Chapter 7

The Cotorsion Dimension of Modules and Rings

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Abstract
In this paper, we introduce a dimension, called the cotorsion dimension, for modules and rings. The relations between the cotorsion dimension and other dimensions are discussed. Various results are developed, some extending known results.

Keywords: Cotorsion dimension; Cotorsion envelope; Flat cover; Perfect ring.

7.1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary.

Let $R$ be a ring and $M$ a right $R$-module. Recall that $M$ is called cotorsion [7] if $\text{Ext}^1_R(F, M) = 0$ for any flat right $R$-module $F$. The class of cotorsion modules contains all pure-injective (hence, injective) modules. A homomorphism $\phi : M \rightarrow C$ with $C$ cotorsion is called a cotorsion preenvelope of $M$ [6, 27] if for any homomorphism $f : M \rightarrow C'$ where $C'$ is cotorsion, there is a homomorphism $g : C \rightarrow C'$ such that $g\phi = f$. Moreover if the only such $g$ are automorphisms of $C$ when $C' = C$ and $f = \phi$, the cotorsion preenvelope $\phi$ is called a cotorsion envelope of $M$. A homomorphism $\phi : F \rightarrow M$ with $F$ flat is called a flat cover of $M$ if for any homomorphism $f : F' \rightarrow M$ where $F'$ is flat, there is a homomorphism $g : F' \rightarrow F$ such that $g\phi = f$, moreover when $F = F'$ and $f = \phi$, the only such $g$ are automorphisms of $F$. It is now well known that all $R$-modules have flat covers for any ring $R$ [2], and it has been proven that every $R$-module has a cotorsion envelope if and only if every $R$-module has a flat cover [27]. Thus all $R$-modules have cotorsion envelopes for arbitrary ring $R$. Note that cotorsion envelopes or flat covers are unique up to isomorphism.

In what follows, for an $R$-module $M$, $E(M)$, $C(M)$ and $F(M)$ stand for the injective envelope, cotorsion envelope and flat cover respectively. We write $M_R$ to indicate a right $R$-module. The projective (resp. injective) dimension of $M$ is denoted by $\text{pd}(M)$ (resp. $\text{id}(M)$). We denote by $r\text{D}(R)$ (resp. $w\text{D}(R)$) the right (resp. the weak) global dimension of a ring $R$. General background material can be found in [1], [9], [23], [27].

We are going to define a dimension, called the cotorsion dimension, for modules and rings. It measures how far away a module is from being cotorsion, and how far away a ring is from being perfect.

Let $R$ be a ring. For any right $R$-module $M$, the cotorsion dimension $\text{cd}(M)$ of $M$ is defined to be the smallest integer $n \geq 0$ such that $\text{Ext}^{n+1}_R(F, M) = 0$ for any flat right $R$-module $F$. If there is no such $n$, set $\text{cd}(M) = \infty$. The right global cotorsion dimension $r\text{cot.D}(R)$ of $R$ is defined as the supremum of the cotorsion dimensions of right $R$-modules. The aim of this paper is to investigate...
these dimensions.

In Section 2, we give the definition and show some of the general results. Let $R$ be a ring. First we prove that $\text{r.cot.D}(R) = \sup\{\text{pd}(F) : F \text{ is a flat right } R\text{-module}\} = \sup\{\text{cd}(F) : F \text{ is a flat right } R\text{-module}\}$ (part of Theorem 7.2.5), which gives rise to some characterizations of right perfect rings (Corollary 7.2.7) and extends [27, Proposition 3.3.1]. Then it is shown that $\text{r.cot.D}(R) \leq 1$ if and only if every quotient module of any cotorsion (or injective) right $R$-module is cotorsion if and only if every pure submodule of any projective right $R$-module is projective (Theorem 7.2.8). This removes the unnecessary hypothesis that $R$ is a commutative domain from [15, Theorem 7.2.7]. For a ring $R$ such that the cotorsion envelope of any projective right $R$-module is projective, we have that $\text{r.cot.D}(R) \leq 1$ if and only if the projectivity of $C(M)$ implies the projectivity of $M$ for any right $R$-module $M$ (Theorem 7.2.10). The relation $\text{rD}(R) \leq \text{wD}(R) + \text{r.cot.D}(R)$ is proven to be true for any ring $R$ (Theorem 7.2.11). Finally, for a left coherent ring $R$, it is shown that $R$ is right perfect if and only if every flat cotorsion right $R$-module is projective (Proposition 7.2.16).

Section 3 is devoted to the cotorsion dimension under change of rings. We first get that if $\varphi : R \to S$ is a surjective ring homomorphism and $S_R$ a flat right $R$-module, then $\text{r.cot.D}(S) \leq \text{r.cot.D}(R)$ (Corollary 7.3.2). Then we prove that if $S$ is an almost excellent extension of $R$, then $\text{r.cot.D}(S) \leq \text{r.cot.D}(R)$, and the equality holds in case $\text{r.cot.D}(R) < \infty$ (Corollary 7.3.4 and Theorem 7.3.5).

In Section 4, some applications in commutative rings are discussed. We start by showing that for a ring $R$ with $\text{cot.D}(R) \leq 1$, $\text{Ext}_R^1(F, M)$ is cotorsion for any flat $R$-module $F$ and any $R$-module $M$ (Proposition 7.4.3), which is motivated by [11, Problem 48, p.462]. Then, for a surjective ring homomorphism $\varphi : R \to S$ with $K = \text{Ker}(\varphi)$ and $S_R$ projective, it is shown that, for any $R$-module $M$, either $\text{cd}(M_R) \leq \sup\{\text{pd}(R/I)_R : I \subseteq K\}$, or $\text{cd}(M_R) = \text{cd}(\text{Hom}_R(S, M))$, where $\text{Hom}_R(S, M)$ may be regarded as an $R$-module or $S$-module (Theorem 7.4.5). As a corollary, we get that a ring $R$ is perfect if and only if there is a quotient ring $S = R/K$ of $R$ such that $S$ is a perfect ring and $R/I$ is a projective $R$-module for any $I \subseteq K$ (Corollary 7.4.7). In the last part of this section, we prove that a ring $R$ is von Neumann regular if and only if $\text{Hom}_R(A, B)$ is injective (or flat) for all cotorsion $R$-modules $A$ and $B$ (Proposition 7.4.10).

