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RELATIVE FLATNESS, MITTAG–LEFFLER MODULES, AND ENDOCOHERENCE

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Let M_R be a right R-module over a ring R with $S = \mathrm{End}(M_R)$. We study the coherence of the left S-module ${}_SM$ relative to a hereditary torsion theory for the category of right R-modules. Various results are developed, many extending known results.

Key Words: Preenvelope; τ-coherent module; τ-M-flat module; τ-Mittag–Leffler module.

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1. INTRODUCTION

Throughout this article, all rings are associative with identity and all modules are unitary. For a ring R, we write Mod-R for the category of all right R-modules. M_R (${}_RM$) denotes a right (left) R-module. As usual, $E(M)$ denotes the injective envelope of M, M^I ($M^{(I)}$) stands for the direct product (sum) of copies of M indexed by a set I. For a module M_R, we denote by $S = \mathrm{End}(M_R)$ the endomorphism ring of M_R and by $\mathrm{Add}\, M_R$ (resp., $\mathrm{add}\, M_R$) the category consisting of all modules isomorphic to direct summands of (finite) direct sums of copies of M_R. The category consisting of all modules isomorphic to direct summands of direct products of copies of M_R is denoted by $\mathrm{Prod}\, M_R$. $\tau = (\mathcal{T}, \mathcal{F})$ always stands for a hereditary torsion theory for Mod-R, and $t(M_R)$ denotes the largest submodule of M_R that belongs to $\mathcal{T}$.

We first recall some known notions and facts which we need in the later sections.

(1) A hereditary torsion theory (Stenström, 1975) $\tau = (\mathcal{T}, \mathcal{F})$ for Mod-R consists of two classes $\mathcal{T}$ and $\mathcal{F}$, the torsion class and the torsionfree class, respectively, such that $\mathrm{Hom}_R(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$, the class $\mathcal{T}$ is closed under submodules, factor modules, extensions and direct sums, the class $\mathcal{F}$ is closed under submodules, injective envelopes, extensions and direct products. For a

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hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, there exists an injective module E_R such that E cogenerates τ, i.e., $\mathcal{F} = \{M_R : M_R \text{ embeds in } E_R^I \text{ for some set } I\}$ (see Stenström, 1975, p. 142).

(2) Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory for Mod-R. A right R-module N is called *τ-finitely generated* (Jones, 1982b) if $N/N' \in \mathcal{T}$ for some finitely generated submodule N' of N, and N is called *τ-finitely presented* if there exists an exact sequence $0 \to K \to F \to N \to 0$ with F finitely generated free and K τ-finitely generated. It is obvious that every module in $\mathcal{T}$ is τ-finitely generated. If N is finitely generated (resp., finitely presented), it is clearly τ-finitely generated (resp., τ-finitely presented). If $\mathcal{T} = \{0\}$, then N is τ-finitely generated (resp., τ-finitely presented) if and only if N is finitely generated (resp., finitely presented). If $\mathcal{T} =$ Mod-R, then N is τ-finitely presented if and only if N is finitely generated.

(3) Let $\mathcal{C}$ be a class of right R-modules and M_R a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope of M* (Enochs and Jenda, 2000) if for any homomorphism $f: M \to F'$ where $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F' = F$ and $f = \phi$, the $\mathcal{C}$-preenvelope ϕ is called a *$\mathcal{C}$-envelope of M*.

(4) Clarke (1976) called M_R an R-Mittag–Leffler module if the canonical map $M \otimes R^J \to M^J$ is a monomorphism for every set J, or equivalently, if for every finitely generated submodule N of M, the inclusion $N \to M$ factors through a finitely presented right R-module (see Goodearl, 1972, Theorem 1 or Clarke, 1976, Theorem 2.4). The concept of R-Mittag–Leffler modules was called *finitely pure-projective modules* by Azumaya (see Azumaya, 1987, Note added in proof, p. 134).

(5) A left R-module ${}_RM$ is called *coherent* if it is finitely presented and every finitely generated submodule of ${}_RM$ is finitely presented. The ring R is left coherent if ${}_RR$ is coherent. Following Angeleri-Hügel (2003), ${}_RM$ is called *Π-coherent* if it is finitely presented and every finitely generated left R-module which is cogenerated by ${}_RM$ is finitely presented. It is clear that the ring R is left Π-coherent in the sense of Camillo (1990) if and only if ${}_RR$ is Π-coherent.

In this article, for a right R-module M_R over a ring R with $S = \mathrm{End}(M_R)$, we mainly study the coherence of the left S-module ${}_SM$ relative to a hereditary torsion theory for the category of right R-modules. Various results are developed, many extending known results.

In Section 2, we introduce the concepts of τ-M-flat modules and τ-Mittag–Leffler modules. Some characterizations and general properties of these modules are given.

In Section 3, for a right R-module M with $S = \mathrm{End}(M_R)$, we consider the coherence of ${}_SM$ relative to a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ for Mod-R. We show that, if M_R is finitely presented, then ${}_SM$ is τ-coherent if and only if all direct products of copies of M_R are τ-M-flat if and only if all direct products of τ-M-flat right R-modules are τ-M-flat if and only if ${}_SM$ is coherent and all direct products of copies of M_R are τ-Mittag–Leffler (Theorem 3.3).

Section 4 is devoted to investigating the relative flatness of injective modules. We show that if M_R is τ-finitely presented, then M_R is injective and every injective right R-module is τ-M-flat if and only if for every τ-finitely presented right R-module, its τ-M-flat envelope exists and coincides with its injective envelope if and only if M_R is injective and every τ-finitely presented right R-module has a monic τ-M-flat-(pre)envelope (Theorem 4.6). Let M_R be τ-finitely presented, it is proven that ${}_SM$ is τ-coherent and submodules of τ-M-flat right R-modules are τ-M-flat if and only if every (τ-finitely presented) right R-module has a τ-M-flat-preenvelope which is an epimorphism if and only if every τ-finitely presented right R-module has an $\operatorname{add} M_R$-preenvelope which is an epimorphism (Theorem 4.7).

In Section 5, we get that, if M_R and ${}_SM$ are finitely presented, then ${}_SM$ is coherent if and only if $U(S)$ is finitely generated for all $U \in M^n$ and $n \geq 1$ if and only if the left annihilator $\operatorname{ann}_{M_n(S)}(Y)$ is a finitely generated left ideal of $M_n(S)$ for any $n \geq 1$ and every element Y of the right R-module M^n if and only if every finitely M-presented right R-module has an $\operatorname{add} M_R$-preenvelope (Theorem 5.1).

The reader should consult Anderson and Fuller (1974), Enochs and Jenda (2000), and Stenström (1975) for background materials in ring theory.

2. RELATIVE FLATNESS AND MITTAG–LEFFLER MODULES

We start with the following definition.

Definition 2.1. Let M_R be a right R-module and $\tau = (\mathcal{T}, \mathcal{F})$ a hereditary torsion theory for Mod-R.

A right R-module N is called *τ-M-flat* (resp., *M-flat*) if every homomorphism $f : K \to N$ with K τ-finitely presented (resp., finitely presented) factors through a module in $\operatorname{add} M_R$.

N_R is called a *τ-Mittag–Leffler module* if every homomorphism $f : K \to N$ with K τ-finitely presented factors through a finitely presented right R-module.

Remark 2.2. (1) By definitions, the class of τ-M-flat (τ-Mittag–Leffler) right R-modules is closed under direct summands and finite direct sums. τ-M-flat right R-modules are always M-flat. R_R-flat right R-modules are exactly flat right R-modules.

(2) If $N \in \operatorname{add} M_R$, then N is τ-M-flat. The converse holds if N is τ-finitely presented.

