WEAK GLOBAL DIMENSION OF COHERENT RINGS

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In this article, we study the weak global dimension of coherent rings in terms of the left $FP$-injective resolutions of modules. Let $R$ be a left coherent ring and $\mathcal{F}$ the class of all $FP$-injective left $R$-modules. It is shown that $wD/lparenoriR/rparenori \leq n$ ($n \geq 1$) if and only if every $n$th $\mathcal{F}$-syzygy of a left $R$-module is $FP$-injective; and $wD/lparenoriR/rparenori \leq n$ ($n \geq 2$) if and only if every $(n-2)$th $\mathcal{F}$-syzygy in a minimal $\mathcal{F}$-resolution of a left $R$-module has an $FP$-injective cover with the unique mapping property. Some results for the weak global dimension of commutative coherent rings are also given.

Key Words: $FP$-injective dimension; (Pre)cover; Syzygy; Weak global dimension.

2000 Mathematics Subject Classification: 16E10; 16E05; 16D50.

1. INTRODUCTION

We first recall some known notions and facts needed in the article.

Let $R$ be a ring. A left $R$-module $M$ is called $FP$-injective (or absolutely pure) (Megibben, 1970; Stenström, 1970) if $\text{Ext}^1(N, M) = 0$ for any finitely presented left $R$-module $N$. The $FP$-injective dimension of $M$, denoted by $FP\text{-id}(M)$, is defined to be the smallest non-negative integer $n$ such that $\text{Ext}^{n+1}(F, M) = 0$ for any finitely presented left $R$-module $F$. If no such $n$ exists, set $FP\text{-id}(M) = \infty$.

Let $\mathcal{C}$ be a class of $R$-modules and $M$ an $R$-module. Following Enochs (1981), we say that a homomorphism $\phi : C \to M$ is a $\mathcal{C}$-precover if $C \in \mathcal{C}$ and the abelian group homomorphism $\text{Hom}(C', \phi) : \text{Hom}(C', C) \to \text{Hom}(C', M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $\phi : C \to M$ is said to be a $\mathcal{C}$-cover if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. A $\mathcal{C}$-cover $\phi : C \to M$ is said to have the unique mapping property (Ding, 1996) if for any homomorphism $f : C' \to M$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g : C' \to C$ such that $\phi g = f$. Dually we have the definitions of a $\mathcal{C}$-(pre)envelope (with the
unique mapping property). \( \mathcal{C} \)-covers (\( \mathcal{C} \)-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

In what follows, we write \( \mathcal{F} \mathcal{I} \) for the class of all FP-injective left \( R \)-modules. It has been recently proven that every left \( R \)-module has an FP-injective (pre)cover over a left coherent ring \( R \) (see Pinzon, 2005), so every left \( R \)-module \( N \) has a left \( \mathcal{F} \mathcal{I} \)-resolution, that is, there is a Hom(\( \mathcal{F} \mathcal{I}, - \)) exact complex \( \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0 \) (not necessarily exact) with each \( F_i \) FP-injective.

Let \( K_0 = N \), \( K_1 = \ker(F_0 \rightarrow N) \), \( K_i = \ker(F_{i-1} \rightarrow F_{i-2}) \) for \( i \geq 2 \). The \( n \)th kernel \( K_n \) \((n \geq 0)\) is called the \( n \)th \( \mathcal{F} \mathcal{I} \)-syzygy of \( N \).

A left \( \mathcal{F} \mathcal{I} \)-resolution \( \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0 \) is said to be minimal if each \( F_i \rightarrow K_i \) is an FP-injective cover.

