Relative copure injective and copure flat modules

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Abstract

Let $R$ be a ring, $n$ a fixed nonnegative integer and $\mathcal{I}_n (\mathcal{F}_n)$ the class of all left (right) $R$-modules of injective (flat) dimension at most $n$. A left $R$-module $M$ (resp., right $R$-module $F$) is called $n$-copure injective (resp., $n$-copure flat) if $\text{Ext}^1(N, M) = 0$ (resp., $\text{Tor}^1(F, N) = 0$) for any $N \in \mathcal{I}_n$. It is shown that a left $R$-module $M$ over any ring $R$ is $n$-copure injective if and only if $M$ is a kernel of an $I_n$-precover $f : A \to B$ of a left $R$-module $B$ with $A$ injective. For a left coherent ring $R$, it is proven that every right $R$-module has an $F_n$-preenvelope, and a finitely presented right $R$-module $M$ is $n$-copure flat if and only if $M$ is a cokernel of an $F_n$-preenvelope $K \to F$ of a right $R$-module $K$ with $F$ flat. These classes of modules are also used to construct cotorsion theories and to characterize the global dimension of a ring under suitable conditions.

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1. Introduction

Let $R$ be a ring. A left $R$-module $M$ is called copure injective [10] if $\text{Ext}^1(N, M) = 0$ for all injective left $R$-modules $N$, and $M$ is called strongly copure injective [10] if $\text{Ext}^i(N, M) = 0$ for all injective left $R$-modules $N$ and all $i \geq 1$. A right $R$-module $F$ is said to be copure flat [11] if $\text{Tor}^1(F, N) = 0$ for all injective left $R$-modules $N$, and $F$ is said to be strongly copure flat [11] or weakly Gorenstein flat [13] if $\text{Tor}_i(F, N) = 0$ for all injective left $R$-modules $N$ and all $i \geq 1$. Copure injective modules and copure flat modules were discovered when studying injective precovers and flat preenvelopes and have been studied by many authors (see [7,10,11,18]).

In Section 2 of this paper, we introduce the concepts of $n$-copure injective modules and $n$-copure flat modules for a fixed nonnegative integer $n$ and show some of their general properties. A left $R$-module $M$ (resp., right $R$-module $F$) is called $n$-copure injective (resp., $n$-copure flat) if $\text{Ext}^1(N, M) = 0$ (resp., $\text{Tor}_1(F, N) = 0$) for any left $R$-module $N$ with $\text{id}(N) \leq n$. We note that $n$-copure injective modules and $n$-copure flat modules coincide with Gorenstein injective and Gorenstein flat modules [12] respectively over an $n$-Gorenstein ring $R$ (i.e., $R$ is a left and right noetherian ring with $\text{id}_R(R) \leq n$ and $\text{id}(R_R) \leq n$). For a fixed nonnegative integer $n$, we denote by $\mathcal{I}_n (\mathcal{F}_n)$ the class of all left (right)
R-modules of injective (flat) dimension at most \( n \). It is shown that a left \( R \)-module \( M \) over any ring \( R \) is \( n \)-coupure injective if and only if \( M \) is a kernel of an \( I_n \)-precover \( f : A \to B \) of a left \( R \)-module \( B \) with \( A \) injective. For a left coherent ring \( R \), we prove that every right \( R \)-module has an \( F_n \)-preenvelope, and a finitely presented right \( R \)-module \( M \) is \( n \)-coupure flat if and only if \( M \) is a kernel of an \( F_n \)-preenvelope \( K \to F \) of a right \( R \)-module \( K \) with \( F \) flat.

Section 3 is devoted to \( n \)-coupure injective modules and \( n \)-coupure flat modules over left noetherian rings with finite left self-injective dimension. Let \( R \) be a left noetherian ring with \( id(R) \leq n \) for an integer \( n \geq 0 \). It is shown that \( M \) is a reduced \( n \)-coupure injective left \( R \)-module if and only if \( M \) is a kernel of an \( I_n \)-cover \( f : A \to B \) with \( A \) injective. It is also shown that \( I(D(R)) \leq n \) (\( n \geq 1 \)) if and only if every \( n \)-coupure injective left \( R \)-module is injective if and only if every \( n \)-coupure flat right \( R \)-module is flat if and only if every \( n \)-coupure injective left \( R \)-module has a monic copure injective cover if and only if every \( n \)-coupure flat right \( R \)-module has an epic flat envelope.

In Section 4, we further investigate some properties of copure injective covers and copure flat envelopes. For a commutative artinian ring \( R \), we prove that \( id(R) \leq 1 \) if and only if every right \( R \)-module has an epic copure flat envelope if and only if every right \( R \)-module has a monic copure injective cover. For a left and right noetherian ring \( R \), it is proven that \( R \) is a 2-Gorenstein ring and every \( R \)-module has a strongly copure injective cover if and only if every \( R \)-module has a strongly copure injective cover with the unique mapping property.

Next we recall some known notions and facts needed in the following.

Let \( C \) be a class of right \( R \)-modules and \( M \) a right \( R \)-module. Following [9], we say that a homomorphism \( \phi : M \to C \) is a \( C \)-preenvelope if \( C \in C \) and the abelian group homomorphism \( \text{Hom}_R(\phi, C') : \text{Hom}(C, C') \to \text{Hom}(M, C') \) is surjective for each \( C' \in C \). A \( C \)-preenvelope \( \phi : M \to C \) is said to be a \( C \)-envelope if every endomorphism \( g : C \to C \) such that \( g\phi = \phi \) is an isomorphism. A \( C \)-envelope \( \phi : M \to C \) is said to have the unique mapping property [8] if for any homomorphism \( f : M \to C' \) with \( C' \in C \), there is a unique homomorphism \( g : C \to C' \) such that \( g\phi = f \). Dually we have the definitions of a \( C \)-precover and a \( C \)-cover (with the unique mapping property). \( C \)-envelopes (\( C \)-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Let \( C \) be a class of \( R \)-modules and \( M \) an \( R \)-module. A left (resp., right) \( C \)-resolution of \( M \) [12] is a \( \text{Hom}(C, -) \) (resp., \( \text{Hom}(-, C) \)) exact complex

\[
\cdots \to C_1 \to C_0 \to M \to 0 \quad \text{(resp., } 0 \to M \to C^0 \to C^1 \to \cdots \]

with each \( C_i, C^i \in \mathcal{C} \).

If \( \cdots \to C_1 \to C_0 \to M \to 0 \) is a left \( C \)-resolution of \( M \), let

\[
K_0 = M, \quad K_1 = \ker(C_0 \to M), \quad K_i = \ker(C_{i-1} \to C_{i-2}) \quad \text{for } i \geq 2.
\]

The \( n \)th kernel \( K_n(n \geq 0) \) is called the \( n \)th \( C \)-syzygy of \( M \).

If \( 0 \to M \to C^0 \to C^1 \to \cdots \) is a right \( C \)-resolution of \( M \), let

\[
L^0 = M, \quad L^1 = \text{coker}(M \to C^0), \quad L^i = \text{coker}(C^{i-2} \to C^{i-1}) \quad \text{for } i \geq 2.
\]

The \( n \)th cokernel \( L^n(n \geq 0) \) is called the \( n \)th \( C \)-cosyzygy of \( M \).

