\
 & & & \mathcal{F}\mathcal{I}_n(L), &
\end{tikzcd}$$

where L is n-FP-projective by Wakamatsu's Lemma in Xu (1996, Lemma 2.1.2). Note that $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$, so $\sigma_L\gamma = 0$ by (7). Therefore $L = \mathrm{im}(\gamma) \subseteq \ker(\sigma_L) = 0$, and hence $M \in \mathcal{F}\mathcal{I}_n$. Thus (6) follows.

(1) ⇒ (5) The proof has appeared in Mao and Ding (2005, Theorem 4.2), and here we include it for completeness. Let M be a 0-FP-projective right R-module. Then M admits a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0.$$

Let N be any right R-module. Since $FP\text{-}id(N) \leq n$, by Stenström (1970, Lemma 3.1), there is an exact sequence

$$0 \to N \to E^0 \to E^1 \to \cdots E^{n-1} \to E^n \to 0,$$

where $E^0, E^1, \ldots, E^n$ are FP-injective. Therefore we form the following double complex

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^n) & \to & \mathrm{Hom}(P_0, E^n) & \to \cdots \to & \mathrm{Hom}(P_n, E^n) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^1) & \to & \mathrm{Hom}(P_0, E^1) & \to \cdots \to & \mathrm{Hom}(P_n, E^1) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathrm{Hom}(M, E^0) & \to & \mathrm{Hom}(P_0, E^0) & \to \cdots \to & \mathrm{Hom}(P_n, E^0) & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & \to & \mathrm{Hom}(P_0, N) & \to \cdots \to & \mathrm{Hom}(P_n, N) & \to \cdots \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

Note that all rows are exact, except for the bottom row since M is 0-FP-projective and all E^i are FP-injective. Also note that all columns are exact except for the left column since all P_i are projective.

Using a spectral sequence argument, we know that the following two complexes

$$0 \to \mathrm{Hom}(P_0, N) \to \mathrm{Hom}(P_1, N) \to \cdots \to \mathrm{Hom}(P_n, N) \to \cdots$$

and

$$0 \to \mathrm{Hom}(M, E^0) \to \mathrm{Hom}(M, E^1) \to \cdots \to \mathrm{Hom}(M, E^n) \to 0$$

have isomorphic homology groups. Thus $\mathrm{Ext}^{n+j}(M, N) = 0$ for all integers $j \geq 1$, and hence $pd(M) \leq n$.

(1) $\Rightarrow$ (8) Let M be an $(n-1)$-FP-projective right R-module and N any right R-module. Since $FP\text{-}id(N) \leq n$, $\mathrm{Ext}^2(M, N) = 0$ by Lemma 3.3. Thus $pd(M) \leq 1$.

(8) $\Rightarrow$ (2) The proof is similar to that of (4) $\Rightarrow$ (2).

(1) $\Rightarrow$ (9) Let M be any right R-module. Write $F = \sum\{N \leq M : FP\text{-}id(N) \leq n-1\}$ and $G = \bigoplus\{N \leq M : FP\text{-}id(N) \leq n-1\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$. Since $FP\text{-}id(K) \leq n$ by (1) and $FP\text{-}id(G) \leq n-1$, we have $FP\text{-}id(F) \leq n-1$. Next, we prove that the inclusion $i : F \to M$ is an $\mathcal{FI}_{n-1}$-cover of M. Let $\psi : F' \to M$ with $F' \in \mathcal{FI}_{n-1}$ be an arbitrary right R-homomorphism. Note that $\psi(F') \leq F$ by the proof above. Define $\zeta : F' \to F$ via $\zeta(x) = \psi(x)$ for $x \in F'$. Then $i\zeta = \psi$, and so $i : F \to M$ is an $\mathcal{FI}_{n-1}$-precover of M. In addition, it is clear that the identity map I_F of F is the only homomorphism $g : F \to F$ such that $ig = i$, and hence (9) follows.