(3) We note that τ-R_R-flat right R-modules are always τ-Mittag–Leffler. A right R-module N is τ-R_R-flat if and only if it is τ-flat in sense of Ding and Chen (1993). Moreover, if M_R is a projective generator in Mod-R, then N is τ-M-flat if and only if N is τ-flat. It is also easy to see that, if M_R is projective, then a τ-M-flat right R-module is τ-flat, and hence it is flat. However, if M_R is not a generator in Mod-R, R_R is clearly τ-flat, but R_R is not τ-M-flat.

(4) Let $\mathcal{T} = \{0\}$. Then every right R-module is τ-Mittag–Leffler. N_R is τ-M-flat if and only if N_R is M-flat.

Let $\mathcal{T} = $ Mod-R. Then τ-Mittag–Leffler right R-modules are precisely R-Mittag–Leffler modules (Clarke, 1976) or finitely pure-projective modules (Azumaya, 1987). N_R is τ-R_R-flat if and only if N_R is f-projective (Jones, 1982a) or finitely projective (Azumaya, 1987).

It is clear that τ-Mittag–Leffler modules are generalizations of both R-Mittag–Leffler modules (Clarke, 1976) and τ-flat modules (Ding and Chen, 1993). The following proposition is also easy to verify.

Proposition 2.3. *Let N be a right R-module. Then*:

(1) *N is τ-M-flat if and only if N is both M-flat and τ-Mittag–Leffler for a finitely presented right R-module M*;
(2) *N is finitely presented if and only if N is both τ-finitely presented and τ-Mittag–Leffler*;
(3) *Every right R-module is τ-Mittag–Leffler if and only if every τ-finitely presented right R-module is finitely presented.*

Recall that a right R-module epimorphism $f : L \to N$ is called *τ-pure* (Ding and Chen, 1993) if for any τ-finitely presented right R-module P, $\mathrm{Hom}_R(P, L) \xrightarrow{f_*} \mathrm{Hom}_R(P, N)$ is epic. Obviously, a τ-pure epimorphism is always pure. But the converse is not true. Indeed, let R be a von Neumann regular ring which is not semisimple Artinian and $\mathcal{T} = \text{Mod-}R$. Then there exists a pure epimorphism which is not τ-pure. However, we have the following proposition.

Proposition 2.4. *Let $f : L \to N$ be a pure epimorphism with $L \in \mathcal{F}$. Then f is τ-pure.*

Proof. Let H be a τ-finitely presented right R-module and $\varphi : H \to N$ any homomorphism. Then there is an exact sequence $0 \to K \to R^n \to H \to 0$, where K is τ-finitely generated, i.e., K has a finitely generated submodule K' such that $K/K' \in \mathcal{T}$. Thus we get an exact sequence $0 \to K/K' \to R^n/K' \xrightarrow{g} H \to 0$. Since R^n/K' is finitely presented and f is pure, there is $\alpha : R^n/K' \to L$ such that $\varphi g = f\alpha$. On the other hand, we have $\mathrm{Hom}_R(K/K', L) = 0$ since $K/K' \in \mathcal{T}$ and $L \in \mathcal{F}$. Thus $K/K' = \ker(g) \leq \ker(\alpha)$, and hence there exists $\gamma : H \to L$ such that $\alpha = \gamma g$. Therefore $f\gamma g = f\alpha = \varphi g$, which implies that $f\gamma = \varphi$ since g is epic, as desired. $\square$

Proposition 2.5. *The following are equivalent for a right R-module N*:

(1) *N is τ-Mittag–Leffler*;
(2) *Every pure epimorphism $f : L \to N$ is τ-pure*;
(3) *There exists a τ-pure epimorphism $f : L \to N$ with L τ-Mittag–Leffler*;
(4) *Given a pure epimorphism $f : L \to C$ and homomorphisms $h : N \to C$, $\alpha : P \to N$ with P τ-finitely presented, there exists a homomorphism $\beta : P \to L$ such that $f\beta = h\alpha$.*

Proof. (1) $\Rightarrow$ (2) Let $f : L \to N$ be a pure epimorphism. Assume that P is a τ-finitely presented right R-module and $\alpha : P \to N$ is any homomorphism. By (1), there exist a finitely presented right R-module H, $g : P \to H$ and $h : H \to N$ such that $\alpha = hg$. Since f is pure and H finitely presented, there exists $\beta : H \to L$ such that $f\beta = h$. So $\alpha = f(\beta g)$, and (2) follows.

(2) $\Rightarrow$ (1) Let P be a τ-finitely presented right R-module and $\alpha : P \to N$ is any homomorphism. By Warfield (1969, Proposition 1) or Dauns (1994, Proposition

18-2.9), there is a pure epimorphism $\gamma : F_i^{(I)} \to N$ with each F_i finitely presented, $i \in I$. By (2), γ is τ-pure. Thus there is $\varphi : P \to F_i^{(I)}$ such that $\gamma\varphi = \alpha$. Since P is finitely generated, so is $\mathrm{im}(\varphi)$. Therefore there exists a finite index set $J \subseteq I$ such that $\mathrm{im}(\varphi) \subseteq F_i^{(J)}$. Note that $F_i^{(J)}$ is finitely presented, hence α factors through a finitely presented right R-module.

(1) $\Leftrightarrow$ (3) is easy to verify.

(2) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (2) holds by letting $C = N$ and h be the identity map. □

Remark 2.6. Note that τ-Mittag–Leffler modules coincide with finitely pure-projective modules when $\mathcal{T} = \text{Mod-}R$. Proposition 7 and Corollary 8 in Azumaya (1987) are particular cases of Proposition 2.5 where $\mathcal{T} = \text{Mod-}R$.

Corollary 2.7. *The following are equivalent for a right R-module N:*

(1) *N is τ-R_R-flat;*
(2) *Every epimorphism $f : L \to N$ is τ-pure;*
(3) *There exists a τ-pure epimorphism $f : L \to N$ with L τ-R_R-flat;*
(4) *Given an epimorphism $f : L \to C$ and homomorphisms $h : N \to C$, $\alpha : P \to N$ with P τ-finitely presented, there exists a homomorphism $\beta : P \to L$ such that $f\beta = h\alpha$.*

Proof. It follows from Propositions 2.3 and 2.5. □

Remark 2.8. We observe that Proposition 12 and Corollary 13 in Azumaya (1987) are consequences of Corollary 2.7 by letting $\mathcal{T} = \text{Mod-}R$ since τ-R_R-flat modules are exactly finitely projective modules in this case.

Next we consider when τ-M-flat modules coincide with M-flat modules for a given module M.

Proposition 2.9. *Let M and N be right R-modules with $M \in \mathcal{F}$. Then N is τ-M-flat if and only if N is M-flat.*

Proof. We only need to show the sufficiency. Let H be a τ-finitely presented right R-module and $\varphi : H \to N$ any homomorphism. By the proof of Proposition 2.4, there is an exact sequence $0 \to K/K' \to R^n/K' \xrightarrow{g} H \to 0$, where K' is a finitely generated submodule of K such that $K/K' \in \mathcal{T}$. Since R^n/K' is finitely presented and N is M-flat, there are $P \in \mathrm{add}\, M_R$ and homomorphisms $\alpha : R^n/K' \to P$, $\beta : P \to N$ such that $\varphi g = \beta\alpha$. On the other hand, we have $\mathrm{Hom}_R(K/K', P) = 0$ since $K/K' \in \mathcal{T}$ and $M \in \mathcal{F}$. So $K/K' = \ker(g) \leq \ker(\alpha)$, and hence there exists $\gamma : H \to P$ such that $\alpha = \gamma g$. Therefore $\beta\gamma g = \beta\alpha = \varphi g$, which implies that $\beta\gamma = \varphi$ since g is epic, as desired. □

Lemma 2.10. *Let M be a right R-module. Then every direct limit of torsionfree τ-M-flat (resp., τ-Mittag–Leffler) right R-modules is τ-M-flat (resp., τ-Mittag–Leffler). In particular, every direct limit of M-flat right R-modules is M-flat.*

Proof. By Jones (1982b, Proposition 2.5), every $f : N \to \varinjlim X_i$ with N τ-finitely presented and $X_i \in \mathcal{F}$, factors through some X_i. So the first statement follows. The last statement holds by letting $\mathcal{T} = \{0\}$. □

Proposition 2.11. *Let M_R be finitely presented. Then the following are equivalent*:

(1) *Every direct limit of τ-M-flat right R-modules is τ-M-flat*;
(2) *Every M-flat right R-module is τ-M-flat*;
(3) *Every M-flat right R-module is τ-Mittag–Leffler.*

Proof. (1) ⇒ (2) By Angeleri-Hügel (2000, Lemma 2.11), every M-flat right R-module A is a direct limit of some modules in $\operatorname{add} M_R$. Since every module in $\operatorname{add} M_R$ is τ-M-flat, A is τ-M-flat by (1).

