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WEAK GLOBAL DIMENSION OF COHERENT RINGS

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In this article, we study the weak global dimension of coherent rings in terms of the left FP-injective resolutions of modules. Let R be a left coherent ring and $\mathscr{F}\mathscr{I}$ the class of all FP-injective left R-modules. It is shown that $wD(R) \leq n$ $(n \geq 1)$ if and only if every nth $\mathscr{F}\mathscr{I}$-syzygy of a left R-module is FP-injective; and $wD(R) \leq n$ $(n \geq 2)$ if and only if every $(n-2)$th $\mathscr{F}\mathscr{I}$-syzygy in a minimal $\mathscr{F}\mathscr{I}$-resolution of a left R-module has an FP-injective cover with the unique mapping property. Some results for the weak global dimension of commutative coherent rings are also given.

Key Words: *FP*-injective dimension; (Pre)cover; Syzygy; Weak global dimension.

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1. INTRODUCTION

We first recall some known notions and facts needed in the article.

Let R be a ring. A left R-module M is called *FP-injective* (or *absolutely pure*) (Megibben, 1970; Stenström, 1970) if $\mathrm{Ext}^1(N, M) = 0$ for any finitely presented left R-module N. The *FP-injective dimension of M*, denoted by *FP-id*(M), is defined to be the smallest non-negative integer n such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for any finitely presented left R-module F. If no such n exists, set *FP-id*$(M) = \infty$.

Let $\mathscr{C}$ be a class of R-modules and M an R-module. Following Enochs (1981), we say that a homomorphism $\phi : C \to M$ is a *$\mathscr{C}$-precover* if $C \in \mathscr{C}$ and the abelian group homomorphism $\mathrm{Hom}(C', \phi) : \mathrm{Hom}(C', C) \to \mathrm{Hom}(C', M)$ is surjective for each $C' \in \mathscr{C}$. A $\mathscr{C}$-precover $\phi : C \to M$ is said to be a *$\mathscr{C}$-cover* if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. A $\mathscr{C}$-cover $\phi : C \to M$ is said to have the *unique mapping property* (Ding, 1996) if for any homomorphism $f: C' \to M$ with $C' \in \mathscr{C}$, there is a unique homomorphism $g : C' \to C$ such that $\phi g = f$. Dually we have the definitions of a $\mathscr{C}$-(pre)envelope (with the

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unique mapping property). $\mathscr{C}$-covers ($\mathscr{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

In what follows, we write $\mathscr{F}\mathscr{I}$ for the class of all *FP*-injective left *R*-modules. It has been recently proven that every left *R*-module has an *FP*-injective (pre)cover over a left coherent ring *R* (see Pinzon, 2005), so every left *R*-module *N* has a *left $\mathscr{F}\mathscr{I}$-resolution*, that is, there is a $\mathrm{Hom}(\mathscr{F}\mathscr{I}, -)$ exact complex $\cdots \to F_1 \to F_0 \to N \to 0$ (not necessarily exact) with each F_i *FP*-injective.

Let $K_0 = N$, $K_1 = \ker(F_0 \to N)$, $K_i = \ker(F_{i-1} \to F_{i-2})$ for $i \geq 2$. The nth kernel K_n $(n \geq 0)$ is called the nth *$\mathscr{F}\mathscr{I}$-syzygy of N*.

A left $\mathscr{F}\mathscr{I}$-resolution $\cdots \to F_1 \to F_0 \to N \to 0$ is said to be *minimal* if each $F_i \to K_i$ is an *FP*-injective cover.