It is well known that every left \( R \)-module has an injective (pre)cover if and only if \( R \) is a left Noetherian ring (see Enochs, 1981). Enochs and Jenda investigated the global dimension of a left Noetherian ring using the left injective resolutions of left \( R \)-modules (see Enochs and Jenda, 1985, 1991, 2000; Jenda, 1986). The main purpose of this article is to generalize their results to the case when \( R \) is a left coherent ring. We shall study the weak global dimension of a left coherent ring \( R \) in terms of the left FP-injective resolutions of left \( R \)-modules. Let \( R \) be a left coherent ring. It is shown that \( wD(R) \leq n \) \((n \geq 1)\) if and only if every \( n \)th \( \mathcal{F} \mathcal{I} \)-syzygy of a left \( R \)-module is FP-injective if and only if every \((n - 1)\)th \( \mathcal{F} \mathcal{I} \)-syzygy of a left \( R \)-module has a FP-injective cover which is a monomorphism; and \( wD(R) \leq n \) \((n \geq 2)\) if and only if every \((n - 2)\)th \( \mathcal{F} \mathcal{I} \)-syzygy of a left \( R \)-module has an FP-injective cover with the unique mapping property. If \( R \) is a commutative coherent ring, we prove that \( wD(R) \leq n \) \((n \geq 2)\) if and only if every \((n - 2)\)th \( \mathcal{F} \mathcal{I} \)-syzygy in a minimal left \( \mathcal{F} \mathcal{I} \)-resolution of a pure-injective \( R \)-module has an (FP-)injective cover with the unique mapping property if and only if \( FP-id(R^+ \otimes M) \leq n - 2 \) for any \( R \)-module \( M \) if and only if \( fd(\text{Hom}(R^+, M)) \leq n - 2 \) for any (pure-injective) \( R \)-module \( M \).

Throughout this article, \( R \) is an associative ring with identity and all modules are unitary. \( wD(R) \) stands for the weak global dimension of a ring \( R \). \( R_{-}M \) denotes a left \( R \)-module. For an \( R \)-module \( M \), the character module \( \text{Hom}_R(M, Q/Z) \) is denoted by \( M^+ \), and \( fd(M) \) is the flat dimension of \( M \). Let \( M \) and \( N \) be two \( R \)-modules. \( \text{Hom}(M, N) \) (resp., \( M \otimes_R N \)) means \( \text{Hom}_R(M, N) \) (resp., \( M \otimes_R N \)), and similarly \( \text{Ext}(M, N) \) and \( \text{Ext}_n(M, N) \) denote \( \text{Ext}(M, N) \) and \( \text{Ext}_n(M, N) \) for an integer \( n \geq 1 \). For unexplained concepts and notations, we refer the reader to Enochs and Jenda (2000), Rotman (1979), and Xu (1996).

### 2. WEAK GLOBAL DIMENSION OF COHERENT RINGS

We begin with the following lemmas.

**Lemma 2.1.** Let \( R \) be a left coherent ring and \( M \) a left \( R \)-module. Then \( FP-id(M) \leq n \) \((n \geq 0)\) if and only if for every left \( \mathcal{F} \mathcal{I} \)-resolution \( \cdots \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0 \) of any left \( R \)-module \( N \), \( \text{Hom}(M, F_n) \rightarrow \text{Hom}(M, K_n) \) is an epimorphism, where \( K_n \) is the \( n \)th \( \mathcal{F} \mathcal{I} \)-syzygy of \( N \).

**Proof.** The proof is modelled on that of Jenda (1986, Lemma 2.2).
We proceed by induction on \( n \). Let \( n = 0 \). If \( M \) is \( \text{FP} \)-injective, then it is clear that \( \text{Hom}(M, F_0) \to \text{Hom}(M, K_0) \) is an epimorphism. Conversely, put \( N = M \). Then \( \text{Hom}(M, F_0) \to \text{Hom}(M, M) \) is an epimorphism. Thus \( M \) is isomorphic to a direct summand of \( F_0 \), and so \( M \) is \( \text{FP} \)-injective.

