Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 208 (2007) 635–646
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Relative copure injective and copure flat modules

Lixin Mao[a,b], Nanqing Ding[a,*]

[a] Department of Mathematics, Nanjing University, Nanjing 210093, PR China
[b] Department of Basic Courses, Nanjing Institute of Technology, Nanjing 211167, PR China

Received 8 June 2005; received in revised form 1 February 2006
Available online 4 April 2006
Communicated by A.V. Geramita

Abstract

Let R be a ring, n a fixed nonnegative integer and $\mathcal{I}_n$ ($\mathcal{F}_n$) the class of all left (right) R-modules of injective (flat) dimension at most n. A left R-module M (resp., right R-module F) is called n-copure injective (resp., n-copure flat) if $\text{Ext}^1(N, M) = 0$ (resp., $\text{Tor}_1(F, N) = 0$) for any $N \in \mathcal{I}_n$. It is shown that a left R-module M over any ring R is n-copure injective if and only if M is a kernel of an $\mathcal{I}_n$-precover $f : A \to B$ of a left R-module B with A injective. For a left coherent ring R, it is proven that every right R-module has an $\mathcal{F}_n$-preenvelope, and a finitely presented right R-module M is n-copure flat if and only if M is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right R-module K with F flat. These classes of modules are also used to construct cotorsion theories and to characterize the global dimension of a ring under suitable conditions.

MSC: 16E10; 16D50; 16D40

1. Introduction

Let R be a ring. A left R-module M is called copure injective [10] if $\text{Ext}^1(N, M) = 0$ for all injective left R-modules N, and M is called strongly copure injective [10] if $\text{Ext}^i(N, M) = 0$ for all injective left R-modules N and all $i \geq 1$. A right R-module F is said to be copure flat [11] if $\text{Tor}_1(F, N) = 0$ for all injective left R-modules N, and F is said to be strongly copure flat [11] or weakly Gorenstein flat [13] if $\text{Tor}_i(F, N) = 0$ for all injective left R-modules N and all $i \geq 1$. Copure injective modules and copure flat modules were discovered when studying injective precovers and flat preenvelopes and have been studied by many authors (see [7,10,11,18]).

In Section 2 of this paper, we introduce the concepts of n-copure injective modules and n-copure flat modules for a fixed nonnegative integer n and show some of their general properties. A left R-module M (resp., right R-module F) is called n-copure injective (resp., n-copure flat) if $\text{Ext}^1(N, M) = 0$ (resp., $\text{Tor}_1(F, N) = 0$) for any left R-module N with $id(N) \leq n$. We note that n-copure injective modules and n-copure flat modules coincide with Gorenstein injective and Gorenstein flat modules [12] respectively over an n-Gorenstein ring R (i.e., R is a left and right noetherian ring with $id({}_RR) \leq n$ and $id(R_R) \leq n$). For a fixed nonnegative integer n, we denote by $\mathcal{I}_n$ ($\mathcal{F}_n$) the class of all left (right)

* Corresponding author. Fax: +86 25 83597130.
E-mail addresses: maolx2@hotmail.com (L. Mao), nqding@nju.edu.cn (N. Ding).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2006.03.002

R-modules of injective (flat) dimension at most n. It is shown that a left R-module M over any ring R is n-copure injective if and only if M is a kernel of an $\mathcal{I}_n$-precover $f : A \to B$ of a left R-module B with A injective. For a left coherent ring R, we prove that every right R-module has an $\mathcal{F}_n$-preenvelope, and a finitely presented right R-module M is n-copure flat if and only if M is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right R-module K with F flat.

Section 3 is devoted to n-copure injective modules and n-copure flat modules over left noetherian rings with finite left self-injective dimension. Let R be a left noetherian ring with $id(_R R) \leq n$ for an integer $n \geq 0$. It is shown that M is a reduced n-copure injective left R-module if and only if M is a kernel of an $\mathcal{I}_n$-cover $f : A \to B$ with A injective. It is also shown that $lD(R) \leq n$ ($n \geq 1$) if and only if every n-copure injective left R-module is injective if and only if every n-copure flat right R-module is flat if and only if every n-copure injective left R-module has a monic injective cover if and only if every n-copure flat right R-module has an epic flat envelope.

In Section 4, we further investigate some properties of copure injective covers and copure flat envelopes. For a commutative artinian ring R, we prove that $id(R) \leq 1$ if and only if every R-module has an epic copure flat envelope if and only if every R-module has a monic copure injective cover. For a left and right noetherian ring R, it is proven that R is a 2-Gorenstein ring and every R-module has a strongly copure injective cover if and only if every R-module has a strongly copure injective cover with the unique mapping property.

Next we recall some known notions and facts needed in the following.

Let $\mathcal{C}$ be a class of right R-modules and M a right R-module. Following [9], we say that a homomorphism $\phi : M \to C$ is a $\mathcal{C}$-preenvelope if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a $\mathcal{C}$-envelope if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. A $\mathcal{C}$-envelope $\phi : M \to C$ is said to have the unique mapping property [8] if for any homomorphism $f : M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g : C \to C'$ such that $g\phi = f$. Dually we have the definitions of a $\mathcal{C}$-precover and a $\mathcal{C}$-cover (with the unique mapping property). $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. A left (resp., right) $\mathcal{C}$-resolution of M [12] is a $\mathrm{Hom}(\mathcal{C}, -)$ (resp., $\mathrm{Hom}(-, \mathcal{C})$) exact complex

$$\cdots \to C_1 \to C_0 \to M \to 0 \text{ (resp., } 0 \to M \to C^0 \to C^1 \to \cdots)$$

with each $C_i, C^i \in \mathcal{C}$.

If $\cdots \to C_1 \to C_0 \to M \to 0$ is a left $\mathcal{C}$-resolution of M, let

$$K_0 = M, \qquad K_1 = \ker(C_0 \to M), \qquad K_i = \ker(C_{i-1} \to C_{i-2}) \quad \text{for } i \geq 2.$$

The nth kernel $K_n (n \geq 0)$ is called the nth $\mathcal{C}$-syzygy of M.

If $0 \to M \to C^0 \to C^1 \to \cdots$ is a right $\mathcal{C}$-resolution of M, let

$$L^0 = M, \qquad L^1 = \mathrm{coker}(M \to C^0), \qquad L^i = \mathrm{coker}(C^{i-2} \to C^{i-1}) \quad \text{for } i \geq 2.$$

The nth cokernel $L^n (n \geq 0)$ is called the nth $\mathcal{C}$-cosyzygy of M.

If $\mathcal{C}$ is the class of projective (resp., injective) modules, then K_n (resp., L^n) is simply called the nth syzygy (resp., cosyzygy).

Let R be a left noetherian ring. Then every left R-module has a left $\mathcal{I}_0$-resolution by [12, Example 8.3.5].

Let R be a left coherent ring. Then every finitely presented right R-module M has a right $\mathcal{F}_0$-resolution $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i finitely generated projective by [12, Example 8.3.10]. So by the nth $\mathcal{F}_0$-cosyzygy of a finitely presented right R-module, we will mean the nth cokernel in such a right $\mathcal{F}_0$-resolution.

