Chapter 7

The Cotorsion Dimension of Modules and Rings

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Abstract In this paper, we introduce a dimension, called the cotorsion dimension, for modules and rings. The relations between the cotorsion dimension and other dimensions are discussed. Various results are developed, some extending known results.

Keywords: Cotorsion dimension; Cotorsion envelope; Flat cover; Perfect ring.

7.1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary.

Let R be a ring and M a right R-module. Recall that M is called *cotorsion* [7] if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat right R-module F. The class of cotorsion modules contains all pure-injective (hence, injective) modules. A homomorphism $\phi : M \to C$ with C cotorsion is called a *cotorsion preenvelope* of M [6, 27] if for any homomorphism $f : M \to C^{'}$ where $C^{'}$ is cotorsion, there is a homomorphism $g : C \to C^{'}$ such that $g\phi = f$. Moreover if the only such g are automorphisms of C when $C^{'} = C$ and $f = \phi$, the cotorsion preenvelope ϕ is called a *cotorsion envelope* of M. A homomorphism $\phi : F \to M$ with F flat is called a *flat cover* of M if for any homomorphism $f : F^{'} \to M$ where $F^{'}$ is flat, there is a homomorphism $g : F^{'} \to F$ such that $\phi g = f$, moreover when $F^{'} = F$ and $f = \phi$, the only such g are automorphisms of F. It is now well known that all R-modules have flat covers for any ring R [2], and it has been proven that every R-module has a cotorsion envelope if and only if every R-module has a flat cover [27]. Thus all R-modules have cotorsion envelopes for arbitrary ring R. Note that cotorsion envelopes or flat covers are unique up to isomorphism.

In what follows, for an R-module M, $E(M)$, $C(M)$ and $F(M)$ stand for the injective envelope, cotorsion envelope and flat cover respectively. We write M_R to indicate a right R-module. The projective (resp. injective) dimension of M is denoted by $\mathrm{pd}(M)$ (resp. $\mathrm{id}(M)$). We denote by $\mathrm{rD}(R)$ (resp. $\mathrm{wD}(R)$) the right (resp. the weak) global dimension of a ring R. General background material can be found in [1], [9], [23], [27].

We are going to define a dimension, called the cotorsion dimension, for modules and rings. It measures how far away a module is from being cortorsion, and how far away a ring is from being perfect.

Let R be a ring. For any right R-module M, the *cotorsion dimension* $\mathrm{cd}(M)$ of M is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for any flat right R-module F. If there is no such n, set $\mathrm{cd}(M) = \infty$. The right *global cotorsion dimension* $\mathrm{r.cot.D}(R)$ of R is defined as the supremum of the cotorsion dimensions of right R-modules. The aim of this paper is to investigate these dimensions.

In Section 2, we give the definition and show some of the general results. Let R be a ring. First we prove that r.cot.D(R) = sup{pd(F): F is a flat right R-module} = sup{cd(F): F is a flat right R-module} (part of Theorem 7.2.5), which gives rise to some characterizations of right perfect rings (Corollary 7.2.7) and extends [27, Proposition 3.3.1]. Then it is shown that r.cot.D$(R) \leq 1$ if and only if every quotient module of any cotorsion (or injective) right R-module is cotorsion if and only if every pure submodule of any projective right R-module is projective (Theorem 7.2.8). This removes the unnecessary hypothesis that R is a commutative domain from [15, Theorem 3.2]. For a ring R such that the cotorsion envelope of any projective right R-module is projective, we have that r.cot.D$(R) \leq 1$ if and only if the projectivity of $C(M)$ implies the projectivity of M for any right R-module M (Theorem 7.2.10). The relation rD$(R) \leq$ wD$(R)+$ r.cot.D(R) is proven to be true for any ring R (Theorem 7.2.11). Finally, for a left coherent ring R, it is shown that R is right perfect if and only if every flat cotorsion right R-module is projective (Proposition 7.2.16).

Section 3 is devoted to the cotorsion dimension under change of rings. We first get that if $\varphi : R \to S$ is a surjective ring homomorphism and S_R a flat right R-module, then r.cot.D$(S) \leq$ r.cot.D(R) (Corollary 7.3.2). Then we prove that if S is an almost excellent extension of R, then r.cot.D$(S) \leq$ r.cot.D(R), and the equality holds in case r.cot.D$(R) < \infty$ (Corollary 7.3.4 and Theorem 7.3.5).

In Section 4, some applications in commutative rings are discussed. We start by showing that for a ring R with cot.D$(R) \leq 1$, $\mathrm{Ext}^1_R(F, M)$ is cotorsion for any flat R-module F and any R-module M (Proposition 7.4.3), which is motivated by [11, Problem 48, p.462]. Then, for a surjective ring homomorphism $\varphi : R \to S$ with $K = \mathrm{Ker}(\varphi)$ and S_R projective, it is shown that, for any R-module M, either cd$(M_R) \leq$ sup{pd$(R/I)_R : I \subseteq K$}, or cd(M_R) = cd$(\mathrm{Hom}_R(S, M))$, where $\mathrm{Hom}_R(S, M)$ may be regarded as an R-module or S-module (Theorem 7.4.5). As a corollary, we get that a ring R is perfect if and only if there is a quotient ring $S = R/K$ of R such that S is a perfect ring and R/I is a projective R-module for any $I \subseteq K$ (Corollary 7.4.7). In the last part of this section, we prove that a ring R is von Neumann regular if and only if $\mathrm{Hom}_R(A, B)$ is injective (or flat) for all cotorsion R-modules A and B (Proposition 7.4.10).

7.2 General results

We start with the following

Proposition 7.2.1 *For any right R-module M and integer $n \geq 0$, the following are equivalent:*

1. *cd$(M) \leq n$.*
2. $\mathrm{Ext}^{n+1}_R(F, M) = 0$ *for any flat right R-module F.*
3. $\mathrm{Ext}^{n+j}_R(F, M) = 0$ *for any flat right R-module F and $j \geq 1$.*
4. *If the sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$ is exact with $C^0, C^1, \ldots, C^{n-1}$ cotorsion, then C^n is also cotorsion.*
5. *cd$(F(M)) \leq n$.*

Proof The proof of (1) ⇔ (2) ⇔ (3) ⇔ (4) is standard homological algebra fare.

(1) ⇔ (5). Let K be the kernel of the flat cover $F(M) \to M$, then we have the exact sequence $0 \to K \to F(M) \to M \to 0$ with K cotorsion. Note that $\mathrm{Ext}^n_R(F, K) = 0$ for all $n \geq 1$ and flat modules F by the proof of [27, Proposition 3.1.2], so the result follows. □

Corollary 7.2.2 *Let M be any right R-module. Then the following are identical:*

1. $cd(M)$.