It is well known that every left *R*-module has an injective (pre)cover if and only if *R* is a left Noetherian ring (see Enochs, 1981). Enochs and Jenda investigated the global dimension of a left Noetherian ring using the left injective resolutions of left *R*-modules (see Enochs and Jenda, 1985, 1991, 2000; Jenda, 1986). The main purpose of this article is to generalize their results to the case when *R* is a left coherent ring. We shall study the weak global dimension of a left coherent ring *R* in terms of the left *FP*-injective resolutions of left *R*-modules. Let *R* be a left coherent ring. It is shown that $wD(R) \leq n$ $(n \geq 1)$ if and only if every nth $\mathscr{F}\mathscr{I}$-syzygy of a left *R*-module is *FP*-injective if and only if every $(n-1)$th $\mathscr{F}\mathscr{I}$-syzygy of a left *R*-module has an *FP*-injective cover which is a monomorphism; and $wD(R) \leq n$ $(n \geq 2)$ if and only if every $(n-2)$th $\mathscr{F}\mathscr{I}$-syzygy in a minimal left $\mathscr{F}\mathscr{I}$-resolution of a left *R*-module has an *FP*-injective cover with the unique mapping property. If *R* is a commutative coherent ring, we prove that $wD(R) \leq n$ $(n \geq 2)$ if and only if every $(n-2)$th $\mathscr{F}\mathscr{I}$-syzygy in a minimal left $\mathscr{F}\mathscr{I}$-resolution of a pure-injective *R*-module has an (*FP*-)injective cover with the unique mapping property if and only if $FP\text{-}id(R^+ \otimes M) \leq n-2$ for any *R*-module *M* if and only if $fd(\mathrm{Hom}(R^+, M)) \leq n-2$ for any (pure-injective) *R*-module *M*.

Throughout this article, *R* is an associative ring with identity and all modules are unitary. $wD(R)$ stands for the weak global dimension of a ring *R*. ${}_RM$ denotes a left *R*-module. For an *R*-module *M*, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, and $fd(M)$ is the flat dimension of *M*. Let *M* and *N* be two *R*-modules. $\mathrm{Hom}(M, N)$ (resp., $M \otimes N$) means $\mathrm{Hom}_R(M, N)$ (resp., $M \otimes_R N$), and similarly $\mathrm{Ext}^n(M, N)$ denotes $\mathrm{Ext}^n_R(M, N)$ for an integer $n \geq 1$. For unexplained concepts and notations, we refer the reader to Enochs and Jenda (2000), Rotman (1979), and Xu (1996).

2. WEAK GLOBAL DIMENSION OF COHERENT RINGS

We begin with the following lemmas.

Lemma 2.1. *Let R be a left coherent ring and M a left R-module. Then FP-id*$(M) \leq n$ $(n \geq 0)$ *if and only if for every left $\mathscr{F}\mathscr{I}$-resolution* $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ *of any left R-module N*, $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ *is an epimorphism, where K_n is the nth $\mathscr{F}\mathscr{I}$-syzygy of N.*

Proof. The proof is modelled on that of Jenda (1986, Lemma 2.2).

We proceed by induction on n. Let $n = 0$. If M is FP-injective, then it is clear that $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, K_0)$ is an epimorphism. Conversely, put $N = M$. Then $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, M)$ is an epimorphism. Thus M is isomorphic to a direct summand of F_0, and so M is FP-injective.

Let $n \geq 1$. By Göbel and Trlifaj (2006, Theorem 4.1.6), there is an exact sequence $0 \to M \to E \to L \to 0$ with E FP-injective and $\mathrm{Ext}^1(L, G) = 0$ for all FP-injective left R-modules G. Thus we have the following exact commutative diagrams:

$$\begin{array}{ccccc}
\mathrm{Hom}(E, F_n) & \longrightarrow & \mathrm{Hom}(E, K_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(M, F_n) & \longrightarrow & \mathrm{Hom}(M, K_n) & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(L, K_n) & \longrightarrow & \mathrm{Hom}(L, F_{n-1}) & \longrightarrow & \mathrm{Hom}(L, K_{n-1}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(E, K_n) & \longrightarrow & \mathrm{Hom}(E, F_{n-1}) & \longrightarrow & \mathrm{Hom}(E, K_{n-1}) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(M, K_n) & \longrightarrow & \mathrm{Hom}(M, F_{n-1}) & \longrightarrow & \mathrm{Hom}(M, K_{n-1}) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Thus $FP\text{-}id(M) \leq n$ if and only if $FP\text{-}id(L) \leq n-1$ by Stenström (1970, Lemma 3.1) if and only if $\mathrm{Hom}(L, F_{n-1}) \to \mathrm{Hom}(L, K_{n-1})$ is an epimorphism by induction if and only if $\mathrm{Hom}(E, K_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the first diagram. □

Lemma 2.2. *Let R be a left coherent ring and M a left R-module. If $\mathrm{Ext}^1(F, M) = 0$ for all FP-injective left R-modules F, then M has an FP-injective cover $E \to M$ with E injective.*

Proof. Let $f : F \to M$ be an FP-injective cover of M. Consider the short exact sequence $0 \to F \stackrel{i}{\to} E \to L \to 0$ with E injective. Note that L is FP-injective by Stenström (1970, Lemma 3.1). So there exists $g : E \to M$ such that $gi = f$ since

$\mathrm{Ext}^1(L, M) = 0$. Thus there exists $h : E \to F$ such that $fh = g$ since f is a cover. Therefore $fhi = f$, and hence hi is an isomorphism. It follows that F is injective. □

We are now in a position to prove the following theorem.