Given a class $\mathcal{L}$ of right R-modules, we will denote by $\mathcal{L}^\perp = \{C : \mathrm{Ext}^1(L, C) = 0 \text{ for all } L \in \mathcal{L}\}$ the right orthogonal class of $\mathcal{L}$, and by ${}^\perp\mathcal{L} = \{C : \mathrm{Ext}^1(C, L) = 0 \text{ for all } L \in \mathcal{L}\}$ the left orthogonal class of $\mathcal{L}$. A pair $(\mathcal{F}, \mathcal{C})$ of classes of right R-modules is called a cotorsion theory [12] if $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called perfect [13] if every right R-module has a $\mathcal{C}$-envelope and an $\mathcal{F}$-cover. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be hereditary [13] if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathcal{F}$, then L' is also in $\mathcal{F}$. By [13, Proposition 1.2], $(\mathcal{F}, \mathcal{C})$ is hereditary if and only if whenever $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in \mathcal{C}$, then C'' is also in $\mathcal{C}$. For example, (the class of all flat right R-modules, the class of all cotorsion right R-modules) is a perfect and hereditary cotorsion theory by [12, Theorem 7.4.4], where a right R-module C is called cotorsion if $\mathrm{Ext}^1(F, C) = 0$ for any flat right R-module F.

Throughout this paper, R is an associative ring with identity and all modules are unitary. We write M_R ($_RM$) to indicate a right (left) R-module. For an R-module M, $E(M)$ denotes the injective envelope of M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, and $\delta_M : M \to M^{++}$ is the evaluation map. $fd(M)$ ($id(M)$) denotes the flat (injective) dimension of M. $lD(R)$ and $wD(R)$ stand for the left global dimension and the weak global dimension of a ring R respectively. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ ($\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ ($\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ ($\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ ($\mathrm{Tor}^R_n(M, N)$) for an integer $n \geq 1$. General background materials can be found in [2,12,17,23].

2. Definitions and general results

We begin with the following

Definition 2.1. Let R be a ring, n a fixed nonnegative integer and $\mathcal{I}_n$ the class of all left R-modules of injective dimension at most n. A left R-module M is called n-copure injective if $\mathrm{Ext}^1(N, M) = 0$ for any $N \in \mathcal{I}_n$. A right R-module F is said to be n-copure flat if $\mathrm{Tor}_1(F, N) = 0$ for any $N \in \mathcal{I}_n$.

Proposition 2.2. *Let R be any ring.*

(1) *If $\mathrm{Ext}^i(N, M) = 0$ for any i with $1 \leq i \leq n+1$ and any injective left R-module N, then every kth cosyzygy of M is $(n-k)$-copure injective for any k with $0 \leq k \leq n$, in particular, M is n-copure injective.*
(2) *If $\mathrm{Tor}_i(M, N) = 0$ for any i with $1 \leq i \leq n+1$ and any injective left R-module N, then every kth syzygy of M is $(n-k)$-copure flat for any k with $0 \leq k \leq n$, in particular, M is n-copure flat.*

Proof. (1) Let k be an integer with $0 \leq k \leq n$, L^k a kth cosyzygy of M, and N a left R-module with $id(N) \leq n-k$. Then $\mathrm{Ext}^1(N, L^k) \cong \mathrm{Ext}^{k+1}(N, M)$. On the other hand, there is an exact sequence $0 \to N \to E^0 \to E^1 \cdots \to E^{n-k} \to 0$ with each E^i injective (for $id(N) \leq n-k$), and so $\mathrm{Ext}^{k+1}(N, M) \cong \mathrm{Ext}^{n+1}(E^{n-k}, M) = 0$ by assumption. Thus $\mathrm{Ext}^1(N, L^k) = 0$, and hence L^k is $(n-k)$-copure injective.

(2) The proof is similar to that of (1). □

Remark 2.3. (1) Obviously, 0-copure injective (0-copure flat) modules are exactly copure injective (copure flat) modules. If $m \geq n$, then m-copure injective (m-copure flat) modules are n-copure injective (n-copure flat).

(2) By [12, Definitions 10.1.1 and 10.3.1] and Proposition 2.2, we have the following implications:

Gorenstein injective modules ⇒ strongly copure injective modules ⇒ n-copure injective modules ⇒ copure injective modules.

Gorenstein flat modules ⇒ strongly copure flat modules ⇒ n-copure flat modules ⇒ copure flat modules.

(3) Let R be an n-Gorenstein ring. For an R-module N, $id(N) \leq n$ if and only if $id(N) < \infty$ by [12, Theorem 9.1.10] or [15, Theorem 2]. Therefore an R-module M is n-copure injective if and only if M is strongly copure injective if and only if M is Gorenstein injective by [12, Corollary 11.2.2]. M is n-copure flat if and only if M is strongly copure flat if and only if M is Gorenstein flat by [12, Theorem 10.3.8].

(4) If R is a 1-Gorenstein ring, then, by [11, Corollary 4.2], any copure injective (copure flat) R-module is strongly copure injective (strongly copure flat), and hence Gorenstein injective (Gorenstein flat).

Next we give some characterizations of n-copure injective modules and n-copure flat modules.

Proposition 2.4. *The following are equivalent for a left R-module M:*

(1) *M is n-copure injective.*
(2) *For every exact sequence $0 \to M \to E \to L \to 0$ with $E \in \mathcal{I}_n$, $E \to L$ is an $\mathcal{I}_n$-precover of L.*
(3) *M is a kernel of an $\mathcal{I}_n$-precover $f : A \to B$ with A injective.*
(4) *M is injective with respect to every exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{I}_n$.*

Proof. (1) ⇒ (2) and (1) ⇒ (4) are clear by definition.

(2) ⇒ (3). Since there exists a short exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ and $E(M) \in \mathcal{I}_n$, then (3) follows from (2).

(3) ⇒ (1). Let M be a kernel of an $\mathcal{I}_n$-precover $f : A \to B$ with A injective. Then we have an exact sequence $0 \to M \to A \to A/M \to 0$. So, for any $N \in \mathcal{I}_n$, the sequence $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to \mathrm{Ext}^1(N, M) \to 0$

is exact. It is easy to verify that $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to 0$ is exact by (3). Thus $\mathrm{Ext}^1(N, M) = 0$, and so (1) follows.