### 7.2 General results

We start with the following

**Proposition 7.2.1** For any right $R$-module $M$ and integer $n \geq 0$, the following are equivalent:

1. $\text{cd}(M) \leq n$.
2. $\text{Ext}_R^{n+1}(F, M) = 0$ for any flat right $R$-module $F$.
3. $\text{Ext}_R^{n+j}(F, M) = 0$ for any flat right $R$-module $F$ and $j \geq 1$.
4. If the sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$ is exact with $C^0, C^1, \ldots, C^{n-1}$ cotorsion, then $C^n$ is also cotorsion.
5. $\text{cd}(F(M)) \leq n$.

**Proof** The proof of (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) is standard homological algebra fare.

(1) $\iff$ (5). Let $K$ be the kernel of the flat cover $F(M) \to M$, then we have the exact sequence $0 \to K \to F(M) \to M \to 0$ with $K$ cotorsion. Note that $\text{Ext}_R^n(F, K) = 0$ for all $n \geq 1$ and flat modules $F$ by the proof of [27, Proposition 3.1.2], so the result follows. \qed
Corollary 7.2.2 Let $M$ be any right $R$-module. Then the following are identical:

1. $\text{cd}(M)$.

2. $\inf \{ k : \text{there exists an exact sequence } 0 \to M \to C^0 \to C^1 \to \cdots \to C^k \to 0, \text{ where each } C^i \text{ is a cotorsion right } R\text{-module}, i = 0, 1, \ldots, k \}.$

3. The integer $n$ such that $M$ admits a minimal cotorsion resolution, i.e., an exact sequence
   
   \[ 0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0, \text{ where each } C^i \text{ is cotorsion, } \text{L}_i = \text{Coker}(C^{i-2} \to C^{i-1}) \to C^i \text{ is a cotorsion envelope of } L^i, C^i \neq 0, i = 0, 1, \ldots, n, C^{-2} = 0, C^{-1} = M. \]

Proof (1) = (2) is straightforward.

(1) $\leq$ (3) is trivial. Assume (1) $< (3) = n$. Let (1) = $k < \infty$. By Proposition 7.2.1, $L^k$ is a cotorsion right $R$-module. Consider the exact sequence $0 \to L^k \to C^k \to L^{k+1} \to 0$, since $L^k \to C^k$ is a cotorsion envelope of $L^k$, it follows that $L^{k+1} = 0$, and hence $C^{k+1} = 0$, a contradiction. Therefore (1) $= (3).$

Proposition 7.2.3 Let $R$ be a ring, $0 \to A \to B \to C \to 0$ an exact sequence of right $R$-modules.
If two of $\text{cd}(A), \text{cd}(B), \text{cd}(C)$ are finite, so is the third. Moreover

1. $\text{cd}(B) \leq \sup \{ \text{cd}(A), \text{cd}(C) \}.$

2. $\text{cd}(A) \leq \sup \{ \text{cd}(B), \text{cd}(C) + 1 \}.$

3. $\text{cd}(C) \leq \sup \{ \text{cd}(B), \text{cd}(A) - 1 \}.$

Proof It is a routine exercise.

The next corollary is an immediate consequence of Proposition 7.2.3.

Corollary 7.2.4 Let $R$ be a ring, $0 \to A \to B \to C \to 0$ an exact sequence of right $R$-modules.
If $B$ is cotorsion, $\text{cd}(A) > 0$, then $\text{cd}(A) = \text{cd}(C) + 1.$

Theorem 7.2.5 Let $R$ be a ring. Then

1. $\text{r.cot}.D(R) = \sup \{ \text{pd}(F) : F \text{ is a flat right } R\text{-module} \}$

2. $\leq \sup \{ \text{pd}(M) : \text{pd}(M) < \infty \}$

3. $\leq \sup \{ \text{id}(P) : P \text{ is a projective right } R\text{-module} \}$

4. $\leq rD(R).$

All equalities hold if $R$ is a von Neumann regular ring.

3. If $r\text{cot}.D(R) < \infty$, then

   \[ r\text{cot}.D(R) = \sup \{ \text{pd}(F) : F \text{ is a flat cotorsion right } R\text{-module} \} \]

   \[ = \sup \{ \text{pd}(C(F)) : F \text{ is a flat right } R\text{-module} \} \]

   \[ = \sup \{ \text{pd}(F(M)) : M \text{ is a cotorsion right } R\text{-module} \} \]

   \[ = \sup \{ \text{cd}(P) : P \text{ is a projective right } R\text{-module} \}. \]
Let $R$ be a ring, then the following are equivalent for an integer $n \geq 0$.

Remark Note that pure injective modules are cotorsion, so [14, Proposition 1.1(a)] (that asserts $\sup[pd(\mathcal{F})]: F$ is a flat right $R$-module] $\leq$ right pure global dimension of the ring $R$) is an immediate consequence of Theorem 7.2.5 (1).
7.2 General results

1. \( \text{r.cot.D}(R) \leq n. \)
2. All flat right \( R \)-modules are of projective dimension \( \leq n. \)
3. All flat right \( R \)-modules are of cotorsion dimension \( \leq n. \)
4. \( \text{r.cot.D}(R) < \infty, \) and all flat cotorsion right \( R \)-modules are of projective dimension \( \leq n. \)
5. \( \text{r.cot.D}(R) < \infty, \) and all projective right \( R \)-modules are of cotorsion dimension \( \leq n. \)
6. \( \text{Ext}^{n+1}_R(M, N) = 0 \) for all flat right \( R \)-modules \( M \) and \( N. \)
7. \( \text{Ext}^{n+j}_R(M, N) = 0 \) for all flat right \( R \)-modules \( M, N \) and \( j \geq 1. \)

Remark The equivalences of (2), (6) and (7) of Corollary 7.2.6 appeared in [9, Theorem 8.4.12] under the hypothesis that \( R \) is left coherent.

By [12, Corollary 10], if every projective right \( R \)-module is cotorsion, then \( R \) is right perfect. So we obtain some characterizations of right perfect rings by specializing Corollary 7.2.6 to the case \( n = 0. \) The equivalences of (2) through (4) in the following corollary are due to Xu [27, Proposition 3.3.1].

Corollary 7.2.7 The following are equivalent for any ring \( R: \)

1. \( \text{r.cot.D}(R) = 0. \)
2. Every right \( R \)-module is cotorsion.
3. \( R \) is right perfect.
4. Every flat right \( R \)-module is cotorsion.
5. Every projective right \( R \)-module is cotorsion.
6. \( \text{r.cot.D}(R) < \infty, \) and every flat cotorsion right \( R \)-module is projective.
7. \( \text{Ext}^1_R(M, N) = 0 \) for all flat right \( R \)-modules \( M \) and \( N. \)

Remark By Corollary 7.2.7, \( r.D(R) \) measures how far away a ring is from being right perfect. It is well known that right perfect rings need not be left perfect (see [1, p.322]), so \( r.D(R) \neq l.D(R) \) in general.