(9) $\Rightarrow$ (6) Let M be any n-FP-projective right R-module. We shall show that $FP\text{-}id(M) \leq n$. Indeed, by Theorem 3.8, there exists an exact sequence $0 \to M \to E \to L \to 0$ with $FP\text{-}id(E) \leq n-1$ and $L \in \mathcal{FP}_{n-1}$. Since L has a monic $\mathcal{FI}_{n-1}$-cover $\phi : F \to L$, there is $\alpha : E \to F$ such that the following exact diagram is commutative.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & F \arrow[d, "\phi"] & \\
0 \arrow[r] & M \arrow[r] & E \arrow[r] \arrow[ur, "\alpha"] & L \arrow[r] & 0
\end{tikzcd}$$

Thus ϕ is epic, and hence it is an isomorphism. Therefore $FP\text{-}id(L) = FP\text{-}id(F) \leq n-1$, and so $FP\text{-}id(M) \leq n$, as desired. □

It is well known that a right coherent ring R is right semi-hereditary if and only if $wD(R) \leq 1$.

By specializing Theorem 4.1 to the case $n = 1$, we have

Corollary 4.2. *Let R be a right coherent ring. Then the following are equivalent*:

(1) *R is right semi-hereditary*;
(2) *Every* 1-*FP-projective right R-module is projective* (*flat*);
(3) *Every* 0-*FP-projective right R-module is of projective* (*flat*) *dimension* ≤ 1;
(4) *Every* 1-*FP-projective right R-module is of FP-injective dimension* ≤ 1;
(5) *Every* (0-*FP-projective*) *right R-module has a monic* $\mathcal{FI}_0$-*cover.*

Corollary 4.3. *Let R be a right coherent ring with $wD(R) < \infty$. Then $wD(R) = r.FP$-$\dim(R) = r.v$-$\dim(R)$.*

Proof. Stenström (1970, Theorem 3.3) shows that $r.FP$-$\dim(R) = wD(R)$. By Remark 3.2 (3), $r.v$-$\dim(R) \leq r.FP$-$\dim(R)$. Conversely, let $r.v$-$\dim(R) = n < \infty$. For any n-FP-projective right R-module M, we have $v_R(M) = \infty$, and so $\mathrm{Ext}^1(M, N) = 0$ for any right R-module N with FP-$id(N) < \infty$, which implies that M is projective since $r.FP$-$\dim(R) < \infty$. Therefore $r.FP$-$\dim(R) \leq n$ by Theorem 4.1 (3). This completes the proof. □

5. RINGS WITH FINITE RIGHT v-DIMENSION

In this section we characterize rings with finite right v-dimension.

Theorem 5.1. *Let R be a right coherent ring and n a fixed nonnegative integer. Then the following are equivalent:*

(1) $r.v$-$\dim(R) \leq n$;
(2) *Every n-FP-projective right R-module is $(n+1)$-FP-projective*;
(3) *Every* nth *syzygy of any finitely presented right R-module is projective relative to each epimorphism $B \to C$, where B is FP-injective and FP-$id(C) \leq n$*;
(4) *Every right R-module M with FP-$id(M) \leq n+1$ has FP-injective dimension $\leq n$*;
(5) *Every right R-module with finite FP-injective dimension has FP-injective dimension $\leq n$*;
(6) *Every nth syzygy of any finitely presented right R-module is $(n+1)$-FP-projective*;
(7) *Every nth syzygy of any finitely presented right R-module is m-FP-projective for any integer $m \geq n+1$*;
(8) $fd(M^+) \leq n$ *for any right R-module M with FP-$id(M) \leq n+1$, where* $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$;
(9) *For any pure submodule N of every right R-module M with FP-$id(M) \leq n+1$, FP-$id(M/N) \leq n$.*

Proof. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) and (7) $\Rightarrow$ (6) are clear.

(2) $\Rightarrow$ (4) holds by Theorem 3.8.

(5) $\Rightarrow$ (1) Let M be a right R-module with $v_R(M) \geq n$, i.e., M is n-FP-projective. For any right R-module N with FP-$id(N) < \infty$, we have $\mathrm{Ext}^1(M, N) = 0$ since FP-$id(N) \leq n$ by (5). So $v_R(M) = \infty$, as desired.

(1) $\Rightarrow$ (7) follows from Corollary 3.6 (1).