(4) $\Rightarrow$ (1). For each $N \in \mathcal{I}_n$, there exists a short exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1(N, M) \to 0$. Note that $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(N, M) = 0$, as desired. □

Corollary 2.5. *Let R be a left noetherian ring. Then every $(n+1)$th $\mathcal{I}_0$-syzygy of any left R-module is n-copure injective.*

Proof. Let $\cdots \to E_1 \to E_0 \to M \to 0$ be a left $\mathcal{I}_0$-resolution of a left R-module M. By [12, Lemma 8.4.34], $E_n \to K_n$ is an $\mathcal{I}_n$-precover, where K_n is the nth $\mathcal{I}_0$-syzygy of M, and so the $(n+1)$th $\mathcal{I}_0$-syzygy K_{n+1} of M is n-copure injective by Proposition 2.4. □

Proposition 2.6. *The following are equivalent for a right R-module M:*

(1) *M is n-copure flat.*
(2) *M^+ is n-copure injective.*
(3) *$M \in {}^{\perp}\mathcal{C}$, where $\mathcal{C} = \{B^+ : B \in \mathcal{I}_n\}$.*
(4) *For every exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with $C \in \mathcal{I}_n$, the functor $M \otimes -$ preserves the exactness.*

Proof. By [4, VI. 5.1] or [17, p. 360], there are the following standard isomorphisms:

$$\mathrm{Ext}^1(N, M^+) \cong \mathrm{Tor}_1(M, N)^+ \cong \mathrm{Ext}^1(M, N^+)$$

for any left R-module N. Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow.

(1) $\Leftrightarrow$ (4) is easy. □

Injective (flat) modules are clearly n-copure injective (n-copure flat). The converse is not true in general. In fact, we have the following

Proposition 2.7. *Let R be a ring. Then*

(1) *A left R-module M is injective if and only if M is n-copure injective and $id(M) \le n+1$.*
(2) *A right R-module N is flat if and only if N is n-copure flat and $fd(N) \le n+1$.*

Proof. (1) "$\Rightarrow$" is trivial.

"$\Leftarrow$". Consider the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Note that $id(E(M)/M) \le n$ since $id(M) \le n+1$. Thus $\mathrm{Ext}^1(E(M)/M, M) = 0$, and hence the above sequence is split. So M is injective.

(2) "$\Rightarrow$" is trivial.

"$\Leftarrow$". Let N be an n-copure flat right R-module with $fd(N) \le n+1$. Then N^+ is n-copure injective by Proposition 2.6. Thus N^+ is injective by (1) since $id(N^+) \le n+1$. Hence N is flat. □

Proposition 2.8. *Let S be a simple R-module over a commutative ring R. Then the following are equivalent:*

(1) *S is n-copure injective.*
(2) *S is n-copure flat.*
(3) *S^+ is n-copure injective.*

Proof. (1) $\Leftrightarrow$ (2). Suppose $\{S_i\}_{i \in I}$ is an irredundant set of representatives of the simple R-modules. Let $E = E(\oplus_{i \in I} S_i)$, then E is an injective cogenerator by [2, Corollary 18.19]. Let $M \in \mathcal{I}_n$. Since E is injective, there is an isomorphism:

$$\mathrm{Ext}^1(M, \mathrm{Hom}(S, E)) \cong \mathrm{Hom}(\mathrm{Tor}_1(M, S), E).$$

Note that $\mathrm{Hom}(S, E) \cong S$ by the proof of [22, Lemma 2.6]. Thus $\mathrm{Ext}^1(M, S) = 0$ if and only if $\mathrm{Tor}_1(M, S) = 0$, and so (1) $\Leftrightarrow$ (2) follows.

(2) $\Leftrightarrow$ (3) holds by Proposition 2.6. □

Proposition 2.9. *Let R be commutative noetherian and M an R-module. Then*

(1) *M is n-copure injective if and only if* $\mathrm{Hom}(F, M)$ *is n-copure injective for all flat R-modules F.*
(2) *M is n-copure flat if and only if $F \otimes M$ is n-copure flat for all flat R-modules F.*

Proof. (1) "$\Leftarrow$" holds by letting $F = R$.

"$\Rightarrow$". Let F be any flat R-module and $E \in \mathcal{I}_n$. There exists an exact sequence $0 \to K \to P \to E \to 0$ with P projective, which yields the exactness of the sequence $0 \to K \otimes F \to P \otimes F \to E \otimes F \to 0$. Note that $E \otimes F \in \mathcal{I}_n$ since R is commutative noetherian. Then we have the following exact sequence

$$\mathrm{Hom}(P \otimes F, M) \to \mathrm{Hom}(K \otimes F, M) \to \mathrm{Ext}^1(E \otimes F, M) = 0,$$

which gives rise to the exactness of the sequence

$$\mathrm{Hom}(P, \mathrm{Hom}(F, M)) \to \mathrm{Hom}(K, \mathrm{Hom}(F, M)) \to 0.$$

On the other hand, the following sequence

$$\mathrm{Hom}(P, \mathrm{Hom}(F, M)) \to \mathrm{Hom}(K, \mathrm{Hom}(F, M)) \to \mathrm{Ext}^1(E, \mathrm{Hom}(F, M)) \to \mathrm{Ext}^1(P, \mathrm{Hom}(F, M)) = 0$$

is exact. Thus $\mathrm{Ext}^1(E, \mathrm{Hom}(F, M)) = 0$, as desired.

(2) "$\Leftarrow$" holds by letting $F = R$.

"$\Rightarrow$". Let F be any flat R-module. We only need to show that $(F \otimes M)^+$ is n-copure injective by Proposition 2.6. In fact, since M^+ is n-copure injective by Proposition 2.6, $(F \otimes M)^+ \cong \mathrm{Hom}(F, M^+)$ is n-copure injective by (1). □

For a left noetherian ring R, [1, Proposition 3.1] shows that every left R-module has an $\mathcal{I}_n$-preenvelope. Let $\mathcal{F}_n$ be the class of all right R-modules of flat dimension at most n, we have

Proposition 2.10. *Let R be a left coherent ring and n a fixed nonnegative integer. Then any right R-module has an $\mathcal{F}_n$-preenvelope. Moreover, suppose M is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right R-module K with F flat, then M is n-copure flat.*

Proof. Let M be a right R-module with $\mathrm{Card}M = \aleph_\beta$. Then, by [12, Lemma 5.3.12], there is an infinite cardinal $\aleph_\alpha$ such that if $F \in \mathcal{F}_n$ and S is a submodule of F with $\mathrm{Card}S \le \aleph_\beta$, there exists a pure submodule G of F with $S \subseteq G$ and $\mathrm{Card}G \le \aleph_\alpha$. Note that the pure exact sequence $0 \to G \to F \to F/G \to 0$ induces the split exact sequence $0 \to (F/G)^+ \to F^+ \to G^+ \to 0$. Thus $G^+ \in \mathcal{I}_n$ since $F^+ \in \mathcal{I}_n$, and so $G \in \mathcal{F}_n$. Therefore M has an $\mathcal{F}_n$-preenvelope by [12, Corollary 6.2.2] since the left coherence of R guarantees that $\mathcal{F}_n$ is closed under direct products.