(2) ⇒ (1) follows from Lemma 2.10.

(2) ⇔ (3) holds by Proposition 2.3(1). □

The next proposition will be used frequently in the sequel.

Proposition 2.12. *Let M be a right R-module. Then*:

(1) *Every pure submodule of a τ-M-flat right R-module is τ-M-flat whenever M_R is pure-projective.*
(2) *Every pure submodule of a τ-Mittag–Leffler right R-module is τ-Mittag–Leffler.*

Proof. (1) Let N be a pure submodule of a τ-M-flat right R-module L and $j : N \to L$ the inclusion. For any τ-finitely presented right R-module P and any homomorphism $f : P \to N$, since L is τ-M-flat, there are $Q \in \operatorname{add} M_R$ and $g : P \to Q$ and $h : Q \to L$ such that $jf = hg$. Note that there is a pure epimorphism $\phi : H \to L$ with H pure-projective by Warfield (1969, Proposition 1) or Dauns (1994, Proposition 18-2.9), and so we have the pullback diagram of j and ϕ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\lambda} & H & \xrightarrow{\pi\phi} & L/N & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\phi}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & N & \xrightarrow{j} & L & \xrightarrow{\pi} & L/N & \longrightarrow & 0.
\end{array}$$

Since Q is pure-projective and ϕ is pure, there exists $l : Q \to H$ such that $h = \phi l$. Therefore we have $\pi\phi lg = \pi hg = \pi jf = 0$, which implies that $lg(P) \subseteq K$ (here λ is regarded as the inclusion). Since P is finitely generated, so is $lg(P)$. Note that j and ϕ are pure, it is easily seen that λ is pure. On the other hand, since H is pure-projective, by Zimmermann (2002, Proposition 1.4(3)), we get a homomorphism $k : H \to K$ such that $klg(p) = lg(p)$ for all $p \in P$. Put $\beta = \alpha kl$, then $\beta \in \operatorname{Hom}_R(Q, N)$, and for all $p \in P$, $\beta g(p) = j\alpha klg(p) = \phi\lambda klg(p) = \phi\lambda lg(p) = \phi lg(p) = hg(p) = jf(p) = f(p)$, *i.e.*, $f = \beta g$. Thus N is τ-M-flat.

(2) can be proven in a similar way as in the proof of (1). □

Let A, B and M be right R-modules with $S = \mathrm{End}(M_R)$. There is a natural homomorphism

$$\sigma = \sigma_{A,B} : \mathrm{Hom}_R(M, A) \otimes_S \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(B, A)$$

defined via $\sigma(f \otimes g)(b) = f(g(b))$ for $f \in \mathrm{Hom}_R(M, A)$, $g \in \mathrm{Hom}_R(B, M)$, $b \in B$.

It is easy to check that $\sigma_{A,B}$ is an isomorphism if $A \in \mathrm{add}\, M_R$ or $B \in \mathrm{add}\, M_R$.

Proposition 2.13. *Let M and A be right R-modules. Then the following are equivalent*:

(1) *A is τ-M-flat*;
(2) *For any τ-finitely presented right R-module B, $\sigma_{A,B}$ is an epimorphism.*

Proof. (1) $\Rightarrow$ (2) Let B be a τ-finitely presented right R-module and $f \in \mathrm{Hom}_R(B, A)$. By (1), f factors through a right R-module M^n, i.e., there exist $g : B \to M^n$ and $h : M^n \to A$ such that $f = hg$. Let $\pi_i : M^n \to M$ be the ith projection and $\lambda_i : M \to M^n$ the ith injection, $i = 1, 2, \ldots, n$. Put $f_i = h\lambda_i$ and $g_i = \pi_i g$. It is easy to check that $f = \sigma_{A,B}(\sum_{i=1}^n f_i \otimes g_i)$, i.e., $\sigma_{A,B}$ is an epimorphism.

(2) $\Rightarrow$ (1) Let B be a τ-finitely presented right R-module and $f \in \mathrm{Hom}_R(B, A)$. By (2), there are $f_i \in \mathrm{Hom}_R(M, A)$ and $g_i \in \mathrm{Hom}_R(B, M)$, $i = 1, 2, \ldots, n$, such that $f = \sigma_{A,B}(\sum_{i=1}^n f_i \otimes g_i)$. Define $g : B \to M^n$ via $g(b) = (g_1(b), g_2(b), \ldots, g_n(b))$ for $b \in B$ and $h : M^n \to A$ via $h(m_1, m_2, \ldots, m_n) = \sum_{i=1}^n f_i(m_i)$ for $m_i \in M$. Then $f = hg$ and (1) follows. □

Proposition 2.14. *Let M be a projective right R-module and $0 \to A \to B \to C \to 0$ a right R-module exact sequence.*

(1) *If A and C are τ-M-flat, then B is τ-M-flat.*
(2) *If B and C are τ-M-flat, then A is τ-M-flat.*

Proof. (1) Let N be a τ-finitely presented right R-module. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(M, A) \otimes_S \mathrm{Hom}(N, M) \arrow[r] \arrow[d, "\sigma_{A,N}"'] & \mathrm{Hom}(M, B) \otimes_S \mathrm{Hom}(N, M) \arrow[r] \arrow[d, "\sigma_{B,N}"'] & \mathrm{Hom}(M, C) \otimes_S \mathrm{Hom}(N, M) \to 0 \arrow[d, "\sigma_{C,N}"'] \\
\mathrm{Hom}(N, A) \arrow[r] & \mathrm{Hom}(N, B) \arrow[r] & \mathrm{Hom}(N, C),
\end{tikzcd}$$

where $\mathrm{Hom}(X, Y)$ means $\mathrm{Hom}_R(X, Y)$ for two right R-modules X and Y. Since A and C are τ-M-flat, $\sigma_{A,N}$ and $\sigma_{C,N}$ are epic by Proposition 2.13. Thus $\sigma_{B,N}$ is epic by Anderson and Fuller (1974, Lemma 3.14), and so B is τ-M-flat by Proposition 2.13 again.

(2) Since M is projective and C is τ-M-flat, then C is flat. Thus A is a pure submodule of B. It follows that A is τ-M-flat by Proposition 2.12 since B is τ-M-flat. □

It is well known that a ring R is right semihereditary if and only if $\operatorname{add} R_R$ is closed under finitely generated submodules if and only if R_R is coherent and submodules of flat right R-modules are flat. The following proposition shows that this classical result on rings can be extended to modules.

Proposition 2.15. *Let M_R be finitely presented. Then the following are equivalent*:

(1) *M_R is coherent, and submodules of τ-M-flat right R-modules are τ-M-flat*;
(2) *$\operatorname{add} M_R$ is closed under finitely generated submodules.*

Proof. (1) $\Rightarrow$ (2) Let N_R be a finitely generated submodule of H with $H \in \operatorname{add} M_R$. Then N_R is finitely presented since H is coherent by (1). But N_R is τ-M-flat by (1), so $N_R \in \operatorname{add} M_R$ by Remark 2.2(2).