Theorem 2.3. *The following are equivalent for a left coherent ring R and $n \geq 1$:*

(1) $wD(R) \leq n$;
(2) *Every nth $\mathcal{FI}$-syzygy of a left R-module is FP-injective*;
(3) *Every nth $\mathcal{FI}$-syzygy of a left R-module has FP-injective dimension* ≤ 1;
(4) *Every $(n-1)$th $\mathcal{FI}$-syzygy of a left R-module has an FP-injective cover which is a monomorphism*;
(5) *For every left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of a left R-module N, $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism for any pure-injective left R-module M, where K_n is the nth $\mathcal{FI}$-syzygy of N*;
(6) *$wD(R) < \infty$ and every left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of a left R-module N is exact at F_i, $i \geq n-1$, where $F_{-1} = N$*;
(7) *$wD(R) < \infty$ and every nth $\mathcal{FI}$-syzygy of a left R-module N has an FP-injective cover which is an epimorphism*;
Moreover, if $n \geq 2$, then the above conditions are equivalent to:
(8) *Every $(n-2)$th $\mathcal{FI}$-syzygy in a minimal left $\mathcal{FI}$-resolution of a left R-module has an FP-injective cover with the unique mapping property.*

Proof. (1) ⇒ (2) Let K_n be an nth $\mathcal{FI}$-syzygy of a left R-module. Then $FP\text{-}id(K_n) \leq wD(R) \leq n$ by Stenström (1970, Theorem 3.3), and so $\mathrm{Hom}(K_n, F_n) \to \mathrm{Hom}(K_n, K_n)$ is an epimorphism by Lemma 2.1. Therefore, K_n is FP-injective.

(2) ⇒ (3) is trivial.

(3) ⇒ (4) Let $f : F_{n-1} \to K_{n-1}$ be an FP-injective precover of K_{n-1} and $K_n = \ker(f)$. Then $FP\text{-}id(K_n) \leq 1$ by (3). So $\mathrm{im}(f)$ is FP-injective by Stenström (1970, Lemma 3.1). Thus the inclusion $\mathrm{im}(f) \to K_{n-1}$ is an FP-injective cover which is a monomorphism.

(4) ⇒ (2) Let $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ be any left $\mathcal{FI}$-resolution of a left R-module N and $K_{n-1} = \ker(F_{n-2} \to F_{n-3})$, $K_n = \ker(F_{n-1} \to F_{n-2}) = \ker(F_{n-1} \to K_{n-1})$. Since K_{n-1} has a monic FP-injective cover $E \to K_{n-1}$ by (4), we have $K_n \oplus E \cong F_{n-1}$ by Enochs and Jenda (2000, Lemma 8.6.3). Hence K_n is FP-injective.

(2) ⇒ (5) For any left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of a left R-module N, $F_n \to K_n$ is a split epimorphism by (2). So $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism for any (pure-injective) left R-module M.

(5) ⇒ (1) Let M be any right R-module. Then M^+ is pure-injective. Hence $FP\text{-}id(M^+) \leq n$ by (5) and Lemma 2.1. Thus $fd(M) \leq n$ by Fieldhouse (1972, Theorem 2.1), and so $wD(R) \leq n$.

(1) $\Rightarrow$ (6) Since $FP\text{-}id(_RR) \leq wD(R) \leq n$, $\mathrm{Hom}(_RR, F_n) \to \mathrm{Hom}(_RR, K_n)$ is an epimorphism by Lemma 2.1, that is, $F_n \to K_n$ is an epimorphism. It follows that $F_n \to F_{n-1} \to F_{n-2}$ is exact. In addition, $FP\text{-}id(_RR) \leq k$ for any $k \geq n+1$ by (1). So $F_k \to F_{k-1} \to F_{k-2}$ is exact, and hence (6) holds.

(6) $\Rightarrow$ (7) is clear since $F_n \to F_{n-1} \to F_{n-2}$ is exact.