If \( C \) is the class of projective (resp., injective) modules, then \( K_n \) (resp., \( L^n \)) is simply called the \( n \)th syzygy (resp., cosyzygy).

Let \( R \) be a left noetherian ring. Then every left \( R \)-module has a left \( I_0 \)-resolution by [12, Example 8.3.5].

Let \( R \) be a left coherent ring. Then every finitely presented right \( R \)-module \( M \) has a right \( F_0 \)-resolution \( 0 \to M \to P^0 \to P^1 \to \cdots \) with each \( P^i \) finitely generated projective by [12, Example 8.3.10]. So by the \( n \)th \( F_0 \)-cosyzygy of a finitely presented right \( R \)-module, we will mean the \( n \)th cokernel in such a right \( F_0 \)-resolution.

Given a class \( L \) of right \( R \)-modules, we denote by \( \mathcal{L}^{\perp} = \{ C : \text{Ext}^1(L, C) = 0 \} \) for all \( L \in \mathcal{L} \) the right orthogonal class of \( L \), and by \( \perp \mathcal{L} = \{ C : \text{Ext}^1(C, L) = 0 \} \) for all \( L \in \mathcal{L} \) the left orthogonal class of \( L \). A pair \( (\mathcal{F}, \mathcal{C}) \) of classes of right \( R \)-modules is called a cotorsion theory [12] if \( \mathcal{F}^{\perp} = \mathcal{C} \) and \( \perp \mathcal{C} = \mathcal{F} \). A cotorsion theory \( (\mathcal{F}, \mathcal{C}) \) is called perfect [13] if every right \( R \)-module has a \( \mathcal{C} \)-envelope and an \( \mathcal{F} \)-cover. A cotorsion theory \( (\mathcal{F}, \mathcal{C}) \) is said to be hereditary [13] if whenever \( 0 \to L' \to L \to L'' \to 0 \) is exact with \( L, L'' \in \mathcal{F} \), then \( L' \) is also in \( \mathcal{F} \). By [13, Proposition 1.2], \( (\mathcal{F}, \mathcal{C}) \) is hereditary if and only if whenever \( 0 \to C' \to C \to C'' \to 0 \) is exact with \( C, C' \in \mathcal{C} \), then \( C'' \) is also in \( \mathcal{C} \).

For example, (the class of all flat right \( R \)-modules, the class of all cotorsion right \( R \)-modules) is a perfect and hereditary cotorsion theory by [12, Theorem 7.4.4], where a right \( R \)-module \( C \) is called cotorsion if \( \text{Ext}^1(F, C) = 0 \) for any flat right \( R \)-module \( F \).
Throughout this paper, \( R \) is an associative ring with identity and all modules are unitary. We write \( M_R \) \((R\text{-}M)\) to indicate a right (left) \( R \)-module. For an \( R \)-module \( M \), \( E(M) \) denotes the injective envelope of \( M \), the character module \( \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \) is denoted by \( M^+ \), and \( \delta_M : M \to M^{++} \) is the evaluation map. \( f \text{id}(M) \) denotes the flat (injective) dimension of \( M \). \( ID(R) \) and \( WD(R) \) stand for the left global dimension and the weak global dimension of a ring \( R \) respectively. Let \( M \) and \( N \) be \( R \)-modules. \( \text{Hom}(M, N) \) \((\text{Ext}^0(M, N))\) means \( \text{Hom}_R(M, N) \) \((\text{Ext}^0_R(M, N))\), and similarly \( M \otimes N \) \((\text{Tor}_{\mathbb{Z}}(M, N))\) denotes \( M \otimes_R N \) \((\text{Tor}_{\mathbb{Z}}^R(M, N))\) for an integer \( n \geq 1 \). General background materials can be found in [2,12,17,23].

2. Definitions and general results

We begin with the following

**Definition 2.1.** Let \( R \) be a ring, \( n \) a fixed nonnegative integer and \( \mathcal{I}_n \) the class of all left \( R \)-modules of injective dimension at most \( n \). A left \( R \)-module \( M \) is called \( n \)-copure injective if \( \text{Ext}^1(N, M) = 0 \) for any \( N \in \mathcal{I}_n \). A right \( R \)-module \( F \) is said to be \( n \)-copure flat if \( \text{Tor}_1(F, N) = 0 \) for any \( N \in \mathcal{I}_n \).

**Proposition 2.2.** Let \( R \) be any ring.

1. If \( \text{Ext}^i(N, M) = 0 \) for any \( i \) with \( 1 \leq i \leq n + 1 \) and any injective left \( R \)-module \( N \), then every \( k \)th cosyzygy of \( M \) is \((n-k)\)-copure injective for any \( k \) with \( 0 \leq k \leq n \), in particular, \( M \) is \( n \)-copure injective.
2. If \( \text{Tor}_i(M, N) = 0 \) for any \( i \) with \( 1 \leq i \leq n + 1 \) and any injective left \( R \)-module \( N \), then every \( k \)th syzygy of \( M \) is \((n-k)\)-copure flat for any \( k \) with \( 0 \leq k \leq n \), in particular, \( M \) is \( n \)-copure flat.

**Proof.** (1) Let \( k \) be an integer with \( 0 \leq k \leq n \), \( L^k \) a \( k \)th cosyzygy of \( M \), and \( N \) a left \( R \)-module with \( \text{id}(N) \leq n - k \). Then \( \text{Ext}^1(N, L^k) \cong \text{Ext}^{k+1}(M, N) \). On the other hand, there is an exact sequence \( 0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-k} \to 0 \) with each \( E^i \) injective (for \( \text{id}(N) \leq n - k \)), and so \( \text{Ext}^{k+1}(M, N) \cong \text{Ext}^{n+1}(E^{n-k}, M) = 0 \) by assumption. Thus \( \text{Ext}^1(N, L^k) = 0 \), and hence \( L^k \) is \((n-k)\)-copure injective.

(2) The proof is similar to that of (1). \( \Box \)

**Remark 2.3.** (1) Obviously, \( 0 \)-copure injective \( (0 \)-copure flat) modules are exactly \( copure \) \((copure flat)\) modules. If \( m \geq n \), then \( m \)-copure injective \((m \)-copure flat) modules are \( n \)-copure injective \((n \)-copure flat).

(2) By [12, Definitions 10.1.1 and 10.3.1] and Proposition 2.2, we have the following implications:

Gorenstein injective modules \( \Rightarrow \) strongly copure injective modules \( \Rightarrow \) \( n \)-copure injective modules \( \Rightarrow \) copure injective modules.

Gorenstein flat modules \( \Rightarrow \) strongly copure flat modules \( \Rightarrow \) \( n \)-copure flat modules \( \Rightarrow \) copure flat modules.