Let \( n \geq 1 \). By Göbel and Trlifaj (2006, Theorem 4.1.6), there is an exact sequence \( 0 \to M \to E \to L \to 0 \) with \( E \) \( \text{FP} \)-injective and \( \text{Ext}^1(L, G) = 0 \) for all \( \text{FP} \)-injective left \( R \)-modules \( G \). Thus we have the following exact commutative diagrams:

\[
\begin{array}{ccc}
\text{Hom}(E, F_n) & \longrightarrow & \text{Hom}(E, K_n) \\
\downarrow & & \downarrow \\
\text{Hom}(M, F_n) & \longrightarrow & \text{Hom}(M, K_n)
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \text{Hom}(L, K_n) & \text{Hom}(L, F_{n-1}) & \text{Hom}(L, K_{n-1}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \text{Hom}(E, K_n) & \text{Hom}(E, F_{n-1}) & \text{Hom}(E, K_{n-1}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \text{Hom}(M, K_n) & \text{Hom}(M, F_{n-1}) & \text{Hom}(M, K_{n-1}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

Thus \( \text{FP-id}(M) \leq n \) if and only if \( \text{FP-id}(L) \leq n - 1 \) by Stenström (1970, Lemma 3.1) if and only if \( \text{Hom}(L, F_{n-1}) \to \text{Hom}(L, K_{n-1}) \) is an epimorphism by induction if and only if \( \text{Hom}(E, K_n) \to \text{Hom}(M, K_n) \) is an epimorphism by the second diagram if and only if \( \text{Hom}(M, F_n) \to \text{Hom}(M, K_n) \) is an epimorphism by the first diagram.

\[\square\]

**Lemma 2.2.** Let \( R \) be a left coherent ring and \( M \) a left \( R \)-module. If \( \text{Ext}^1(F, M) = 0 \) for all \( \text{FP} \)-injective left \( R \)-modules \( F \), then \( M \) has an \( \text{FP} \)-injective cover \( E \to M \) with \( E \) injective.

**Proof.** Let \( f : F \to M \) be an \( \text{FP} \)-injective cover of \( M \). Consider the short exact sequence \( 0 \to F \overset{i}{\to} E \to L \to 0 \) with \( E \) injective. Note that \( L \) is \( \text{FP} \)-injective by Stenström (1970, Lemma 3.1). So there exists \( g : E \to M \) such that \( gi = f \) since
\[
\text{Ext}^1(L, M) = 0. \text{ Thus there exists } h : E \to F \text{ such that } fh = g \text{ since } f \text{ is a cover. Therefore } fhi = f, \text{ and hence } hi \text{ is an isomorphism. It follows that } F \text{ is injective. }
\]

We are now in a position to prove the following theorem.

**Theorem 2.3.** The following are equivalent for a left coherent ring ring \( R \) and \( n \geq 1 \):

1. \( wD(R) \leq n \);
2. Every nth \( \mathcal{I} \)-syzygy of a left \( R \)-module is FP-injective;
3. Every nth \( \mathcal{I} \)-syzygy of a left \( R \)-module has FP-injective dimension \( \leq 1 \);
4. Every \((n - 1)\)th \( \mathcal{I} \)-syzygy of a left \( R \)-module has an FP-injective cover which is a monomorphism;
5. For every left \( \mathcal{I} \)-resolution \( \ldots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0 \) of a left \( R \)-module \( N \), \( \text{Hom}(M, F_n) \to \text{Hom}(M, K_n) \) is an epimorphism for any pure-injective left \( R \)-module \( M \), where \( K_n \) is the nth \( \mathcal{I} \)-syzygy of \( N \);
6. \( wD(R) < \infty \) and every left \( \mathcal{I} \)-resolution \( \ldots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0 \) of a left \( R \)-module \( N \) is exact at \( F_i \), for \( i \geq n - 1 \), where \( F_{-1} = N \);
7. \( wD(R) < \infty \) and every nth \( \mathcal{I} \)-syzygy of a left \( R \)-module \( N \) has an FP-injective cover which is an epimorphism;
   Moreover, if \( n \geq 2 \), then the above conditions are equivalent to:
8. Every \( (n - 2) \)th \( \mathcal{I} \)-syzygy in a minimal left \( \mathcal{I} \)-resolution of a left \( R \)-module has an FP-injective cover with the unique mapping property.

