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RELATIVE COTORSION MODULES AND RELATIVE FLAT MODULES

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Let R be a ring, M a right R-module, and n a fixed non-negative integer. M is called n-cotorsion if $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for any flat right R-module N. M is said to be n-flat if $\mathrm{Ext}_R^1(M, N) = 0$ for any n-cotorsion right R-module N. We prove that $(\mathscr{F}_n, \mathscr{C}_n)$ is a complete hereditary cotorsion theory, where $\mathscr{F}_n$ (resp. $\mathscr{C}_n$) denotes the class of all n-flat (resp. n-cotorsion) right R-modules. Several applications are given.

Key Words: Cotorsion theory; n-cotorsion module; n-flat module.

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1. NOTATION

In this section, we shall recall some known notions and definitions which we need in the later sections.

Throughout this article, R is an associative ring with identity and all modules are unitary R-modules. $rD(R)$ stands for the right global dimension of R. For a right R-module M, $pd(M)$ and $id(M)$ denote the projective and injective dimensions of M, respectively. If $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution of M, let $K_0 = M$, $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$ for $i \geq 2$. The nth kernel K_n $(n \geq 0)$ is called the *nth syzygy of M*. Dually, we have the nth cosyzygy L^n of M using an injective resolution of M. For two right R-modules M, N, $\mathrm{Hom}(M, N)$ $(\mathrm{Ext}^n(M, N))$ means $\mathrm{Hom}_R(M, N)$ $(\mathrm{Ext}_R^n(M, N))$ for an integer $n \geq 1$.

Given a class $\mathscr{L}$ of right R-modules, we denote by $\mathscr{L}^{\perp} = \{C : \mathrm{Ext}^1(L, C) = 0 \text{ for all } L \in \mathscr{L}\}$ the right orthogonal class of $\mathscr{L}$, and by ${}^{\perp}\mathscr{L} = \{C : \mathrm{Ext}^1(C, L) = 0 \text{ for all } L \in \mathscr{L}\}$ the left orthogonal class of $\mathscr{L}$.

Let $\mathscr{C}$ be a class of right R-modules and M a right R-module. A homomorphism $\phi : M \to F$ with $F \in \mathscr{C}$ is called a *$\mathscr{C}$-preenvelope of M* (Enochs, 1981) if for any homomorphism $f: M \to F'$ with $F' \in \mathscr{C}$, there is a homomorphism

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$g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F' = F$ and $f = \phi$, the $\mathscr{C}$-preenvelope ϕ is called a *$\mathscr{C}$-envelope of M*.

A $\mathscr{C}$-envelope $\phi : M \to F$ is said to have the unique mapping property (Ding, 1996) if for any homomorphism f: $M \to F'$ with $F' \in \mathscr{C}$, there is a unique homomorphism $g : F \to F'$ such that $g\phi = f$.

Following Enochs and Jenda (2000, Definition 7.1.6), a monomorphism $\alpha : M \to C$ with $C \in \mathscr{C}$ is said to be a *special $\mathscr{C}$-preenvelope of M* if $\operatorname{coker}(\alpha) \in {}^{\perp}\mathscr{C}$. Dually, we have the definitions of a (special) $\mathscr{C}$-precover and a $\mathscr{C}$-cover (with the unique mapping property). Special $\mathscr{C}$-preenvelopes (resp. special $\mathscr{C}$-precovers) are obviously $\mathscr{C}$-preenvelopes (resp. $\mathscr{C}$-precovers). $\mathscr{C}$-envelopes ($\mathscr{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(\mathscr{F}, \mathscr{C})$ of classes of right R-modules is called a *cotorsion theory* (Enochs and Jenda, 2000) if $\mathscr{F}^{\perp} = \mathscr{C}$ and ${}^{\perp}\mathscr{C} = \mathscr{F}$. A cotorsion theory $(\mathscr{F}, \mathscr{C})$ is called *complete* (Trlifaj, 2000) if every right R-module has a special $\mathscr{C}$-preenvelope, and every right R-module has a special $\mathscr{F}$-precover. A cotorsion theory $(\mathscr{F}, \mathscr{C})$ is said to be *hereditary* (Enochs et al., 2004) if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathscr{F}$, then L' is also in $\mathscr{F}$.

For further concepts and notations, we refer the reader to (Anderson and Fuller, 1974; Enochs and Jenda, 2000; Rotman, 1979; Xu, 1996).

2. INTRODUCTION

Recall that a right R-module C is called *cotorsion* (Enochs, 1984) provided that $\operatorname{Ext}^1(F, C) = 0$ for any flat right R-module F. E.E. Enochs proved that (the class of flat right R-modules, the class of cotorsion right R-modules) is a complete cotorsion theory over any ring R, thus proving the celebrated Flat Cover Conjecture (FCC): Every module over any ring has a flat cover (Bican et al., 2001) (and hence every module has a cotorsion envelope). The main purpose of this article is to extend the flat cotorsion theory to a more general setting. Some applications are given.