(3) Let \( R \) be an \( n \)-Gorenstein ring. For an \( R \)-module \( N \), \( \text{id}(N) \leq n \) if and only if \( \text{id}(N) < \infty \) by [12, Theorem 9.1.10] or [15, Theorem 2]. Therefore an \( R \)-module \( M \) is \( n \)-copure injective if and only if \( M \) is \( n \)-copure injective if and only if \( M \) is Gorenstein injective by [12, Corollary 11.2.2]. \( M \) is \( n \)-copure flat if and only if \( M \) is strongly copure flat if and only if \( M \) is Gorenstein flat by [12, Theorem 10.3.8].

(4) If \( R \) is a 1-Gorenstein ring, then, by [11, Corollary 4.2], any copure injective \((copure flat)\) \( R \)-module is strongly copure injective \((strongly copure flat)\), and hence Gorenstein injective \( (\text{Gorenstein flat}) \).

Next we give some characterizations of \( n \)-copure injective modules and \( n \)-copure flat modules.

**Proposition 2.4.** The following are equivalent for a left \( R \)-module \( M \):

1. \( M \) is \( n \)-copure injective.
2. For every exact sequence \( 0 \to M \to E \to L \to 0 \) with \( E \in \mathcal{I}_n \), \( E \to L \) is an \( \mathcal{I}_n \)-precover of \( L \).
3. \( M \) is a kernel of an \( \mathcal{I}_n \)-precover \( f : A \to B \) with \( A \) injective.
4. \( M \) is injective with respect to every exact sequence \( 0 \to A \to B \to C \to 0 \) with \( C \in \mathcal{I}_n \).

**Proof.** (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (4) are clear by definition.

(2) \( \Rightarrow \) (3). Since there exists a short exact sequence \( 0 \to M \to E(M) \to E(M)/M \to 0 \) and \( E(M) \in \mathcal{I}_n \), then (3) follows from (2).

(3) \( \Rightarrow \) (1). Let \( M \) be a kernel of an \( \mathcal{I}_n \)-precover \( f : A \to B \) with \( A \) injective. Then we have an exact sequence \( 0 \to M \to A \to A/M \to 0 \). So, for any \( N \in \mathcal{I}_n \), the sequence \( \text{Hom}(N, A) \to \text{Hom}(N, A/M) \to \text{Ext}^1(N, M) \to 0 \).
is exact. It is easy to verify that \( \text{Hom}(N, A) \to \text{Hom}(N, A/M) \to 0 \) is exact by (3). Thus \( \text{Ext}^1(N, M) = 0 \), and so (1) follows.

(4) \( \Rightarrow \) (1). For each \( N \in \mathcal{I}_n \), there exists a short exact sequence \( 0 \to K \to P \to N \to 0 \) with \( P \) projective, which induces an exact sequence \( \text{Hom}(P, M) \to \text{Hom}(K, M) \to \text{Ext}^1(N, M) \to 0 \). Note that \( \text{Hom}(P, M) \to \text{Hom}(K, M) \to 0 \) is exact by (4). Hence \( \text{Ext}^1(N, M) = 0 \), as desired. \( \square \)

**Corollary 2.5.** Let \( R \) be a left noetherian ring. Then every \( (n + 1) \)th \( \mathcal{I}_0 \)-syzygy of any left \( R \)-module is \( n \)-copure injective.

**Proof.** Let \( \cdots \to E_1 \to E_0 \to M \to 0 \) be a left \( \mathcal{I}_0 \)-resolution of a left \( R \)-module \( M \). By [12, Lemma 8.4.34], \( E_n \to K_n \) is an \( \mathcal{I}_n \)-precover, where \( K_n \) is the \( n \)th \( \mathcal{I}_0 \)-syzygy of \( M \), and so the \( (n + 1) \)th \( \mathcal{I}_0 \)-syzygy \( K_{n+1} \) of \( M \) is \( n \)-copure injective by Proposition 2.4. \( \square \)

**Proposition 2.6.** The following are equivalent for a right \( R \)-module \( M \):

(1) \( M \) is \( n \)-copure flat.
(2) \( M^+ \) is \( n \)-copure injective.
(3) \( M \in \mathcal{I}^C \), where \( \mathcal{I}^C = \{ B^+ : B \in \mathcal{I}_n \} \).
(4) For every exact sequence \( 0 \to A \to B \to C \to 0 \) of left \( R \)-modules with \( C \in \mathcal{I}_n \), the functor \( M \otimes - \) preserves the exactness.

**Proof.** By [4, VI. 5.1] or [17, p. 360], there are the following standard isomorphisms:

\[
\text{Ext}^1(N, M^+) \cong \text{Tor}_1(M, N)^+ \cong \text{Ext}^1(M, N^+)
\]

for any left \( R \)-module \( N \). Thus (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) follow.

(1) \( \Leftrightarrow \) (4) is easy. \( \square \)

Injective (flat) modules are clearly \( n \)-copure injective (\( n \)-copure flat). The converse is not true in general. In fact, we have the following

**Proposition 2.7.** Let \( R \) be a ring. Then

(1) A left \( R \)-module \( M \) is injective if and only if \( M \) is \( n \)-copure injective and \( \text{id}(M) \leq n + 1 \).
(2) A right \( R \)-module \( N \) is flat if and only if \( N \) is \( n \)-copure flat and \( f d(N) \leq n + 1 \).

**Proof.** (1) “\( \Rightarrow \)” is trivial.

“\( \Leftarrow \)” Consider the exact sequence \( 0 \to M \to E(M) \to E(M)/M \to 0 \). Note that \( \text{id}(E(M)/M) \leq n + 1 \) since \( \text{id}(M) \leq n + 1 \). Thus \( \text{Ext}^1(E(M)/M, M) = 0 \), and hence the above sequence is split. So \( M \) is injective.

(2) “\( \Rightarrow \)” is trivial.

“\( \Leftarrow \)” Let \( N \) be an \( n \)-copure flat right \( R \)-module with \( f d(N) \leq n + 1 \). Then \( N^+ \) is \( n \)-copure injective by Proposition 2.6. Thus \( N^+ \) is injective by (1) since \( \text{id}(N^+) \leq n + 1 \). Hence \( N \) is flat. \( \square \)

**Proposition 2.8.** Let \( S \) be a simple \( R \)-module over a commutative ring \( R \). Then the following are equivalent:

(1) \( S \) is \( n \)-copure injective.
(2) \( S \) is \( n \)-copure flat.
(3) \( S^+ \) is \( n \)-copure injective.

**Proof.** (1) \( \Leftrightarrow \) (2). Suppose \( \{ S_i \}_{i \in I} \) is an irredundant set of representatives of the simple \( R \)-modules. Let \( E = E(\bigoplus_{i \in I} S_i) \), then \( E \) is an injective cogenerator by [2, Corollary 18.19]. Let \( M \in \mathcal{I}_n \). Since \( E \) is injective, there is an isomorphism:

\[
\text{Ext}^1(M, \text{Hom}(S, E)) \cong \text{Hom}(\text{Tor}_1(M, S), E).
\]

Note that \( \text{Hom}(S, E) \cong S \) by the proof of [22, Lemma 2.6]. Thus \( \text{Ext}^1(M, S) = 0 \) if and only if \( \text{Tor}_1(M, S) = 0 \), and so (1) \( \Leftrightarrow \) (2) follows.