**Proof.** (1) \( \Rightarrow \) (2) Let \( K_n \) be an nth \( \mathcal{I} \)-syzygy of a left \( R \)-module. Then \( \text{FP-id}(K_n) \leq wD(R) \leq n \) by Stenström (1970, Theorem 3.3), and so \( \text{Hom}(K_n, F_n) \to \text{Hom}(K_n, K_n) \) is an epimorphism by Lemma 2.1. Therefore, \( K_n \) is FP-injective.

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (4) Let \( f : F_{n-1} \to K_{n-1} \) be an FP-injective precover of \( K_{n-1} \) and \( K_n = \ker(f) \). Then \( \text{FP-id}(K_n) \leq 1 \) by (3). So \( \text{im}(f) \) is FP-injective by Stenström (1970, Lemma 3.1). Thus the inclusion \( \text{im}(f) \to K_{n-1} \) is an FP-injective cover which is a monomorphism.

(4) \( \Rightarrow \) (2) Let \( \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0 \) be any left \( \mathcal{I} \)-resolution of a left \( R \)-module \( N \) and \( K_{n-1} = \ker(F_{n-2} \to F_{n-3}) \), \( K_n = \ker(F_{n-1} \to F_{n-2}) = \ker(F_{n-1} \to K_{n-1}) \). Since \( K_{n-1} \) has a monic FP-injective cover \( E \to K_{n-1} \) by (4), we have \( K_n \oplus E \cong F_{n-1} \) by Enochs and Jenda (2000, Lemma 8.6.3). Hence \( K_n \) is FP-injective.

(2) \( \Rightarrow \) (5) For any left \( \mathcal{I} \)-resolution \( \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0 \) of a left \( R \)-module \( N \), \( F_n \to K_n \) is a split epimorphism by (2). So \( \text{Hom}(M, F_n) \to \text{Hom}(M, K_n) \) is an epimorphism for any (pure-injective) left \( R \)-module \( M \).

(5) \( \Rightarrow \) (1) Let \( M \) be any right \( R \)-module. Then \( M^+ \) is pure-injective. Hence \( \text{FP-id}(M^+) \leq n \) by (5) and Lemma 2.1. Thus \( \text{fd}(M) \leq n \) by Fieldhouse (1972, Theorem 2.1), and so \( wD(R) \leq n \).
PROOF

(2) By Lemma 2.1.1. Thus $F_k \to R$ is left hereditary if and only if every left $R$-module has an injective cover with the unique mapping property; which is a monomorphism.

(3) Let $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{E}$ be a direct system of left $R$-modules with each $\mathcal{F}_i$ $\mathcal{FP}$-injective. By hypothesis, $\lim_{i=0} \mathcal{F}_i$ has a left $\mathcal{FP}$-injective cover $\mathcal{I}_i$, so there exists $\mathcal{E}_i$ an $\mathcal{FP}$-injective cover with the unique mapping property; which is a monomorphism.

(4) For any $\alpha$, $\beta$ and $\gamma$, $\mathcal{F}_\alpha \oplus \mathcal{F}_\beta \cong \mathcal{F}_{\alpha+\beta}$ and $\mathcal{I}_\alpha \oplus \mathcal{I}_\beta \cong \mathcal{I}_{\alpha+\beta}$. Therefore, $\mathcal{K}_\alpha$ is $\mathcal{FP}$-injective, as required.