Now suppose M is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right R-module K with F flat. Let $L = \mathrm{im}(K \to F)$, then $0 \to L \to F \to M \to 0$ is exact and $L \to F$ is an $\mathcal{F}_n$-preenvelope of L. Note that $E^+ \in \mathcal{F}_n$ for any $E \in \mathcal{I}_n$ since R is left coherent. Thus we obtain an exact sequence $\mathrm{Hom}(F, E^+) \to \mathrm{Hom}(L, E^+) \to 0$, which gives rise to the exactness of $(F \otimes E)^+ \to (L \otimes E)^+ \to 0$. So the sequence $0 \to L \otimes E \to F \otimes E$ is exact. But the flatness of F implies the exactness of $0 \to \mathrm{Tor}_1(M, E) \to L \otimes E \to F \otimes E$, and hence $\mathrm{Tor}_1(M, E) = 0$. This completes the proof. □

Corollary 2.11. *Let R be a left coherent ring. Then every $(n + 1)$th $\mathcal{F}_0$-cosyzygy of any finitely presented right R-module is n-copure flat.*

Proof. Let M be a finitely presented right R-module and $0 \to M \to F^0 \to F^1 \to \cdots$ any right $\mathcal{F}_0$-resolution of M with each F^i finitely generated projective. By [12, Remark 8.4.35] or [5, Lemma 2.1], $L^n \to F^n$ is an $\mathcal{F}_n$-preenvelope, where L^n is the nth $\mathcal{F}_0$-cosyzygy of M. Thus the $(n + 1)$th $\mathcal{F}_0$-cosyzygy L^{n+1} is n-copure flat by Proposition 2.10. □

Theorem 2.12. *Let R be a left coherent ring and M a finitely presented right R-module. Then M is n-copure flat if and only if M is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right R-module K with F flat.*

Proof. "$\Leftarrow$" follows from Proposition 2.10.

"$\Rightarrow$". Since M is finitely presented, there is an exact sequence $0 \to K \to P \to M \to 0$ with P finitely generated projective and K finitely generated. We claim that $K \to P$ is an $\mathcal{F}_n$-preenvelope. In fact, for any $F \in \mathcal{F}_n$, we have $F^+ \in \mathcal{I}_n$. Thus $\mathrm{Tor}_1(M, F^+) = 0$, and so we get the exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & K \otimes F^+ \arrow[r, "\alpha"] \arrow[d, "\sigma_K"'] & P \otimes F^+ \arrow[d, "\sigma_P"'] \\ & \mathrm{Hom}(K, F)^+ \arrow[r, "\theta"] & \mathrm{Hom}(P, F)^+. \end{tikzcd}$$

On the other hand, there exists an exact sequence $Q \to K \to 0$ with Q finitely generated projective since K is finitely generated. Then we have the exact commutative diagram:

$$\begin{tikzcd} Q \otimes F^+ \arrow[r] \arrow[d, "\sigma_Q"'] & K \otimes F^+ \arrow[r] \arrow[d, "\sigma_K"'] & 0 \\ \mathrm{Hom}(Q, F)^+ \arrow[r] & \mathrm{Hom}(K, F)^+ \arrow[r] & 0. \end{tikzcd}$$

Note that σ_Q is an isomorphism by [17, Lemma 3.59], so σ_K is epic. Thus θ is a monomorphism since σ_P is an isomorphism, and hence the sequence $\mathrm{Hom}(P, F) \to \mathrm{Hom}(K, F) \to 0$ is exact, as desired. $\square$

In what follows, $\mathcal{CI}_n$ ($\mathcal{CF}_n$) stands for the class of all n-copure injective left R-modules (n-copure flat right R-modules).

If R is an n-Gorenstein ring, then an R-module is n-copure flat if and only if it is Gorenstein flat by Remark 2.3 (3). So every R-module over an n-Gorenstein ring R has a $\mathcal{CF}_n$-cover by [12, Theorem 11.7.3]. In fact, every right R-module over any ring R has a $\mathcal{CF}_n$-cover as shown by the following proposition.

Proposition 2.13. *Let R be any ring and n a fixed nonnegative integer. Then $(\mathcal{CF}_n, \mathcal{CF}_n^{\perp})$ is a perfect cotorsion theory. Moreover, the following are equivalent:*

(1) *$(\mathcal{CF}_n, \mathcal{CF}_n^{\perp})$ is a hereditary cotorsion theory.*
(2) *$\mathrm{Tor}_2(F, N) = 0$ for any $F \in \mathcal{CF}_n$ and any $N \in \mathcal{I}_n$.*
(3) *$\mathrm{Tor}_j(F, N) = 0$ for any $F \in \mathcal{CF}_n$, any $N \in \mathcal{I}_n$ and any $j \geq 1$.*

If R is a left noetherian ring with $id({}_R R) \leq n + 1$, then the above conditions are also equivalent to:

(4) *Every n-copure flat right R-module is m-copure flat for any $m \geq n$.*
(5) *Every n-copure flat right R-module is $(n + 1)$-copure flat.*

Proof. By [21, Lemma 1.11 and Theorem 2.8], $(\mathcal{CF}_n, \mathcal{CF}_n^{\perp})$ is a perfect cotorsion theory.

(1) $\Rightarrow$ (2). Let $F \in \mathcal{CF}_n$. Then there is an exact sequence $0 \to K \to P \to F \to 0$ with P projective. Thus $K \in \mathcal{CF}_n$ since $(\mathcal{CF}_n, \mathcal{CF}_n^{\perp})$ is hereditary. Hence $\mathrm{Tor}_2(F, N) = 0$ for any $N \in \mathcal{I}_n$.

(2) $\Rightarrow$ (3). Let $F \in \mathcal{CF}_n$ and $N \in \mathcal{I}_n$. Then $\mathrm{Tor}_1(F, N) = 0$ by definition and $\mathrm{Tor}_j(F, N) = 0$ for any $j \geq 2$ by induction.

(3) $\Rightarrow$ (1) is easy.

(3) $\Rightarrow$ (4). Let $F \in \mathcal{CF}_n$ and $M \in \mathcal{I}_m$ with $m > n$. Then there is an exact sequence

$$0 \to M \to E^0 \to E^1 \to \cdots E^{m-n-1} \to L^{m-n} \to 0$$

with each E^i injective. Note that $L^{m-n} \in \mathcal{I}_n$, thus, by (3), we have

$$\mathrm{Tor}_1(F, M) \cong \mathrm{Tor}_2(F, L^1) \cong \cdots \cong \mathrm{Tor}_{m-n}(F, L^{m-n-1}) \cong \mathrm{Tor}_{m-n+1}(F, L^{m-n}) = 0,$$

where each L^i is an ith cosyzygy of M, $i = 1, 2, \ldots, m - n$. Hence $F \in \mathcal{CF}_m$.

(4) $\Rightarrow$ (5) is trivial.

(5) ⇒ (2). Let $F \in \mathcal{CF}_n$ and $N \in \mathcal{I}_n$. There is an exact sequence $0 \to K \to P \to N \to 0$ with P projective. Note that $P \in \mathcal{I}_{n+1}$ by hypothesis, and so $K \in \mathcal{I}_{n+1}$. But $\mathrm{Tor}_1(F, K) = 0$ since $F \in \mathcal{CF}_{n+1}$ by (5), and hence $\mathrm{Tor}_2(F, N) = 0$. □

Recall that a ring R is called right semi-artinian [20] if every non-zero cyclic right R-module has non-zero socle. R is said to be left IF [6] if every injective left R-module is flat.