2. $\inf\{k$: *there exists an exact sequence* $0 \to M \to C^0 \to C^1 \to \cdots \to C^k \to 0$, *where each* C^i *is a cotorsion right* R*-module,* $i = 0, 1, \ldots, k\}$.

3. *The integer* n *such that* M *admits a minimal cotorsion resolution, i.e., an exact sequence* $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$, *where each* C^i *is cotorsion,* $L^i = \mathrm{Coker}(C^{i-2} \to C^{i-1}) \to C^i$ *is a cotorsion envelope of* L^i, $C^i \neq 0$, $i = 0, 1, \ldots, n$, $C^{-2} = 0, C^{-1} = M$.

Proof (1) = (2) is straightforward.

(1) $\leq$ (3) is trivial. Assume (1) $<$ (3) $= n$. Let (1) $= k < \infty$. By Proposition 7.2.1, L^k is a cotorsion right R-module. Consider the exact sequence $0 \to L^k \to C^k \to L^{k+1} \to 0$, since $L^k \to C^k$ is a cotorsion envelope of L^k, it follows that $L^{k+1} = 0$, and hence $C^{k+1} = 0$, a contradiction. Therefore (1) = (3). □

Proposition 7.2.3 *Let R be a ring,* $0 \to A \to B \to C \to 0$ *an exact sequence of right R-modules. If two of* $cd(A)$, $cd(B)$, $cd(C)$ *are finite, so is the third. Moreover*

1. $cd(B) \leq \sup\{cd(A), cd(C)\}$.

2. $cd(A) \leq \sup\{cd(B), cd(C) + 1\}$.

3. $cd(C) \leq \sup\{cd(B), cd(A) - 1\}$.

Proof It is a routine exercise. □

The next corollary is an immediate consequence of Proposition 7.2.3.

Corollary 7.2.4 *Let R be a ring,* $0 \to A \to B \to C \to 0$ *an exact sequence of right R-modules. If B is cotorsion,* $cd(A) > 0$, *then* $cd(A) = cd(C) + 1$.

Theorem 7.2.5 *Let R be a ring. Then*

1. $r.cot.D(R)$
   $= \sup\{pd(F)$: F *is a flat right* R*-module*$\}$
   $= \sup\{cd(F)$: F *is a flat right* R*-module*$\}$.

2. $r.cot.D(R)$
   $\leq \sup\{pd(M): pd(M) < \infty\}$
   $\leq \sup\{id(P)$: P *is a projective right* R*-module*$\}$
   $\leq rD(R)$.
   All equalities hold if R is a von Neumann regular ring.

3. *If* $r.cot.D(R) < \infty$, *then*
   $r.cot.D(R)$
   $= \sup\{pd(F)$: F *is a flat cotorsion right* R*-module*$\}$
   $= \sup\{pd(C(F))$: F *is a flat right* R*-module*$\}$
   $= \sup\{pd(F(M))$: M *is a cotorsion right* R*-module*$\}$
   $= \sup\{cd(P)$: P *is a projective right* R*-module*$\}$.

Proof (1). First, we show that r.cot.D$(R) \leq \sup\{\mathrm{pd}(F)$: F is a flat right R-module$\}$. We may assume $\sup\{\mathrm{pd}(F)$: F is a flat right R-module$\} = m < \infty$. Let M be any right R-module. It follows that $\mathrm{Ext}_R^{m+1}(F, M) = 0$ for any flat right R-module F since $\mathrm{pd}(F) \leq m$, so $\mathrm{cd}(M) \leq m$. Thus r.cot.D$(R) \leq m$.

It is clear that $\sup\{\mathrm{cd}(F)$: F is a flat right R-module$\} \leq$ r.cot.D(R). Next we shall show that $\sup\{\mathrm{pd}(F)$: F is a flat right R-module$\} \leq \sup\{\mathrm{cd}(F)$: F is a flat right R-module$\}$. In fact, we may assume that $\sup\{\mathrm{cd}(F)$: F is a flat right R-module$\} = n < \infty$. Let M be any flat right R-module, N any right R-module. There exists an exact sequence $0 \to K \to F(N) \to N \to 0$. By [27, Lemma 2.1.1], K is cotorsion. We have the following exact sequence

$$\mathrm{Ext}_R^{n+1}(M, F(N)) \to \mathrm{Ext}_R^{n+1}(M, N) \to \mathrm{Ext}_R^{n+2}(M, K) = 0.$$

Note that $\mathrm{Ext}_R^{n+1}(M, F(N)) = 0$ since $\mathrm{cd}(F(N)) \leq n$. So $\mathrm{Ext}_R^{n+1}(M, N) = 0$, which implies $\mathrm{pd}(M) \leq n$, as desired.

(2). By (1), r.cot.D$(R) \leq \sup\{\mathrm{pd}(M)$: $\mathrm{pd}(M) < \infty\}$ follows from [13, Proposition 6]. The last inequality is obvious. Next we shall show $\sup\{\mathrm{pd}(M)$: $\mathrm{pd}(M) < \infty\} \leq \sup\{\mathrm{id}(P)$: P is a projective right R-module$\}$. In fact, we may assume $\sup\{\mathrm{id}(P)$: P is a projective right R-module$\}$ $= m < \infty$. Let M be any right R-module with $\mathrm{pd}(M) = n < \infty$. We claim that $n \leq m$. Otherwise, let $n > m$. For any right R-module N, there exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces the exact sequence

$$\mathrm{Ext}_R^{n}(M, P) \to \mathrm{Ext}_R^{n}(M, N) \to \mathrm{Ext}_R^{n+1}(M, K).$$

Note that $\mathrm{Ext}_R^{n}(M, P) = 0$ since $\mathrm{id}(P) \leq m < n$, and $\mathrm{Ext}_R^{n+1}(M, K) = 0$ since $\mathrm{pd}(M) = n$. Thus $\mathrm{Ext}_R^{n}(M, N) = 0$, and hence $\mathrm{pd}(M) \leq n - 1$, this is a contradiction. So $n \leq m$, as required.

The last statement is obvious.