(2) \( \Leftrightarrow \) (3) holds by Proposition 2.6. \( \square \)
Proposition 2.9. Let $R$ be commutative noetherian and $M$ an $R$-module. Then

1. $M$ is $n$-copure injective if and only if $\text{Hom}(F, M)$ is $n$-copure injective for all flat $R$-modules $F$.
2. $M$ is $n$-copure flat if and only if $F \otimes M$ is $n$-copure flat for all flat $R$-modules $F$.

Proof. (1) “$\Rightarrow$” holds by letting $F = R$.

“$\Leftarrow$”. Let $F$ be any flat $R$-module and $E \in \mathcal{I}_n$. There exists an exact sequence $0 \to K \to P \to E \to 0$ with $P$ projective, which yields the exactness of the sequence $0 \to K \otimes F \to P \otimes F \to E \otimes F \to 0$. Note that $E \otimes F \in \mathcal{I}_n$ since $R$ is commutative noetherian. Then we have the following exact sequence

$$\text{Hom}(P \otimes F, M) \to \text{Hom}(K \otimes F, M) \to \text{Ext}^1(E \otimes F, M) = 0,$$

which gives rise to the exactness of the sequence

$$\text{Hom}(P, \text{Hom}(F, M)) \to \text{Hom}(K, \text{Hom}(F, M)) \to 0.$$

On the other hand, the following sequence

$$\text{Hom}(P, \text{Hom}(F, M)) \to \text{Hom}(K, \text{Hom}(F, M)) \to \text{Ext}^1(E, \text{Hom}(F, M)) \to \text{Ext}^1(P, \text{Hom}(F, M)) = 0$$

is exact. Thus $\text{Ext}^1(E, \text{Hom}(F, M)) = 0$, as desired.

(2) “$\Leftarrow$” holds by letting $F = R$.

“$\Rightarrow$”. Let $F$ be any flat $R$-module. We only need to show that $(F \otimes M)^+$ is $n$-copure injective by Proposition 2.6. In fact, since $M^+$ is $n$-copure injective by Proposition 2.6, $(F \otimes M)^+ \cong \text{Hom}(F, M^+)$ is $n$-copure injective by (1). \hfill \Box

For a left noetherian ring $R$, [1, Proposition 3.1] shows that every left $R$-module has an $\mathcal{I}_n$-preenvelope. Let $\mathcal{F}_n$ be the class of all right $R$-modules of flat dimension at most $n$, we have

Proposition 2.10. Let $R$ be a left coherent ring and $n$ a fixed nonnegative integer. Then any right $R$-module has an $\mathcal{F}_n$-preenvelope. Moreover, suppose $M$ is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right $R$-module $K$ with $F$ flat, then $M$ is $n$-copure flat.

Proof. Let $M$ be a right $R$-module with $\text{Card}M = \aleph$. Then, by [12, Lemma 5.3.12], there is an infinite cardinal $\aleph_\alpha$ such that if $F \in \mathcal{F}_\alpha$ and $S$ is a submodule of $F$ with $\text{Card}S \leq \aleph_\beta$, there exists a pure submodule $G$ of $F$ with $S \subseteq G$ and $\text{Card}G \leq \aleph_\alpha$. Note that the pure exact sequence $0 \to G \to F \to F/G \to 0$ induces the split exact sequence $0 \to (F/G)^+ \to F^+ \to G^+ \to 0$. Thus $G^+ \in \mathcal{I}_n$ since $F^+ \in \mathcal{I}_n$, and so $G \in \mathcal{F}_n$. Therefore $M$ has an $\mathcal{F}_n$-preenvelope by [12, Corollary 6.2.2] since the left coherence of $R$ guarantees that $\mathcal{F}_n$ is closed under direct products.

Now suppose $M$ is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right $R$-module $K$ with $F$ flat. Let $L = \text{im}(K \to F)$, then $0 \to L \to F \to M \to 0$ is exact and $L \to F$ is an $\mathcal{F}_n$-preenvelope of $L$. Note that $E^+ \in \mathcal{F}_n$ for any $E \in \mathcal{I}_n$ since $R$ is left coherent. Thus we obtain an exact sequence $\text{Hom}(F, E^+) \to \text{Hom}(L, E^+) \to 0$, which gives rise to the exactness of $(F \otimes E)^+ \to (L \otimes E)^+ \to 0$. So the sequence $0 \to L \otimes E \to F \otimes E$ is exact. But the flatness of $F$ implies the exactness of $0 \to \text{Tor}_1(M, E) \to L \otimes E \to F \otimes E$, and hence $\text{Tor}_1(M, E) = 0$. This completes the proof. \hfill \Box

Corollary 2.11. Let $R$ be a left coherent ring. Then every $(n + 1)$th $\mathcal{F}_0$-cosyzygy of any finitely presented right $R$-module is $n$-copure flat.

Proof. Let $M$ be a finitely presented right $R$-module and $0 \to M \to F^0 \to F^1 \to \cdots$ any right $\mathcal{F}_0$-resolution of $M$ with each $F^i$ finitely generated projective. By [12, Remark 8.4.35] or [5, Lemma 2.1], $L^n \to F^n$ is an $\mathcal{F}_n$-preenvelope, where $L^n$ is the $n$th $\mathcal{F}_0$-cosyzygy of $M$. Thus the $(n + 1)$th $\mathcal{F}_0$-cosyzygy $L^{n+1}$ is $n$-copure flat by Proposition 2.10. \hfill \Box

Theorem 2.12. Let $R$ be a left coherent ring and $M$ a finitely presented right $R$-module. Then $M$ is $n$-copure flat if and only if $M$ is a cokernel of an $\mathcal{F}_n$-preenvelope $K \to F$ of a right $R$-module $K$ with $F$ flat.
Proposition 2.10
Every n-copure flat right R-module is m-copure flat for any m ≥ 1.

Remark 2.3
Let R be any ring and n a fixed nonnegative integer. Then (CIRn, CIRn⊥) is a perfect cotorsion theory. Moreover, the following are equivalent:

1. (CIRn, CIRn⊥) is a hereditary cotorsion theory.
2. Torj(F, N) = 0 for any F ∈ CIRn and any N ∈ IRn.
3. Torj(F, N) = 0 for any F ∈ CIRn, any N ∈ IRn and any j ≥ 1.

If R is a left noetherian ring with id(R) ≤ n + 1, then the above conditions are also equivalent to:

4. Every n-copure flat right R-module is m-copure flat for any m ≥ n.
5. Every n-copure flat right R-module is (n + 1)-copure flat.

Proof. By [21, Lemma 1.11 and Theorem 2.8], (CIRn, CIRn⊥) is a perfect cotorsion theory.

(1) ⇒ (2). Let F ∈ CIRn. Then there is an exact sequence 0 → K → P → F → 0 with P projective. Thus K ∈ CIRn since (CIRn, CIRn⊥) is hereditary. Hence Tor2(F, N) = 0 for any N ∈ IRn.