Let \( R \) be a ring. It is well known that \( r.D(R) \leq 1 \) if and only if every quotient module of any injective right \( R \)-module is injective. Here we prove that \( r.D(R) \leq 1 \) if and only if every quotient module of any cotorsion right \( R \)-module is cotorsion as shown in the following theorem.

Theorem 7.2.8 Let \( R \) be a ring, then the following are equivalent:

1. \( r.D(R) \leq 1. \)
2. All flat right \( R \)-modules are of projective dimension \( \leq 1. \)
3. All flat right \( R \)-modules are of cotorsion dimension \( \leq 1. \)
4. Every quotient module of any injective right \( R \)-module is cotorsion.
5. Every quotient module of any cotorsion right \( R \)-module is cotorsion.
6. Every pure submodule module of any projective right $R$-module is projective.

**Proof** (1) $\Rightarrow$ (4). Let $E$ be any injective right $R$-module and $K$ a submodule of $E$. The exactness of the sequence $0 \rightarrow K \rightarrow E \rightarrow E/K \rightarrow 0$ induces the exact sequence

$$0 = \text{Ext}^1_R(F, E) \rightarrow \text{Ext}_R^1(F, E/K) \rightarrow \text{Ext}^2_R(F, K),$$

where $F$ is a flat right $R$-module. Note that $\text{Ext}^2_R(F, K) = 0$ by (1) and Proposition 7.2.1, so $\text{Ext}^1_R(F, E/K) = 0$, as required.

(4) $\Rightarrow$ (1). Let $M$ be any injective right $R$-module. Then there exists an exact sequence $0 \rightarrow M \rightarrow E \rightarrow E/M \rightarrow 0$ with $E$ injective. Thus $\text{cd}(M) \leq 1$ since $E/M$ is cotorsion, and hence $r.cot.D(R) \leq 1$.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Corollary 7.2.6.

(2) $\Rightarrow$ (6). Let $M$ be a projective right $R$-module and $N$ a pure submodule of $M$. Then $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ is exact. Note that $M/N$ is flat and hence $\text{pd}(M/N) \leq 1$ by (2). Thus $N$ is projective.

(6) $\Rightarrow$ (2). Let $M$ be any flat right $R$-module. There exists an exact sequence $0 \rightarrow N \rightarrow P \rightarrow M \rightarrow 0$ with $P$ projective. Note that $N$ is a pure submodule of $P$, so $N$ is projective. It follows that $\text{pd}(M) \leq 1$.

(5) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (5). Let $M$ be any cotorsion right $R$-module and $N$ any submodule of $M$. There exists an exact sequence $0 \rightarrow N \rightarrow E(N) \rightarrow L \rightarrow 0$. Consider the following pushout diagram

```
0 0
\downarrow \downarrow \downarrow \downarrow
0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0
```

```
0 \rightarrow E(N) \rightarrow H \rightarrow M/N \rightarrow 0
```

```
L \rightarrow L
```

```
0 0
```

By (4), $L$ is cotorsion. Since $M$ is cotorsion, $H$ is cotorsion by [27, Proposition 3.1.2]. Note that $E(N)$ is cotorsion, it follows that $M/N$ is cotorsion by [27, Proposition 3.1.2] again.

We note that the equivalences of (2), (4), (5) and (6) in the previous theorem have recently been proven for commutative domains ([15, Theorem 3.2]).

By [27, Theorem 3.3.2], a ring $R$ is von Neumann regular if and only if every cotorsion right $R$-module is flat. Replacing “flat” with “projective”, we have the following

**Proposition 7.2.9** Let $R$ be a ring. Then the following are equivalent:

1. $R$ is a semisimple Artinian ring.
2. Every cotorsion right $R$-module is projective.
3. $r.cot.D(R) \leq 1$ and the cotorsion envelope of every simple right $R$-module is projective.

**Proof** (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1). It is easy to see that $R$ is quasi-Frobenius and von Neumann regular, and hence (1) follows.

(3) $\Rightarrow$ (1). By (3), every simple right $R$-module $M$ is a pure submodule of a projective right $R$-module, and hence $M$ is projective by Theorem 7.2.8. So (1) follows.

\qed
We know that the cotorsion envelope of any flat right $R$-module is always flat. Rothmaler [22] has discussed when the pure-injective envelope of any flat right $R$-module is flat. It is natural to consider the condition that the cotorsion (pure-injective) envelope of any projective right $R$-module is projective. For a ring with this condition, we have the following

**Theorem 7.2.10** Let $R$ be a ring such that the cotorsion envelope of any projective right $R$-module is projective. Then the following are equivalent:

1. $rcot.D(R) \leq 1$.

2. The projectivity of $C(M)$ implies the projectivity of $M$ for any right $R$-module $M$.

If “cotorsion envelope” is replaced with “pure-injective envelope”, the result still holds.

**Proof** (1) $\Rightarrow$ (2). Assume $M$ is a right $R$-module such that $C(M)$ is projective. Note that $M$ is a pure submodule of $C(M)$, so $M$ is projective by Theorem 7.2.8.

(2) $\Rightarrow$ (1). Let $M$ be a pure submodule of a projective right $R$-module $P$, it is enough to show that $M$ is projective by Theorem 7.2.8. In fact, there is an exact sequence

$$0 \to M \xrightarrow{f} P \xrightarrow{L} 0,$$

where $L$ is flat. By the defining property of cotorsion envelope, there exists $g : C(M) \to C(P)$ such that the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & P \\
\phi \downarrow & & \downarrow \psi \\
C(M) & \xrightarrow{g} & C(P)
\end{array}
$$

commutes, i.e., $g\phi = \psi f$. Consider the pushout diagram of $f$ and $\phi$:

$$
\begin{array}{ccc}
0 & \xrightarrow{f} & M \\
\phi \downarrow & & \downarrow \alpha \\
0 & \xrightarrow{\gamma} & C(M)
\end{array}
\begin{array}{ccc}
& & 0 \\
& & \downarrow \beta \\
& & 0
\end{array}
\begin{array}{ccc}
P & \xrightarrow{\psi} & C(P) \\
\gamma \downarrow & & \downarrow \sigma \\
K & \xrightarrow{\beta} & C(M)
\end{array}
$$

Note that the second row is split, so there is $\beta : K \to C(M)$ such that $\beta\alpha = 1$. It follows that $\beta\gamma : P \to C(M)$ factors through $\psi$. Hence there is $\sigma : C(P) \to C(M)$ such that the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\psi} & C(P) \\
\gamma \downarrow & & \downarrow \sigma \\
K & \xrightarrow{\beta} & C(M)
\end{array}
$$

commutes, i.e., $\alpha\psi = \beta\gamma$. Then $\sigma g\phi = \sigma \psi f = \beta\gamma f = \beta\alpha\phi = \phi$. The defining property of cotorsion envelope now implies that $\sigma g$ is an automorphism of $C(M)$. Therefore $C(M)$ is isomorphic to a direct summand of $C(P)$. Since $C(P)$ is projective by hypothesis, $C(M)$ is projective. So $M$ is projective by (2), as required.