(3). The inequalities r.cot.D$(R) \geq \sup\{\mathrm{pd}(F)$: F is a flat cotorsion right R-module$\}$ $\geq \sup\{\mathrm{pd}(C(F))$: F is a flat right R-module$\}$ are clear since cotorsion envelopes of flat modules are always flat. Next we shall show that r.cot.D$(R) \leq \sup\{\mathrm{pd}(C(F))$: F is a flat right R-module$\}$. Assume $\sup\{\mathrm{pd}(C(F))$: F is a flat right R-module$\} = m < \infty$. For any flat right R-module F, $\mathrm{cd}(F) = t < \infty$ since r.cot.D$(R) < \infty$. Thus, by Corollary 7.2.2, M admits a minimal cotorsion resolution

$$0 \to F \to C^0 \to C^1 \to \cdots \to C^{t-1} \to C^t \to 0.$$

Note that each C^i is a cotorsion envelope of the flat right R-module L^i, $i = 0, 1, \ldots, t$. By hypothesis, $\mathrm{pd}(C^i) \leq m$, $i = 0, 1, \ldots, t$. Therefore $\mathrm{pd}(F) \leq m$. So r.cot.D$(R) = \sup\{\mathrm{pd}(F)$: F is a flat right R-module$\} \leq m$. Thus r.cot.D$(R) = \sup\{\mathrm{pd}(M)$: M is flat$\} \geq \sup\{\mathrm{pd}(F(M))$: M is cotorsion$\}$ $\geq \sup\{\mathrm{pd}(M)$: M is flat cotorsion$\}$ = r.cot.D(R), and hence r.cot.D$(R) = \sup\{\mathrm{pd}(F(M))$: M is a cotorsion right R-module$\}$ follows.

Now we prove that r.cot.D$(R) = \sup\{\mathrm{cd}(P)$: P is a projective right R-module$\}$. Let $\sup\{\mathrm{cd}(P)$: P is a projective right R-module$\} = n < \infty$. For any flat right R-module F, $\mathrm{pd}(F) = m < \infty$ since r.cot.D$(R) < \infty$. Thus there exists an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to F \to 0,$$

where P_i is projective, $i = 0, 1, \ldots, m$. Thus $\mathrm{cd}(F) \leq n$ by hypothesis and Proposition 7.2.3. This completes the proof. □

Remark Note that pure injective modules are cotorsion, so [14, Proposition 1.1(a)] (that asserts $\sup\{\mathrm{pd}(F)$: F is a flat right R-module$\} \leq$ right pure global dimension of the ring R) is an immediate consequence of Theorem 7.2.5 (1).

Corollary 7.2.6 *Let R be a ring, then the following are equivalent for an integer $n \geq 0$.*

1. r.cot.D$(R) \leq n$.
2. *All flat right R-modules are of projective dimension $\leq n$.*
3. *All flat right R-modules are of cotorsion dimension $\leq n$.*
4. r.cot.D$(R) < \infty$*, and all flat cotorsion right R-modules are of projective dimension $\leq n$.*
5. r.cot.D$(R) < \infty$*, and all projective right R-modules are of cotorsion dimension $\leq n$.*
6. $\mathrm{Ext}_R^{n+1}(M, N) = 0$ *for all flat right R-modules M and N.*
7. $\mathrm{Ext}_R^{n+j}(M, N) = 0$ *for all flat right R-modules M, N and $j \geq 1$.*

Remark The equivalences of (2), (6) and (7) of Corollary 7.2.6 appeared in [9, Theorem 8.4.12] under the hypothesis that R is left coherent.

By [12, Corollary 10], if every projective right R-module is cotorsion, then R is right perfect. So we obtain some characterizations of right perfect rings by specializing Corollary 7.2.6 to the case $n = 0$. The equivalences of (2) through (4) in the following corollary are due to Xu [27, Proposition 3.3.1].

Corollary 7.2.7 *The following are equivalent for any ring R:*

1. r.cot.D$(R) = 0$.
2. *Every right R-module is cotorsion.*
3. *R is right perfect.*
4. *Every flat right R-module is cotorsion.*
5. *Every projective right R-module is cotorsion.*
6. r.cot.D$(R) < \infty$*, and every flat cotorsion right R-module is projective.*
7. $\mathrm{Ext}_R^1(M, N) = 0$ *for all flat right R-modules M and N.*

Remark By Corollary 7.2.7, r.cot.D(R) measures how far away a ring is from being right perfect. It is well known that right perfect rings need not be left perfect (see [1, p.322]), so r.cot.D$(R) \neq$ l.cot.D(R) in general.

Let R be a ring. It is well known that rD$(R) \leq 1$ if and only if every quotient module of any injective right R-module is injective. Here we prove that r.cot.D$(R) \leq 1$ if and only if every quotient module of any cotorsion right R-module is cotorsion as shown in the following theorem.

Theorem 7.2.8 *Let R be a ring, then the following are equivalent:*

1. r.cot.D$(R) \leq 1$.
2. *All flat right R-modules are of projective dimension ≤ 1.*
3. *All flat right R-modules are of cotorsion dimension ≤ 1.*
4. *Every quotient module of any injective right R-module is cotorsion.*
5. *Every quotient module of any cotorsion right R-module is cotorsion.*

6. *Every pure submodule module of any projective right R-module is projective.*

Proof (1) $\Rightarrow$ (4). Let E be any injective right R-module and K a submodule of E. The exactness of the sequence $0 \to K \to E \to E/K \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}_R^1(F, E) \to \mathrm{Ext}_R^1(F, E/K) \to \mathrm{Ext}_R^2(F, K),$$

where F is a flat right R-module. Note that $\mathrm{Ext}_R^2(F, K) = 0$ by (1) and Proposition 7.2.1, so $\mathrm{Ext}_R^1(F, E/K) = 0$, as required.

(4) $\Rightarrow$ (1). Let M be any right R-module. Then there exists an exact sequence $0 \to M \to E \to E/M \to 0$ with E injective. Thus $\mathrm{cd}(M) \leq 1$ since E/M is cotorsion, and hence $\mathrm{r.cot.D}(R) \leq 1$.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Corollary 7.2.6.

(2) $\Rightarrow$ (6). Let M be a projective right R-module and N a pure submodule of M. Then $0 \to N \to M \to M/N \to 0$ is exact. Note that M/N is flat and hence $\mathrm{pd}(M/N) \leq 1$ by (2). Thus N is projective.

(6) $\Rightarrow$ (2). Let M be any flat right R-module. There exists an exact sequence $0 \to N \to P \to M \to 0$ with P projective. Note that N is a pure submodule of P, so N is projective. It follows that $\mathrm{pd}(M) \leq 1$.

(5) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (5). Let M be any cotorsion right R-module and N any submodule of M. There exists an exact sequence $0 \to N \to E(N) \to L \to 0$. Consider the following pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E(N) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & M/N \arrow[r] & 0 \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

By (4), L is cotorsion. Since M is cotorsion, H is cotorsion by [27, Proposition 3.1.2]. Note that $E(N)$ is cotorsion, it follows that M/N is cotorsion by [27, Proposition 3.1.2] again. $\square$

We note that the equivalences of (2), (4), (5) and (6) in the previous theorem have recently been proven for commutative domains ([15, Theorem 3.2]).