(2) $\Rightarrow$ (1) Since M_R is finitely presented, every finitely generated submodule of M_R is finitely presented by (2). So M_R is coherent.

Now let A be a submodule of a τ-M-flat module B and $i : A \to B$ the inclusion. For any τ-finitely presented right R-module L and any homomorphism $f : L \to A$, there exist $C \in \operatorname{add} M_R$ and homomorphisms $g : L \to C$, $h : C \to B$ such that $if = hg$. Since $\operatorname{im}(g)$ is finitely generated, $\operatorname{im}(g) \in \operatorname{add} M_R$ by (2). Define $\alpha : \operatorname{im}(g) \to A$ by $\alpha(g(x)) = f(x)$ for $x \in A$. It is easy to see that α is well defined and $f = \alpha\beta$, where $\beta : L \to \operatorname{im}(g)$ is defined by $\beta(x) = g(x)$ for $x \in L$. Therefore A is τ-M-flat. □

3. RELATIVE ENDOCOHERENCE

Definition 3.1. Let M_R be a right R-module and $\tau = (\mathscr{T}, \mathscr{F})$ a hereditary torsion theory for Mod-R.

${}_SM$ is called *τ-coherent* if M_R is τ-finitely presented and ${}_S\operatorname{Hom}_R(A, M)$ is a finitely generated left S-module for any τ-finitely presented right R-module A.

Remark 3.2. (1) By Angeleri-Hügel (2003, Lemma 3), ${}_SM$ is τ-coherent if and only if M_R is τ-finitely presented and any τ-finitely presented right R-module has an $\operatorname{add}M_R$-preenvelope. So it follows that ${}_SM$ is τ-coherent if and only if M_R is τ-finitely presented and any τ-finitely presented right R-module has a τ-M-flat-preenvelope.

(2) Let M_R be a finitely presented right R-module. If ${}_SM$ is τ-coherent, then ${}_SM$ is coherent by Angeleri-Hügel (2003, Theorem 2(2)). Moreover, ${}_SM$ is coherent if and only if S is left coherent and ${}_SM$ is finitely presented by Angeleri-Hügel (2003, Theorem 2(2) and Proposition 5(1)).

(3) Let $\mathscr{T} = \{0\}$. Then ${}_SM$ is τ-coherent if and only if ${}_SM$ is coherent and M_R is finitely presented by Angeleri-Hügel (2003, Theorem 2(2)).

(4) Let $\mathscr{T} =$ Mod-R. Then ${}_SM$ is τ-coherent if and only if ${}_SM$ is Π-coherent and M_R is finitely generated if and only if every finitely generated right R-module has an $\operatorname{add} M_R$-preenvelope and M_R is finitely generated by Angeleri-Hügel (2003, Theorem 2(1)).

(5) A ring R is left τ-coherent in sense of Ding and Chen (1993) if and only if ${}_RR$ is τ-coherent by Ding and Chen (1993, Theorem 3.10).

Theorem 3.3. *Let M_R be finitely presented. Then the following are equivalent*:

(1) ${}_SM$ *is* τ*-coherent*;
(2) *The left S-module* ${}_S\mathrm{Hom}_R(A, M)$ *is finitely presented for any* τ*-finitely presented right R-module A*;
(3) *Every right R-module has a* τ*-M-flat-preenvelope*;
(4) *All direct products of copies of* M_R *are* τ*-M-flat*;
(5) *All direct products of* τ*-M-flat right R-modules are* τ*-M-flat*;
(6) ${}_SM$ *is coherent and all direct products of copies of* M_R *are* τ*-Mittag–Leffler*;
(7) ${}_SM$ *is coherent and all direct products of* N_i *with* $N_i \in \mathrm{Add}\, M_R$ *are* τ*-Mittag–Leffler*;
(8) *The right R-module* $\mathrm{Hom}_S(P, M)$ *is* τ*-M-flat for any projective left S-module P.*

Proof. (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1), (5) $\Rightarrow$ (4), and (7) $\Rightarrow$ (6) are trivial.

(1) $\Rightarrow$ (2) Let A be a τ-finitely presented right R-module. Then there is an epimorphism $\alpha : F \to A$ with F a finitely generated free right R-module, which induces a right R-module exact sequence $0 \to \mathrm{Hom}_R(A, M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(F, M)$. By Remark 3.2(2), ${}_SM$ is coherent and S is left coherent. Thus $\mathrm{Hom}_R(F, M)$ is a coherent left S-module, and so $\mathrm{Hom}_R(A, M)$ is finitely presented since it is finitely generated by (1).

(4) $\Rightarrow$ (1) Let A be a τ-finitely presented right R-module. For every index set I, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(M, M^I) \otimes_S \mathrm{Hom}_R(A, M) \arrow[d, "\varphi"'] \arrow[dr, "\sigma_{M^I,A}"] & \\
(\mathrm{Hom}_R(A, M))^I \arrow[r, "\theta"] & \mathrm{Hom}_R(A, M^I)
\end{tikzcd}$$

where θ is an isomorphism, and φ is a canonical homomorphism. By Proposition 2.13, $\sigma_{M^I,A}$ is epic since M^I is τ-M-flat. Thus φ is epic, and hence $\mathrm{Hom}_R(A, M)$ is a finitely generated left S-module by Stenström (1975, Lemma 13.1, p. 41).

(1) $\Rightarrow$ (5) Let $\{M_i\}_{i\in I}$ be a family of τ-M-flat right R-modules and N any τ-finitely presented right R-module. For any homomorphism $f_i : N \to M_i$, since M_i is τ-M-flat, there exist $F_i \in \mathrm{add}\, M_R$ and homomorphisms $g_i : N \to F_i$, $h_i : F_i \to M_i$ such that $f_i = h_i g_i$. Since N has an $\mathrm{add}\, M_R$-preenvelope $f : N \to F$ by (1), there is $k_i : F \to F_i$ such that $g_i = k_i f$. Hence $f_i = (h_i k_i) f$. It follows that the sequence $\mathrm{Hom}_R(F, M_i) \to \mathrm{Hom}_R(N, M_i) \to 0$ is exact. Thus we get the exact sequence

$$(\mathrm{Hom}_R(F, M_i))^I \to (\mathrm{Hom}_R(N, M_i))^I \to 0.$$

Note that $(\mathrm{Hom}_R(F, M_i))^I \cong \mathrm{Hom}_R(F, M_i^I)$ and $(\mathrm{Hom}_R(N, M_i))^I \cong \mathrm{Hom}_R(N, M_i^I)$, thus every homomorphism from N to M_i^I factors through F. So (5) follows.

(5) $\Rightarrow$ (3) Let N be any right R-module. By Enochs and Jenda (2000, Lemma 5.3.12), there is a cardinal number $\aleph_\alpha$ such that for any R-homomorphism $f : N \to L$ with L τ-M-flat, there is a pure submodule Q of L such that $\mathrm{Card}(Q) \leq \aleph_\alpha$

and $f(N) \subseteq Q$. Note that Q is τ-M-flat by Proposition 2.12(1), and so N has a τ-M-flat preenvelope by (5) and Enochs and Jenda (2000, Proposition 6.2.1).

(1) $\Rightarrow$ (6) ${}_SM$ is coherent by Remark 3.2(2). Note that (1) $\Leftrightarrow$ (5) by the preceding proof, thus all products of copies of M_R are τ-M-flat, and hence τ-Mittag–Leffler by Proposition 2.3 since M_R is finitely presented.