Proposition 2.14. *Let R be a ring and n a fixed nonnegative integer. Then the following are equivalent:*

(1) *R is a left IF ring.*
(2) *Every left R-module M with $M \in \mathcal{I}_n$ is flat.*
(3) *Every cotorsion left R-module is n-copure injective.*
(4) *Every right R-module is n-copure flat.*
(5) *Every right R-module M with $M \in \mathcal{CF}_n^{\perp}$ is injective.*
(6) *$(\mathcal{CF}_n, \mathcal{CF}_n^{\perp})$ is a hereditary cotorsion theory, and every right R-module M with $M \in \mathcal{CF}_n^{\perp}$ is n-copure flat.*
If R is right semi-artinian, then the above conditions are equivalent to:
(7) *Every simple right R-module is n-copure flat.*

Proof. (1) ⇔ (2) ⇔ (4) ⇒ (6), (7) are clear by definition.

(2) ⇔ (3) holds by the flat cotorsion theory.

(4) ⇔ (5) follows from Proposition 2.13.

(6) ⇒ (4). Let M be any right R-module. By Proposition 2.13 and Wakamatsu's Lemma [23, Section 2.1], there is a short exact sequence $0 \to M \to E \to L \to 0$ with $E \in \mathcal{CF}_n^{\perp}$ and $L \in \mathcal{CF}_n$. Then $E \in \mathcal{CF}_n$ by (6), and hence $M \in \mathcal{CF}_n$ since $(\mathcal{CF}_n, \mathcal{CF}_n^{\perp})$ is hereditary

(7) ⇒ (2). Let I be a maximal right ideal of R and $M \in \mathcal{I}_n$. Then we have $\mathrm{Tor}_1(R/I, M) = 0$ by (7). Thus $\mathrm{Ext}^1(R/I, M^+) = 0$ since $\mathrm{Ext}^1(R/I, M^+) \cong \mathrm{Tor}_1(R/I, M)^+$. So M^+ is injective with respect to any maximal right ideal of R. Hence M^+ is injective by [19, Lemma 4] since R is right semi-artinian. Thus M is flat. □

Proposition 2.15. *The following are equivalent for a ring R and a fixed nonnegative integer n:*

(1) *R is a QF ring.*
(2) *Every left R-module is n-copure injective.*
(3) *$(\mathcal{I}_n, \mathcal{CI}_n)$ is a perfect hereditary cotorsion theory, and every left R-module M with $M \in \mathcal{I}_n$ is n-copure injective.*
(4) *$(\mathcal{I}_0, \mathcal{CI}_0)$ is a cotorsion theory.*

Proof. (1) ⇔ (2) ⇒ (3) and (1) ⇒ (4) are clear.

(3) ⇒ (2). Let M be any left R-module. By (3) and Wakamatsu's Lemma, there is a short exact sequence $0 \to L \to F \to M \to 0$ with $F \in \mathcal{I}_n$ and $L \in \mathcal{CI}_n$. Then $F \in \mathcal{CI}_n$ by (3), and hence $M \in \mathcal{CI}_n$ since $(\mathcal{I}_n, \mathcal{CI}_n)$ is hereditary.

(4) ⇒ (1). R is a QF ring since every projective left R-module is injective. □

Proposition 2.13 shows that every right R-module has a $\mathcal{CF}_n$-cover. We end this section by the following

Theorem 2.16. *The following are equivalent for any ring R and any integer $n \geq 0$:*

(1) *The injective envelope $E(M)$ is n-copure flat for any n-copure flat right R-module M.*
(2) *The $\mathcal{CF}_n$-cover $F(I)$ is injective for any injective right R-module I.*

Proof. (1) ⇒ (2). Let I be an injective right R-module, $\varepsilon : F(I) \to I$ the $\mathcal{CF}_n$-cover of I, and $\lambda : F(I) \to E(F(I))$ the injective envelope. Then there exists $\theta : E(F(I)) \to I$ such that $\theta\lambda = \varepsilon$. On the other hand, since $E(F(I))$ is n-copure flat by (1), there exists $\beta : E(F(I)) \to F(I)$ such that $\varepsilon\beta = \theta$. Thus $\varepsilon\beta\lambda = \varepsilon$, and hence $\beta\lambda$ is an isomorphism. This means that $F(I)$ is a direct summand of $E(F(I))$ and so it is injective.

(2) ⇒ (1). Let M be an n-copure flat right R-module, $\lambda : M \to E(M)$ the injective envelope, and $\varepsilon : F(E(M)) \to E(M)$ the $\mathcal{CF}_n$-cover of $E(M)$. Then there exists $\alpha : M \to F(E(M))$ such that $\varepsilon\alpha = \lambda$. On the other hand, since

$F(E(M))$ is injective by (2), there exists $\gamma : E(M) \to F(E(M))$ such that $\gamma\lambda = \alpha$. Thus $\varepsilon\gamma\lambda = \lambda$, and so $\varepsilon\gamma$ is an isomorphism. It follows that $E(M)$ is n-copure flat. □

3. Left noetherian rings with $id({}_R R) \leq n$

Recall that a left R-module M (resp., right R-module N) is called strongly cotorsion (resp., strongly torsionfree) [23,18] if $\mathrm{Ext}^1(F, M) = 0$ (resp., $\mathrm{Tor}_1(N, F) = 0$) for any left R-module F with $fd(F) < \infty$.

If R is an n-Gorenstein ring, then an R-module is n-copure injective if and only if it is Gorenstein injective by Remark 2.3 (3), and so $(\mathcal{I}_n, \mathcal{CI}_n)$ is a perfect cotorsion theory by [12, Theorem 11.3.2]. For left noetherian rings with finite left self-injective dimension, we have

Lemma 3.1. *Let n be a fixed nonnegative integer. Then the following hold for a left noetherian ring R with $id({}_R R) \leq n$:*

(1) *$(\mathcal{I}_n, \mathcal{CI}_n)$ is a perfect cotorsion theory.*
(2) *Every n-copure injective left R-module is strongly cotorsion, and every n-copure flat right R-module is strongly torsionfree.*
(3) *If R is an n-Gorenstein ring, then $(\mathcal{I}_n, \mathcal{CI}_n)$ is a hereditary cotorsion theory.*
Moreover, every strongly cotorsion left R-module is n-copure injective, and every strongly torsionfree right R-module is n-copure flat.

Proof. (1) Since R is left noetherian, $\mathcal{I}_n$ is closed under well ordered inductive limits by [3, Theorem 1.1], so (1) follows from [1, Theorem 2.8] and [12, Theorem 7.2.6].

(2) Let $fd(F) < \infty$, then $F \in \mathcal{I}_n$ since every flat left R-module has injective dimension at most n by [12, Proposition 9.1.2]. Thus (2) follows.

(3) holds by [15, Theorem 2]. □

Recall that an R-module M is called reduced [10] if M has no nonzero injective submodule.