By [27, Theorem 3.3.2], a ring R is von Neumann regular if and only if every cotorsion right R-module is flat. Replacing "flat" with "projective", we have the following

Proposition 7.2.9 *Let R be a ring. Then the following are equivalent:*

1. *R is a semisimple Artinian ring.*
2. *Every cotorsion right R-module is projective.*
3. *$\mathrm{r.cot.D}(R) \leq 1$ and the cotorsion envelope of every simple right R-module is projective.*

Proof (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1). It is easy to see that R is quasi-Frobenius and von Neumann regular, and hence (1) follows.

(3) $\Rightarrow$ (1). By (3), every simple right R-module M is a pure submodule of a projective right R-module, and hence M is projective by Theorem 7.2.8. So (1) follows. $\square$

We know that the cotorsion envelope of any flat right R-module is always flat. Rothmaler [22] has discussed when the pure-injective envelope of any flat right R-module is flat. It is natural to consider the condition that the cotorsion (pure-injective) envelope of any projective right R-module is projective. For a ring with this condition, we have the following

Theorem 7.2.10 *Let R be a ring such that the cotorsion envelope of any projective right R-module is projective. Then the following are equivalent:*

1. *$r.cot.D(R) \leq 1$.*
2. *The projectivity of $C(M)$ implies the projectivity of M for any right R-module M.*

If "cotorsion envelope" is replaced with "pure-injective envelope", the result still holds.

Proof (1) $\Rightarrow$ (2). Assume M is a right R-module such that $C(M)$ is projective. Note that M is a pure submodule of $C(M)$, so M is projective by Theorem 7.2.8.

(2) $\Rightarrow$ (1). Let M be a pure submodule of a projective right R-module P, it is enough to show that M is projective by Theorem 7.2.8. In fact, there is an exact sequence

$$0 \longrightarrow M \xrightarrow{f} P \longrightarrow L \longrightarrow 0\,,$$

where L is flat. By the defining property of cotorsion envelope, there exists $g : C(M) \to C(P)$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi"'] & P \arrow[d, "\psi"] \\ C(M) \arrow[r, "g"'] & C(P) \end{tikzcd}$$

commutes, i.e., $g\phi = \psi f$. Consider the pushout diagram of f and ϕ:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "\phi"'] & P \arrow[r] \arrow[d, "\gamma"'] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C(M) \arrow[r, "\alpha"'] & K \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

Note that the second row is split, so there is $\beta : K \to C(M)$ such that $\beta\alpha = 1$. It follows that $\beta\gamma : P \to C(M)$ factors through ψ. Hence there is $\sigma : C(P) \to C(M)$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\gamma"'] & C(P) \arrow[d, "\sigma"] \\ K \arrow[r, "\beta"'] & C(M) \end{tikzcd}$$

commutes, i.e., $\sigma\psi = \beta\gamma$. Then $\sigma g\phi = \sigma\psi f = \beta\gamma f = \beta\alpha\phi = \phi$. The defining property of cotorsion envelope now implies that σg is an automorphism of $C(M)$. Therefore $C(M)$ is isomorphic to a direct summand of $C(P)$. Since $C(P)$ is projective by hypothesis, $C(M)$ is projective. So M is projective by (2), as required.

The last statement can be proven similarly. $\square$

It is well known that $\mathrm{rD}(R) = \mathrm{wD}(R)$ when R is right perfect; $\mathrm{rD}(R) = \mathrm{r.cot.D}(R)$ when R is von Neumann regular by Theorem 7.2.5 (2). In general, we have the following inequality.

Theorem 7.2.11 *Let R be a ring, then $rD(R) \leq r.cot.D(R) + wD(R)$.*

Proof We may assume that both $\mathrm{r.cot.D}(R)$ and $\mathrm{wD}(R)$ are finite. Let $\mathrm{r.cot.D}(R) = m < \infty$ and $\mathrm{wD}(R) = n < \infty$. Suppose M is a right R-module, then M admits a flat resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Let $K_i = \mathrm{Ker}(F_i \to F_{i-1})$, $i = 0, 1, 2, \ldots, n-1$, $F_{-1} = M$, $F_n = K_{n-1}$. Then we have the following short exact sequences

$$0 \to F_n \to F_{n-1} \to K_{n-2} \to 0,$$

$$0 \to K_{n-2} \to F_{n-2} \to K_{n-3} \to 0,$$

$$\cdots\cdots,$$

$$0 \to K_0 \to F_0 \to M \to 0.$$

Note that $\mathrm{pd}(K_{n-2}) \leq 1 + \sup\{\mathrm{pd}(F_n), \mathrm{pd}(F_{n-1})\}$ by [23, Lemma 9.26]. Since $\mathrm{pd}(F_i) \leq m$, $i = 0, 1, \ldots, n$, it follows that $\mathrm{pd}(K_{n-2}) \leq 1 + m$, $\mathrm{pd}(K_{n-3}) \leq 2 + m, \cdots, \mathrm{pd}(M) \leq n + m$. This completes the proof. □

Remark In general, the inequality in Theorem 7.2.11 may be strict. Indeed, if R is right Noetherian, but not right perfect (e.g. the integer ring $\mathbb{Z}$), then $\mathrm{rD}(R) = wD(R)$ (see [23, Theorem 9.22]) and $\mathrm{r.cot.D}(R) \neq 0$. In this case, the inequality is strict. It is easy to verify that, if R is right Noetherian, then $\mathrm{rD}(R) = \mathrm{r.cot.D}(R) + \mathrm{wD}(R)$ if and only if R is right Artinian.

Recall that a ring R is called an *n-Gorenstein ring* if R is a left and right Noetherian ring with $\mathrm{id}({}_RR) \leq n$ and $\mathrm{id}(R_R) \leq n$ for an integer $n \geq 0$. For this ring, we have the following

Proposition 7.2.12 *If R is an n-Gorenstein ring, then $r.cot.D(R) \leq n$ and $l.cot.D(R) \leq n$.*

Proof Recall that a right R-module M is called FP-injective if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented right R-modules N. Note that a right R-module M is FP-injective if and only if M is injective when R is right Noetherian. It follows that $\mathrm{r.cot.D}(R) = \sup\{\mathrm{cd}(M)$: M is a flat right R-module$\} \leq \sup\{\mathrm{id}(M)$: M is a flat right R-module$\} = \mathrm{id}(R_R) \leq n$ by [5, Theorem 3.8]. The inequality $\mathrm{l.cot.D}(R) \leq n$ can be proven similarly. □

Corollary 7.2.13 *[8, Corollary 3.4]. If R is a 1-Gorenstein ring, then every quotient module of each injective right (left) R-module is cotorsion.*

Proof It follows from Proposition 7.2.12 and Theorem 7.2.8. □

For an exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if B and C are both cotorsion, we know $\mathrm{cd}(A) \leq 1$ by Proposition 7.2.3 (2). However A need not be cotorsion in general (see [27, p.75]). Next we discuss when A is cotorsion if B and C are.