(2) ⇒ (3). Let F ∈ CIRn and N ∈ IRn. Then Tor1(F, N) = 0 by definition and Torj(F, N) = 0 for any j ≥ 2 by induction.

(3) ⇒ (1) is easy.

(3) ⇒ (4). Let F ∈ CIRn and M ∈ IRm with m > n. Then there is an exact sequence

0 → M → E0 → E1 → ··· → Em−n−1 → Lm−n → 0

with each Ei injective. Note that Lm−n ∈ IRn, thus, by (3), we have

Tor1(F, M) ≅ Tor2(F, L1) ≅ ··· ≅ Torm−n(F, Lm−n−1) ≅ Torm−n+1(F, Lm−n) = 0,

where each Li is an i-th cosyzygy of M, i = 1, 2, ···, m − n. Hence F ∈ CIRm.

(4) ⇒ (5) is trivial.
(5) $\Rightarrow$ (2). Let $F \in \mathcal{CF}_n$ and $N \in \mathcal{I}_n$. There is an exact sequence $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$ with $P$ projective. Note that $P \in \mathcal{I}_{n+1}$ by hypothesis, and so $K \in \mathcal{I}_{n+1}$. But $\text{Tor}_1(F, K) = 0$ since $F \in \mathcal{CF}_{n+1}$ by (5), and hence $\text{Tor}_2(F, N) = 0$. \hfill \Box

Recall that a ring $R$ is called right semi-artinian [20] if every non-zero cyclic right $R$-module has non-zero socle. $R$ is said to be left IF [6] if every injective left $R$-module is flat.

**Proposition 2.14.** Let $R$ be a ring and $n$ a fixed nonnegative integer. Then the following are equivalent:

1. $R$ is a left IF ring.
2. Every left $R$-module $M$ with $M \in \mathcal{I}_n$ is flat.
3. Every cotorsion left $R$-module is $n$-copure injective.
4. Every right $R$-module is $n$-copure flat.
5. Every right $R$-module $M$ with $M \in \mathcal{CF}_n^\perp$ is injective.
6. $(\mathcal{CF}_n, \mathcal{CF}_n^\perp)$ is a hereditary cotorsion theory, and every right $R$-module $M$ with $M \in \mathcal{CF}_n^\perp$ is $n$-copure flat.

If $R$ is right semi-artinian, then the above conditions are equivalent to:

7. Every simple right $R$-module is $n$-copure flat.

**Proof.** (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Rightarrow$ (6), (7) are clear by definition.

(2) $\Rightarrow$ (3) holds by the flat cotorsion theory.

(4) $\Leftrightarrow$ (5) follows from Proposition 2.13.

(6) $\Rightarrow$ (4). Let $M$ be any right $R$-module. By Proposition 2.13 and Wakamatsu’s Lemma [23, Section 2.1], there is a short exact sequence $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ with $E \in \mathcal{CF}_n^\perp$ and $L \in \mathcal{CF}_n$. Then $E \in \mathcal{CF}_n$ by (6), and hence $M \in \mathcal{CF}_n$ since $(\mathcal{CF}_n, \mathcal{CF}_n^\perp)$ is hereditary.

(7) $\Rightarrow$ (2). Let $I$ be a maximal right ideal of $R$ and $M \in \mathcal{I}_n$. Then we have $\text{Tor}_1(R/I, M) = 0$ by (7). Thus $\text{Ext}^1(R/I, M^+) = 0$ since $\text{Ext}^1(R/I, M^+) \cong \text{Tor}_1(R/I, M)^+$. So $M^+$ is injective with respect to any maximal right ideal of $R$. Hence $M^+$ is injective by [19, Lemma 4] since $R$ is right semi-artinian. Thus $M$ is flat. \hfill \Box

**Proposition 2.15.** The following are equivalent for a ring $R$ and a fixed nonnegative integer $n$:

1. $R$ is a QF ring.
2. Every left $R$-module is $n$-copure injective.
3. $(\mathcal{I}_n, \mathcal{CI}_n)$ is a perfect hereditary cotorsion theory, and every left $R$-module $M$ with $M \in \mathcal{I}_n$ is $n$-copure injective.
4. $(\mathcal{I}_0, \mathcal{CI}_0)$ is a cotorsion theory.

**Proof.** (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are clear.

(3) $\Rightarrow$ (2). Let $M$ be any left $R$-module. By (3) and Wakamatsu’s Lemma, there is a short exact sequence $0 \rightarrow L \rightarrow F \rightarrow M \rightarrow 0$ with $F \in \mathcal{I}_n$ and $L \in \mathcal{CI}_n$. Then $F \in \mathcal{CI}_n$ by (3), and hence $M \in \mathcal{CI}_n$ since $(\mathcal{I}_n, \mathcal{CI}_n)$ is hereditary.

(4) $\Rightarrow$ (1). $R$ is a QF ring since every projective left $R$-module is injective. \hfill \Box

Proposition 2.13 shows that every right $R$-module has a $\mathcal{CF}_n$-cover. We end this section by the following

**Theorem 2.16.** The following are equivalent for any ring $R$ and any integer $n \geq 0$:

1. The injective envelope $E(M)$ is $n$-copure flat for any $n$-copure flat right $R$-module $M$.
2. The $\mathcal{CF}_n$-cover $F(I)$ is injective for any injective right $R$-module $I$.

**Proof.** (1) $\Rightarrow$ (2). Let $I$ be an injective right $R$-module, $\varepsilon : F(I) \rightarrow I$ the $\mathcal{CF}_n$-cover of $I$, and $\lambda : F(I) \rightarrow E(F(I))$ the injective envelope. Then there exists $\theta : E(F(I)) \rightarrow I$ such that $\theta \varepsilon = \lambda$. On the other hand, since $E(F(I))$ is $n$-copure flat by (1), there exists $\beta : E(F(I)) \rightarrow F(I)$ such that $\varepsilon \beta = \theta$. Thus $\varepsilon \beta \lambda = \varepsilon$, and hence $\beta \lambda$ is an isomorphism. This means that $F(I)$ is a direct summand of $E(F(I))$ and so it is injective.

(2) $\Rightarrow$ (1). Let $M$ be an $n$-copure flat right $R$-module, $\lambda : M \rightarrow E(M)$ the injective envelope, and $\varepsilon : F(E(M)) \rightarrow E(M)$ the $\mathcal{CF}_n$-cover of $E(M)$. Then there exists $\alpha : M \rightarrow F(E(M))$ such that $\varepsilon \alpha = \lambda$. On the other hand, since
\[ F(E(M)) \] is injective by (2), there exists \( \gamma : E(M) \to F(E(M)) \) such that \( \gamma \lambda = \alpha \). Thus \( \varepsilon \gamma \lambda = \lambda \), and so \( \varepsilon \gamma \) is an isomorphism. It follows that \( E(M) \) is \( n \)-copure flat. \( \square \)

3. Left noetherian rings with \( id(R) \leq n \)

Recall that a left \( R \)-module \( M \) (resp., right \( R \)-module \( N \)) is called strongly cotorsion (resp., strongly torsionfree) \([23, 18]\) if \( \text{Ext}^1(F, M) = 0 \) (resp., \( \text{Tor}_1(N, F) = 0 \)) for any left \( R \)-module \( F \) with \( f d(F) < \infty \).