(7) $\Rightarrow$ (1) Note that $FP\text{-}id(_RR) \leq n$ by (7) and Lemma 2.1. Thus $wD(R) = FP\text{-}id(_RR) \leq n$ by Stenström (1970, Proposition 3.5) since $wD(R) < \infty$.

(4) $\Rightarrow$ (8) Suppose that $F_{n-3} \to F_{n-4} \to \cdots \to F_1 \to F_0 \to N \to 0$ is a partial minimal left $\mathcal{FI}$-resolution of a left R-module N with $K_{n-2} = \ker(F_{n-3} \to F_{n-4})$. Let $f : F_{n-2} \to K_{n-2}$ be the FP-injective cover of K_{n-2}. Then $K_{n-1} = \ker(F_{n-2} \to F_{n-3})$ has a monic FP-injective cover $i : F_{n-1} \to K_{n-1}$ by (4). Note that $\mathrm{Ext}^1(G, K_{n-1}) = 0$ for any FP-injective left R-module G by Wakamatsu's Lemma (see Xu, 1996, Lemma 2.1.1). Thus F_{n-1} is injective by Lemma 2.2. But K_{n-1} has no nonzero injective submodule by Xu (1996, Corollary 1.2.8). Thus $F_{n-1} = 0$, and hence $\mathrm{Hom}(F, K_{n-1}) \cong \mathrm{Hom}(F, F_{n-1}) = 0$ for any FP-injective left R-module F. Therefore, the exact sequence $0 \to K_{n-1} \to F_{n-2} \xrightarrow{f} K_{n-2}$ yields the desired exactness of $0 \to \mathrm{Hom}(F, F_{n-2}) \xrightarrow{f_*} \mathrm{Hom}(F, K_{n-2}) \to 0$.

(8) $\Rightarrow$ (2) Let $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ be any left $\mathcal{FI}$-resolution of a left R-module N with $K_n = \ker(F_{n-1} \to F_{n-2})$. By (8), N has a left $\mathcal{FI}$-resolution of the form $0 \to E_{n-2} \to E_{n-3} \to \cdots \to E_1 \to E_0 \to N \to 0$. Thus by Enochs and Jenda (2000, Corollary 8.6.4), we have $K_n \oplus F_{n-2} \oplus E_{n-3} \oplus \cdots \cong F_{n-1} \oplus E_{n-2} \oplus F_{n-3} \oplus \cdots$. Therefore, K_n is FP-injective, as required. □

It is known that R is a left Noetherian ring with left global dimension at most 2 if and only if every left R-module has an injective cover with the unique mapping property (see Enochs and Jenda, 2000, Remark 8.4.18); R is a left Noetherian and left hereditary ring if and only if every left R-module has an injective cover which is a monomorphism (see Enochs and Jenda, 1991, Corollary 3.4 or Rada and Saorin, 1998, Corollary 4.12). Now we have the following corollary.

Corollary 2.4. *Let R be a ring. Then*:

(1) *R is left coherent and $wD(R) \leq 2$ if and only if every left R-module has an FP-injective cover with the unique mapping property*;
(2) *R is left semihereditary if and only if every left R-module has an FP-injective cover which is a monomorphism.*

Proof. (1) The necessity follows from Theorem 2.3. For the sufficiency, we shall prove that the class of FP-injective left R-modules is closed under direct limits. Let $\{C_i, \varphi_j^i\}$ be a direct system of left R-modules with each C_i FP-injective. By hypothesis, $\varinjlim C_i$ has an FP-injective cover $\alpha : E \to \varinjlim C_i$ with the unique mapping property. Let $\alpha_i : C_i \to \varinjlim C_i$ satisfy $\alpha_i = \alpha_j \varphi_j^i$ whenever $i \leq j$. Then there exists $f_i : C_i \to E$ such that $\alpha_i = \alpha f_i$ for any i. It follows that $\alpha f_i = \alpha f_j \varphi_j^i$, and so $f_i = f_j \varphi_j^i$ whenever $i \leq j$. Therefore, by the definition of direct limits, there exists $\beta : \varinjlim C_i \to E$ such that $f_i = \beta \alpha_i$. So $(\alpha\beta)\alpha_i = \alpha(\beta\alpha_i) = \alpha f_i = \alpha_i$ for any i. Therefore $\alpha\beta = 1_{\varinjlim C_i}$

by the definition of direct limits, and hence $\varinjlim C_i$ is a direct summand of E. So $\varinjlim C_i$ is FP-injective. Thus R is a left coherent ring by Stenström (1970, Theorem 3.2), and so $wD(R) \leq 2$ by Theorem 2.3.