(5) It is known that $R$ is a left Noetherian ring with left global dimension at most 2 if and only if every left $R$-module has an injective cover with the unique mapping property (see Enochs and Jenda, 2000, Remark 8.4.18). $R$ is a left Noetherian and left hereditary ring if and only if every left $R$-module has an injective cover which is a monomorphism (see Enochs and Jenda, 1991, Corollary 3.4 or Rada and Saorín, 1998, Corollary 4.12). Now we have the following corollary.

**Corollary 2.4.** Let $R$ be a ring. Then:

1. $R$ is left coherent and $wD(R) \leq 2$ if and only if every left $R$-module has an $\mathcal{FP}$-injective cover with the unique mapping property;
2. $R$ is left semihereditary if and only if every left $R$-module has an $\mathcal{FP}$-injective cover which is a monomorphism.

**Proof.** (1) The necessity follows from Theorem 2.3. For the sufficiency, we shall prove that the class of $FP$-injective left $R$-modules is closed under direct limits. Let $\{C_i, \varphi_j^i\}$ be a direct system of left $R$-modules with each $C_i$ $\mathcal{FP}$-injective. By hypothesis, $\lim_{i=0} C_i$ has an $\mathcal{FP}$-injective cover $x : E \to \lim_{i=0} C_i$ with the unique mapping property. Let $x_i : C_i \to \lim_{i=0} C_i$ satisfy $x_i = \varphi_j^i x_j$ whenever $i \leq j$. Then there exists $f_i : C_i \to E$ such that $x_i = x f_i$ for any $i$. It follows that $x f_i = \varphi_j^i x_j$, and so $f_i = \varphi_j^i x_j$, whenever $i \leq j$. Therefore, by the definition of direct limits, there exists $\beta : \lim_{i=0} C_i \to E$ such that $f_i = \beta x_i$. So $(x \beta) x_i = \beta(\beta x_i) = x f_i = x_i$ for any $i$. Therefore $x \beta = 1_{\lim_{i=0} C_i}$.
by the definition of direct limits, and hence \( \lim C_i \) is a direct summand of \( E \). So \( \lim C_i \) is \( FP \)-injective. Thus \( R \) is a left coherent ring by Stenström (1970, Theorem 3.2), and so \( wD(R) \leq 2 \) by Theorem 2.3.

(2) The necessity comes from Theorem 2.3. Conversely, every left \( R \)-module has an \( FP \)-injective cover with the unique mapping property since every left \( R \)-module has a monic \( FP \)-injective cover. So \( R \) is left coherent by (1), and hence \( wD(R) \leq 1 \) by Theorem 2.3. Thus \( R \) is left semihereditary.

□

Remark 2.5. (1) Corollary 2.4 (1) may be viewed as the dual of Asensio Mayor and Martínez Hernández (1993, Proposition 2.1) which says that a ring \( R \) is left coherent and \( wD(R) \leq 2 \) if and only if every left \( R \)-module has a flat envelope with the unique mapping property.

(2) Corollary 2.4 (2) has been proven in Chen and Ding (1996) and Rada and Saorin (1998) in a different way.

Next we shall study the weak global dimension of commutative coherent rings.

Lemma 2.6. Let \( R \) be a commutative ring. Then \( R \) is a coherent ring if and only if \( \text{Hom}(R^+, E) \) is flat for any injective \( R \)-module \( E \).

Proof. If \( R \) is coherent, then \( \text{Hom}(R^+, E) \) is flat by Matlis (1982, Theorem 1) (for \( R^+ \) is injective).

Conversely, we shall show that any direct product \( \Pi R \) of \( R \) is flat. Indeed, since \( R \) is a pure submodule of \( R^{++} \), \( \Pi R \) is a pure submodule of \( R^{++} \) by Cheatham and Stone (1981, Lemma 1(2)). Note that \( \Pi R^{++} \cong \Pi \text{Hom}(R^+ \otimes R, \mathbb{Q}/\mathbb{Z}) \cong \Pi \text{Hom}(R^+, R^+) \cong \text{Hom}(R^+, \Pi R^+) \). Thus \( \Pi R^{++} \) is flat by hypothesis, and hence \( \Pi R \) is flat. So \( R \) is coherent.