In Section 3, we introduce the concepts of *n-cotorsion modules*, *n-flat modules*, and *σ-dimensions* of modules and rings. Let M be a right R-module and n a fixed non-negative integer. M is called *n-cotorsion* if $\operatorname{Ext}^{n+1}(N, M) = 0$ for any flat right R-module N. M is said to be *n-flat* if $\operatorname{Ext}^1(M, N) = 0$ for any n-cotorsion right R-module N. We define $\sigma_R(M) = \sup\{n : M \text{ is } n\text{-flat}\}$ and $\sigma_R(M) = -1$ if $\operatorname{Ext}^1(M, N) \neq 0$ for some cotorsion right R-module N. The right σ-dimension $r.\sigma\text{-dim}(R)$ of a ring R is defined to be the least non-negative integer n such that $\sigma_R(M) \geq n$ implies $\sigma_R(M) = \infty$ for any right R-module M. If no such n exists, set $r.\sigma\text{-dim}(R) = \infty$. We prove that $(\mathscr{F}_n, \mathscr{C}_n)$ is a complete hereditary cotorsion theory, where $\mathscr{F}_n$ (resp. $\mathscr{C}_n$) denotes the class of all n-flat (resp. n-cotorsion) right modules (see Theorem 3.9).

Section 4 consists of some applications. Let n be a fixed non-negative integer. It is proven that every right R-module is n-cotorsion if and only if all (n-)flat right R-modules are n-cotorsion if and only if every n-flat right R-module is projective if and only if all flat right R-modules are of projective dimension $\leq n$ if and only if every (n-flat) right R-module has a $\mathscr{C}_n$-envelope with the unique mapping property if and only if $pd(M) \leq m$ for some m with $0 \leq m \leq n$ and any $(n-m)$-flat right R-module M (see Theorem 4.1). Dually, we give characterizations of those rings such that every right R-module is n-flat. It is shown that every right R-module is n-flat if and only if every cyclic right R-module is n-flat if and only if every n-cotorsion right R-module is n-flat if and only if every n-cotorsion right R-module is injective if and only if every (n-cotorsion) right R-module M has an $\mathscr{F}_n$-cover with the unique mapping property (see Theorem 4.5). We also characterize those rings with finite right σ-dimension. It is proven that $r.\ \sigma\text{-dim}(R) \leq n$ if and only if every n-flat right R-module is $(n+1)$-flat if and only if every $(n+1)$-cotorsion right R-module is n-cotorsion if and only if every nth syzygy of any flat right R-module is $(n+1)$-flat (see Theorem 4.7).

3. DEFINITION AND GENERAL RESULTS

We start with the following definition.

Definition 3.1. Let M be a right R-module and n a fixed non-negative integer.

M is called *n-cotorsion* if $\mathrm{Ext}^{n+1}(N, M) = 0$ for any flat right R-module N.

M is said to be *n-flat* if $\mathrm{Ext}^1(M, N) = 0$ for any n-cotorsion right R-module N.

$\sigma_R(M) = \sup\{n : M \text{ is } n\text{-flat}\}$ and $\sigma_R(M) = -1$ if $\mathrm{Ext}^1(M, N) \neq 0$ for some cotorsion right R-module N.

The right σ-dimension $r.\ \sigma\text{-dim}(R)$ of a ring R is defined to be the least non-negative integer n such that $\sigma_R(M) \geq n$ implies $\sigma_R(M) = \infty$ for any right R-module M. If no such n exists, set $r.\ \sigma\text{-dim}(R) = \infty$.

Remark 3.2. It is clear that a right R-module M is 0-cotorsion (resp. 0-flat) if and only if M is cotorsion (resp. flat). Let n be a non-negative integer. Then any cotorsion right R-module is n-cotorsion by the proof of Xu (1996, Proposition 3.1.2), any projective right R-module is n-flat and any n-flat right R-module is flat. But none of the converses of these implications is true in general as shown below.

Proposition 3.3. *The following are equivalent for a right R-module M and an integer* $n \geq 0$:

(1) *M is n-cotorsion*;
(2) *If the sequence* $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$ *is exact with* C^0, $C^1, \ldots, C^{n-1}$ *cotorsion, then* C^n *is also cotorsion*;
(3) *There exists an exact sequence* $0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to C^n \to 0$ *with* $C^0, C^1, \ldots, C^{n-1}, C^n$ *cotorsion*;
(4) *The flat cover of M is n-cotorsion.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is standard homological algebra fare.

(1) $\Leftrightarrow$ (4) Let K be the kernel of the flat cover $N \to M$, then we have the exact sequence $0 \to K \to N \to M \to 0$ with K cotorsion by Wakamatsu's Lemma (Xu, 1996, Lemma 2.1.1). Note that $\mathrm{Ext}^j(F, K) = 0$ for all $j \geq 1$ and flat modules F by the proof of Xu (1996, Proposition 3.1.2), so (1) $\Leftrightarrow$ (4) follows. $\square$

Remark 3.4. Let m and n be integers with $m \geq n > 0$. Proposition 3.3 shows that any n-cotorsion right R-module is m-cotorsion. It follows that any m-flat right R-module is n-flat.