The last statement can be proven similarly. \qed
It is well known that \( rD(R) = wD(R) \) when \( R \) is right perfect; \( rD(R) = r.cot.D(R) \) when \( R \) is von Neumann regular by Theorem 7.2.5 (2). In general, we have the following inequality.

**Theorem 7.2.11** Let \( R \) be a ring, then \( rD(R) \leq r.cot.D(R) + wD(R) \).

**Proof** We may assume that both \( r.cot.D(R) \) and \( wD(R) \) are finite. Let \( r.cot.D(R) = m < \infty \) and \( wD(R) = n < \infty \). Suppose \( M \) is a right \( R \)-module, then \( M \) admits a flat resolution

\[
0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.
\]

Let \( K_i = \text{Ker}(F_i \to F_{i-1}) \), \( i = 0, 1, 2, \ldots, n-1 \), \( F_{-1} = M \), \( F_n = K_{n-1} \). Then we have the following short exact sequences

\[
0 \to F_n \to F_{n-1} \to K_{n-2} \to 0,
\]

\[
0 \to K_{n-2} \to F_{n-2} \to K_{n-3} \to 0,
\]

\[
\cdots \cdots
\]

\[
0 \to K_0 \to F_0 \to M \to 0.
\]

Note that \( \text{pd}(K_{n-2}) \leq 1 + \sup \{\text{pd}(F_i) \mid i = 0, 1, \ldots, n \} \) by [23, Lemma 9.26]. Since \( \text{pd}(F_i) \leq m \), \( i = 0, 1, \ldots, n \), it follows that \( \text{pd}(K_{n-2}) \leq 1 + m \), \( \text{pd}(K_{n-3}) \leq 2 + m \), \( \cdots \), \( \text{pd}(M) \leq n + m \). This completes the proof. \( \square \)

**Remark** In general, the inequality in Theorem 7.2.11 may be strict. Indeed, if \( R \) is right Noetherian, but not right perfect (e.g. the integer ring \( \mathbb{Z} \)), then \( rD(R) = wD(R) \) (see [23, Theorem 9.22]) and \( r.cot.D(R) \neq 0 \). In this case, the inequality is strict. It is easy to verify that, if \( R \) is right Noetherian, then \( rD(R) = r.cot.D(R) + wD(R) \) if and only if \( R \) is right Artinian.

Recall that a ring \( R \) is called an \( n \)-Gorenstein ring if \( R \) is a left and right Noetherian ring with \( \text{id}(R) \leq n \) and \( \text{id}(R) \leq n \) for an integer \( n \geq 0 \). For this ring, we have the following

**Proposition 7.2.12** If \( R \) is an \( n \)-Gorenstein ring, then \( r.cot.D(R) \leq n \) and \( l.cot.D(R) \leq n \).

**Proof** Recall that a right \( R \)-module \( M \) is called \( FP \)-injective if \( \text{Ext}^1_R(N, M) = 0 \) for all finitely presented right \( R \)-modules \( N \). Note that a right \( R \)-module \( M \) is \( FP \)-injective if and only if \( M \) is injective when \( R \) is right Noetherian. It follows that \( r.cot.D(R) = \sup \{\text{cd}(M) \mid M \text{ is a flat right } R\text{-module} \} \leq \sup \{\text{id}(M) \mid M \text{ is a flat right } R\text{-module} \} = \text{id}(R) \leq n \) by [5, Theorem 3.8]. The inequality \( l.cot.D(R) \leq n \) can be proven similarly. \( \square \)

**Corollary 7.2.13** [8, Corollary 3.4]. If \( R \) is a 1-Gorenstein ring, then every quotient module of each injective right (left) \( R \)-module is cotorsion.

**Proof** It follows from Proposition 7.2.12 and Theorem 7.2.8. \( \square \)

For an exact sequence \( 0 \to A \to B \to C \to 0 \) of right \( R \)-modules, if \( B \) and \( C \) are both cotorsion, we know \( \text{cd}(A) \leq 1 \) by Proposition 7.2.3 (2). However \( A \) need not be cotorsion in general (see [27, p.75]). Next we discuss when \( A \) is cotorsion if \( B \) and \( C \) are.

**Proposition 7.2.14** Let \( R \) be a ring. Then the following are equivalent:

1. The cotorsion envelope of every flat right \( R \)-module is projective.
2. The flat cover of every cotorsion right \( R \)-module is projective.
3. Every flat cotorsion right \( R \)-module is projective.
4. Every flat right $R$-module is a pure submodule of some projective right $R$-module.

**Proof** (1) $\implies$ (4). Let $F$ be a flat right $R$-module. There exists an exact sequence $0 \to F \to C(F) \to L \to 0$. By (1), $C(F)$ is projective. Note that $L$ is flat, so the exact sequence is pure, and (4) follows.

(4) $\implies$ (3). Let $F$ be a flat cotorsion right $R$-module. By (4), there exists a projective right $R$-module $P$ and a pure exact sequence $0 \to F \to P \to L \to 0$. Note that $L$ is flat. It follows that the exact sequence is split. Thus $F$ is projective.

(2) $\Leftrightarrow$ (3) $\implies$ (1) are easy.

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**Proposition 7.2.15** Let $R$ be a ring satisfying the equivalent conditions in Proposition 7.2.14.

1. Assume $0 \to A \to B \to C \to 0$ is an exact sequence of right $R$-modules, then if two of $A$, $B$, $C$ are cotorsion, so is the third.

2. $r\cot.D(R) = 0$ or $r\cot.D(R) = \infty$.

**Proof** It is clear that (1) implies (2). We now prove (1).

It is enough to show that $A$ is cotorsion if $B$ and $C$ are cotorsion by [27, Proposition 3.1.2]. Let $F$ be any flat right $R$-module. By Proposition 7.2.14, there exists a pure exact sequence $0 \to F \to P \to L \to 0$ with $P$ projective. Note that $L$ is flat. The exact sequence $0 \to A \to B \to C \to 0$ gives rise to the following exact sequence

$$\text{Ext}^1_R(L, C) \to \text{Ext}^2_R(L, A) \to \text{Ext}^2_R(L, B),$$

which implies $\text{Ext}^2_R(L, A) = 0$ since the first term and the last term are both zero by hypothesis. In addition, the exact sequence $0 \to F \to P \to L \to 0$ yields the following exact sequence

$$\text{Ext}^1_R(P, A) \to \text{Ext}^1_R(F, A) \to \text{Ext}^2_R(L, A).$$

Note that the first term and the last term are both zero, so $\text{Ext}^1_R(F, A) = 0$. This completes the proof.