Proposition 7.2.14 *Let R be a ring. Then the following are equivalent:*

1. *The cotorsion envelope of every flat right R-module is projective.*
2. *The flat cover of every cotorsion right R-module is projective.*
3. *Every flat cotorsion right R-module is projective.*

4. *Every flat right R-module is a pure submodule of some projective right R-module.*

Proof (1) $\Rightarrow$ (4). Let F be a flat right R-module. There exists an exact sequence $0 \to F \to C(F) \to L \to 0$. By (1), $C(F)$ is projective. Note that L is flat, so the exact sequence is pure, and (4) follows.

(4) $\Rightarrow$ (3). Let F be a flat cotorsion right R-module. By (4), there exists a projective right R-module P and a pure exact sequence $0 \to F \to P \to L \to 0$. Note that L is flat. It follows that the exact sequence is split. Thus F is projective.

(2) $\Leftrightarrow$ (3) $\Rightarrow$ (1) are easy. □

Proposition 7.2.15 *Let R be a ring satisfying the equivalent conditions in Proposition 7.2.14.*

1. *Assume $0 \to A \to B \to C \to 0$ is an exact sequence of right R-modules, then if two of A, B, C are cotorsion, so is the third.*

2. *r.cot.D$(R) = 0$ or r.cot.D$(R) = \infty$.*

Proof It is clear that (1) implies (2). We now prove (1).

It is enough to show that A is cotorsion if B and C are cotorsion by [27, Proposition 3.1.2]. Let F be any flat right R-module. By Proposition 7.2.14, there exists a pure exact sequence $0 \to F \to P \to L \to 0$ with P projective. Note that L is flat. The exact sequence $0 \to A \to B \to C \to 0$ gives rise to the following exact sequence

$$\mathrm{Ext}^1_R(L, C) \to \mathrm{Ext}^2_R(L, A) \to \mathrm{Ext}^2_R(L, B),$$

which implies $\mathrm{Ext}^2_R(L, A) = 0$ since the first term and the last term are both zero by hypothesis. In addition, the exact sequence $0 \to F \to P \to L \to 0$ yields the following exact sequence

$$\mathrm{Ext}^1_R(P, A) \to \mathrm{Ext}^1_R(F, A) \to \mathrm{Ext}^2_R(L, A).$$

Note that the first term and the last term are both zero, so $\mathrm{Ext}^1_R(F, A) = 0$. This completes the proof. □

We end this section with the following result which is of independent interest.

Recall that a ring R is called *left coherent* if every finitely generated left ideal is finitely presented.

Proposition 7.2.16 *Let R be a left coherent ring, then the following are equivalent:*

1. *R is right perfect.*

2. *R is a ring satisfying the equivalent conditions in Proposition 7.2.14.*

Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). For any family $\{R_i\}_{i\in I}$, where each $R_i \cong R$ is a right R-module, $\prod_{i\in I} R_i$ is a flat right R-module since R is left coherent. Hence we have an exact sequence

$$0 \to \prod_{i\in I} R_i \to C(\prod_{i\in I} R_i) \to L \to 0,$$

where $C(\prod_{i\in I} R_i)$ and L are flat by [27, Theorem 3.4.2]. By hypothesis, $C(\prod_{i\in I} R_i)$ is projective. Thus $\prod_{i\in I} R_i$ is a pure submodule of a projective right R-module, and hence it is a pure submodule of a free right R-module. It follows that R is a right perfect ring by [4, Theorem 3.1]. □

7.3 Cotorsion dimension under change of rings

We begin with the following

Proposition 7.3.1 *Let $\varphi : R \to S$ be a surjective ring homomorphism.*

1. *If M_S is a right S-module, then $cd(M_R) \leq cd(M_S)$. Moreover, if S_R is a flat right R-module, then $cd(M_S) = cd(M_R)$.*

2. *If S_R is a flat right R-module, and M_R is a cotorsion right R-module, then* $\mathrm{Hom}_R(S, M)$ *is a cotorsion right S-module, and hence a cotorsion right R-module.*

Proof (1). We may assume $\mathrm{cd}(M_S) = n < \infty$. Then there exists an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0,$$

where each C^i is a cotorsion right S-module, $i = 0, 1, \ldots, n$. By [27, Proposition 3.3.3], each C^i is also cotorsion as a right R-module. So $\mathrm{cd}(M_R) \leq n$.

If S_R is a flat right R-module, we claim $\mathrm{cd}(M_S) \leq \mathrm{cd}(M_R)$. In fact, we may assume $\mathrm{cd}(M_R) = n < \infty$. Let F be a flat right S-module, then F is a flat right R-module. Thus $\mathrm{Ext}_S^{n+1}(F_S, M_S) = \mathrm{Ext}_R^{n+1}(F_R, M_R) = 0$ by [23, Theorem 11.65]. Therefore $\mathrm{cd}(M_S) \leq n$, and hence $\mathrm{cd}(M_S) = \mathrm{cd}(M_R)$.

(2). By hypothesis, $\mathrm{Ext}_R^1(S, M) = 0$. Let X be a flat right S-module, then X is a flat right R-module. Thus

$$\mathrm{Ext}_S^1(X, \mathrm{Hom}_R(S, M)) = \mathrm{Ext}_R^1(X, M) = 0$$

by [24, Lemma 3.1]. Therefore $\mathrm{Hom}_R(S, M)$ is a cotorsion right S-module, and hence a cotorsion right R-module by [27, Proposition 3.3.3]. □

Corollary 7.3.2 *Let $\varphi : R \to S$ be a surjective ring homomorphism and S_R a flat right R-module, then r.cot.D$(S) \leq$ r.cot.D(R).*

Recall that a ring S is said to be an *almost excellent extension* of a ring R [28, 29] if the following conditions are satisfied:

1. S is a finite *normalizing extension* of a ring R [25], that is, R and S have the same identity and there are elements $s_1, \cdots, s_n \in S$ such that $S = Rs_1 + \cdots + Rs_n$ and $Rs_i = s_i R$ for all $i = 1, \cdots, n$.

2. ${}_RS$ is flat and S_R is projective.

3. S is right R-projective, that is, if M_S is a submodule of N_S and M_R is a direct summand of N_R, then M_S is a direct summand of N_S.

Further, S is an *excellent extension* of R if S is an almost excellent extension of R and S is free with basis $s_1, \cdots, s_n$ as both a right and a left R-module with $s_1 = 1_R$. The concept of excellent extension was introduced by Passman [18] and named by Bonami [3]. Examples of excellent extensions include finite matrix rings [18], and crossed product $R * G$ where G is a finite group with $|G|^{-1} \in R$ [19]. The notion of almost excellent extensions was introduced and studied in [28] as a non-trivial generalization of excellent extensions.