(6) $\Rightarrow$ (9) Let N be a pure submodule of a right R-module M with FP-$id(M) \leq n+1$. Then the pure exact sequence $0 \to N \to M \to M/N \to 0$ gives rise to the split exact sequence $0 \to (M/N)^+ \to M^+ \to N^+ \to 0$. Therefore $(M/N)^+$ is a direct summand of M^+. By Fieldhouse (1972, Theorem 2.2), we have $fd(M^+) = FP$-$id(M) \leq n+1$. So $fd((\mathrm{M/N})^+) \leq n+1$, and hence FP-$id(M/N) \leq n+1$. Let K be any finitely presented right R-module and K_n an nth syzygy of K. Then $\mathrm{Ext}^1(K_n, M/N) = 0$ by (6), and so $\mathrm{Ext}^{n+1}(K, M/N) = 0$, which implies that FP-$id(M/N) \leq n$.

(9) $\Rightarrow$ (4) holds by letting $N = 0$.

(6) $\Leftrightarrow$ (8) Let N be a finitely presented right R-module and N_n an nth syzygy of N. By Rotman (1979, Theorem 9.51) and the remark following it, $\mathrm{Tor}_{n+1}(N, M^+) \cong (\mathrm{Ext}^{n+1}(N, M))^+$ for any right R-module M. Since $\mathrm{Ext}^{n+1}(N, M) \cong \mathrm{Ext}^1(N_n, M)$, the equivalence follows.

(3) $\Rightarrow$ (4) Let M be a right R-module with $FP\text{-}id(M) \leq n+1$. There exists an exact sequence $0 \to M \to E \to N \to 0$ with E injective and $FP\text{-}id(N) \leq n$. Suppose that K is a finitely presented right R-module and K_n an nth syzygy of K. Then $\mathrm{Ext}^1(K_n, M) = 0$ by (3), and hence $\mathrm{Ext}^{n+1}(K, M) = 0$, which means $FP\text{-}id(M) \leq n$.

(4) $\Rightarrow$ (3) Let $f : B \to C$ be an epimorphism and $A = \ker(f)$, where B is FP-injective and $FP\text{-}id(C) \leq n$. The exactness of $0 \to A \to B \to C \to 0$ shows that $FP\text{-}id(A) \leq n+1$, and so $FP\text{-}id(A) \leq n$ by (4). Let N_n be an nth syzygy of a finitely presented right R-module N. Then N_n is n-FP-projective by Corollary 3.6 (1), and so $\mathrm{Ext}^1(N_n, A) = 0$. Thus (3) follows. $\square$

Let $n = 0$ in Theorem 5.1. One gets

Corollary 5.2. *Let R be a right coherent ring. Then the following are equivalent*:

(1) $r.v\text{-}\dim(R) = 0$;
(2) *Every* 0-*FP-projective right R-module is* 1-*FP-projective*;
(3) *Every finitely presented right R-module is projective relative to each epimorphism $B \to C$, where B and C are FP-injective*;
(4) *For any short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if B and C are FP-injective, then A is FP-injective*;
(5) *Every right R-module with finite FP-injective dimension is FP-injective*;
(6) *Every finitely presented right R-module is* 1-*FP-projective*;
(7) *Every finitely presented right R-module is m-FP-projective for any integer $m \geq 1$*;
(8) *M^+ is flat for any right R-module M with $FP\text{-}id(M) \leq 1$*;
(9) *For any pure submodule N of every right R-module M with $FP\text{-}id(M) \leq 1$, the quotient M/N is FP-injective.*

Remark 5.3. (1) Recall that a ring R is said to be right *IF* if every injective right R-module is flat (see Colby, 1975). A right coherent and right *IF* ring R satisfies the equivalent conditions in Corollary 5.2. In fact, every finitely presented right R-module M is a submodule of a finitely generated free right R-module by Colby (1975, Theorem 1), so M is n-FP-projective for any $n \geq 1$ by Corollary 3.6 (1).

(2) By Remark 3.2 (3), $r.v\text{-}\dim(R) \leq r.FP\text{-}\dim(R)$. The inequality may be strict. In fact, let $R = \mathbb{Z}_4$. Then R is a QF ring with $wD(R) = \infty$ by Rotman (1979, Exercise 9.2 and Theorem 9.22), and so $FP\text{-}\dim(R) = \infty$ by Stenström (1970, Theorem 3.3). On the other hand, since the class of (FP-)injective modules over a QF ring coincides with the class of projective modules, $r.v\text{-}\dim(R) = 0$ by Corollary 5.2 (4). This example also shows that Corollary 4.3 does not hold for a right coherent ring R with $wD(R) = \infty$.