We end this section with the following result which is of independent interest.

Recall that a ring $R$ is called *left coherent* if every finitely generated left ideal is finitely presented.

**Proposition 7.2.16** Let $R$ be a left coherent ring, then the following are equivalent:

1. $R$ is right perfect.

2. $R$ is a ring satisfying the equivalent conditions in Proposition 7.2.14.

**Proof** (1) $\implies$ (2) is trivial.

(2) $\implies$ (1). For any family $\{R_i\}_{i \in I}$, where each $R_i \cong R$ is a right $R$-module, $\prod_{i \in I} R_i$ is a flat right $R$-module since $R$ is left coherent. Hence we have an exact sequence

$$0 \to \prod_{i \in I} R_i \to C(\prod_{i \in I} R_i) \to L \to 0,$$

where $C(\prod_{i \in I} R_i)$ and $L$ are flat by [27, Theorem 3.4.2]. By hypothesis, $C(\prod_{i \in I} R_i)$ is projective. Thus $\prod_{i \in I} R_i$ is a pure submodule of a projective right $R$-module, and hence it is a pure submodule of a free right $R$-module. It follows that $R$ is a right perfect ring by [4, Theorem 3.1].
7.3 Cotorsion dimension under change of rings

We begin with the following

Proposition 7.3.1 Let \( \varphi : R \to S \) be a surjective ring homomorphism.

1. If \( M_S \) is a right \( S \)-module, then \( \text{cd}(M_R) \leq \text{cd}(M_S) \). Moreover, if \( S_R \) is a flat right \( R \)-module, then \( \text{cd}(M_S) = \text{cd}(M_R) \).

2. If \( S_R \) is a flat right \( R \)-module, and \( M_R \) is a cotorsion right \( R \)-module, then \( \text{Hom}_R(S, M) \) is a cotorsion right \( S \)-module, and hence a cotorsion right \( R \)-module.

Proof (1). We may assume \( \text{cd}(M_S) = n < \infty \). Then there exists an exact sequence

\[
0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0,
\]

where each \( C^i \) is a cotorsion right \( S \)-module, \( i = 0, 1, \ldots, n \). By [27, Proposition 3.3.3], each \( C^i \) is also cotorsion as a right \( R \)-module. So \( \text{cd}(M_R) \leq n \).

If \( S_R \) is a flat right \( R \)-module, we claim \( \text{cd}(M_S) \leq \text{cd}(M_R) \). In fact, we may assume \( \text{cd}(M_R) = n < \infty \). Let \( F \) be a flat right \( S \)-module, then \( F \) is a flat right \( R \)-module. Thus \( \text{Ext}^n_{S_R}(F_S, M_S) = \text{Ext}^n_{S_R}(F_R, M_R) = 0 \) by [23, Theorem 11.65]. Therefore \( \text{cd}(M_S) \leq n \), and hence \( \text{cd}(M_S) = \text{cd}(M_R) \).

(2). By hypothesis, \( \text{Ext}^1_R(S, M) = 0 \). Let \( X \) be a flat right \( S \)-module, then \( X \) is a flat right \( R \)-module. Thus

\[
\text{Ext}^1_R(X, \text{Hom}_R(S, M)) = \text{Ext}^1_R(X, M) = 0
\]

by [24, Lemma 3.1]. Therefore \( \text{Hom}_R(S, M) \) is a cotorsion right \( S \)-module, and hence a cotorsion right \( R \)-module by [27, Proposition 3.3.3].

\[ \square \]

Corollary 7.3.2 Let \( \varphi : R \to S \) be a surjective ring homomorphism and \( S_R \) a flat right \( R \)-module, then \( \text{rcot.D}(S) \leq \text{rcot.D}(R) \).

Recall that a ring \( S \) is said to be an almost excellent extension of a ring \( R \) [28, 29] if the following conditions are satisfied:

1. \( S \) is a finite normalizing extension of a ring \( R \) [25], that is, \( R \) and \( S \) have the same identity and there are elements \( s_1, \ldots, s_n \in S \) such that \( S = Rs_1 + \cdots + Rs_n \) and \( Rs_i = s_i R \) for all \( i = 1, \ldots, n \).

2. \( R S \) is flat and \( S_R \) is projective.

3. \( S \) is right \( R \)-projective, that is, if \( M_S \) is a submodule of \( N_S \) and \( M_R \) is a direct summand of \( N_R \), then \( M_S \) is a direct summand of \( N_S \).

Further, \( S \) is an excellent extension of \( R \) if \( S \) is an almost excellent extension of \( R \) and \( S \) is free with basis \( s_1, \ldots, s_n \) as both a right and a left \( R \)-module with \( s_1 = 1_R \). The concept of excellent extension was introduced by Passman [18] and named by Bonami [3]. Examples of excellent extensions include finite matrix rings [18], and crossed product \( R \ast G \) where \( G \) is a finite group with \( |G|^{-1} \in R \) [19]. The notion of almost excellent extensions was introduced and studied in [28] as a non-trivial generalization of excellent extensions.

Let \( S \) be a finite normalizing extension (in particular, an (almost) excellent extension) of a ring \( R \). It is well known that \( R \) is right perfect if and only if \( S \) is right perfect [21, Corollary 7]. It seems natural to generalize descent of right perfectness to cotorsion dimension in the case when \( S \) is an (almost) excellent extension of a ring \( R \) and this is the main goal of the rest of this section.
7.3 Cotorsion dimension under change of rings

**Theorem 7.3.3** Let $S$ be an almost excellent extension of a ring $R$ and $M_S$ a right $S$-module. Then

1. $\text{cd}(M_S) = \text{cd}(M_R) = \text{cd}(\text{Hom}_R(S, M))$.

2. $M_S$ is cotorsion if and only if $M_R$ is cotorsion if and only if $\text{Hom}_R(S, M)$ is a cotorsion right $S$-module.