Let S be a finite normalizing extension (in particular, an (almost) excellent extension) of a ring R. It is well known that R is right perfect if and only if S is right perfect [21, Corollary 7]. It seems natural to generalize descent of right perfectness to cotorsion dimension in the case when S is an (almost) excellent extension of a ring R and this is the main goal of the rest of this section.

Theorem 7.3.3 *Let S be an almost excellent extension of a ring R and M_S a right S-module. Then*

1. $cd(M_S) = cd(M_R) = cd(\mathrm{Hom}_R(S, M))$.
2. *M_S is cotorsion if and only if M_R is cotorsion if and only if $\mathrm{Hom}_R(S, M)$ is a cotorsion right S-module.*

Proof (1). We first prove that $\mathrm{cd}(M_S) \leq \mathrm{cd}(M_R)$. We may assume that $\mathrm{cd}(M_R) = n < \infty$. Let N_S be a flat right S-module. Then N_R is a flat right R-module by [29, Lemma 1.2 (3)]. Note that $\mathrm{Ext}_R^{n+1}(N, M) \cong \mathrm{Ext}_S^{n+1}(N \otimes_R S, M)$ by [23, Theorem 11.65]. Since $\mathrm{Ext}_R^{n+1}(N, M) = 0$, $\mathrm{Ext}_S^{n+1}(N \otimes_R S, M) = 0$. Thus $\mathrm{Ext}_S^{n+1}(N, M) = 0$ by [29, Lemma 1.1 (1)], and so $\mathrm{cd}(M_S) \leq n$.

Conversely, suppose $\mathrm{cd}(M_S) = n < \infty$. Let N_R be a flat right R-module. Then $N \otimes_R S$ is a flat right S-module, and so $\mathrm{Ext}_R^{n+1}(N \otimes_R S, M) = 0$. Thus, by the above isomorphism, we get $\mathrm{Ext}_R^{n+1}(N, M) = 0$, and hence $\mathrm{cd}(M_R) \leq n$.

By [16, Lemma 2.16], if E_R is a cotorsion right R-module, then $\mathrm{Hom}_R(S, E)$ is a cotorsion right S-module. Hence $\mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq \mathrm{cd}(M_R)$ by Corollary 7.2.2. Since M_S is isomorphic to a direct summand of $\mathrm{Hom}_R(S, M)$ by [29, Lemma 1.1 (2)], $\mathrm{cd}(M_S) \leq \mathrm{cd}(\mathrm{Hom}_R(S, M))$. So (1) holds.

(2) follows from (1). □

Corollary 7.3.4 *Let R and S be rings.*

1. *If S is an almost excellent extension of R, then $r.cot.D(S) \leq r.cot.D(R)$.*
2. *If S is an excellent extension of R, then $r.cot.D(S) = r.cot.D(R)$.*

Proof (1) follows from Theorem 7.3.3.

(2). Since S is an excellent extension of R, R is an R-bimodule direct summand of S. Let ${}_RS_R = R \oplus T$, and M_R be any right R-module. Observe that $\mathrm{Hom}_R(S, M) \cong \mathrm{Hom}_R(R, M) \oplus \mathrm{Hom}_R(T, M)$. Therefore

$$\mathrm{cd}(M_R) \leq \mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq \mathrm{r.cot.D}(S)$$

by Theorem 7.3.3 (1), and hence $\mathrm{r.cot.D}(R) \leq \mathrm{r.cot.D}(S)$. So (2) follows from (1). □

Theorem 7.3.5 *Let S be an almost excellent extension of a ring R. If $r.cot.D(R) < \infty$, then $r.cot.D(S) = r.cot.D(R)$.*

Proof It is enough to show that $\mathrm{r.cot.D}(R) \leq \mathrm{r.cot.D}(S)$ by Corollary 7.3.4. Suppose $\mathrm{r.cot.D}(R) = n < \infty$. Then there exists a right R-module M such that $\mathrm{cd}(M_R) = n$. Define a right R-homomorphism $\alpha : \mathrm{Hom}_R(S, M) \to M$ via $\alpha(f) = f(1)$ for any $f \in \mathrm{Hom}_R(S, M)$. Since S_R is projective, the epimorphism $\pi : M \to M/\mathrm{im}(\alpha)$ induces the epimorphism $\pi_* : \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, M/\mathrm{im}(\alpha))$. Let $f \in \mathrm{Hom}_R(S, M)$ and $s \in S$. Then $\pi_*(f)(s) = \pi(f(s)) = \pi((fs)(1)) = \pi(\alpha(fs)) = 0$, and so $\ker(\pi_*) = \mathrm{Hom}_R(S, M)$. It follows that $\mathrm{Hom}_R(S, M/\mathrm{im}(\alpha)) = 0$, and hence $M/\mathrm{im}(\alpha) = 0$ by [25, Proposition 2.1]. Thus α is epic, and so we have a right R-module exact sequence $0 \to K \to \mathrm{Hom}_R(S, M) \to M \to 0$. By Propositon 7.2.3 (3), we have $n = \mathrm{cd}(M_R) \leq \sup\{\mathrm{cd}(\mathrm{Hom}_R(S, M)), \mathrm{cd}(K_R) - 1\} \leq \mathrm{r.cot.D}(R) = n$. Since $\mathrm{cd}(K_R) - 1 \leq n - 1$, then $\mathrm{cd}(\mathrm{Hom}_R(S, M)) = n$. On the other hand, $\mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq \mathrm{r.cot.D}(S)$ by Theorem 7.3.3. Therefore $\mathrm{r.cot.D}(R) \leq \mathrm{r.cot.D}(S)$, as desired. □

7.4 Applications in commutative rings

In this section, all rings are assumed to be commutative. We need the following lemma which will be frequently used in the sequel.

Lemma 7.4.1 *Let R be a ring and M an R-module, then the following are equivalent:*

1. *M is cotorsion.*
2. $\mathrm{Hom}_R(F, M)$ *is a cotorsion R-module for any flat R-module F.*
3. $\mathrm{Hom}_R(P, M)$ *is a cotorsion R-module for any projective R-module P.*

   Moreover, if the class of cotorsion R-modules is closed under direct sums, then the above conditions are also equivalent to

4. *$P \otimes_R M$ is a cotorsion R-module for any projective R-module P.*

Proof (1) ⇒ (2). Let N, F be two flat R-modules. There exists an exact sequence $0 \to K \to G \to N \to 0$ with G projective, which yields the exactness of the sequence $0 \to K \otimes_R F \to G \otimes_R F \to N \otimes_R F \to 0$. Note that $N \otimes_R F$ is flat. We have the following exact sequence

$$\mathrm{Hom}_R(G \otimes_R F, M) \to \mathrm{Hom}_R(K \otimes_R F, M) \to \mathrm{Ext}^1_R(N \otimes_R F, M) = 0,$$

which gives rise to the exactness of the sequence

$$\mathrm{Hom}_R(G, \mathrm{Hom}_R(F, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M)) \to 0.$$

On the other hand, the following sequence

$$\begin{aligned}\mathrm{Hom}_R(G, \mathrm{Hom}_R(F, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M)) \to& \\ \mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M)) \to \mathrm{Ext}^1_R(G, \mathrm{Hom}_R(F, M)) &= 0\end{aligned}$$

is exact. Thus $\mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M)) = 0$, and (2) follows.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) follows by letting $P = R$.