(3) Let R be a commutative ring. The λ-dimension $\lambda_R(M)$ of an R-module M and the λ-dimension λ-dim(R) of the ring R have been widely studied (see Couchot, 2003 and Vasconcelos, 1976). It is well known that R is noetherian if and only if λ-dim$(R) = 0$, and R is coherent if and only if λ-dim$(R) \leq 1$. However the λ-dimension is completely different from the ν-dimension defined here. In fact, let $R = \mathbb{Z}$, the ring of integers. Then λ-dim$(R) = 0$. It is easy to see that ν-dim$(R) \leq 1$. However, in the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective, but $\mathbb{Z}$ is not injective, so ν-dim$(R) \neq 0$ by Corollary 5.2 (4). Thus ν-dim$(R) = 1$. On the other hand, there is a finitely generated R-module M which is not 1-*FP*-projective by Corollary 5.2 (6). Thus $\nu_R(M) = 0$ while $\lambda_R(M) = \infty$.

6. RINGS SATISFYING EVERY MODULE IS *n*-*FP*-PROJECTIVE

It is easy to see that a ring R is right noetherian if and only if every right R-module is 0-*FP*-projective if and only if $\nu_R(M) \geq 0$ for every right R-module M, and R is semisimple artinian if and only if every right R-module is ∞-projective.

Next we shall give characterizations of those rings satisfying every right R-module is n-*FP*-projective for a fixed nonnegative integer n.

Theorem 6.1. *Let R be a right coherent ring and n a fixed nonnegative integer. Then the following are equivalent*:

(1) *Every right R-module is n-FP-projective*;
(2) *Every finitely generated right R-module is n-FP-projective*;
(3) *Every cyclic right R-module is n-FP-projective*;
(4) *Every right R-module of FP-injective dimension $\leq n$ is n-FP-projective*;
(5) *Every right R-module of FP-injective dimension $\leq n$ is injective*;
(6) $\mathrm{Ext}^1(M, N) = 0$ *for all right R-modules M and N with FP-id$(M) \leq n$ and FP-id$(N) \leq n$*;
(7) $\mathrm{Ext}^i(M, N) = 0$ *for all $i \geq 1$ and all right R-modules M and N with FP-id$(M) \leq n$ and FP-id$(N) \leq n$*;
(8) *Every right R-module M (with FP-id$(M) \leq n$) has an $\mathscr{FP}_n$-cover with the unique mapping property.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (8) and (1) $\Rightarrow$ (4) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are obvious. (1) $\Leftrightarrow$ (5) follows from Theorem 3.8.

(8) $\Rightarrow$ (4) Let M be any right R-module with *FP-id*$(M) \leq n$. There is the following exact commutative diagram

$$\begin{tikzcd}
 & \mathscr{FP}_n(K) \arrow[d, "\epsilon_K"'] \arrow[dr, "\alpha\epsilon_K"] \arrow[drr, "0"] & & & \\
0 \arrow[r] & K \arrow[r, "\alpha"'] \arrow[d] & \mathscr{FP}_n(M) \arrow[r, "\epsilon_M"'] & M \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

with $K \in \mathscr{FI}_n$. Note that $\epsilon_M \alpha \epsilon_K = 0 = \epsilon_M 0$, so $\alpha\epsilon_K = 0$ by (8). Therefore $K = \mathrm{im}(\epsilon_K) \subseteq \ker(\alpha) = 0$, and so M is n-*FP*-projective, as required.

(4) $\Rightarrow$ (1) For any right R-module M, by Theorem 3.8, there is a short exact sequence $0 \to M \to N \to L \to 0$, where $FP\text{-}id(N) \leq n$ and L is n-*FP*-projective. Since N is n-*FP*-projective by (4), M is n-*FP*-projective by Proposition 3.5 (1). Hence (1) follows.

(3) $\Rightarrow$ (5) Let M be any right R-module with $FP\text{-}id(M) \leq n$ and I any right ideal of R. Then $\mathrm{Ext}^1(R/I, M) = 0$ by (3). Thus M is injective, as desired. ☐

Remark 6.2. By Theorem 6.1, if $n \geq 1$, then every right R-module is n-*FP*-projective if and only if every right R-module is 1-*FP*-projective if and only if every right R-module with finite *FP*-injective dimension is injective if and only if $v_R(M) = \infty$ for every right R-module M. Thus, right noetherian rings can be classified into three mutually exclusive types: (a) semisimple artinian rings; (b) rings R such that $wD(R) \neq 0$ and every right R-module is 1-*FP*-projective; (c) rings R for which there is a right R-module N with $v_R(N) = 0$.