If \( R \) is an \( n \)-Gorenstein ring, then an \( R \)-module is \( n \)-copure injective if and only if it is Gorenstein injective by Remark 2.3 (3), and so \( (\mathcal{I}_n, \mathcal{CI}_n) \) is a perfect cotorsion theory by \([12, \text{Theorem 11.3.2}]\). For left noetherian rings with finite left self-injective dimension, we have

**Lemma 3.1.** Let \( n \) be a fixed nonnegative integer. Then the following hold for a left noetherian ring \( R \) with \( id(R) \leq n \):

1. \( (\mathcal{I}_n, \mathcal{CI}_n) \) is a perfect cotorsion theory.
2. Every \( n \)-copure injective left \( R \)-module is strongly cotorsion, and every \( n \)-copure flat right \( R \)-module is strongly torsionfree.
3. If \( R \) is an \( n \)-Gorenstein ring, then \( (\mathcal{I}_n, \mathcal{CI}_n) \) is a hereditary cotorsion theory.

Moreover, every strongly cotorsion left \( R \)-module is \( n \)-copure injective, and every strongly torsionfree right \( R \)-module is \( n \)-copure flat.

**Proof.** (1) Since \( R \) is left noetherian, \( \mathcal{I}_n \) is closed under well ordered inductive limits by \([3, \text{Theorem 1.1}]\), so (1) follows from \([1, \text{Theorem 2.8}]\) and \([12, \text{Theorem 7.2.6}]\).

(2) Let \( f d(F) < \infty \), then \( F \in \mathcal{I}_n \) since every flat left \( R \)-module has injective dimension at most \( n \) by \([12, \text{Proposition 9.1.2}]\). Thus (2) follows.

(3) holds by \([15, \text{Theorem 2}]\). \( \square \)

Recall that an \( R \)-module \( M \) is called reduced \([10]\) if \( M \) has no nonzero injective submodule.

**Proposition 3.2.** Let \( R \) be a left noetherian ring with \( id(R) \leq n \) and \( n \geq 0 \). Then the following are equivalent for a left \( R \)-module \( M \):

1. \( M \) is a reduced \( n \)-copure injective left \( R \)-module.
2. \( M \) is a kernel of an \( \mathcal{I}_n \)-cover \( f : A \to B \) with \( A \) injective.

**Proof.** (1) \( \Rightarrow \) (2). By Proposition 2.4, the natural map \( \pi : E(M) \to E(M)/M \) is an \( \mathcal{I}_n \)-precover. Thus \( E(M) \) has no nonzero direct summand \( K \) contained in \( M \) since \( M \) is reduced. Note that \( E(M)/M \) has an \( \mathcal{I}_n \)-cover by Lemma 3.1. It follows that \( \pi : E(M) \to E(M)/M \) is an \( \mathcal{I}_n \)-cover by \([23, \text{Corollary 1.2.8}]\), and hence (2) follows.

(2) \( \Rightarrow \) (1). Let \( M \) be a kernel of an \( \mathcal{I}_n \)-cover \( \alpha : A \to B \) with \( A \) injective. By Proposition 2.4, \( M \) is \( n \)-copure injective. Now let \( K \) be an injective submodule of \( M \). Suppose \( A = K \oplus L \), \( p : A \to L \) is the projection and \( i : L \to A \) is the inclusion. It is easy to see that \( \alpha(ip) = \alpha \) since \( \alpha(K) = 0 \). Therefore \( ip \) is an isomorphism, and hence \( i \) is epic. Thus \( A = L, K = 0 \), and so \( M \) is reduced. \( \square \)

In order to prove the next main result, we need the following lemma which is of independent interest.

**Lemma 3.3.** Let \( R \) be a left noetherian ring with \( id(R) \leq n \) and \( n \geq 1 \).

1. If \( M \) is an \( (n - 1) \)-copure injective left \( R \)-module, then there is an exact sequence \( 0 \to K \to E \to M \to 0 \) such that \( E \) is injective and \( K \) is \( n \)-copure injective.
2. If \( N \) is an \( (n - 1) \)-copure flat right \( R \)-module, then there is an exact sequence \( 0 \to N \to F \to L \to 0 \) such that \( F \) is flat and \( L \) is \( n \)-copure flat.
Proof. (1) Consider the following pushout diagram:

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & & \downarrow \\
0 & N & P & M & 0 \\
\downarrow & & & \downarrow & \\
0 & N & E(P) & Q & 0 \\
\downarrow & & \downarrow & & \downarrow \\
C & C & \downarrow & & \\
\downarrow & & \downarrow & & \\
0 & 0 & & & \\
\end{array}
\]

where \( P \) is projective and \( P \rightarrow E(P) \) is an injective envelope. Note that \( id(C) \leq n - 1 \) since \( id(P) \leq n \). So \( \text{Ext}^1(C, M) = 0 \) (for \( M \) is \((n - 1)\)-copure injective), then the sequence \( 0 \rightarrow M \rightarrow Q \rightarrow C \rightarrow 0 \) is split. Therefore \( M \) is a quotient of \( E(P) \).

Now suppose \( \alpha : E \rightarrow M \) is an injective cover of \( M \), then \( \alpha \) is epic. Thus we have the exact sequence \( 0 \rightarrow K \rightarrow E \rightarrow M \rightarrow 0 \). Note that \( K \) is copure injective by [9, Lemma 2.1]. We claim that \( K \) is also \( n \)-copure injective. In fact, let \( X \in \mathcal{I}_n \). Consider the exact sequence \( 0 \rightarrow X \rightarrow E(X) \rightarrow D \rightarrow 0 \). Then \( D \in \mathcal{I}_{n-1} \). Thus we get the induced exact sequence

\[
0 = \text{Ext}^1(D, M) \rightarrow \text{Ext}^2(D, K) \rightarrow \text{Ext}^2(D, E) = 0.
\]

Therefore \( \text{Ext}^2(D, K) = 0 \). On the other hand, the short exact sequence \( 0 \rightarrow X \rightarrow E(X) \rightarrow D \rightarrow 0 \) induces the exactness of the sequence

\[
0 = \text{Ext}^1(E(X), K) \rightarrow \text{Ext}^1(X, K) \rightarrow \text{Ext}^2(D, K) = 0.
\]

Therefore \( \text{Ext}^1(X, K) = 0 \), as desired.

(2) Let \( N \) be an \((n - 1)\)-copure flat right \( R \)-module. Then \( N^+ \) is \((n - 1)\)-copure injective by Proposition 2.6. Thus there is an exact sequence \( E \rightarrow N^+ \rightarrow 0 \) with \( E \) injective by (1), which in turn yields the exactness of \( 0 \rightarrow N^+ \rightarrow E^+ \). So \( N \) embeds in a flat right \( R \)-module (for \( E^+ \) is flat).