(6) $\Rightarrow$ (1) We shall show that any τ-finitely presented right R-module has an addM_R-preenvelope. Let N_R be τ-finitely presented. Then the product map $f: N \to M^J$ induced by all maps in $J = \mathrm{Hom}_R(N, M)$ is a Prod(M)-preenvelope. Thus, by (6), there exist a finitely presented right R-module L and homomorphisms $g: N \to L$, $k: L \to M^J$ such that $f = kg$. Note that L has an addM_R-preenvelope $h: L \to M^n$ since ${}_SM$ is coherent. It is easy to verify that $hg: N \to M^n$ is an addM_R-preenvelope of N.

(6) $\Rightarrow$ (7) Let $\{N_i\}_{i\in I} \subseteq \mathrm{Add}\, M_R$ with I an index set. Then N_i is a direct summand of $M^{(J_i)}$ for some index set J_i. Since $M^{(J_i)}$ is a pure submodule of M^{J_i} by Cheatham and Stone (1981, Lemma 1(1)), N_i is pure in M^{J_i}. Thus $\prod_{i\in I} N_i$ is a pure submodule of $\prod_{i\in I} M^{J_i}$ by Cheatham and Stone (1981, Lemma 1(2)). So the result follows from Proposition 2.12(2).

(4) $\Rightarrow$ (8) For any projective left S-module P, there is a projective left S-module Q and an index set I such that $P \oplus Q \cong S^{(I)}$. So we have

$$\mathrm{Hom}_S(P, M) \oplus \mathrm{Hom}_S(Q, M) \cong \mathrm{Hom}_S(S^{(I)}, M) \cong M_R^I.$$

Thus $\mathrm{Hom}_S(P, M)$ is τ-M-flat by (4) and Remark 2.2(1).

(8) $\Rightarrow$ (4) is obvious by choosing P to be $S^{(I)}$ for any index set I. $\square$

By specializing Theorem 3.3 to the case $\mathcal{T} = \{0\}$, we have the following corollary.

Corollary 3.4. *Let M_R be finitely presented. Then the following are equivalent*:

(1) *${}_SM$ is coherent*;
(2) *The left S-module ${}_S\mathrm{Hom}_R(A, M)$ is finitely presented for any finitely presented right R-module A*;
(3) *Every right R-module has an M-flat-preenvelope*;
(4) *All direct products of copies of M_R are M-flat*;
(5) *All direct products of M-flat right R-modules are M-flat*;
(6) *The right R-module $\mathrm{Hom}_S(P, M)$ is M-flat for any projective left S-module P.*

Remark 3.5. (1) Angeleri-Hügel (2000, Proposition 3.26) asserts that for a finitely presented right R-module M, ${}_SM$ is Π-coherent if and only if S is left coherent, ${}_SM$ is finitely presented and all products of copies of M_R are R-Mittag–Leffler modules. It is an immediate consequence of Theorem 3.3 since ${}_SM$ is coherent if and only if S is left coherent and ${}_SM$ is finitely presented by Remark 3.2(2).

(2) Theorem 3.10 in Ding and Chen (1993) is a special case of Theorem 3.3 where $M_R = R_R$.

Corollary 3.6. *Let M_R be finitely presented and $M_R \in \mathcal{F}$. Then ${}_SM$ is τ-coherent if and only if ${}_SM$ is coherent.*

Proof. It follows from Proposition 2.9, Theorem 3.3, and Corollary 3.4. □

Recall that a right R-module N is called *FP-injective* (Stenström, 1970) if $\mathrm{Ext}^1_R(F, N) = 0$ for all finitely presented right R-modules F.

Proposition 3.7. *Let M_R be finitely generated projective. Consider the following conditions:*

(1) *N^+ is τ-M-flat for every FP-injective left R-module N;*
(2) *N^+ is τ-M-flat for every injective left R-module N;*
(3) *N^{++} is τ-M-flat for every M-flat right R-module N;*
(4) *${}_SM$ is τ-coherent, and every M-flat right R-module is τ-M-flat,*

where $N^+ = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). If $M = R$, then (4) $\Rightarrow$ (1) holds.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Let N be an M-flat right R-module. Then N is flat since M_R is finitely generated projective, and hence N^+ is injective by Rotman (1979, Theorem 3.52). So N^{++} is τ-M-flat by (2).

(3) $\Rightarrow$ (4) Let F be an M-flat right R-module. Then F^{++} is τ-M-flat by (3). Note that F is a pure submodule of F^{++}, so F is τ-M-flat by Proposition 2.12(1). In addition, for any index set I, the pure exact sequence $0 \to (M^+)^{(I)} \to (M^+)^I$ induces a split exact sequence $((M^+)^I)^+ \to ((M^+)^{(I)})^+ \to 0$. Thus $((M^+)^{(I)})^+$ is isomorphic to a direct summand of $((M^+)^I)^+$. Note that $((M^+)^{(I)})^+ \cong (M^{++})^I$ and $((M^+)^I)^+ \cong (M^{(I)})^{++}$. Since $(M^{(I)})^{++}$ is τ-M-flat by (3), so is $(M^{++})^I$. Note that M^I is a pure submodule of $(M^{++})^I$ by Cheatham and Stone (1981, Lemma 1(2)), so M^I is τ-M-flat, and hence ${}_SM$ is τ-coherent by Theorem 3.3.

(4) $\Rightarrow$ (1) For any FP-injective left R-module N, N^+ is flat by Fieldhouse (1972, Theorem 2.2). Thus (1) follows from (4). □

4. RELATIVE FLATNESS OF INJECTIVE MODULES

Proposition 4.1. *Let E_R be an injective right R-module that cogenerates $\tau = (\mathcal{T}, \mathcal{F})$, and M_R a right R-module. Then the following are equivalent:*

(1) *Every τ-finitely presented torsionfree right R-module embeds in L with $L \in \mathrm{add}\, M_R$ (resp., with L finitely presented);*
(2) *All direct products of copies of E_R are τ-M-flat (resp., τ-Mittag–Leffler);*
(3) *Every injective torsionfree right R-module is τ-M-flat (resp., τ-Mittag–Leffler);*
(4) *Every injective envelope of any (τ-finitely presented) torsionfree right R-module is τ-M-flat (resp., τ-Mittag–Leffler).*

Proof. (1) $\Rightarrow$ (2) Suppose that N is a τ-finitely presented right R-module, and $f : N \to E^I$ is a homomorphism with I an index set. Let $i : t(N) \to N$ be the inclusion, and $\pi : N \to N/t(N)$ the canonical map. Note that $fi \in \mathrm{Hom}_R(\mathcal{T}, \mathcal{F}) = 0$ since $E^I \in \mathcal{F}$. Thus $t(N) \subseteq \ker(f)$, and so there exists $g : N/t(N) \to E^I$ such that $g\pi = f$. However $N/t(N)$ is torsionfree and τ-finitely presented by Jones (1982b, Corollary 2.6) since N is τ-finitely presented and $t(N)$ is τ-finitely generated. Thus there is a monomorphism $h : N/t(N) \to L$ with $L \in \mathrm{add}\, M_R$ (resp., with L finitely presented) by (1). By the injectivity of E^I, there exists a homomorphism $j : L \to E^I$ such that $jh = g$. Hence $f = j(h\pi)$, and (2) follows.

(2) $\Rightarrow$ (3) follows from the fact that any direct summand of a τ-M-flat (resp., τ-Mittag–Leffler) module is τ-M-flat (resp., τ-Mittag–Leffler).

(3) $\Rightarrow$ (4) is clear since $\mathcal{F}$ is closed under injective envelopes.

(4) $\Rightarrow$ (1) is obvious since every module embeds in its injective envelope. $\square$

Remark 4.2. We note that Proposition 2.1 in Jones (1982a) can be obtained by Propositions 4.1 and 2.9.