(2) The necessity comes from Theorem 2.3. Conversely, every left R-module has an FP-injective cover with the unique mapping property since every left R-module has a monic FP-injective cover. So R is left coherent by (1), and hence $wD(R) \leq 1$ by Theorem 2.3. Thus R is left semihereditary. □

Remark 2.5. (1) Corollary 2.4 (1) may be viewed as the dual of Asensio Mayor and Martínez Hernández (1993, Proposition 2.1) which says that a ring R is left coherent and $wD(R) \leq 2$ if and only if every left R-module has a flat envelope with the unique mapping property.

(2) Corollary 2.4 (2) has been proven in Chen and Ding (1996) and Rada and Saorin (1998) in a different way.

Next we shall study the weak global dimension of commutative coherent rings.

Lemma 2.6. *Let R be a commutative ring. Then R is a coherent ring if and only if* $\mathrm{Hom}(R^+, E)$ *is flat for any injective R-module E.*

Proof. If R is coherent, then $\mathrm{Hom}(R^+, E)$ is flat by Matlis (1982, Theorem 1) (for R^+ is injective).

Conversely, we shall show that any direct product ΠR of R is flat. Indeed, since R is a pure submodule of R^{++}, ΠR is a pure submodule of ΠR^{++} by Cheatham and Stone (1981, Lemma 1(2)). Note that $\Pi R^{++} \cong \Pi\mathrm{Hom}(R^+ \otimes R, \mathbb{Q}/\mathbb{Z}) \cong \Pi\mathrm{Hom}(R^+, R^+) \cong \mathrm{Hom}(R^+, \Pi R^+)$. Thus ΠR^{++} is flat by hypothesis, and hence ΠR is flat. So R is coherent. □

Theorem 2.7. *The following are equivalent for a commutative coherent ring R and an integer $n \geq 2$:*

(1) $wD(R) \leq n$;
(2) *Every $(n-2)$th $\mathcal{FI}$-syzygy in a minimal left $\mathcal{FI}$-resolution of a pure-injective R-module has an FP-injective cover with the unique mapping property*;
(3) *Every $(n-2)$th $\mathcal{FI}$-syzygy in a minimal left $\mathcal{FI}$-resolution of a pure-injective R-module has an injective cover with the unique mapping property*;
(4) $FP\text{-}id(R^+ \otimes M) \leq n-2$ *for any R-module M*;
(5) $fd(\mathrm{Hom}(R^+, M)) \leq n-2$ *for any (pure-injective) R-module M*;
(6) $wD(R) < \infty$ *and* $FP\text{-}id(\mathrm{Hom}(R^+, E)) \leq n$ *for any injective R-module E*;
(7) $wD(R) < \infty$ *and* $fd(R^+ \otimes M) \leq n$ *for any flat R-module M.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 2.3.

(2) $\Rightarrow$ (3) Let $F_{n-3} \to \cdots \to F_1 \to F_0 \to M \to 0$ be a partial minimal left $\mathcal{FI}$-resolution of a pure-injective R-module M. By (2), $K_{n-2} = \ker(F_{n-3} \to F_{n-4})$ has an FP-injective cover $f : F_{n-2} \to K_{n-2}$ with the unique mapping property.

Suppose $n = 2$. There is a pure exact sequence $0 \to F_0 \xrightarrow{i} E \to L \to 0$ with E injective. Since M is pure-injective, there exists $g : E \to M$ such that $gi = f$. It is easy to check that $g : E \to M$ is an FP-injective precover of M. So F_0 is injective by Enochs and Jenda (2000, Proposition 5.1.2).

Suppose $n > 2$. Then $\mathrm{Ext}^1(G, K_{n-2}) = 0$ for any FP-injective R-module G by Wakamatsu's lemma. So F_{n-2} is injective by Lemma 2.2.