Lemma 3.5. *Let R be a ring, m and n non-negative integers. If M is an n-flat right R-module, then $\mathrm{Ext}^{j+1}(M, N) = 0$ for any integer $j \geq m$ and any $(m+n)$-cotorsion right R-module N.*

Proof. For every $(m+n)$-cotorsion right R-module N, it is easy to verify that the mth cosyzygy L^m of N is n-cotorsion. Therefore, $\mathrm{Ext}^{m+1}(M, N) \cong \mathrm{Ext}^1(M, L^m) = 0$ since M is n-flat, and the result follows by induction. □

Proposition 3.6. *Let R be a ring and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules.*

(1) *If $\sigma_R(C) \geq 0$, then $\sigma_R(A) \geq inf\{\sigma_R(B), \sigma_R(C)+1\}$.*
(2) *$\sigma_R(B) \geq inf\{\sigma_R(A), \sigma_R(C)\}$.*
(3) *If $B = A \oplus C$, then $\sigma_R(A \oplus C) = inf\{\sigma_R(A), \sigma_R(C)\}$.*

Proof. The exact sequence $0 \to A \to B \to C \to 0$ gives rise to the exactness of the sequence $\mathrm{Ext}^1(C, N) \to \mathrm{Ext}^1(B, N) \to \mathrm{Ext}^1(A, N) \to \mathrm{Ext}^2(C, N)$ for any right R-module N. Now the result follows from Lemma 3.5 by a standard homological algebra argument. □

Let $\mathscr{C}_n$ be the class of all n-cotorsion right R-modules. We have the following proposition.

Proposition 3.7. *Let n be a non-negative integer and R a ring such that every projective right R-module is n-cotorsion. Then the following are equivalent for a right R-module M:*

(1) *M is n-flat;*
(2) *M is projective with respect to every exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathscr{C}_n$;*
(3) *For every exact sequence $0 \to K \to F \to M \to 0$ with $F \in \mathscr{C}_n$, $K \to F$ is a $\mathscr{C}_n$-preenvelope of K;*
(4) *M is a cokernel of a $\mathscr{C}_n$-preenvelope $K \to F$ with F projective;*
(5) *For any projective resolution $\widetilde{F} = \cdots \to F_1 \to F_0 \to M \to 0$, $\mathrm{Hom}(\widetilde{F}, N)$ is exact for all right R-modules N with $N \in \mathscr{C}_n$.*

Proof. (1) ⇒ (2) and (1) ⇒ (3) are straightforward.

(2) ⇒ (1) For every $N \in \mathscr{C}_n$, apply (2) to a short exact sequence $0 \to N \to E \to L \to 0$ with E injective.

(3) ⇒ (4) Let $0 \to K \to F \to M \to 0$ be an exact sequence with F projective. Note that $F \in \mathscr{C}_n$ by hypothesis, thus $K \to F$ is a $\mathscr{C}_n$-preenvelope by (3), and so (4) follows.

(4) ⇒ (1) By (4), there is an exact sequence $0 \to K \to F \to M \to 0$, where $K \to F$ is a $\mathscr{C}_n$-preenvelope with F projective. Hence there is an exact sequence $\mathrm{Hom}(F, N) \to \mathrm{Hom}(K, N) \to \mathrm{Ext}^1(M, N) \to 0$ for each $N \in \mathscr{C}_n$. Note that $\mathrm{Hom}(F, N) \to \mathrm{Hom}(K, N) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(M, N) = 0$, as desired.

(1) $\Rightarrow$ (5) Let $N \in \mathscr{C}_n$ and $\cdots \to F_1 \to F_0 \to M \to 0$ be a projective resolution of M. Since M is n-flat, $\mathrm{Ext}^j(M, N) = 0$ for any integer $j \geq 1$ by Lemma 3.5. Therefore the sequence

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(F_0, N) \to \mathrm{Hom}(F_1, N) \to \cdots$$

is exact.

(5) $\Rightarrow$ (1) Let

$$\cdots \to F_1 \to F_0 \to M \to 0$$

be a projective resolution of M. Since $\mathrm{Hom}(F_0, N) \to \mathrm{Hom}(F_1, N) \to \mathrm{Hom}(F_2, N)$ is exact for all $N \in \mathscr{C}_n$ by (5), $\mathrm{Ext}^1(M, N) = 0$. $\square$

Recall that a family $(M_\alpha)_{\alpha<\lambda}$ (λ is an ordinal number) of submodules of a right R-module M is called a *continuous chain of submodules* (Enochs and Jenda, 2000, p. 160) if $M_\alpha \leq M_\beta$ whenever $\alpha \leq \beta < \lambda$ and if $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$ whenever $\beta < \lambda$ is a limit ordinal.

Denote by $\mathscr{F}_n$ the class of all n-flat right R-modules. Then $\mathscr{F}_n = {}^{\perp}\mathscr{C}_n$. Motivated by the important fact that $(\mathscr{F}_0, \mathscr{C}_0)$ is a complete cotorsion theory, we shall show that $(\mathscr{F}_n, \mathscr{C}_n)$ is also a complete cotorsion theory for any fixed non-negative integer n. To this aim, we need the following homological lemma which generalizes Eklof and Trlifaj (2001, Lemma 1) or Enochs and Jenda (2000, Theorem 7.3.4).

Lemma 3.8. *Let M and N be right R-modules, n a positive integer and M the union of a continuous chain of submodules $(M_\alpha)_{\alpha<\lambda}$. If* $\mathrm{Ext}^n(M_0, N) = 0$ *and* $\mathrm{Ext}^n(M_{\alpha+1}/M_\alpha, N) = 0$ *whenever $\alpha + 1 < \lambda$, then* $\mathrm{Ext}^n(M, N) = 0$.

Proof. If $n = 1$, it is exactly Eklof and Trlifaj (2001, Lemma 1). So we may assume $n > 1$ and the result holds for $n - 1$.