Theorem 4.3. *Let E_R be an injective right R-module that cogenerates $\tau = (\mathcal{T}, \mathcal{F})$, and M_R τ-finitely presented. Consider the following conditions*:

(1) *${}_SM$ is τ-coherent, and every τ-finitely presented torsionfree right R-module embeds in L with $L \in \mathrm{add}\, M_R$*;
(2) *${}_SM$ is τ-coherent, and all direct products of copies of E_R are τ-M-flat*;
(3) *${}_SM$ is τ-coherent, and every injective torsionfree right R-module is τ-M-flat*;
(4) *${}_SM$ is τ-coherent, and every injective envelope of any (τ-finitely presented) torsionfree right R-module is τ-M-flat*;
(5) *Every τ-finitely presented torsionfree right R-module has a τ-M-flat-preenvelope which is a monomorphism*;
(6) *Every τ-finitely presented torsionfree right R-module has an $\mathrm{add}\, M_R$-preenvelope which is a monomorphism.*

Then (1) *through* (4) *are equivalent, and* (4) $\Rightarrow$ (5) $\Rightarrow$ (6). *Moreover* (6) $\Rightarrow$ (1) *in case $M_R \in \mathcal{F}$.*

Proof. The equivalences of (1) through (4) follow from Proposition 4.1.

(4) $\Rightarrow$ (5) Since ${}_SM$ is τ-coherent, every τ-finitely presented torsionfree right R-module N has a τ-M-flat-preenvelope $f : N \to L$. Since the injective envelope $E(N)$ of N is torsionfree, $E(N)$ is τ-M-flat by (4). Therefore f is a monomorphism.

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (1) It is enough to show that ${}_SM$ is τ-coherent, i.e., every τ-finitely presented right R-module has an $\mathrm{add}\, M_R$-preenvelope. Let N_R be τ-finitely presented. Since $N/t(N)$ is torsionfree and τ-finitely presented, $N/t(N)$ has an $\mathrm{add}\, M_R$-preenvelope $f : N/t(N) \to Q$ by (6). We claim that $f\pi$ is an $\mathrm{add}\, M_R$-preenvelope of N, where $\pi : N \to N/t(N)$ is the canonical map. In fact, for any $g : N \to M$, there exists $j : N/t(N) \to M$ such that $j\pi = g$ since $M_R \in \mathcal{F}$ and $t(N) \subseteq \ker(g)$. Thus there is $h : Q \to M$ such that $hf = j$, and so $h(f\pi) = g$. This completes the proof. $\square$

If we omit the "torsionfree" condition in Theorem 4.3, then we have the following theorem.

Theorem 4.4. *Let M_R be τ-finitely presented. Then the following are equivalent*:

(1) ${}_SM$ *is* τ*-coherent, and every* τ*-finitely presented right* R*-module embeds in* L *with* $L \in \operatorname{add} M_R$;
(2) ${}_SM$ *is* τ*-coherent, and every injective right* R*-module is* τ*-*M*-flat*;
(3) ${}_SM$ *is* τ*-coherent, and the injective envelope of each* τ*-finitely presented right* R*-module is* τ*-*M*-flat*;
(4) *Every* τ*-finitely presented right* R*-module has a monic* τ*-*M*-flat-preenvelope*;
(5) *Every* τ*-finitely presented right* R*-module has a monic* $\operatorname{add} M_R$*-preenvelope*;
(6) ${}_SM$ *is* τ*-coherent, and the injective envelope of every simple right* R*-module is* τ*-*M*-flat*;
(7) ${}_SM$ *is* τ*-coherent, and the injective envelope of every finitely cogenerated right* R*-module is* τ*-*M*-flat*;
(8) ${}_SM$ *is* τ*-coherent, and each* τ*-finitely presented right* R*-module is cogenerated by* M_R;
(9) ${}_SM$ *is* τ*-coherent, and every right* R*-module is a submodule of some* τ*-*M*-flat right* R*-module.*

Proof. The proofs of the equivalences of (1) through (5) are similar to those of Theorem 4.3.

(2) $\Rightarrow$ (6) is trivial.

(6) $\Leftrightarrow$ (7) By Kasch (1982, Theorem 9.4.3), N_R is finitely cogenerated if and only if $E(N) = E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_n)$, where $S_1, S_2, \ldots, S_n$ are simple right R-modules. So (6) $\Leftrightarrow$ (7) follows from Remark 2.2(1).

(6) $\Rightarrow$ (8) Let N_R be a τ-finitely presented right R-module. It is enough to show that for any $0 \neq m \in N$, there exists $f : N \to M$ such that $f(m) \neq 0$. In fact, there is a maximal submodule K of mR, and so mR/K is simple. By the injectivity of $E(mR/K)$, there exists $j : N \to mR/K$ such that $j\imath = i\pi$, where $\imath : mR \to N$ and $i : mR/K \to E(mR/K)$ are the inclusions, and $\pi : mR \to mR/K$ is the natural map. Note that $j(m) = j\imath(m) = i\pi(m) \neq 0$. On the other hand, since $E(mR/K)$ is τ-M-flat by (6), there exist $n \in \mathbb{N}$, $g : N \to M^n$ and $h : M^n \to E(mR/K)$ such that $j = hg$. Therefore $g(m) = (x_1, x_2, \ldots, x_n) \neq 0$. Let $x_i \neq 0$, and $p_i : M^n \to M$ be the ith projection. Then $p_i g(m) \neq 0$.

(8) $\Rightarrow$ (1) By the proof of Theorem 3.3, any direct product of M_R is τ-M-flat, so every τ-finitely presented right R-module embeds in a τ-M-flat right R-module, and hence embeds in L with $L \in \operatorname{add} M_R$.

(2) $\Rightarrow$ (9) is clear since every right R-module is a submodule of its injective envelope.

(9) $\Rightarrow$ (2) Since every injective right R-module A is a direct summand of some τ-M-flat right R-module B by (9), A is τ-M-flat by Remark 2.2(1). □

Remark 4.5. (1) Recall that a ring R is called *right IF* (Colby, 1975) if every injctive right R-module is flat. R is called *left FC* (Damiano, 1979) if ${}_RR$ is FP-injective and coherent. It is well known that R is left FC if and only if R is left coherent and right IF (see Jain, 1973, Theorem 3.10). Specializing Theorem 4.4 to the case that $M_R = R_R$ and $\mathscr{T} = 0$ gives various characterizations of a left FC ring.

(2) If M_R is finitely presented and FP-injective, and every injective right R-module is τ-M-flat, then the equivalent conditions in Theorem 4.4 hold. In fact, for any index set I, $E(M_R^I)$ is τ-M-flat, and M_R^I is a pure submodule of $E(M_R^I)$. Thus M_R^I is τ-M-flat by Proposition 2.12, and so ${}_SM$ is τ-coherent by Theorem 3.3. In particular, a right FP-injective right IF ring is left coherent.

The following theorem extends Theorem 12 in Asensio Mayor and Martinez Hernandez (1990).

Theorem 4.6. *Let M_R be τ-finitely presented. Then the following are equivalent*:

(1) *M_R is injective, and every injective right R-module is τ-M-flat*;
(2) *For every τ-finitely presented right R-module, its τ-M-flat-envelope exists and coincides with its injective envelope*;
(3) *M_R is injective, and every τ-finitely presented right R-module has a monic τ-M-flat-(pre)envelope*;
(4) *M_R is injective, and the injective envelope of each τ-finitely presented right R-module is τ-M-flat* (*in* $\operatorname{add} M_R$).