**Proof** (1). We first prove that $\text{cd}(M_S) \leq \text{cd}(M_R)$. We may assume that $\text{cd}(M_R) = n < \infty$. Let $N_S$ be a flat right $S$-module. Then $N_R$ is a flat right $R$-module by [29, Lemma 1.2 (3)]. Note that $\text{Ext}^{n+1}_R(N, M) \cong \text{Ext}^{n+1}_S(N \otimes_R S, M)$ by [23, Theorem 11.65]. Since $\text{Ext}^{n+1}_R(N, M) = 0$, $\text{Ext}^{n+1}_S(N \otimes_R S, M) = 0$. Thus $\text{Ext}^{n+1}_S(N, M) = 0$ by [29, Lemma 1.1 (1)], and so $\text{cd}(M_S) \leq n$.

Conversely, suppose $\text{cd}(M_S) = n < \infty$. Let $N_R$ be a flat right $R$-module. Then $N \otimes_R S$ is a flat right $S$-module, and so $\text{Ext}^{n+1}_R(N \otimes_R S, M) = 0$. Thus, by the above isomorphism, we get $\text{Ext}^{n+1}_S(N, M) = 0$, and hence $\text{cd}(M_R) \leq n$.

By [16, Lemma 2.16], if $E_R$ is a cotorsion right $R$-module, then $\text{Hom}_R(S, E)$ is a cotorsion right $S$-module. Hence $\text{cd}(\text{Hom}_R(S, M)) \leq \text{cd}(M_R)$ by Corollary 7.2.2. Since $M_S$ is isomorphic to a direct summand of $\text{Hom}_R(S, M)$ by [29, Lemma 1.1 (2)], $\text{cd}(M_S) \leq \text{cd}(\text{Hom}_R(S, M))$. So (1) holds.

(2) follows from (1). \qed

**Corollary 7.3.4** Let $R$ and $S$ be rings.

1. If $S$ is an almost excellent extension of $R$, then $r.\text{cot}.D(S) \leq r.\text{cot}.D(R)$.

2. If $S$ is an excellent extension of $R$, then $r.\text{cot}.D(S) = r.\text{cot}.D(R)$.

**Proof** (1) follows from Theorem 7.3.3.

(2). Since $S$ is an excellent extension of $R$, $R$ is an $R$-bimodule direct summand of $S$. Let $\_S = R \oplus T$, and $M_R$ be any right $R$-module. Observe that $\text{Hom}_R(S, M) \cong \text{Hom}_R(R, M) \oplus \text{Hom}_R(T, M)$. Therefore

$$\text{cd}(M_R) \leq \text{cd}(\text{Hom}_R(S, M)) \leq r.\text{cot}.D(S)$$

by Theorem 7.3.3 (1), and hence $r.\text{cot}.D(R) \leq r.\text{cot}.D(S)$. So (2) follows from (1). \qed

**Theorem 7.3.5** Let $S$ be an almost excellent extension of a ring $R$. If $r.\text{cot}.D(R) < \infty$, then $r.\text{cot}.D(S) = r.\text{cot}.D(R)$.

**Proof** It is enough to show that $r.\text{cot}.D(S) \leq r.\text{cot}.D(S)$ by Corollary 7.3.4. Suppose $r.\text{cot}.D(R) = n < \infty$. Then there exists a right $R$-module $M$ such that $\text{cd}(M_R) = n$. Define a right $R$-homomorphism $\alpha : \text{Hom}_R(S, M) \to M$ via $\alpha(f) = f(1)$ for any $f \in \text{Hom}_R(S, M)$. Since $S_R$ is projective, the epimorphism $\pi : M \to M/\text{im}(\alpha)$ induces the epimorphism $\pi_S : \text{Hom}_R(S, M) \to \text{Hom}_R(S, M/\text{im}(\alpha))$. Let $f \in \text{Hom}_R(S, M)$ and $s \in S$. Then $\pi_S(f)(s) = \pi(f(s)) = \pi((f(s))(1)) = \pi(\alpha(f(s))) = 0$, and so $\ker(\pi_S) = \text{Hom}_R(S, M)$. It follows that $\text{Hom}_R(S, M/\text{im}(\alpha)) = 0$, and hence $M/\text{im}(\alpha) = 0$ by [25, Proposition 2.1]. Thus $\alpha$ is epic, and so we have a right $R$-module exact sequence $0 \to K \to \text{Hom}_R(S, M) \to M \to 0$. By Proposition 7.2.3 (3), we have $n = \text{cd}(M_R) \leq \sup\{\text{cd}(\text{Hom}_R(S, M)), \, \text{cd}(K_R) - 1\} \leq r.\text{cot}.D(R) = n$. Since $\text{cd}(K_R) - 1 \leq n - 1$, then $\text{cd}(\text{Hom}_R(S, M)) = n$. On the other hand, $\text{cd}(\text{Hom}_R(S, M)) \leq r.\text{cot}.D(S)$ by Theorem 7.3.3. Therefore $r.\text{cot}.D(R) \leq r.\text{cot}.D(S)$, as desired. \qed
7.4 Applications in commutative rings

In this section, all rings are assumed to be commutative. We need the following lemma which will be frequently used in the sequel.

**Lemma 7.4.1** Let \( R \) be a ring and \( M \) an \( R \)-module, then the following are equivalent:

1. \( M \) is cotorsion.
2. \( \text{Hom}_R(F, M) \) is a cotorsion \( R \)-module for any flat \( R \)-module \( F \).
3. \( \text{Hom}_R(P, M) \) is a cotorsion \( R \)-module for any projective \( R \)-module \( P \).

Moreover, if the class of cotorsion \( R \)-modules is closed under direct sums, then the above conditions are also equivalent to

4. \( P \otimes_R M \) is a cotorsion \( R \)-module for any projective \( R \)-module \( P \).

**Proof** (1) \( \Rightarrow \) (2). Let \( N, F \) be two flat \( R \)-modules. There exists an exact sequence \( 0 \to K \to G \to N \to 0 \) with \( G \) projective, which yields the exactness of the sequence \( 0 \to K \otimes_R F \to G \otimes_R F \to N \otimes_R F \to 0 \). Note that \( N \otimes_R F \) is flat. We have the following exact sequence

\[
\text{Hom}_R(G \otimes_R F, M) \to \text{Hom}_R(K \otimes_R F, M) \to \text{Ext}^1_R(N \otimes_R F, M) = 0,
\]

which gives rise to the exactness of the sequence

\[
\text{Hom}_R(G, \text{Hom}_R(F, M)) \to \text{Hom}_R(K, \text{Hom}_R(F, M)) \to 0.
\]

On the other hand, the following sequence

\[
\text{Hom}_R(G, \text{Hom}_R(F, M)) \to \text{Hom}_R(K, \text{Hom}_R(F, M)) \to \text{Ext}^1_R(N, \text{Hom}_R(F, M)) \to \text{Ext}^1_R(G, \text{Hom}_R(F, M)) = 0
\]

is exact. Thus \( \text{Ext}^1_R(N, \text{Hom}_R(F, M)) = 0 \), and (2) follows.