Now let \( \beta : N \rightarrow F \) be a flat preenvelope of \( N \), then \( \beta \) is monic. So we have the exact sequence \( 0 \rightarrow N \rightarrow F \rightarrow L \rightarrow 0 \). Note that \( L \) is copure flat by Proposition 2.10. Applying an argument similar to that in the proof of (1), we can prove that \( L \) is also \( n \)-copure flat. \( \square \)

Let \( R \) be a left noetherian ring. It is known that \( R \) is a left hereditary ring if and only if every right \( R \)-module has an epic flat envelope if and only if every \( n \)-copure left \( R \)-module is injective if and only if every \( n \)-copure flat right \( R \)-module is flat (see [10, Corollary 2.4] and [7, Theorem 4.5]). Here we get

Theorem 3.4. Let \( R \) be a left noetherian ring with \( \text{id}(R)R \leq n \) and \( n \geq 1 \). Then the following are equivalent:

1. \( \text{id}(R) < \infty \).
2. \( \text{id}(R) \leq n \).
3. Every \((n - 1)\)-copure injective left \( R \)-module is injective.
4. Every \( n \)-copure injective left \( R \)-module is injective.
5. Every \( n \)-copure injective left \( R \)-module has a monic injective cover.
6. Every \((n - 1)\)-copure injective left \( R \)-module has a monic \( \mathcal{I}_{n-1} \)-cover.
7. Every \((n - 1)\)-copure flat right \( R \)-module is flat.
8. Every cotorsion right \( R \)-module belongs to \( CF_n^+ \).
9. Every \( n \)-copure flat right \( R \)-module is flat.
10. Every (finitely presented) \( n \)-copure flat right \( R \)-module has an epic flat envelope.
11. Every right \( R \)-module has an epic \( \mathcal{F}_{n-1} \)-envelope.
Every cotorsion $R$-module has an epic copure flat envelope.

By [3, Proposition 4.2], $I(D(R)) \leq n$ since $I(D(R)) < \infty$.

(2) $\Rightarrow$ (6). Note that $\mathcal{I}_{n-1}$ is closed under direct sums and quotients by (2). So (6) follows from [14, Proposition 4].

(6) $\Rightarrow$ (2). Let $M$ be any left $R$-module. By Lemma 3.1 and Wakamatsu’s Lemma, there is a short exact sequence $0 \rightarrow L \rightarrow F \rightarrow M \rightarrow 0$, where $F \in \mathcal{I}_n$ and $L \in CI_n$. Note that $L$ is $(n - 1)$-cogrounded injective, and so $L$ has a monic $\mathcal{I}_{n-1}$-cover by (6). But $L$ is a quot of an injective left $R$-module by Lemma 3.3 (1). Thus $L \in \mathcal{I}_{n-1}$, and hence $M \in \mathcal{I}_n$.

(4) $\Rightarrow$ (3) and (9) $\Rightarrow$ (7) follow from Lemma 3.3.

(4) $\Rightarrow$ (9) holds by Proposition 2.6.

(5) $\Rightarrow$ (1). Let $M$ be a left $R$-module. For any left $\mathcal{I}_0$-resolution $\cdots \rightarrow E_1 \rightarrow E_0 \rightarrow M \rightarrow 0$, the $(n + 1)$th $\mathcal{I}_0$-syzygy $K_{n+1}$ of $M$ is $n$-cogrounded injective by Corollary 2.5. Thus $K_{n+1}$ has a monic injective cover by (5), but $K_{n+1}$ is a quotient of an injective left $R$-module by Lemma 3.3 (1). Hence $K_{n+1}$ is injective. Therefore $I(D(R)) \leq n + 3 < \infty$ by [12, Corollary 8.4.17].

(8) $\Leftrightarrow$ (9) comes from Proposition 2.13.

(11) $\Rightarrow$ (1). By Corollary 2.11, the $(n + 1)$th $\mathcal{F}_0$-syzygy $L^{n+1}$ of any finitely presented right $R$-module $M$ is $n$-cogrounded flat. Thus $L^{n+1}$ embeds in a flat right $R$-module by Lemma 3.3 (2). But $L^{n+1}$ has an epic flat envelope by (10). Therefore $L^{n+1}$ is flat, and hence projective. So $I(D(R)) \leq n + 3 < \infty$ by [12, Corollary 8.4.28].

4. On cogrounded injective covers and copure flat envelopes

Enochs and Jenda have shown that every left $R$-module has a strongly cogrounded injective precover over a left noetherian ring $R$ (see [11, Theorem 2.2]). Here we have

**Proposition 4.1.** Let $R$ be a commutative artinian ring. Then $M^+$ has a strongly cogrounded injective precover for any $R$-module $M$.

**Proof.** By [11, Theorem 2.5], $M$ has a strongly cogrounded flat preenvelope $f : M \rightarrow N$. We shall show that $f^+ : N^+ \rightarrow M^+$ is a strongly cogrounded injective precover of $M^+$. Indeed, let $\psi : H \rightarrow M^+$ be any homomorphism with $H$ strongly cogrounded injective. Since $H^+$ is strongly cogrounded flat by [11, Lemma 3.6], there exists $g : N \rightarrow H^+$ such that $gf = \psi^+ \delta_M$. Thus $f^+g^+ = \delta^+_M\psi^+$. Note that $\psi^+\delta_H = \delta_M^+\psi$, then by [2, Proposition 20.14], we have $f^+(g^+\delta_H) = \delta^+_M(\psi^+\delta_H) = (\delta^+_M\delta_M^+)(\psi) = \psi$. Hence $f^+$ is a strongly cogrounded injective precover.

**Theorem 4.2.** The following are equivalent for a commutative artinian ring $R$:

1. $id(R) \leq 1$.
2. Every $R$-module has an epic cogrounded flat envelope.
3. Every cotorsion $R$-module has an epic cogrounded flat envelope.
4. Every $R$-module has a monic cogrounded injective cover.

**Proof.** (1) $\Rightarrow$ (2). Since $R$ is a commutative artinian ring, any $R$-module $M$ has a strongly cogrounded flat preenvelope $f : M \rightarrow N$ by [11, Theorem 2.5]. But $R$ is a 1-Gorenstein ring by (1), so any cogrounded flat module is strongly cogrounded flat by [11, Corollary 4.2]. Thus $f$ is also a cogrounded flat preenvelope. Note that $im(f)$ is cogrounded flat by [11, Corollary 4.2], hence $f : M \rightarrow im(f)$ is an epic cogrounded flat envelope.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). By [11, Corollary 4.2], we shall show that any submodule $N$ of any cogrounded flat $R$-module $M$ is cogrounded flat. Since $M/N$ has a flat cover $f : F \rightarrow M/N$, we get an exact sequence $0 \rightarrow C \rightarrow F \rightarrow M/N \rightarrow 0$ with $C$ cotorsion by Wakamatsu’s Lemma. By (3), $C$ has an epic cogrounded flat envelope. Thus $C$ is cogrounded flat since $C$ embeds in a cogrounded $R$-module. So, for any injective $R$-module $E$, we get an induced exact sequence

$$0 = \text{Tor}_2(F, E) \rightarrow \text{Tor}_2(M/N, E) \rightarrow \text{Tor}_1(C, E) = 0.$$
Hence $\text{Tor}_2(M/N, E) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

$$0 = \text{Tor}_2(M/N, E) \to \text{Tor}_1(N, E) \to \text{Tor}_1(M, E) = 0.$$ 

Therefore $\text{Tor}_1(N, E) = 0$, as desired.