Proof. (1) $\Rightarrow$ (2) Let N_R be τ-finitely presented. By (1), $E(N)$ is τ-M-flat. We claim that the inclusion $i : N \to E(N)$ is a τ-M-flat-envelope of N. In fact, for any τ-M-flat right R-module F and any homomorphism $f : N \to F$, f factors through a module L in $\operatorname{add} M_R$, i.e., there exist $g : N \to L$ and $h : L \to F$ such that $f = hg$. Since M_R is injective, L is injective. Therefore there is $j : E(N) \to L$ such that $g = ji$. Thus $f = h(ji) = (hj)i$, which means that i is a τ-M-flat-preenvelope, and hence i is τ-M-flat envelope of N since i is an injective envelope.

(2) $\Rightarrow$ (3) M_R is injective since $M_R \cong E(M_R)$. The rest is clear.

(3) $\Rightarrow$ (4) Let N_R be τ-finitely presented. By (3), N_R has a monic τ-M-flat-preenvelope $\alpha : N \to F$. Since F is τ-M-flat, α factors through a module L in $\operatorname{add} M_R$, i.e., there exist $g : N \to L$ and $h : L \to F$ such that $\alpha = hg$. Note that g is monic and L is injective. Thus $E(N)$ is isomorphic to a direct summand of L, and hence $E(N) \in \operatorname{add} M_R$.

(4) $\Rightarrow$ (1) Let Q_R be any injective right R-module. For any τ-finitely presented right R-module N_R and any homomorphism $f : N \to Q$, there exists $g : E(N) \to Q$ such that $f = gi$, where $i : N \to E(N)$ is the inclusion. Since $E(N)$ is τ-M-flat by (4), Q is τ-M-flat. □

It was shown in Enochs and Jenda (1991, Theorem 3.1) that a ring R is left semihereditary if and only if every finitely presented right R-module has a projective preenvelope which is an epimorphism. This result is a particular case of the following theorem where $M_R = R_R$ and $\mathscr{T} = 0$.

Theorem 4.7. *Let M_R be τ-finitely presented. Then the following are equivalent*:

(1) *${}_SM$ is τ-coherent, and submodules of τ-M-flat right R-modules are τ-M-flat*;
(2) *Every τ-finitely presented right R-module has a τ-M-flat-(pre)envelope which is an epimorphism*;
(3) *Every τ-finitely presented right R-module has an addM_R-(pre)envelope which is an epimorphism.*

Proof. (1) $\Rightarrow$ (2) Let N_R be τ-finitely presented. Then N has a τ-M-flat-preenvelope $f : N \to F$ since ${}_SM$ is τ-coherent. However im(f) is τ-M-flat by (1), it follows that $f : N \to \mathrm{im}(f)$ is a τ-M-flat-(pre)envelope which is an epimorphism.

(2) $\Rightarrow$ (3) Let N_R be τ-finitely presented. Then N has an epic τ-M-flat-(pre)envelope $f : N \to F$. By definition, f factors through a module L in add$\,M_R$, i.e., there exist $g : N \to L$ and $h : L \to F$ such that $f = hg$. On the other hand, since L is τ-M-flat, there exists $\alpha : F \to L$ such that $g = \alpha f$. Thus $f = h\alpha f$, and so $h\alpha = 1$ since f is epic. Hence $F \in \mathrm{add}\,M_R$ and (3) follows.

(3) $\Rightarrow$ (1) ${}_SM$ is clearly τ-coherent by definition. Now suppose that N is a submodule of L with L τ-M-flat, and $\iota : N \to L$ is the inclusion. For any τ-finitely presented right R-module K and $\alpha \in \mathrm{Hom}_R(K, N)$, $\iota\alpha$ factors through a module H in add$\,M_R$, i.e., there exist $g : K \to H$ and $h : H \to L$ such that $\iota\alpha = hg$. By (3), K has an epic add$\,M_R$-preenvelope $\beta : K \to Q$ with $Q \in \mathrm{add}M_R$. Thus there exists $\gamma : Q \to H$ such that $g = \gamma\beta$, which implies that $\ker(\beta) \subseteq \ker(\alpha)$ and so there exists $\varphi : Q \to N$ such that $\alpha = \varphi\beta$, i.e., N is τ-M-flat. □

5. ANNIHILATORS AND ENDOCOHERENCE

In this section, we shall give characterizations of (Π-)coherent modules in terms of annihilators.

In what follows, for a right R-module M with $S = \mathrm{End}(M_R)$ and a positive integer n, we write $M^{n\times n}$ for the set of all $n \times n$ matrices whose entries are elements of M. We regard each element of M^n as a vector with entries in M, and regard it as a row vector or column vector according to the context. If R is a ring, then $R^{n\times n} = M_n(R)$, the ring of $n \times n$-matrices over R. It is clear that $M^{n\times n}$ is a left $M_n(S)$-right $M_n(R)$-bimodule. By Anderson and Fuller (1974, Proposition 13.2), $M_n(S) \cong \mathrm{End}(M_R^n)$.

A right R-module N is called *finitely M-generated* (resp., *finitely M-presented*) if there is an exact sequence $M^n \to N \to 0$ (resp., $M^m \to M^n \to N \to 0$) with $m, n \in \mathbb{N}$.

Let M_R be a right R-module and $U \in M^{n\times m}$. Using the idea of Azumaya (1995), we define

$$U(S) = \{s \in S : (s, s_2, \dots, s_n)U = 0 \text{ for some } s_2, \dots, s_n \in S\}.$$

Then $U(S)$ is a left ideal of S.

Theorem 5.1. *Let M_R and ${}_SM$ be finitely presented. Then the following are equivalent*:

(1) ${}_SM$ *is coherent*;
(2) $U(S)$ *is finitely generated for all* $U \in M^n$ *and* $n \geq 1$;
(3) $U(S)$ *is finitely generated for all* $U \in M^{n\times m}$ *and* $n, m \geq 1$;
(4) $U(S)$ *is finitely generated for all* $U \in M^{n\times n}$ *and* $n \geq 1$;
(5) *The left annihilator* $\mathrm{ann}_{M_n(S)}(X)$ *is a finitely generated left ideal of* $M_n(S)$ *for any* $n \geq 1$ *and any finitely generated submodule* X *of the right* R*-module* M^n;
(6) *The left annihilator* $\mathrm{ann}_{M_n(S)}(Y)$ *is a finitely generated left ideal of* $M_n(S)$ *for any* $n \geq 1$ *and every element* Y *of the right* R*-module* M^n;
(7) *The left annihilator* $\mathrm{ann}_{M_n(S)}(L)$ *is a finitely generated left ideal of* $M_n(S)$ *for any* $n \geq 1$ *and any finitely generated submodule* L *of the right* $M_n(R)$*-module* $M^{n\times n}$;
(8) *The left annihilator* $\mathrm{ann}_{M_n(S)}(N)$ *is a finitely generated left ideal of* $M_n(S)$ *for any* $n \geq 1$ *and every element* N *of the right* $M_n(R)$*-module* $M^{n\times n}$;
(9) *Every finitely* M*-presented right* R*-module has an* $addM_R$*-preenvelope.*

Proof. (1) $\Rightarrow$ (9), (3) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6), and (7) $\Rightarrow$ (8) are obvious.

(1) $\Rightarrow$ (2) Let $U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in M^n$, where $u_i \in M, i = 1, 2, \ldots, n$. Put $I_1 = Su_1 + Su_2 + \cdots + Su_n$ and $I_2 = Su_2 + \cdots + Su_n$. Then $I_1 = Su_1 + I_2$. Define $\alpha : S \to I_1/I_2$ via $\alpha(s) = su_1 + I_2$. Obviously, α is epic and $\ker(\alpha) = U(S)$. Thus $S/U(S) \cong I_1/I_2$. Since ${}_SM$ is coherent, I_1/I_2 is finitely presented. So $U(S)$ is finitely generated.