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1) follows by letting \( P = R \).

The last statement is easy to verify. \( \square \)

**Corollary 7.4.2** Let \( R \) be a ring such that the class of cotorsion \( R \)-modules is closed under direct sums. Then the following are equivalent:

1. The cotorsion envelope of any projective \( R \)-module is always projective.
2. \( C(R_R) \) is projective.

**Proof** (1) \( \Rightarrow \) (2) is trivial.

(2) \( \Rightarrow \) (1). Consider the exact sequence \( 0 \to R \to C(R_R) \to N \to 0 \). Let \( M \) be any projective \( R \)-module, then \( 0 \to R \otimes_R M \to C(R_R) \otimes_R M \to N \otimes_R M \to 0 \) is also exact. Note that \( C(R_R) \otimes_R M \) is projective, and cotorsion by Lemma 7.4.1. It follows that \( M \to C(R_R) \otimes_R M \) is a cotorsion preenvelope of \( M \) since \( N \otimes_R M \) is flat. Hence \( C(M) \) is projective since it is a direct summand of \( C(R_R) \otimes_R M \) by [9, Proposition 6.1.2]. \( \square \)
The next proposition shows that if $R$ is a Dedekind domain, then $\Ext^1_R(B, C)$ is cotorsion for all $R$-modules $B$ and $C$, which may be viewed as an answer to [11, Problem 48, p.462].

**Proposition 7.4.3** Let $R$ be a ring.

1. If $D(R) \leq 1$ (i.e., $R$ is a hereditary ring), then $\Ext^1_R(B, C)$ is cotorsion for all $R$-modules $B$ and $C$.

2. If $\cot D(R) \leq 1$, then $\Ext^1_R(F, M)$ is cotorsion for any flat $R$-module $F$ and any $R$-module $M$.

**Proof** (1) follows from the isomorphism

$$\Ext^1_R(\Tor^R_1(A, B), C) \cong \Ext^1_R(A, \Ext^1_R(B, C))$$

for all $R$-modules $A$, $B$ and $C$ (see [23, p.343]).

(2). Let $M$ be any $R$-module. By hypothesis, there exists an exact sequence $0 \to M \to C^0 \to C^1 \to 0$, where $C^0$ and $C^1$ are cotorsion. So the sequence $\Hom_R(F, C^1) \to \Ext^1_R(F, M) \to \Ext^1_R(F, C^0) = 0$ is exact for any flat $R$-module $F$. By Lemma 7.4.1, $\Hom_R(F, C^1)$ is cotorsion, and hence $\Ext^1_R(F, M)$ is cotorsion by Theorem 7.2.8.

We omit the proof of the next proposition which can be deduced easily from Lemma 7.4.1.

**Proposition 7.4.4** Let $R$ be a ring and $M$ an $R$-module. Then the following are equivalent:

1. $\cd(M) \leq n$.

2. $\cd(\Hom_R(P, M)) \leq n$ for any projective $R$-module $P$.

We are now in a position to prove the following

**Theorem 7.4.5** Let $\varphi : R \to S$ be a surjective ring homomorphism with $K = \Ker(\varphi)$. If $S_R$ is projective, then, for any $R$-module $M$, either $\cd(M_R) \leq \sup\{\pd(R/I)_R : I \subseteq K\}$, or $\cd(M_R) = \cd(\Hom_R(S, M))$ and the exactness of $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$,

where each $C^i$ is a cotorsion $R$-module, $i = 1, 2, \ldots, n-1$. Thus

$$\cd(M_R) = \cd(C^n) + n$$

by Corollary 7.2.4, and

$$\Ext^j_R(R/I, C^n) \cong \Ext^{j+n}_R(R/I, M) = 0$$

for all $j > 0$, and all $I \subseteq K$.

We claim that $\cd(C^n) = \cd(\Hom_R(S, C^n))$.

In fact, $\cd(\Hom_R(S, C^n)) \leq \cd(C^n)$ by Proposition 7.4.4. We only need to show that $\cd(C^n) \leq \cd(\Hom_R(S, C^n))$. Note that $C^n \cong \Hom_R(R, C^n)$ and the exactness of $0 \to K \to R \to S \to 0$ induces an exact sequence $0 \to \Hom_R(S, C^n) \to \Hom_R(R, C^n) \to \Hom_R(K, C^n) \to 0$. It is enough to show that $\Hom_R(K, C^n)$ is an injective $R$-module by Proposition 7.2.3 (1).

Let $L$ be any ideal of $R$. The exactness of $0 \to K/LK \to R/LK \to R/K \to 0$ gives an exact sequence

$$\Ext^1_R(R/LK, C^n) \to \Ext^1_R(K/LK, C^n) \to \Ext^2_R(R/K, C^n).$$
Since \( \text{Ext}_R^1(R/LK, C^n) = \text{Ext}_R^2(R/K, C^n) = 0 \) by the first part of the proof, \( \text{Ext}_R^1(K/LK, C^n) = 0 \). Hence the exact sequence \( 0 \to LK \to K \to K/LK \to 0 \) yields the exactness of
\[
\text{Hom}_R(K, C^n) \to \text{Hom}_R(LK, C^n) \to 0.
\]

Note that
\[
\text{Hom}_R(R, \text{Hom}_R(K, C^n)) \cong \text{Hom}_R(K, C^n),
\]
\[
\text{Hom}_R(L, \text{Hom}_R(K, C^n)) \cong \text{Hom}_R(L \otimes K, C^n) \cong \text{Hom}_R(LK, C^n).
\]
The last isomorphism holds by the flatness of \( K \). Thus the sequence
\[
\text{Hom}_R(R, \text{Hom}_R(K, C^n)) \to \text{Hom}_R(L, \text{Hom}_R(K, C^n)) \to 0
\]
is exact, and so \( \text{Hom}_R(K, C^n) \) is \( R \)-injective, as required.