(1) $\Leftrightarrow$ (4). We first show that the class of copure injective $R$-modules over a commutative artinian ring $R$ is closed under direct sums. Indeed, let $\{M_j\}_{j \in J}$ be a family of copure injective $R$-modules and $E$ an injective $R$-module. Since $R$ is artinian, then $E = \bigoplus_{i \in I} E(S_i)$, where each $S_i$ is simple. Note that $E(S_i)$ is finitely generated by [16, Theorem 3.11], so by [2, Exercise 16.3, p. 189], we get

$$\text{Ext}^1\left(E, \bigoplus_{j \in J} M_j\right) = \prod_{i \in I} \text{Ext}^1\left(E(S_i), \bigoplus_{j \in J} M_j\right) = \prod_{i \in I} \bigoplus_{j \in J} \text{Ext}^1(E(S_i), M_j) = 0.$$

Hence $\text{Tor}_1(E, \bigoplus_{j \in J} M_j) = 0$ if and only if each $\bigoplus_{j \in J} M_j$ is copure injective. Since $R$ is 1-Gorenstein and only if the class of copure injective $R$-modules is closed under quotients by $[11, Corollary 4.2]$, (1) $\Leftrightarrow$ (4) follows from [14, Proposition 4].

It is well known that $R$ is a left noetherian ring with $\text{ID}(R) \leq 2$ if and only if every left $R$-module has an injective cover with the unique mapping property. Here we have

**Theorem 4.3.** The following are equivalent for a left and right noetherian ring $R$:

1. $R$ is a 2-Gorenstein ring, and every (left and right) $R$-module has a strongly copure injective cover.
2. Every (left and right) $R$-module has a strongly copure injective cover with the unique mapping property.

**Proof.** (1) $\Rightarrow$ (2). Let $M$ be any (left and right) $R$-module. Then $M$ has a strongly copure injective cover $f : F \to M$ by (1). It is enough to show that, for any strongly copure injective $R$-module $G$ and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/im(g) \to M$ such that $\beta \pi = f$ since $\text{im}(g) \subseteq \text{ker}(f)$, where $\pi : F \to F/im(g)$ is the natural map. Since $R$ is 2-Gorenstein, $\text{Ext}^i(E, \text{ker}(g)) = 0$ for any $i \geq 3$ and any injective $R$-module $E$ by [11, Lemma 3.1 and Theorem 4.1]. It follows that $F/im(g)$ is strongly copure injective. Thus there exists $\alpha : F/im(g) \to F$ such that $\beta = f\alpha$, and so we get the exact commutative diagram:

$$\begin{array}{ccc}
0 & \to & \text{ker}(g) \\
\downarrow{f} & & \downarrow{\beta} \\
G & \to & F \\
\downarrow{g} & & \downarrow{\pi} \\
0 & \to & F/im(g)
\end{array}$$

Therefore $f\alpha \pi = f$, and hence $\alpha\pi$ is an isomorphism. Therefore $\pi$ is monic, and so $g = 0$.

(2) $\Rightarrow$ (1). Let $M$ be any (left and right) $R$-module. Then we have the exact sequence $0 \to M \to E^0 \xrightarrow{\psi} E^1 \xrightarrow{\psi} N \to 0$, where $E^0, E^1$ are injective. Let $\theta : H \to N$ be a strongly copure injective cover with the unique mapping property. Then there exists $\tau : E^1 \to H$ such that $\psi = \theta \tau$. Thus $\theta \tau \varphi = \psi \varphi = 0 = \theta 0$, and hence $\tau \varphi = 0$, which implies that $\text{ker}(\psi) = \text{im}(\varphi) \subseteq \text{ker}(\tau)$. Therefore there exists $\gamma : N \to H$ such that $\gamma \psi = \tau$, and so we get the exact commutative diagram:

$$\begin{array}{ccc}
0 & \to & M \\
\downarrow{\tau} & & \downarrow{\psi} \\
E^0 & \xrightarrow{\psi} & E^1 \\
\downarrow{\theta} & & \downarrow{\gamma} \\
N & \to & 0
\end{array}$$

Thus $\theta \gamma \psi = \psi$, and so $\theta \gamma = 1_N$ since $\psi$ is epic. It follows that $N$ is isomorphic to a direct summand of $H$, and hence $N$ is strongly copure injective. So $R$ is 2-Gorenstein by [11, Lemma 3.1 and Theorem 4.1].

**Remark 4.4.** If we replace “strongly copure injective cover” with “strongly copure flat envelope” in Theorem 4.3, the result still holds by [11, Lemma 3.3 and Theorem 4.1] and a proof dual to that of Theorem 4.3.

For an arbitrary class $\mathcal{C}$, it is not true in general that the direct product of $\mathcal{C}$-covers is a $\mathcal{C}$-cover (even if $\mathcal{C}$ is closed under direct products). We conclude this paper with the following
Proposition 4.5. Let R be a 2-Gorenstein ring. Suppose that \( \alpha_i : L_i \to M_i \) is a strongly copure injective cover for each \( i \in I \), then \( \prod \alpha_i : \prod L_i \to \prod M_i \) is a strongly copure injective cover.

Proof. By Theorem 4.3, every \( \alpha_i \) is a strongly copure injective cover with the unique mapping property. Consider the exact sequence \( 0 \to \ker(\alpha_i) \to L_i \to M_i \). For any strongly copure injective \( R \)-module \( L \), we have the exact sequence

\[
0 \to \text{Hom}(L, \ker(\alpha_i)) \to \text{Hom}(L, L_i) \to \text{Hom}(L, M_i).
\]

Thus \( \text{Hom}(L, \ker(\alpha_i)) = 0 \) since \( 0 \to \text{Hom}(L, L_i) \to \text{Hom}(L, M_i) \) is exact.

Note that the class of strongly copure injective \( R \)-modules is closed under direct products, and so \( \prod \alpha_i : \prod L_i \to \prod M_i \) is a strongly copure injective precover by [23, Theorem 1.2.9]. Since \( R \) is a 2-Gorenstein ring, strongly copure injective modules coincide with Gorenstein injective modules by Remark 2.3 (3). So \( \prod M_i \) admits a strongly copure injective cover by [12, Theorem 11.1.3]. On the other hand, we claim that \( \prod L_i \) has no nonzero direct summand contained in \( \prod \ker \alpha_i \). Indeed, let \( K \) be a direct summand of \( \prod L_i \) and \( K \subseteq \prod \ker \alpha_i \). Then \( K \) is strongly copure injective, and hence

\[
\text{Hom}(K, \prod \ker \alpha_i) \cong \prod \text{Hom}(K, \ker \alpha_i) = 0.
\]

Thus \( K = 0 \). It follows that \( \prod \alpha_i : \prod L_i \to \prod M_i \) is a strongly copure injective cover by [23, Corollary 1.2.8].

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