The last statement is easy to verify. □

Corollary 7.4.2 *Let R be a ring such that the class of cotorsion R-modules is closed under direct sums. Then the following are equivalent:*

1. *The cotorsion envelope of any projective R-module is always projective.*
2. *$C(R_R)$ is projective.*

Proof (1) ⇒ (2) is trivial.

(2) ⇒ (1). Consider the exact sequence $0 \to R \to C(R_R) \to N \to 0$. Let M be any projective R-module, then $0 \to R \otimes_R M \to C(R_R) \otimes_R M \to N \otimes_R M \to 0$ is also exact. Note that $C(R_R) \otimes_R M$ is projective, and cotorsion by Lemma 7.4.1. It follows that $M \to C(R_R) \otimes_R M$ is a cotorsion preenvelope of M since $N \otimes_R M$ is flat. Hence $C(M)$ is projective since it is a direct summand of $C(R_R) \otimes_R M$ by [9, Proposition 6.1.2]. □

The next proposition shows that if R is a Dedekind domain, then $\mathrm{Ext}_R^1(B, C)$ is cotorsion for all R-modules B and C, which may be viewed as an answer to [11, Problem 48, p.462].

Proposition 7.4.3 *Let R be a ring.*

1. *If $D(R) \leq 1$ (i.e., R is a hereditary ring), then $\mathrm{Ext}_R^1(B, C)$ is cotorsion for all R-modules B and C.*
2. *If cot.$D(R) \leq 1$, then $\mathrm{Ext}_R^1(F, M)$ is cotorsion for any flat R-module F and any R-module M.*

Proof (1) follows from the isomorphism

$$\mathrm{Ext}_R^1(\mathrm{Tor}_1^R(A, B), C) \cong \mathrm{Ext}_R^1(A, \mathrm{Ext}_R^1(B, C))$$

for all R-modules A, B and C (see [23, p.343]).

(2). Let M be any R-module. By hypothesis, there exists an exact sequence $0 \to M \to C^0 \to C^1 \to 0$, where C^0 and C^1 are cotorsion. So the sequence $\mathrm{Hom}_R(F, C^1) \to \mathrm{Ext}_R^1(F, M) \to \mathrm{Ext}_R^1(F, C^0) = 0$ is exact for any flat R-module F. By Lemma 7.4.1, $\mathrm{Hom}_R(F, C^1)$ is cotorsion, and hence $\mathrm{Ext}_R^1(F, M)$ is cotorsion by Theorem 7.2.8. □

We omit the proof of the next proposition which can be deduced easily from Lemma 7.4.1.

Proposition 7.4.4 *Let R be a ring and M an R-module. Then the following are equivalent:*

1. *$cd(M) \leq n$.*
2. *$cd(\mathrm{Hom}_R(P, M)) \leq n$ for any projective R-module P.*

We are now in a position to prove the following

Theorem 7.4.5 *Let $\varphi : R \to S$ be a surjective ring homomorphism with $K = Ker(\varphi)$. If S_R is projective, then, for any R-module M, either $cd(M_R) \leq sup\{pd(R/I)_R : I \subseteq K\}$, or $cd(M_R) = cd(\mathrm{Hom}_R(S, M))$, where $\mathrm{Hom}_R(S, M)$ may be regarded as an R-module or S-module.*

Proof Let $\sup\{\mathrm{pd}(R/I)_R : I \subseteq K\} = n$. We may assume $n < \infty$.

Suppose $\mathrm{cd}(M_R) > n$. We shall show that $\mathrm{cd}(M_R) = \mathrm{cd}(\mathrm{Hom}_R(S, M))$.

In fact, there exists an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0,$$

where each C^i is a cotorsion R-module, $i = 1, 2, \ldots, n-1$. Thus

$$\mathrm{cd}(M_R) = \mathrm{cd}(C^n) + n$$

by Corollary 7.2.4, and

$$\mathrm{Ext}_R^j(R/I, C^n) \cong \mathrm{Ext}_R^{n+j}(R/I, M) = 0$$

for all $j > 0$, and all $I \subseteq K$.

We claim that $\mathrm{cd}(C^n) = \mathrm{cd}(\mathrm{Hom}_R(S, C^n))$.

In fact, $\mathrm{cd}(\mathrm{Hom}_R(S, C^n)) \leq \mathrm{cd}(C^n)$ by Proposition 7.4.4. We only need to show that $\mathrm{cd}(C^n) \leq \mathrm{cd}(\mathrm{Hom}_R(S, C^n))$. Note that $C^n \cong \mathrm{Hom}_R(R, C^n)$ and the exactness of $0 \to K \to R \to S \to 0$ induces an exact sequence $0 \to \mathrm{Hom}_R(S, C^n) \to \mathrm{Hom}_R(R, C^n) \to \mathrm{Hom}_R(K, C^n) \to 0$. It is enough to show that $\mathrm{Hom}_R(K, C^n)$ is an injective R-module by Proposition 7.2.3 (1).

Let L be any ideal of R. The exactness of $0 \to K/LK \to R/LK \to R/K \to 0$ gives an exact sequence

$$\mathrm{Ext}_R^1(R/LK, C^n) \to \mathrm{Ext}_R^1(K/LK, C^n) \to \mathrm{Ext}_R^2(R/K, C^n).$$

Since $\mathrm{Ext}_R^1(R/LK, C^n) = \mathrm{Ext}_R^2(R/K, C^n) = 0$ by the first part of the proof, $\mathrm{Ext}_R^1(K/LK, C^n) = 0$. Hence the exact sequence $0 \to LK \to K \to K/LK \to 0$ yields the exactness of

$$\mathrm{Hom}_R(K, C^n) \to \mathrm{Hom}_R(LK, C^n) \to 0.$$

Note that

$$\mathrm{Hom}_R(R, \mathrm{Hom}_R(K, C^n)) \cong \mathrm{Hom}_R(K, C^n),$$
$$\mathrm{Hom}_R(L, \mathrm{Hom}_R(K, C^n)) \cong \mathrm{Hom}_R(L \otimes K, C^n) \cong \mathrm{Hom}_R(LK, C^n).$$

The last isomorphism holds by the flatness of K. Thus the sequence

$$\mathrm{Hom}_R(R, \mathrm{Hom}_R(K, C^n)) \to \mathrm{Hom}_R(L, \mathrm{Hom}_R(K, C^n)) \to 0$$

is exact, and so $\mathrm{Hom}_R(K, C^n)$ is R-injective, as required.