□

Theorem 2.7. The following are equivalent for a commutative coherent ring \( R \) and an integer \( n \geq 2 \):

1. \( wD(R) \leq n \);
2. Every \( (n - 2) \)th \( \mathcal{F} \)-syzygy in a minimal left \( \mathcal{F} \)-resolution of a pure-injective \( R \)-module has an \( FP \)-injective cover with the unique mapping property;
3. Every \( (n - 2) \)th \( \mathcal{F} \)-syzygy in a minimal left \( \mathcal{F} \)-resolution of a pure-injective \( R \)-module has an injective cover with the unique mapping property;
4. \( FP-\text{id}(R^+ \otimes M) \leq n - 2 \) for any \( R \)-module \( M \);
5. \( fd(\text{Hom}(R^+, M)) \leq n - 2 \) for any (pure-injective) \( R \)-module \( M \);
6. \( wD(R) < \infty \) and \( FP-\text{id}(\text{Hom}(R^+, E)) \leq n \) for any injective \( R \)-module \( E \);
7. \( wD(R) < \infty \) and \( fd(R^+ \otimes M) \leq n \) for any flat \( R \)-module \( M \).

Proof. (1) \( \Rightarrow \) (2) follows from Theorem 2.3.

(2) \( \Rightarrow \) (3) Let \( F_{n-3} \to \cdots \to F_1 \to F_0 \to M \to 0 \) be a partial minimal left \( \mathcal{F} \)-resolution of a pure-injective \( R \)-module \( M \). By (2), \( K_{n-2} = \ker(F_{n-3} \to F_{n-4}) \) has an \( FP \)-injective cover \( f: F_{n-2} \to K_{n-2} \) with the unique mapping property.
Suppose \( n = 2 \). There is a pure exact sequence \( 0 \to F_0 \to E \to L \to 0 \) with \( E \) injective. Since \( M \) is pure-injective, there exists \( g : E \to M \) such that \( gi = f \). It is easy to check that \( g : E \to M \) is an \( FP \)-injective precover of \( M \). So \( F_0 \) is injective by Enochs and Jenda (2000, Proposition 5.1.2).

Suppose \( n > 2 \). Then \( \text{Ext}^1(G, K_{n-2}) = 0 \) for any \( FP \)-injective \( R \)-module \( G \) by Wakamatsu’s lemma. So \( F_{n-2} \) is injective by Lemma 2.2.

(3) \( \Rightarrow \) (4) Let \( M \) be any \( R \)-module and \( F_{n-3} \to \cdots \to F_1 \to F_0 \to M^+ \to 0 \) be a minimal left \( \mathcal{FJ} \)-resolution of \( M^+ \). Then \( K_{n-2} = \ker(F_{n-3} \to F_{n-4}) \) has an injective cover \( F_{n-2} \to K_{n-2} \) with the unique mapping property by (3). Thus we get the following exact sequence

\[
0 \to \text{Hom}(R^+, F_{n-2}) \to \text{Hom}(R^+, F_{n-3}) \to \cdots \\
\to \text{Hom}(R^+, F_0) \to \text{Hom}(R^+, M^+) \to 0.
\]

Since each \( \text{Hom}(R^+, F_i) \) is flat by Lemma 2.6, \( fd(\text{Hom}(R^+, M^+)) \leq n - 2 \). Note that \( \text{Hom}(R^+, M^+) \cong (R^+ \otimes M)^+ \). Thus, by Fieldhouse (1972, Theorem 2.2), we have

\[
FP - id(R^+ \otimes M) = fd((R^+ \otimes M)^+) = fd(\text{Hom}(R^+, M^+)) \leq n - 2.
\]

(4) \( \Rightarrow \) (1) Let \( M \) be a finitely presented \( R \)-module. Then \( R^+ \otimes M \cong (\text{Hom}(M, R))^+ \) by Colby (1975, Lemma 2), and so

\[
fd(\text{Hom}(M, R)) = FP - id(\text{Hom}(M, R)^+) = FP - id(R^+ \otimes M) \leq n - 2
\]

by (4) and Fieldhouse (1972, Theorem 2.1). Therefore, \( wD(R) \leq n \) by Jones and Teply (1982, Lemma 4).