Put $M_\lambda = M$. We use the principle of transfinite induction on $\beta \leq \lambda$. Suppose $\beta < \lambda$ and $\mathrm{Ext}^n(M_\alpha, N) = 0$ for all $\alpha < \beta$. We shall argue that $\mathrm{Ext}^n(M_\beta, N) = 0$. By hypothesis, this is true for $\beta = 0$.

If β is not a limit ordinal, we have the exact sequence

$$0 \to M_{\beta-1} \to M_\beta \to M_\beta/M_{\beta-1} \to 0,$$

which induces the exact sequence

$$0 = \mathrm{Ext}^n(M_\beta/M_{\beta-1}, N) \to \mathrm{Ext}^n(M_\beta, N) \to \mathrm{Ext}^n(M_{\beta-1}, N) = 0$$

by assumption. Thus $\mathrm{Ext}^n(M_\beta, N) = 0$.

If β is a limit ordinal, then $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$. We have the exact sequence

$$0 \to K_\beta \to F_\beta \to M_\beta \to 0,$$

where F_β is a free right R-module which bases on elements of M_β, and $K_\beta = \ker(F_\beta \to M_\beta)$. For any $\gamma < \beta$, there exists an exact sequence

$$0 \to K_\gamma \to F_\gamma \to M_\gamma \to 0,$$

where F_γ is a free right R-module which bases on elements of M_γ, and $K_\gamma = \ker(F_\gamma \to M_\gamma)$. It is easy to see that F_β is the union of a continuous chain of submodules $(F_\gamma)_{\gamma<\beta}$, and so K_β is the union of a continuous chain of submodules $(K_\gamma)_{\gamma<\beta}$. We claim that $\mathrm{Ext}^{n-1}(K_{\gamma+1}/K_\gamma, N) = 0$ whenever $\gamma + 1 < \lambda$. In fact, consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_\gamma & \longrightarrow & F_\gamma & \longrightarrow & M_\gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{\gamma+1} & \longrightarrow & F_{\gamma+1} & \longrightarrow & M_{\gamma+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{\gamma+1}/K_\gamma & \longrightarrow & F_{\gamma+1}/F_\gamma & \longrightarrow & M_{\gamma+1}/M_\gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

By 3×3 Lemma (Rotman, 1979, Exercise 6.16, p. 175), the bottom row is exact. So we get the induced exact sequence

$$0 = \mathrm{Ext}^{n-1}(F_{\gamma+1}/F_\gamma, N) \to \mathrm{Ext}^{n-1}(K_{\gamma+1}/K_\gamma, N) \to \mathrm{Ext}^{n}(M_{\gamma+1}/M_\gamma, N) = 0$$

by hypothesis since $F_{\gamma+1}/F_\gamma$ is free. Hence $\mathrm{Ext}^{n-1}(K_{\gamma+1}/K_\gamma, N) = 0$. It is clear that $\mathrm{Ext}^{n-1}(K_0, N) = 0$, then $\mathrm{Ext}^{n-1}(K_\beta, N) = 0$ by hypothesis. Thus $\mathrm{Ext}^n(M_\beta, N) = 0$ follows from the exactness of $\mathrm{Ext}^{n-1}(K_\beta, N) \to \mathrm{Ext}^n(M_\beta, N) \to \mathrm{Ext}^n(F_\beta, N)$. So $\mathrm{Ext}^n(M, N) = 0$ (for $M = M_\lambda$). □

Theorem 3.9. *Let R be a ring and $n \geq 0$. Then $(\mathscr{F}_n, \mathscr{C}_n)$ is a complete hereditary cotorsion theory.*

Proof. Let F be a flat right R-module. By Enochs and Jenda (2000, Lemma 5.3.12), if Card $R \leq \aleph_\beta$, then for each $x \in F$, there is a pure submodule S of F with $x \in S$ such that Card $S \leq \aleph_\beta$. So we can write F as a union of a continuous chain $(F_\alpha)_{\alpha<\lambda}$ of pure submodules of F such that Card $F_0 \leq \aleph_\beta$ and Card $F_{\alpha+1}/F_\alpha \leq \aleph_\beta$ whenever $\alpha + 1 < \lambda$. By Lemma 3.8, if N is a right R-module such that $\mathrm{Ext}^{n+1}(F_0, N) = 0$ and $\mathrm{Ext}^{n+1}(F_{\alpha+1}/F_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^{n+1}(F, N) = 0$. Denote by F_I the nth syzygy module of F. Then $\mathrm{Ext}^{n+1}(F, N) = 0$ if and only if $\mathrm{Ext}^1(F_I, N) = 0$. Let X be the set of representatives of nth syzygy modules of all flat right R-modules G

with Card $G \leq \aleph_\beta$. Then $\mathscr{C}_n = X^\perp$, and so $(\mathscr{F}_n, \mathscr{C}_n)$ is complete by Enochs and Jenda (2000, Theorem 7.4.1).