Recall that the right *FP*-projective dimension $rfpD(R)$ of a ring R is defined as $\sup\{fpd(M) : M$ is a finitely generated right R-module$\}$ (see Mao and Ding, 2005). We conclude this paper with the following result which is of independent interest.

Theorem 6.3. *Let R be a right coherent ring. Then the following are equivalent*:

(1) $rfpD(R) \leq 1$ *and* $\mathscr{FP}_0$ *is closed under direct products*;
(2) *Every right R-module has an epic* $\mathscr{FP}_0$*-envelope.*

Proof. (1) $\Rightarrow$ (2) Let M be any right R-module and $\{M_i\}_{i\in I}$ the set of all the submodules of M with $M/M_i \in \mathscr{FP}_0$. The index set in the following statements is I. Let $\pi : M \to M/\cap M_i$ be the natural map. It is clear that $\alpha : M/\cap M_i \to M/M_i$ defined by $x + \cap M_i \mapsto x + M_i$ induces a monomorphism $\beta : M/\cap M_i \to \prod M/M_i$. Note that $\prod M/M_i \in \mathscr{FP}_0$, and so $M/\cap M_i \in \mathscr{FP}_0$ by Mao and Ding (2005, Proposition 3.7).

Now let $N \in \mathscr{FP}_0$ and $\delta : M \to N$ be any homomorphism. Since $M/\ker(\delta) \cong \mathrm{im}(\delta) \leq N \in \mathscr{FP}_0$, $M/\ker(\delta) \in \mathscr{FP}_0$. Thus $\cap M_i \leq \ker(\delta)$, and so there is $\xi : M/\cap M_i \to N$ such that $\xi\pi = \delta$. Thus π is an $\mathscr{FP}_0$-preenvelope of M. Since π is epic, π is an $\mathscr{FP}_0$-envelope of M.

(2) $\Rightarrow$ (1) For any family $\{M_i\}_{i\in I} \subseteq \mathscr{FP}_0$, $\prod M_i$ has an epic $\mathscr{FP}_0$-envelope $\alpha : \prod M_i \to M$ by (2). Let $\pi_i : \prod M_i \to M_i$ be the canonical projection. Then there is $\beta_i : M \to M_i$ such that $\beta_i\alpha = \pi_i$. On the other hand, there is $\gamma : M \to \prod M_i$ such that $\pi_i\gamma = \beta_i$. Therefore $\pi_i\gamma\alpha = \beta_i\alpha = \pi_i$, and so $\gamma\alpha = I_{\prod M_i}$. Thus α is monic, and hence $\prod M_i \cong M \in \mathscr{FP}_0$.

Now let M be any right R-module. Then, by Theorem 3.8, there exists an exact sequence $0 \longrightarrow K \xrightarrow{\varphi} N \longrightarrow M \longrightarrow 0$, where K is *FP*-injective and $N \in \mathscr{FP}_0$. By (2), K has an epic $\mathscr{FP}_0$-envelope $\gamma : K \to L$. Since $N \in \mathscr{FP}_0$, there is $\theta : L \to N$

such that the following exact diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\varphi"] \arrow[d, "\gamma"] & N \arrow[r] & M \arrow[r] & 0 \\ & L \arrow[ur, "\theta"'] \arrow[d] & & & \\ & 0 & & & \end{tikzcd}$$

It follows that γ is monic since φ is monic. Therefore γ is an isomorphism, and so $K \cong L \in \mathcal{FP}_0$. Thus $fpd(M) \leq 1$ by Mao and Ding (2005, Proposition 3.1), and hence $rfpD(R) \leq 1$. This completes the proof. $\square$

Remark 6.4. The proof of Theorem 6.3 shows that, if R is a right coherent ring such that $\mathcal{FP}_0$ is closed under direct products, then $rfpD(R) \leq 1$ if and only if every *FP*-injective right R-module has an epic $\mathcal{FP}_0$-envelope.

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