(1) \( \Rightarrow \) (5) Let \( M \) be any \( R \)-module. Then there is an exact sequence \( 0 \to M \to E^0 \to E^1 \) with \( E^0 \) and \( E^1 \) injective. So we obtain the exact sequence

\[
0 \to \text{Hom}(R^+, M) \to \text{Hom}(R^+, E^0) \to \text{Hom}(R^+, E^1) \to L \to 0.
\]

Thus \( fd(\text{Hom}(R^+, M)) \leq n - 2 \) since \( fd(L) \leq n \) by (1) and each \( \text{Hom}(R^+, E^i) \) is flat by Lemma 2.6.

(5) \( \Rightarrow \) (4) For any \( R \)-module \( M \), we have \( FP-id(R^+ \otimes M) = fd((R^+ \otimes M)^+) = fd(\text{Hom}(R^+, M^+)) \leq n - 2 \) by (5).

(1) \( \Rightarrow \) (6) Let \( E \) be any injective \( R \)-module. Then \( (\text{Hom}(R^+, E))^+ \) is injective since \( \text{Hom}(R^+, E) \) is flat by Lemma 2.6. Thus there exists a split exact sequence \( 0 \to (\text{Hom}(R^+, E))^+ \to \Pi R^+ \). Note that \( fd(R^+) \leq n \) by (1), and so \( fd(\Pi R^+) \leq n \). Thus \( fd(\text{Hom}(R^+, E))^+ \leq n \), and hence \( FP-id(\text{Hom}(R^+, E)) = fd(\text{Hom}(R^+, E))^+ \leq n \).

(6) \( \Rightarrow \) (7) Let \( M \) be any flat \( R \)-module. Then we have \( FP-id((R^+ \otimes M)^+) = FP-id(\text{Hom}(R^+, M^+)) \leq n \) by (6). Thus \( fd(R^+ \otimes M) = FP-id((R^+ \otimes M)^+) \leq n \).

(7) \( \Rightarrow \) (1) Note that \( FP-id(R) = fd(R^+) = fd(R^+ \otimes R) \leq n \) by (7). So \( wD(R) = FP-id(R) \leq n \) by Stenström (1970, Proposition 3.5) since \( wD(R) < \infty \).
Letting \( n = 2 \) in Theorem 2.7, we have the following corollary.

**Corollary 2.8.** The following are equivalent for a commutative coherent ring \( R \):

1. \( wD(R) \leq 2 \);
2. Every pure-injective \( R \)-module has an (FP-)injective cover with the unique mapping property;
3. \( R^+ \otimes M \) is FP-injective for any \( R \)-module \( M \);
4. \( \text{Hom}(R^+, M) \) is flat for any (pure-injective) \( R \)-module \( M \).

We conclude the article with the following corollary.

**Corollary 2.9.** Let \( R \) be a commutative ring. Then \( R \) is coherent and \( wD(R) \leq 2 \) if and only if \( \text{Hom}(R^+, M) \) is flat for any (pure-injective) \( R \)-module \( M \).

**Proof.** The result follows from Lemma 2.6 and Corollary 2.8. \( \square \)

**ACKNOWLEDGMENTS**

This research was partially supported by SRFDP (No. 20050284015), NSFC (No. 10771096), Science Research Fund of Nanjing Institute of Technology (No. KXJ07061), Jiangsu 333 Project, and Jiangsu Qinglan Project. The authors would like to thank the referee for the helpful comments and suggestions.

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