On the other hand, let $0 \to A \to B \to C \to 0$ be exact with $A, B \in \mathscr{C}_n$. Then $C \in \mathscr{C}_n$ by Remark 3.4. So $(\mathscr{F}_n, \mathscr{C}_n)$ is hereditary by Enochs et al. (2004, Proposition 1.2). □

Remark 3.10. (1) Let n and m be non-negative integers such that $n < m$. If M is m-flat (resp. n-cotorsion), then M is n-flat (resp. m-cotorsion). However, the converse is not true in general. In fact, by Pierce (1967, Corollary 5.2), there exists a von Neumann regular ring R of right global dimension m. Then the class of all right R-modules $= \mathscr{C}_m \neq \mathscr{C}_n$ (for cotorsion modules coincide with injective modules by Xu, 1996, Theorem 3.3.2), and so there exists an n-flat right R-module which is not m-flat (and hence not projective) by Theorem 3.9.

(2) We do not know whether every right R-module has an $\mathscr{F}_n$-cover or a $\mathscr{C}_n$-envelope ($n \geq 1$) although every right R-module has a flat cover and a cotorsion envelope. Nevertheless, if $\mathscr{F}_n$ is closed under direct limits, then every right R-module has an $\mathscr{F}_n$-cover and a $\mathscr{C}_n$-envelope by Theorem 3.9 and Enochs and Jenda (2000, Theorem 7.2.6).

4. APPLICATIONS

In what follows, let $\sigma_M : M \to \mathscr{C}_n(M)$ (resp. $\epsilon_M : \mathscr{F}_n(M) \to M$) denote the $\mathscr{C}_n$-envelope (resp. $\mathscr{F}_n$-cover) of a right R-module M. Following Trlifaj (2000, Theorem 3.7) or Enochs and Jenda (2000, Theorem 7.4.6), $(\mathscr{P}_n, \mathscr{P}_n^\perp)$ is a complete cotorsion theory, where $\mathscr{P}_n$ stands for the class of all right modules of projective dimension $\leq n$. It is easy to verify that $M \in \mathscr{P}_n^\perp$ ($n \geq 1$) if and only if M is injective with respect to every right R-module exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathscr{P}_{n-1}$ and B projective.

It is well known that a ring R is right perfect if and only if every (flat) right R-module is cotorsion if and only if every flat right R-module is projective if and only if every (flat) right R-module has a cotorsion envelope with the unique mapping property (see Xu, 1996, Proposition 3.3.1 and Mao and Ding, 2005, Proposition 2.18). Now we have the following theorem.

Theorem 4.1. *Let R be a ring and n a fixed non-negative integer. Then the following are equivalent*:

(1) *Every right R-module is n-cotorsion*;
(2) *Every (n-)flat right R-module is n-cotorsion*;
(3) *Every n-flat right R-module is projective*;
(4) *$M \in \mathscr{P}_n$ for every flat right R-module M*;
(5) *Every right R-module M with $M \in \mathscr{P}_n^\perp$ is cotorsion*;
(6) $\mathrm{Ext}^{n+1}(M, N) = 0$ *for all flat right R-modules M and N*;
(7) $\mathrm{Ext}^{n+j}(M, N) = 0$ *for all flat right R-modules M, N and $j \geq 1$*;
(8) *Every (n-flat) right R-module has a $\mathscr{C}_n$-envelope with the unique mapping property*;
(9) *$M \in \mathscr{P}_m$ for any m with $0 \leq m \leq n$ and any $(n-m)$-flat right R-module M*;

(10) *$M \in \mathscr{P}_m$ for some m with $0 \leq m \leq n$ and any $(n-m)$-flat right R-module M;*
(11) *M is $(n-m)$-cotorsion for any m with $0 \leq m \leq n$ and any $M \in \mathscr{P}_m^{\perp}$;*
(12) *M is $(n-m)$-cotorsion for some m with $0 \leq m \leq n$ and any $M \in \mathscr{P}_m^{\perp}$.*

Moreover if $n \geq 1$ and the class of $(n-1)$-cotorsion right R-modules is closed under direct sums, then the above conditions are also equivalent to

(13) *Every ($(n-1)$-flat) right R-module M has a monic $\mathscr{C}_{n-1}$-cover.*

Proof. (1) $\Rightarrow$ (6) $\Rightarrow$ (2), (6) $\Leftrightarrow$ (7), (1) $\Rightarrow$ (8), and (9) $\Rightarrow$ (10) are obvious.

(1) $\Leftrightarrow$ (3) is clear since $(\mathscr{F}_n, \mathscr{C}_n)$ is a cotorsion theory by Theorem 3.9.

(4) $\Leftrightarrow$ (5), (9) $\Leftrightarrow$ (11), and (10) $\Leftrightarrow$ (12) follow from Theorem 3.9 and the fact that $(\mathscr{P}_k, \mathscr{P}_k^{\perp})$ is a cotorsion theory for any $k \geq 0$.

(8) $\Rightarrow$ (2) Let M be an n-flat right R-module. There is the following exact commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r, "\sigma_M"] \arrow[drr, "0"'] & \mathscr{C}_n(M) \arrow[r, "\gamma"] \arrow[dr, "\sigma_L\gamma"] & L \arrow[r] \arrow[d, "\sigma_L"] & 0 \\
 & & & \mathscr{C}_n(L) &
\end{tikzcd}$$

where L is n-flat by Wakamatsu's Lemma (Xu, 1996, Lemma 2.1.2). Note that $\sigma_L \gamma \sigma_M = 0 = 0\sigma_M$, so $\sigma_L \gamma = 0$ by (8). Therefore, $L = \mathrm{im}(\gamma) \subseteq \ker(\sigma_L) = 0$, and hence $M \in \mathscr{C}_n$. Thus (2) follows.