Proposition 3.2. *Let R be a left noetherian ring with $id({}_R R) \leq n$ and $n \geq 0$. Then the following are equivalent for a left R-module M:*

(1) *M is a reduced n-copure injective left R-module.*
(2) *M is a kernel of an $\mathcal{I}_n$-cover $f : A \to B$ with A injective.*

Proof. (1) ⇒ (2). By Proposition 2.4, the natural map $\pi : E(M) \to E(M)/M$ is an $\mathcal{I}_n$-precover. Thus $E(M)$ has no nonzero direct summand K contained in M since M is reduced. Note that $E(M)/M$ has an $\mathcal{I}_n$-cover by Lemma 3.1. It follows that $\pi : E(M) \to E(M)/M$ is an $\mathcal{I}_n$-cover by [23, Corollary 1.2.8], and hence (2) follows.

(2) ⇒ (1). Let M be a kernel of an $\mathcal{I}_n$-cover $\alpha : A \to B$ with A injective. By Proposition 2.4, M is n-copure injective. Now let K be an injective submodule of M. Suppose $A = K \oplus L$, $p : A \to L$ is the projection and $i : L \to A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$ since $\alpha(K) = 0$. Therefore ip is an isomorphism, and hence i is epic. Thus $A = L$, $K = 0$, and so M is reduced. □

In order to prove the next main result, we need the following lemma which is of independent interest.

Lemma 3.3. *Let R be a left noetherian ring with $id({}_R R) \leq n$ and $n \geq 1$.*

(1) *If M is an $(n-1)$-copure injective left R-module, then there is an exact sequence $0 \to K \to E \to M \to 0$ such that E is injective and K is n-copure injective.*
(2) *If N is an $(n-1)$-copure flat right R-module, then there is an exact sequence $0 \to N \to F \to L \to 0$ such that F is flat and L is n-copure flat.*

Proof. (1) Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & E(P) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & & C \arrow[r, equal] \arrow[d] & C \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where P is projective and $P \to E(P)$ is an injective envelope. Note that $id(C) \leq n - 1$ since $id(P) \leq n$. So $\mathrm{Ext}^1(C, M) = 0$ (for M is $(n-1)$-copure injective), then the sequence $0 \to M \to Q \to C \to 0$ is split. Therefore M is a quotient of $E(P)$.

Now suppose $\alpha : E \to M$ is an injective cover of M, then α is epic. Thus we have the exact sequence $0 \to K \to E \to M \to 0$. Note that K is copure injective by [9, Lemma 2.1]. We claim that K is also n-copure injective. In fact, let $X \in \mathcal{I}_n$. Consider the exact sequence $0 \to X \to E(X) \to D \to 0$. Then $D \in \mathcal{I}_{n-1}$. Thus we get the induced exact sequence

$$0 = \mathrm{Ext}^1(D, M) \to \mathrm{Ext}^2(D, K) \to \mathrm{Ext}^2(D, E) = 0.$$

Therefore $\mathrm{Ext}^2(D, K) = 0$. On the other hand, the short exact sequence $0 \to X \to E(X) \to D \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Ext}^1(E(X), K) \to \mathrm{Ext}^1(X, K) \to \mathrm{Ext}^2(D, K) = 0.$$

Therefore $\mathrm{Ext}^1(X, K) = 0$, as desired.

(2) Let N be an $(n-1)$-copure flat right R-module. Then N^+ is $(n-1)$-copure injective by Proposition 2.6. Thus there is an exact sequence $E \to N^+ \to 0$ with E injective by (1), which in turn yields the exactness of $0 \to N^{++} \to E^+$. So N embeds in a flat right R-module (for E^+ is flat).

Now let $\beta : N \to F$ be a flat preenvelope of N, then β is monic. So we have the exact sequence $0 \to N \to F \to L \to 0$. Note that L is copure flat by Proposition 2.10. Applying an argument similar to that in the proof of (1), we can prove that L is also n-copure flat. □

Let R be a left noetherian ring. It is known that R is a left hereditary ring if and only if every right R-module has an epic flat envelope if and only if every left R-module has a monic injective cover if and only if every copure injective left R-module is injective if and only if every copure flat right R-module is flat (see [10, Corollary 2.4] and [7, Theorem 4.5]). Here we get

Theorem 3.4. *Let R be a left noetherian ring with $id({}_RR) \leq n$ and $n \geq 1$. Then the following are equivalent:*

(1) $lD(R) < \infty$.
(2) $lD(R) \leq n$.
(3) *Every $(n-1)$-copure injective left R-module is injective.*
(4) *Every n-copure injective left R-module is injective.*
(5) *Every n-copure injective left R-module has a monic injective cover.*
(6) *Every $((n-1)$-copure injective) left R-module has a monic $\mathcal{I}_{n-1}$-cover.*
(7) *Every $(n-1)$-copure flat right R-module is flat.*
(8) *Every cotorsion right R-module belongs to $\mathcal{CF}_n^{\perp}$.*
(9) *Every n-copure flat right R-module is flat.*
(10) *Every (finitely presented) n-copure flat right R-module has an epic flat envelope.*
(11) *Every right R-module has an epic $\mathcal{F}_{n-1}$-envelope.*

Proof. (2) ⇒ (4), (3) ⇒ (4) ⇒ (5), and (7) ⇒ (9) ⇒ (10) are clear.

(1) ⇒ (2). By [3, Proposition 4.2], $lD(R) = id({}_R R) \leq n$ since $lD(R) < \infty$.

(2) ⇒ (6). Note that $\mathcal{I}_{n-1}$ is closed under direct sums and quotients by (2). So (6) follows from [14, Proposition 4].

(6) ⇒ (2). Let M be any left R-module. By Lemma 3.1 and Wakamatsu's Lemma, there is a short exact sequence $0 \to L \to F \to M \to 0$, where $F \in \mathcal{I}_n$ and $L \in \mathcal{CI}_n$. Note that L is $(n-1)$-copure injective, and so L has a monic $\mathcal{I}_{n-1}$-cover by (6). But L is a quotient of an injective left R-module by Lemma 3.3 (1). Thus $L \in \mathcal{I}_{n-1}$, and hence $M \in \mathcal{I}_n$.

(4) ⇒ (3) and (9) ⇒ (7) follow from Lemma 3.3.

(4) ⇒ (9) holds by Proposition 2.6.

(5) ⇒ (1). Let M be a left R-module. For any left $\mathcal{I}_0$-resolution $\cdots \to E_1 \to E_0 \to M \to 0$, the $(n+1)$th $\mathcal{I}_0$-syzygy K_{n+1} of M is n-copure injective by Corollary 2.5. Thus K_{n+1} has a monic injective cover by (5), but K_{n+1} is a quotient of an injective left R-module by Lemma 3.3 (1). Hence K_{n+1} is injective. Therefore $lD(R) \leq n+3 < \infty$ by [12, Corollary 8.4.17].

(8) ⇔ (9) comes from Proposition 2.13.