On the other hand, since S_R is projective, we have the following exact sequence

$$0 \to \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, C^0) \to$$
$$\mathrm{Hom}_R(S, C^1) \to \cdots \to \mathrm{Hom}_R(S, C^{n-1}) \to \mathrm{Hom}_R(S, C^n) \to 0,$$

where each $\mathrm{Hom}_R(S, C^i)$, $i = 1, 2, \ldots, n-1$, is a cotorsion R-module by Proposition 7.3.1 (2). Note that

$$\mathrm{cd}(\mathrm{Hom}_R(S, C^n)) = \mathrm{cd}(C^n) = \mathrm{cd}(M_R) - n > 0.$$

Thus $\mathrm{cd}(\mathrm{Hom}_R(S, M)) > n$, and so

$$\mathrm{cd}(\mathrm{Hom}_R(S, M)) = \mathrm{cd}(\mathrm{Hom}_R(S, C^n)) + n$$

by Corollary 7.2.4. It follows that $\mathrm{cd}(M_R) = \mathrm{cd}(\mathrm{Hom}_R(S, M))$, where $\mathrm{Hom}_R(S, M)$ may be regarded as an R-module or S-module by Proposition 7.3.1 (1). □

Corollary 7.4.6 *Let $\varphi : R \to S$ be a surjective ring homomorphism with $K = Ker(\varphi)$. If S_R is projective, then either cot.D$(R) \leq sup\{pd(R/I)_R : I \subseteq K\}$, or cot.D$(R)$ = cot.D(S).*

Proof Let $\sup\{\mathrm{pd}(R/I)_R : I \subseteq K\} = n$. If $\mathrm{cd}(M_R) \leq n$ for every R-module M_R, then cot.D$(R) \leq n$. If there is M_R such that $\mathrm{cd}(M_R) > n$, then $\mathrm{cd}(M_R) = \mathrm{cd}(\mathrm{Hom}_R(S, M)) \leq$ cot.D(S) by Theorem 7.4.5, and so cot.D$(R) \leq$ cot.D(S). Note that cot.D$(S) \leq$ cot.D(R) by Corollary 7.3.2. So cot.D(R) = cot.D(S). □

Corollary 7.4.7 *A ring R is perfect if and only if there is a quotient ring $S = R/K$ of R such that S is a perfect ring and R/I is a projective R-module for any $I \subseteq K$.*

Corollary 7.4.8 *Let K be a maximal ideal of a ring R such that R/K is a projective R-module, then cot.D$(R) \leq sup\{pd(R/I)_R : I \subseteq K\}$.*

Proposition 7.4.9 *Let P be any prime ideal of a ring R, then cot.D$(R_P) \leq$ cot.D(R), where R_P is the localization of R at P.*

Proof We may assume cot.D$(R) = n < \infty$. Let M be any flat R_P-module. Since R_P is a flat R-module, then M is a flat R-module. Thus $\mathrm{pd}(M_R) \leq n$. There exists a projective resolution of M_R

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0,$$

which induces an R_P-module exact sequence

$$0 \to (F_n)_P \to (F_{n-1})_P \to \cdots \to (F_1)_P \to (F_0)_P \to M_P \to 0.$$

Note that each $(F_i)_P$ is a projective R_P-module, $i = 0, 1, \ldots, n$, it follows that $\mathrm{pd}(M_P)_{R_P} \leq n$. Since $(M_P)_{R_P} \cong M_{R_P}$, $\mathrm{pd}(M_{R_P}) \leq n$. Thus cot.D$(R_P) \leq n$, as required. □

It is well known that R is a coherent ring if and only if $\mathrm{Hom}_R(A, B)$ is flat for all injective R-modules A and B ([17]). By [5, Corollary 3.22], R is an IF ring (the ring for which every injective R-module is flat) if and only if $\mathrm{Hom}_R(A, B)$ is injective for all injective R-modules A and B. Continuing this style of charactering rings by properties of homormophism modules of certain special R-modules, we conclude this paper with the following easy results for completeness.

Proposition 7.4.10 *Let R be a ring, then the following are equivalent:*

1. *R is a von Neumann regular ring.*
2. *For each cotorsion R-module A, $\mathrm{Hom}_R(A, B)$ is injective for all cotorsion (or injective) R-modules B.*
3. *For each cotorsion R-module A, $\mathrm{Hom}_R(A, B)$ is flat for all cotorsion (or injective) R-modules B.*

Proof (1) $\Rightarrow$ (2). Let A and B be cotorsion, then $\mathrm{Hom}_R(A, B)$ is cotorsion by Lemma 7.4.1 (for A is flat by (1)). Thus $\mathrm{Hom}_R(A, B)$ is injective by [27, Theorem 3.3.2].

(2) $\Rightarrow$ (1). Let A be a cotorsion R-module. (2) implies that $\mathrm{Hom}_R(A, -)$ preserves injectives. Thus A is flat by [10, Proposition 11.35], and (1) follows from [27, Theorem 3.3.2].

(1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let S be any simple R-module. Then S is cotorsion by [16, Lemma 2.14]. Let $E = E(\oplus_{i\in I} S_i)$, where $\{S_i\}_{i\in I}$ is an irredundant set of representatives of the simple R-modules. Then E is an injective cogenerator by [1, Corollary 18.19]. Note that $\mathrm{Hom}_R(S, E)$ is flat by (3) and $\mathrm{Hom}_R(S, E) \cong S$ by the proof of [26, Lemma 2.6]. Thus S is flat, and hence R is regular by [20, 3.3]. □

Proposition 7.4.11 *Let R be a ring, then the following are equivalent:*

1. *R is a semisimple Artinian ring.*
2. *For each cotorsion R-module A, $\mathrm{Hom}_R(A, B)$ is projective for all cotorsion (or injective) R-modules B.*

Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Let S be any simple R-module. By (2) and the proof of (3) $\Rightarrow$ (1) in Proposition 7.4.10, S is projective. So R is semisimple Artinian. □

Remark We wonder what kind of commutative rings is characterized by the condition that every homomorphism module of cotorsion modules is cotorsion. This kind of rings, of course, contains perfect rings and von Neumann regular rings. It is easy to verify that a ring R is of this kind if and only if $\mathrm{Hom}_R(A, B)$ is cotorsion for all R-modules A and all cotorsion R-modules B.

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