(2) $\Rightarrow$ (1) Let M be a right R-module. By Theorem 3.9, M has a special $\mathscr{F}_n$-precover, and hence there is a short exact sequence $0 \to K \to N \to M \to 0$, where K is n-cotorsion and N is n-flat. Since N is n-cotorsion by (2), M is n-cotorsion by Remark 3.4. So (1) follows.

(1) $\Rightarrow$ (9) Let M be an $(n-m)$-flat right R-module and N any right R-module. Since N is n-cotorsion, $\mathrm{Ext}^{m+1}(M, N) = 0$ by Lemma 3.5. Thus $pd(M) \leq m$.

(10) $\Rightarrow$ (4) Let M be a flat right R-module and K_{n-m} an $(n-m)$th syzygy of M. Then, for any $(n-m)$-cotorsion right R-module N, $\mathrm{Ext}^1(K_{n-m}, N) \cong \mathrm{Ext}^{n-m+1}(M, N) = 0$, and so K_{n-m} is $(n-m)$-flat. Thus $pd(K_{n-m}) \leq m$ by (10), and hence $pd(M) \leq n$, as desired.

(4) $\Rightarrow$ (1) Let M be any right R-module. It follows that $\mathrm{Ext}^{n+1}(F, M) = 0$ for any flat right R-module F since $pd(F) \leq n$, so M is n-cotorsion.

(1) $\Rightarrow$ (13) Let M be any right R-module. Write $F = \sum\{N \leq M : N \text{ is } (n-1)\text{-cotorsion}\}$ and $G = \bigoplus\{N \leq M : N \text{ is } (n-1)\text{-cotorsion}\}$. Then there exists an exact sequence $0 \to K \to G \to F \to 0$. Since K is n-cotorsion by (1) and G is $(n-1)$-cotorsion by hypothesis, we have F is $(n-1)$-cotorsion. Next we prove that

the inclusion $i : F \to M$ is a $\mathscr{C}_{n-1}$-cover of M. Let $\psi : F' \to M$ with $F' \in \mathscr{C}_{n-1}$ be an arbitrary right R-homomorphism. Note that $\psi(F') \leq F$ by the proof above. Define $\zeta : F' \to F$ via $\zeta(x) = \psi(x)$ for $x \in F'$. Then $i\zeta = \psi$, and so $i : F \to M$ is a $\mathscr{C}_{n-1}$-precover of M. In addition, it is clear that the identity map 1_F of F is the only homomorphism $g : F \to F$ such that $ig = i$, and hence (13) follows.

(13) $\Rightarrow$ (2) Let M be any n-flat right R-module. We shall show that M is n-cotorsion. Indeed, by Theorem 3.9, there exists an exact sequence $0 \to M \to E \to L \to 0$ with $E \in \mathscr{C}_{n-1}$ and $L \in \mathscr{F}_{n-1}$. Since L has a monic $\mathscr{C}_{n-1}$-cover $\phi : F \to L$, there is $\alpha : E \to F$ such that the following exact diagram is commutative

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ & & & F \arrow[d, "\phi"] & \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r] \arrow[ur, "\alpha"] & L \arrow[r] & 0 \end{tikzcd}$$

Thus ϕ is epic, and hence it is an isomorphism. Therefore, L is $(n-1)$-cotorsion, and so M is n-cotorsion, as desired. $\square$

By specializing Theorem 4.1 to the case $n = 1$, we have the following theorem.

Theorem 4.2. *The following are equivalent for a ring R:*

(1) *Every right R-module is* 1*-cotorsion*;
(2) *Every* (1-)*flat right R-module is* 1*-cotorsion*;
(3) *Every* 1*-flat right R-module is projective*;
(4) $pd(M) \leq 1$ *for every flat right R-module M*;
(5) *Every right R-module M with $M \in \mathscr{P}_1^{\perp}$ is cotorsion*;
(6) *Every quotient module of any injective right R-module is cotorsion*;
(7) *Every quotient module of any cotorsion right R-module is cotorsion*;
(8) *Every pure submodule of any projective right R-module is projective.*

Proof. The equivalence of (1) through (5) follows from Theorem 4.1.

(4) $\Rightarrow$ (8) Let M be a projective right R-module and N a pure submodule of M. Then $0 \to N \to M \to M/N \to 0$ is exact. Note that M/N is flat and hence $pd(M/N) \leq 1$ by (4). Thus N is projective.

(8) $\Rightarrow$ (4) Let M be any flat right R-module. There exists an exact sequence $0 \to N \to P \to M \to 0$ with P projective. Note that N is a pure submodule of P, so N is projective. It follows that $pd(M) \leq 1$.

(6) $\Rightarrow$ (7) Let M be any cotorsion right R-module and N any submodule of M. There exists an exact sequence $0 \to N \to E(N) \to L \to 0$. Consider the

following pushout diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E(N) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & M/N \arrow[r] & 0. \\ & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\ & 0 & 0 & & \end{tikzcd}$$

By (6), L is cotorsion. Since M is cotorsion, H is cotorsion by Xu (1996, Proposition 3.1.2). Note that $E(N)$ is cotorsion, it follows that M/N is cotorsion by Xu (1996, Proposition 3.1.2) again.

(7) $\Rightarrow$ (1) Let M be any right R-module. Then there exists an exact sequence $0 \to M \to E \to E/M \to 0$ with E injective. Thus M is 1-cotorsion since E/M is cotorsion.