On the other hand, since \( S_R \) is projective, we have the following exact sequence
\[
0 \to \text{Hom}_R(S, M) \to \text{Hom}_R(S, C^n) \to \\
\text{Hom}_R(S, C^1) \to \cdots \to \text{Hom}_R(S, C^n) \to 0,
\]
where each \( \text{Hom}_R(S, C^i) \), \( i = 1, 2, \ldots, n - 1 \), is a cotorsion \( R \)-module by Proposition 7.3.1 (2). Note that
\[
\text{cd}(\text{Hom}_R(S, C^n)) = \text{cd}(C^n) = \text{cd}(M_R) - n > 0.
\]
Thus \( \text{cd}(\text{Hom}_R(S, M)) > n \), and so
\[
\text{cd}(\text{Hom}_R(S, M)) = \text{cd}(\text{Hom}_R(S, C^n)) + n
\]
by Corollary 7.2.4. It follows that \( \text{cd}(M_R) = \text{cd}(\text{Hom}_R(S, M)) \), where \( \text{Hom}_R(S, M) \) may be regarded as an \( R \)-module or \( S \)-module by Proposition 7.3.1 (1).

**Corollary 7.4.6** Let \( \varphi : R \to S \) be a surjective ring homomorphism with \( K = \text{Ker}(\varphi) \). If \( S_R \) is projective, then either \( \cotorsion{D}(R) \leq \sup \{ \text{pd}(R/I)_R : I \subseteq K \} \), or \( \cotorsion{D}(R) = \cotorsion{D}(S) \).

**Proof** Let \( \sup \{ \text{pd}(R/I)_R : I \subseteq K \} = n \). If \( \text{cd}(M_R) \leq n \) for every \( R \)-module \( M_R \), then \( \cotorsion{D}(R) \leq n \). If there is \( M_R \) such that \( \text{cd}(M_R) > n \), then \( \text{cd}(M_R) = \text{cd}(\text{Hom}_R(S, M)) \leq \cotorsion{D}(S) \) by Theorem 7.4.5, and so \( \cotorsion{D}(R) \leq \cotorsion{D}(S) \). Note that \( \cotorsion{D}(S) \leq \cotorsion{D}(R) \) by Corollary 7.3.2. So \( \cotorsion{D}(R) = \cotorsion{D}(S) \).

**Corollary 7.4.7** A ring \( R \) is perfect if and only if there is a quotient ring \( S = R/K \) of \( R \) such that \( S \) is a perfect ring and \( R/I \) is a projective \( R \)-module for any \( I \subseteq K \).

**Corollary 7.4.8** Let \( K \) be a maximal ideal of a ring \( R \) such that \( R/K \) is a projective \( R \)-module, then \( \cotorsion{D}(R) \leq \sup \{ \text{pd}(R/I)_R : I \subseteq K \} \).

**Proposition 7.4.9** Let \( P \) be any prime ideal of a ring \( R \), then \( \cotorsion{D}(R_P) \leq \cotorsion{D}(R) \), where \( R_P \) is the localization of \( R \) at \( P \).

**Proof** We may assume \( \cotorsion{D}(R) = n < \infty \). Let \( M \) be any flat \( R_P \)-module. Since \( R_P \) is a flat \( R \)-module, then \( M \) is a flat \( R \)-module. Thus \( \text{pd}(M_R) \leq n \). There exists a projective resolution of \( M_R \)
\[
0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,
\]
which induces an \( R_P \)-module exact sequence
\[
0 \to (F_n)_P \to (F_{n-1})_P \to \cdots \to (F_1)_P \to (F_0)_P \to M_P \to 0.
\]
Note that each \( (F_i)_P \) is a projective \( R_P \)-module, \( i = 0, 1, \ldots, n \), it follows that \( \text{pd}(M_P)_{R_P} \leq n \). Since \( (M_P)_{R_P} \cong M_{R_P}, \text{pd}(M_{R_P}) \leq n \). Thus \( \cotorsion{D}(R_P) \leq n \), as required.
It is well known that $R$ is a coherent ring if and only if $\text{Hom}_R(A, B)$ is flat for all injective $R$-modules $A$ and $B$ ([17]). By [5, Corollary 3.22], $R$ is an IF ring (the ring for which every injective $R$-module is flat) if and only if $\text{Hom}_R(A, B)$ is injective for all injective $R$-modules $A$ and $B$. Continuing this style of charactering rings by properties of homomorphism modules of certain special $R$-modules, we conclude this paper with the following easy results for completeness.

**Proposition 7.4.10** Let $R$ be a ring, then the following are equivalent:

1. $R$ is a von Neumann regular ring.
2. For each cotorsion $R$-module $A$, $\text{Hom}_R(A, B)$ is injective for all cotorsion (or injective) $R$-modules $B$.
3. For each cotorsion $R$-module $A$, $\text{Hom}_R(A, B)$ is flat for all cotorsion (or injective) $R$-modules $B$.

**Proof**

(1) $\Rightarrow$ (2). Let $A$ and $B$ be cotorsion, then $\text{Hom}_R(A, B)$ is cotorsion by Lemma 7.4.1 (for $A$ is flat by (1)). Thus $\text{Hom}_R(A, B)$ is injective by [27, Theorem 3.3.2].

(2) $\Rightarrow$ (1). Let $A$ be a cotorsion $R$-module. (2) implies that $\text{Hom}_R(A, -)$ preserves injectives. Thus $A$ is flat by [10, Proposition 11.35], and (1) follows from [27, Theorem 3.3.2].

(1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $S$ be any simple $R$-module. Then $S$ is cotorsion by [16, Lemma 2.14]. Let $E = \text{Ext}(\oplus_{i \in I} S_i)$, where $\{S_i\}_{i \in I}$ is an irredundant set of representatives of the simple $R$-modules. Then $E$ is an injective cogenerator by [1, Corollary 18.19]. Note that $\text{Hom}_R(S, E)$ is flat by (3) and $\text{Hom}_R(S, E) \cong S$ by the proof of [26, Lemma 2.6]. Thus $S$ is flat, and hence $R$ is regular by [20, 3.3].

**Proposition 7.4.11** Let $R$ be a ring, then the following are equivalent:

1. $R$ is a semisimple Artinian ring.
2. For each cotorsion $R$-module $A$, $\text{Hom}_R(A, B)$ is projective for all cotorsion (or injective) $R$-modules $B$.

**Proof**

(1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let $S$ be any simple $R$-module. By (2) and the proof of (3) $\Rightarrow$ (1) in Proposition 7.4.10, $S$ is projective. So $R$ is semisimple Artinian.

**Remark** We wonder what kind of commutative rings is characterized by the condition that every homomorphism module of cotorsion modules is cotorsion. This kind of rings, of course, contains perfect rings and von Neumann regular rings. It is easy to verify that a ring $R$ is of this kind if and only if $\text{Hom}_R(A, B)$ is cotorsion for all $R$-modules $A$ and all cotorsion $R$-modules $B$.

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**Acknowledgements**

This research was partially supported by Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20020284009), EYTP and NNSF of China (No. 10331030) and the Nanjing Institute of Technology of China.
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