(3) $\Rightarrow$ (4) Let M be any R-module and $F_{n-3} \to \cdots \to F_1 \to F_0 \to M^+ \to 0$ be a minimal left $\mathcal{FI}$-resolution of M^+. Then $K_{n-2} = \ker(F_{n-3} \to F_{n-4})$ has an injective cover $F_{n-2} \to K_{n-2}$ with the unique mapping property by (3). Thus we get the following exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(R^+, F_{n-2}) &\to \mathrm{Hom}(R^+, F_{n-3}) \to \cdots \\ &\to \mathrm{Hom}(R^+, F_0) \to \mathrm{Hom}(R^+, M^+) \to 0. \end{aligned}$$

Since each $\mathrm{Hom}(R^+, F_i)$ is flat by Lemma 2.6, $fd(\mathrm{Hom}(R^+, M^+)) \leq n-2$. Note that $\mathrm{Hom}(R^+, M^+) \cong (R^+ \otimes M)^+$. Thus, by Fieldhouse (1972, Theorem 2.2), we have

$$FP - id(R^+ \otimes M) = fd((R^+ \otimes M)^+) = fd(\mathrm{Hom}(R^+, M^+)) \leq n-2.$$

(4) $\Rightarrow$ (1) Let M be a finitely presented R-module. Then $R^+ \otimes M \cong (\mathrm{Hom}(M, R))^+$ by Colby (1975, Lemma 2), and so

$$fd(\mathrm{Hom}(M, R)) = FP - id(\mathrm{Hom}(M, R)^+) = FP - id(R^+ \otimes M) \leq n-2$$

by (4) and Fieldhouse (1972, Theorem 2.1). Therefore, $wD(R) \leq n$ by Jones and Teply (1982, Lemma 4).

(1) $\Rightarrow$ (5) Let M be any R-module. Then there is an exact sequence $0 \to M \to E^0 \to E^1$ with E^0 and E^1 injective. So we obtain the exact sequence

$$0 \to \mathrm{Hom}(R^+, M) \to \mathrm{Hom}(R^+, E^0) \to \mathrm{Hom}(R^+, E^1) \to L \to 0.$$

Thus $fd(\mathrm{Hom}(R^+, M)) \leq n-2$ since $fd(L) \leq n$ by (1) and each $\mathrm{Hom}(R^+, E^i)$ is flat by Lemma 2.6.

(5) $\Rightarrow$ (4) For any R-module M, we have $FP\text{-}id(R^+ \otimes M) = fd((R^+ \otimes M)^+) = fd(\mathrm{Hom}(R^+, M^+)) \leq n-2$ by (5).

(1) $\Rightarrow$ (6) Let E be any injective R-module. Then $(\mathrm{Hom}(R^+, E))^+$ is injective since $\mathrm{Hom}(R^+, E)$ is flat by Lemma 2.6. Thus there exists a split exact sequence $0 \to (\mathrm{Hom}(R^+, E))^+ \to \Pi R^+$. Note that $fd(R^+) \leq n$ by (1), and so $fd(\Pi R^+) \leq n$. Thus $fd(\mathrm{Hom}(R^+, E))^+ \leq n$, and hence $FP\text{-}id(\mathrm{Hom}(R^+, E)) = fd(\mathrm{Hom}(R^+, E))^+ \leq n$.

(6) $\Rightarrow$ (7) Let M be any flat R-module. Then we have $FP\text{-}id((R^+ \otimes M)^+) = FP\text{-}id(\mathrm{Hom}(R^+, M^+)) \leq n$ by (6). Thus $fd(R^+ \otimes M) = FP\text{-}id((R^+ \otimes M)^+) \leq n$.

(7) $\Rightarrow$ (1) Note that $FP\text{-}id(R) = fd(R^+) = fd(R^+ \otimes R) \leq n$ by (7). So $wD(R) = FP\text{-}id(R) \leq n$ by Stenström (1970, Proposition 3.5) since $wD(R) < \infty$. $\square$

Letting $n = 2$ in Theorem 2.7, we have the following corollary.

Corollary 2.8. *The following are equivalent for a commutative coherent ring R:*

(1) $wD(R) \leq 2$;
(2) *Every pure-injective R-module has an (FP-)injective cover with the unique mapping property*;
(3) $R^+ \otimes M$ *is FP-injective for any R-module M*;
(4) $\mathrm{Hom}(R^+, M)$ *is flat for any (pure-injective) R-module M.*

We conclude the article with the following corollary.

Corollary 2.9. *Let R be a commutative ring. Then R is coherent and $wD(R) \leq 2$ if and only if* $\mathrm{Hom}(R^+, M)$ *is flat for any (pure-injective) R-module M.*

Proof. The result follows from Lemma 2.6 and Corollary 2.8. □

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