(1) $\Rightarrow$ (6) Let E be any injective right R-module and K a submodule of E. The exactness of the sequence $0 \to K \to E \to E/K \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}^1(F, E) \to \mathrm{Ext}^1(F, E/K) \to \mathrm{Ext}^2(F, K),$$

where F is a flat right R-module. Since $\mathrm{Ext}^2(F, K) = 0$ by (1), then $\mathrm{Ext}^1(F, E/K) = 0$, as required. □

Recall that R is said to be a *semisimple Artinian ring* (Anderson and Fuller, 1974) if it is a direct sum of a finite number of simple Artinian rings.

Corollary 4.3. *The following are equivalent for a ring R:*

(1) *Every right R-module is 1-cotorsion and the cotorsion envelope of every simple right R-module is projective*;
(2) *R is a semisimple Artinian ring.*

Proof. (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2) By (1) and Wakamatsu's Lemma, every simple right R-module M is a pure submodule of a projective right R-module, and hence M is projective by Theorem 4.2(8). So (2) follows. □

Proposition 4.4. *The following are equivalent for a ring R:*

(1) *Every right R-module is 2-cotorsion, and every flat right R-module has a cotorsion cover*;
(2) *Every flat right R-module has a cotorsion cover with the unique mapping property.*

Proof. (1) $\Rightarrow$ (2) Let M be any flat right R-module. Then M has a cotorsion cover $f : F \to M$ by (1). It is enough to show that, for any cotorsion right R-module G and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\mathrm{im}(g) \to M$ such that $\beta\pi = f$ since $\mathrm{im}(g) \subseteq \ker(f)$, where $\pi : F \to F/\mathrm{im}(g)$ is the natural map. Note that $F/\mathrm{im}(g)$ is cotorsion by Proposition 3.3 since $\ker(g)$ is 2-cotorsion. Thus there exists $\alpha : F/\mathrm{im}(g) \to F$ such that $\beta = f\alpha$, and so we get the following exact commutative diagram:

$$\begin{tikzcd}
 & & & M & \\
0 \arrow[r] & \ker(g) \arrow[r, "i"] & G \arrow[r, "g"] \arrow[ur, "0"] & F \arrow[u, "f"] \arrow[r, shift left, "\pi"] & F/\mathrm{im}(g) \arrow[r] \arrow[ul, "\beta"'] \arrow[l, shift left, "\alpha"] & 0.
\end{tikzcd}$$

Thus $f\alpha\pi = f$, and hence $\alpha\pi$ is an isomorphism since f is a cover. Therefore π is monic, and $g = 0$.

(2) $\Rightarrow$ (1) Let M be any right R-module. Then we have the exact sequences

$$0 \longrightarrow M \xrightarrow{\sigma_M} C_0(M) \longrightarrow C_0(M)/M \longrightarrow 0,$$
$$0 \longrightarrow C_0(M)/M \longrightarrow C \xrightarrow{\psi} N \longrightarrow 0,$$

where $C = \mathscr{C}_0(\mathscr{C}_0(M)/M)$, and N is flat. Thus we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\sigma_M} C_0(M) \xrightarrow{\psi} C \xrightarrow{\psi} N \longrightarrow 0$$

Let $\theta : H \to N$ be a cotorsion cover with the unique mapping property. Then there exists $\delta : C \to H$ such that $\psi = \theta\delta$. Thus $\theta\delta\varphi = \psi\varphi = 0 = \theta 0$, and hence $\delta\varphi = 0$, which implies that $\ker(\psi) = \mathrm{im}(\varphi) \subseteq \ker(\delta)$. Therefore, there exists $\gamma : N \to H$ such that $\gamma\psi = \delta$, and so we get the following exact commutative diagram:

$$\begin{tikzcd}
 & & & & H \arrow[d, shift right, "\theta"'] & \\
0 \arrow[r] & M \arrow[r] & \mathscr{C}_0(M) \arrow[r, "\varphi"] & C \arrow[r, "\psi"] \arrow[ur, "\delta"] & N \arrow[r] \arrow[u, shift right, "\gamma"'] & 0.
\end{tikzcd}$$

Thus $\theta\gamma\psi = \psi$, and so $\theta\gamma = 1_N$ since ψ is epic. It follows that N is isomorphic to a direct summand of H, and hence N is cotorsion. So M is 2-cotorsion by Proposition 3.3. □

It is well known that a ring R is von Neumann regular if and only if every (cyclic) right R-module is flat if and only if every cotorsion right R-module is flat if and only if every cotorsion right R-module is injective if and only if every (cotorsion) right R-module has a flat cover with the unique mapping property (see Xu, 1996, Theorem 3.3.2 and Mao and Ding, 2005, Proposition 2.19). Next we shall give characterizations of those rings such that every right R-module is n-flat for a fixed non-negative integer n.