(10) ⇒ (1). By Corollary 2.11, the $(n+1)$th $\mathcal{F}_0$-cosyzygy L^{n+1} of any finitely presented right R-module M is n-copure flat. Thus L^{n+1} embeds in a flat right R-module by Lemma 3.3 (2). But L^{n+1} has an epic flat envelope by (10). Therefore L^{n+1} is flat, and hence projective. So $lD(R) \leq n+3 < \infty$ by [12, Corollary 8.4.28].

(11) ⇒ (2). Let M be a right R-module. Consider the exact sequence $0 \to K \to P \to M \to 0$ with P projective. Note that K has an epic $\mathcal{F}_{n-1}$-envelope by (11), then $K \in \mathcal{F}_{n-1}$, and hence $M \in \mathcal{F}_n$. Therefore $lD(R) = wD(R) \leq n$.

(2) ⇒ (11). Let M be a right R-module. Then M has an $\mathcal{F}_{n-1}$-preenvelope $\alpha : M \to N$ by Proposition 2.10. It follows that $\mathrm{im}(\alpha) \in \mathcal{F}_{n-1}$ since $wD(R) \leq n$ by (2). Thus $M \to \mathrm{im}(\alpha)$ is an epic $\mathcal{F}_{n-1}$-envelope. □

4. On copure injective covers and copure flat envelopes

Enochs and Jenda have shown that every left R-module has a strongly copure injective preenvelope over a left noetherian ring R (see [11, Theorem 2.2]). Here we have

Proposition 4.1. *Let R be a commutative artinian ring. Then M^+ has a strongly copure injective precover for any R-module M.*

Proof. By [11, Theorem 2.5], M has a strongly copure flat preenvelope $f : M \to N$. We shall show that $f^+ : N^+ \to M^+$ is a strongly copure injective precover of M^+. Indeed, let $\psi : H \to M^+$ be any homomorphism with H strongly copure injective. Since H^+ is strongly copure flat by [11, Lemma 3.6], there exists $g : N \to H^+$ such that $gf = \psi^+\delta_M$. Thus $f^+g^+ = \delta_M^+\psi^{++}$. Note that $\psi^{++}\delta_H = \delta_{M^+}\psi$, then by [2, Proposition 20.14], we have $f^+(g^+\delta_H) = \delta_M^+(\psi^{++}\delta_H) = (\delta_M^+\delta_{M^+})\psi = \psi$. Hence f^+ is a strongly copure injective precover. □

Theorem 4.2. *The following are equivalent for a commutative artinian ring R:*

(1) *$id(R) \leq 1$.*
(2) *Every R-module has an epic copure flat envelope.*
(3) *Every cotorsion R-module has an epic copure flat envelope.*
(4) *Every R-module has a monic copure injective cover.*

Proof. (1) ⇒ (2). Since R is a commutative artinian ring, any R-module M has a strongly copure flat preenvelope $f : M \to N$ by [11, Theorem 2.5]. But R is a 1-Gorenstein ring by (1), so any copure flat module is strongly copure flat by [11, Corollary 4.2]. Thus f is also a copure flat preenvelope. Note that $\mathrm{im}(f)$ is copure flat by [11, Corollary 4.2], hence $f : M \to \mathrm{im}(f)$ is an epic copure flat envelope.

(2) ⇒ (3) is trivial.

(3) ⇒ (1). By [11, Corollary 4.2], we shall show that any submodule N of any copure flat R-module M is copure flat. Since M/N has a flat cover $f : F \to M/N$, we get an exact sequence $0 \to C \to F \to M/N \to 0$ with C cotorsion by Wakamatsu's Lemma. By (3), C has an epic copure flat envelope. Thus C is copure flat since C embeds in a flat R-module. So, for any injective R-module E, we get an induced exact sequence

$$0 = \mathrm{Tor}_2(F, E) \to \mathrm{Tor}_2(M/N, E) \to \mathrm{Tor}_1(C, E) = 0.$$

Hence $\mathrm{Tor}_2(M/N, E) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Tor}_2(M/N, E) \to \mathrm{Tor}_1(N, E) \to \mathrm{Tor}_1(M, E) = 0.$$

Therefore $\mathrm{Tor}_1(N, E) = 0$, as desired.

(1) $\Leftrightarrow$ (4). We first show that the class of copure injective R-modules over a commutative artinian ring R is closed under direct sums. Indeed, let $\{M_j\}_{j\in J}$ be a family of copure injective R-modules and E an injective R-module. Since R is artinian, then $E = \oplus_{i\in I} E(S_i)$, where each S_i is simple. Note that $E(S_i)$ is finitely generated by [16, Theorem 3.11], so by [2, Exercise 16.3, p. 189], we get

$$\mathrm{Ext}^1\left(E, \bigoplus_{j\in J} M_j\right) = \prod_{i\in I}\mathrm{Ext}^1\left(E(S_i), \bigoplus_{j\in J} M_j\right) = \prod_{i\in I}\bigoplus_{j\in J}\mathrm{Ext}^1(E(S_i), M_j) = 0.$$

Thus $\oplus_{j\in J} M_j$ is copure injective. Since R is 1-Gorenstein if and only if the class of copure injective R-modules is closed under quotients by [11, Corollary 4.2], (1) $\Leftrightarrow$ (4) follows from [14, Proposition 4]. □

It is well known that R is a left noetherian ring with $lD(R) \le 2$ if and only if every left R-module has an injective cover with the unique mapping property. Here we have

Theorem 4.3. *The following are equivalent for a left and right noetherian ring R:*

(1) *R is a 2-Gorenstein ring, and every (left and right) R-module has a strongly copure injective cover.*
(2) *Every (left and right) R-module has a strongly copure injective cover with the unique mapping property.*

Proof. (1) $\Rightarrow$ (2). Let M be any (left and right) R-module. Then M has a strongly copure injective cover $f : F \to M$ by (1). It is enough to show that, for any strongly copure injective R-module G and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\mathrm{im}(g) \to M$ such that $\beta\pi = f$ since $\mathrm{im}(g) \subseteq \ker(f)$, where $\pi : F \to F/\mathrm{im}(g)$ is the natural map. Since R is 2-Gorenstein, $\mathrm{Ext}^i(E, \ker(g)) = 0$ for any $i \ge 3$ and any injective R-module E by [11, Lemma 3.1 and Theorem 4.1]. It follows that $F/\mathrm{im}(g)$ is strongly copure injective. Thus there exists $\alpha : F/\mathrm{im}(g) \to F$ such that $\beta = f\alpha$, and so we get the exact commutative diagram:

$$\begin{tikzcd}
 & & & M & & \\
0 \arrow[r] & \ker(g) \arrow[r, "i"] & G \arrow[r, "g"] \arrow[ur, "0"] & F \arrow[u, "f"] \arrow[r, shift left, "\pi"] & F/\mathrm{im}(g) \arrow[ul, "\beta"'] \arrow[l, shift left, "\alpha"] \arrow[r] & 0.
\end{tikzcd}$$

Thus $f\alpha\pi = f$, and hence $\alpha\pi$ is an isomorphism. Therefore π is monic, and so $g = 0$.