(2) $\Rightarrow$ (1) Let $I_1 = Su_1 + Su_2 + \cdots + Su_n$ be a finitely generated submodule of ${}_SM$. Let $I_2 = Su_2 + \cdots + Su_n, I_3 = Su_3 + \cdots + Su_n, \ldots, I_n = Su_n$. By the proof of (1) $\Rightarrow$ (2), we have $I_n, I_{n-1}/I_n, I_{n-2}/I_{n-1}, \ldots, I_1/I_2$ are finitely presented. Therefore, I_1 is finitely presented, and so (1) follows.

(1) $\Rightarrow$ (3) Since ${}_SM$ is coherent, ${}_SM^m$ is coherent for any $m \geq 1$. Thus $U(S)$ is finitely generated for all $U \in M^{n\times m}$ with $n \geq 1$ by the equivalence of (1) and (2).

(4) $\Rightarrow$ (1) is easy to verify.

(8) $\Rightarrow$ (4) Let $U \in M^{n\times n}$. Then $\mathrm{ann}_{M_n(S)}(U)$ is finitely generated by (8). Suppose that $\mathrm{ann}_{M_n(S)}(U) = M_n(S)A_1 + M_n(S)A_2 + \cdots + M_n(S)A_t$ with $A_k = (a_{ij}^{(k)}) \in \mathrm{ann}_{M_n(S)}(U), k = 1, 2, \ldots, t$. Since $A_kU = 0$, $a_{j1}^{(k)} \in U(S), k = 1, 2, \ldots, t, j = 1, 2, \ldots, n$.

For any $x \in U(S)$, then $(x, x_2, \ldots, x_n)U = 0$ for some $x_2, \ldots, x_n \in S$. Let

$$B = \begin{pmatrix} x & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $BU = 0$, and so $B \in \mathrm{ann}_{M_n(S)}(U)$. Thus there exists $C_k = (c_{ij}^{(k)}) \in M_n(S)$, $k = 1, 2, \ldots, t$, such that $B = C_1A_1 + C_2A_2 + \cdots + C_tA_t$, which shows that

$$x = \sum_{k=1}^{t}\sum_{j=1}^{n} c_{1j}^{k} a_{j1}^{(k)}.$$

Therefore $U(S)$ is finitely generated.

(6) $\Rightarrow$ (2) follows from the proof of (8) $\Rightarrow$ (4)

(7) $\Rightarrow$ (5) Let X be a finitely generated submodule of the right R-module M^n. It is easy to see that X^n is a finitely generated submodule of the right $M_n(R)$-module $M^{n\times n}$ and $\mathrm{ann}_{M_n(S)}(X) = \mathrm{ann}_{M_n(S)}(X^n)$. So $\mathrm{ann}_{M_n(S)}(X)$ is finitely generated by (7).

(5) $\Rightarrow$ (7) Let L be a finitely generated submodule of the right $M_n(R)$-module $M^{n\times n}$ and $K = \{\alpha : (\alpha, \alpha_2, \ldots, \alpha_n) \in L\}$. Then $L \cong K^n$ as right R-modules and K is a finitely generated submodule of the right R-module M^n. Therefore, $\mathrm{ann}_{M_n(S)}(K)$ is a finitely generated left ideal of $M_n(S)$ by (5), and so is $\mathrm{ann}_{M_n(S)}(L)$ (for $L \cong K^n$).

(9) $\Rightarrow$ (1) By Angeleri-Hügel (2003, Proposition 5(1)), S is left coherent. So ${}_SM$ is coherent since ${}_SM$ is finitely presented.

In the rest of the proof, let $p_k : M^m \to M$ (resp., $\lambda_k : M \to M^m$) be the kth canonical projection (resp., injection) and $\lambda : M \to M^n$ (resp., $p : M^n \to M$) the first canonical injection (resp., projection).

(5) $\Rightarrow$ (9) Let N be a finitely M-presented right R-module. Then there is a right R-module exact sequence

$$0 \to K \to M^n \xrightarrow{g} N \to 0,$$

where K is finitely M-generated and hence is finitely generated. Thus $\mathrm{ann}_{M_n(S)}(K)$ is a finitely generated left ideal of $M_n(S)$ by (5). Suppose that $f_1, f_2, \ldots, f_m$ is a generating set of $\mathrm{ann}_{M_n(S)}(K)$. Then K is contained in the kernel of the product map $f : M^n \to M^{nm}$ induced by the f_i (we set $\pi_i f = f_i$, where $\pi_i : M^{nm} \to M^n$ is the ith canonical projection, $i = 1, 2, \ldots, m$), and hence there is a map $h : N \to M^{nm}$ such that $f = hg$. We claim that h is an $\mathrm{add}\, M_R$-preenvelope. In fact, for any homomorphism $\psi : N \to M$, it is obvious that $\lambda\psi g \in \mathrm{ann}_{M_n(S)}(K)$. Let $\lambda\psi g = \sum_{i=1}^m t_i f_i$ for some $t_i \in M_n(S)$, $i = 1, 2, \ldots, m$. Then $\psi g = p\sum_{i=1}^m t_i f_i = p\sum_{i=1}^m t_i \pi_i f = p\sum_{i=1}^m t_i \pi_i hg$. Since g is epic, $\psi = (p\sum_{i=1}^m t_i \pi_i)h$. It follows that h is an $\mathrm{add}M_R$-preenvelope.

(1) $\Rightarrow$ (5) Let X be a finitely generated submodule of the right R-module M^n. Consider the right R-module exact sequence

$$0 \to X \xrightarrow{i} M^n \xrightarrow{\pi} M^n/X \to 0,$$

where i is the inclusion and π is the natural map. Since M^n is finitely presented and X is finitely generated, M^n/X is finitely presented. Thus M^n/X has an $\mathrm{add}\, M_R$-preenvelope $\alpha : M^n/X \to M^m$ by (1). Put $\beta_k = \lambda p_k \alpha\pi \in M_n(S)$. It is clear that $\beta_k \in \mathrm{ann}_{M_n(S)}(X)$, $k = 1, 2, \ldots, m$.

On the other hand, for any $f \in \mathrm{ann}_{M_n(S)}(X)$, there is a right R-homomorphism $\gamma : M^n/X \to M^n$ such that $\gamma\pi = f$. Since α is an $\mathrm{add}\, M_R$-preenvelope, there exists $\phi : M^m \to M^n$ such that $\phi\alpha = \gamma$. Thus $f = \phi\alpha\pi = \sum_{k=1}^m \phi\lambda_k p\lambda p_k \alpha\pi = \sum_{k=1}^m \phi\lambda_k p\beta_k \in \sum_{k=1}^m M_n(S)\beta_k$, which implies that $\mathrm{ann}_{M_n(S)}(X) = \sum_{k=1}^m M_n(S)\beta_k$, as desired. $\square$

Corollary 5.2. *Let M_R and ${}_SM$ be finitely presented. Then the following are equivalent*:

(1) ${}_S M$ *is* Π*-coherent*;
(2) $U(S)$ *is finitely generated for any* $U \in (M^J)^n$, *any* $n \geq 1$ *and any index set* J;
(3) *Every finitely* M*-generated right* R*-module has an* $addM_R$*-preenvelope*;
(4) *The left annihilator* $ann_{M_n(S)}(X)$ *is a finitely generated left ideal of* $M_n(S)$ *for any submodule* X *of the right* R*-module* M^n *and any* $n \geq 1$;
(5) *The left annihilator* $ann_{M_n(S)}(X)$ *is a finitely generated left ideal of* $M_n(S)$ *for any submodule* X *of the right* R*-module* $M^{n\times n}$ *and any* $n \geq 1$.

Proof. (1) $\Leftrightarrow$ (2) holds by the definition of Π-coherent modules and the proof of (1) $\Leftrightarrow$ (2) in Theorem 5.1. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from Angeleri-Hügel (2000, Proposition 3.16). The proof of (4) $\Leftrightarrow$ (5) is similar to that of (5) $\Leftrightarrow$ (7) in Theorem 5.1. □

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