(2) $\Rightarrow$ (1). Let M be any (left and right) R-module. Then we have the exact sequence $0 \longrightarrow M \longrightarrow E^0 \xrightarrow{\varphi} E^1 \xrightarrow{\psi} N \longrightarrow 0$, where E^0, E^1 are injective. Let $\theta : H \to N$ be a strongly copure injective cover with the unique mapping property. Then there exists $\tau : E^1 \to H$ such that $\psi = \theta\tau$. Thus $\theta\tau\varphi = \psi\varphi = 0 = \theta 0$, and hence $\tau\varphi = 0$, which implies that $\ker(\psi) = \mathrm{im}(\varphi) \subseteq \ker(\tau)$. Therefore there exists $\gamma : N \to H$ such that $\gamma\psi = \tau$, and so we get the exact commutative diagram:

$$\begin{tikzcd}
 & & & & H \arrow[d, shift right, "\theta"'] & \\
0 \arrow[r] & M \arrow[r] & E^0 \arrow[r, "\varphi"] & E^1 \arrow[r, "\psi"] \arrow[ur, "\tau"] & N \arrow[u, shift right, "\gamma"'] \arrow[r] & 0.
\end{tikzcd}$$

Thus $\theta\gamma\psi = \psi$, and so $\theta\gamma = 1_N$ since ψ is epic. It follows that N is isomorphic to a direct summand of H, and hence N is strongly copure injective. So R is 2-Gorenstein by [11, Lemma 3.1 and Theorem 4.1]. □

Remark 4.4. If we replace "strongly copure injective cover" with "strongly copure flat envelope" in Theorem 4.3, the result still holds by [11, Lemma 3.3 and Theorem 4.1] and a proof dual to that of Theorem 4.3.

For an arbitrary class $\mathcal{C}$, it is not true in general that the direct product of $\mathcal{C}$-covers is a $\mathcal{C}$-cover (even if $\mathcal{C}$ is closed under direct products). We conclude this paper with the following

Proposition 4.5. *Let R be a 2-Gorenstein ring. Suppose that $\alpha_i : L_i \to M_i$ is a strongly copure injective cover for each $i \in I$, then $\prod \alpha_i : \prod L_i \to \prod M_i$ is a strongly copure injective cover.*

Proof. By Theorem 4.3, every α_i is a strongly copure injective cover with the unique mapping property. Consider the exact sequence $0 \to \ker(\alpha_i) \to L_i \to M_i$. For any strongly copure injective R-module L, we have the exact sequence

$$0 \to \mathrm{Hom}(L, \ker(\alpha_i)) \to \mathrm{Hom}(L, L_i) \to \mathrm{Hom}(L, M_i).$$

Thus $\mathrm{Hom}(L, \ker \alpha_i) = 0$ since $0 \to \mathrm{Hom}(L, L_i) \to \mathrm{Hom}(L, M_i)$ is exact.

Note that the class of strongly copure injective R-modules is closed under direct products, and so $\prod \alpha_i : \prod L_i \to \prod M_i$ is a strongly copure injective precover by [23, Theorem 1.2.9]. Since R is a 2-Gorenstein ring, strongly copure injective modules coincide with Gorenstein injective modules by Remark 2.3 (3). So $\prod M_i$ admits a strongly copure injective cover by [12, Theorem 11.1.3]. On the other hand, we claim that $\prod L_i$ has no nonzero direct summand contained in $\prod \ker \alpha_i$. Indeed, let K be a direct summand of $\prod L_i$ and $K \subseteq \prod \ker \alpha_i$. Then K is strongly copure injective, and hence

$$\mathrm{Hom}\left(K, \prod \ker \alpha_i\right) \cong \prod \mathrm{Hom}(K, \ker \alpha_i) = 0.$$

Thus $K = 0$. It follows that $\prod \alpha_i : \prod L_i \to \prod M_i$ is a strongly copure injective cover by [23, Corollary 1.2.8]. □

Acknowledgements

This research was partially supported by SRFDP (No. 20050284015), NSFC (No. 10331030), NSF of Jiangsu Province of China (No. BK 2005207) and the Nanjing Institute of Technology of China. The authors would like to thank the referee for the very helpful comments and suggestions.

References

[1] S.T. Aldrich, E.E. Enochs, O.M.G. Jenda, L. Oyonarte, Envelopes and covers by modules of finite injective and projective dimensions, J. Algebra 242 (2001) 447–459.
[2] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York, 1974.
[3] H. Bass, Injective dimension in noetherian rings, Trans. Amer. Math. Soc. 102 (1962) 18–29.
[4] H. Cartan, S. Eilenberg, Homological Algebra, Princeton Univ. Press, Princeton, 1956.
[5] J.L. Chen, N.Q. Ding, On the exactness of flat resolvents, Comm. Algebra 22 (1994) 4013–4021.
[6] R.R. Colby, Rings which have flat injective modules, J. Algebra 35 (1975) 239–252.
[7] N.Q. Ding, J.L. Chen, On copure flat modules and flat resolvents, Comm. Algebra 24 (1996) 1071–1081.
[8] N.Q. Ding, On envelopes with the unique mapping property, Comm. Algebra 24 (4) (1996) 1459–1470.
[9] E.E. Enochs, Injective and flat covers, envelopes and resolvents, Israel J. Math. 39 (1981) 189–209.
[10] E.E. Enochs, O.M.G. Jenda, Copure injective modules, Quaest. Math. 14 (1991) 401–409.
[11] E.E. Enochs, O.M.G. Jenda, Copure injective resolutions, flat resolutions and dimensions, Comment. Math. 34 (1993) 203–211.
[12] E.E. Enochs, O.M.G. Jenda, Relative Homological Algebra, Walter de Gruyter, Berlin, New York, 2000.
[13] E.E. Enochs, O.M.G. Jenda, J.A. Lopez-Ramos, The existence of Gorenstein flat covers, Math. Scand. 94 (2004) 46–62.
[14] J.R. García Rozas, B. Torrecillas, Relative injective covers, Comm. Algebra 22 (1994) 2925–2940.
[15] Y. Iwanaga, On rings with finite self-injective dimension I, Tsukuba J. Math. 4 (1980) 107–113.
[16] E. Matlis, Injective modules over noetherian rings, Pacific J. Math. 8 (1958) 511–528.
[17] J.J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.
[18] R. Sazeedeh, Strongly torsion free, copure flat and Matlis reflexive modules, J. Pure Appl. Algebra 192 (2004) 265–274.
[19] P.F. Smith, Injective modules and prime ideals, Comm. Algebra 9 (1981) 989–999.
[20] B. Stenström, Rings of Quotients, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
[21] J. Trlifaj, Covers, Envelopes, and Cotorsion Theories; Lecture notes for the workshop, "Homological Methods in Module Theory", Cortona, September 10–16, 2000.
[22] R. Ware, Endomorphism rings of projective modules, Trans. Amer. Math. Soc. 155 (1971) 233–256.
[23] J. Xu, Flat Covers of Modules, in: Lecture Notes in Math., 1634, Springer-Verlag, Berlin-Heidelberg-New York, 1996.