Theorem 4.5. *Let R be a ring and n a fixed non-negative integer. Then the following are equivalent*:

(1) *Every right R-module is n-flat*;
(2) *Every finitely generated right R-module is n-flat*;
(3) *Every cyclic right R-module is n-flat*;
(4) *Every n-cotorsion right R-module is n-flat*;
(5) *Every n-cotorsion right R-module is injective*;
(6) $\mathrm{Ext}^1(M, N) = 0$ *for all n-cotorsion right R-modules M, N*;
(7) $\mathrm{Ext}^i(M, N) = 0$ *for all $i \geq 1$ and all n-cotorsion right R-modules M, N*;
(8) *Every (n-cotorsion) right R-module M has an $\mathscr{F}_n$-cover with the unique mapping property.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (8) and (1) $\Rightarrow$ (4) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are obvious. (1) $\Leftrightarrow$ (5) follows from Theorem 3.9.

(8) $\Rightarrow$ (4) Let M be any n-cotorsion right R-module. There is the following exact commutative diagram

$$\begin{tikzcd}
& \mathscr{F}_n(K) \arrow[d, "\epsilon_K"'] \arrow[dr, "\alpha\epsilon_K"] \arrow[drr, "0"] & & & \\
0 \arrow[r] & K \arrow[r, "\alpha"'] \arrow[d] & \mathscr{F}_n(M) \arrow[r, "\epsilon_M"'] & M \arrow[r] & 0 \\
& 0 & & &
\end{tikzcd}$$

with $K \in \mathscr{C}_n$. Note that $\epsilon_M \alpha \epsilon_K = 0 = \epsilon_M 0$, so $\alpha\epsilon_K = 0$ by (8). Therefore $K = \mathrm{im}(\epsilon_K) \subseteq \ker(\alpha) = 0$, and so M is n-flat, as required.

(4) $\Rightarrow$ (1) For any right R-module M, by Theorem 3.9, there is a short exact sequence $0 \to M \to N \to L \to 0$, where N is n-cotorsion and L is n-flat. Since N is n-flat by (4), M is n-flat by Proposition 3.6(1). Hence (1) follows.

(3) $\Rightarrow$ (5) Let M be any n-cotorsion right R-module and I any right ideal of R. Then $\mathrm{Ext}^1(R/I, M) = 0$ by (3). Thus M is injective, as desired. $\square$

Remark 4.6. By Theorem 4.5, if $n \geq 1$, then every right R-module is n-flat if and only if every right R-module is 1-flat if and only if for any non-negative integer m, every m-cotorsion right R-module is injective if and only if $\sigma_R(M) = \infty$ for every right R-module M. Thus von Neumann regular rings can be classified into three mutually exclusive types: (a) semisimple Artinian rings; (b) rings R such that $rD(R) \neq 0$ and every right R-module is 1-flat; (c) rings R for which there is a right R-module N with $\sigma_R(N) = 0$.

Now, we argue when $\mathscr{F}_n$ (resp. $\mathscr{C}_n$) coincides with $\mathscr{F}_{n+1}$ (resp. $\mathscr{C}_{n+1}$).

Theorem 4.7. *Let R be a ring and n a fixed non-negative integer. Then the following are equivalent*:

(1) $r.\ \sigma\text{-}\mathrm{dim}(R) \leq n$;
(2) *Every n-flat right R-module is $(n+1)$-flat*;

(3) *Every nth syzygy of any flat right R-module is projective relative to each epimorphism $B \to C$, where B is cotorsion and C is n-cotorsion*;
(4) *Every $(n+1)$-cotorsion right R-module is n-cotorsion*;
(5) *For any non-negative integer m, every m-cotorsion right R-module is n-cotorsion*;
(6) *Every nth syzygy of any flat right R-module is $(n+1)$-flat*;
(7) *Every nth syzygy of any flat right R-module is m-flat for any integer $m \geq n+1$.*

Proof. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5), and (1) $\Rightarrow$ (7) $\Rightarrow$ (6) are clear.

(2) $\Leftrightarrow$ (4) holds by Theorem 3.9.

(5) $\Rightarrow$ (1) Let M be a right R-module with $\sigma_R(M) \geq n$, i.e., M is n-flat. For any non-negative integer m and any m-cotorsion right R-module N, we have $\mathrm{Ext}^1(M, N) = 0$ since N is n-cotorsion by (5). So $\sigma_R(M) = \infty$, as desired.

(3) $\Rightarrow$ (4) Let M be an $(n+1)$-cotorsion right R-module. There exists an exact sequence $0 \to M \to E \to N \to 0$ with E injective and N n-cotorsion. Suppose that K is a flat right R-module and K_n an nth syzygy of K. Then $\mathrm{Ext}^1(K_n, M) = 0$ by (3), and hence $\mathrm{Ext}^{n+1}(K, M) = 0$, which means that M is n-cotorsion.

(4) $\Rightarrow$ (3) Let $f: B \to C$ be an epimorphism and $A = \ker(f)$, where B is cotorsion and C is n-cotorsion. The exactness of $0 \to A \to B \to C \to 0$ shows that A is $(n+1)$-cotorsion, and so A is n-cotorsion by (4). Let N_n be an nth syzygy of a flat right R-module N. Then N_n is n-flat, and so $\mathrm{Ext}^1(N_n, A) = 0$. Thus (3) follows.

(6) $\Rightarrow$ (4) Let M be an $(n+1)$-cotorsion right R-module, K any flat right R-module and K_n an nth syzygy of K. Then $\mathrm{Ext}^1(K_n, M) = 0$ by (6), and so $\mathrm{Ext}^{n+1}(K, M) = 0$, which implies that M is n-cotorsion. □

Let $n = 0$ in Theorem 4.7. One gets the following corollary.

Corollary 4.8. *The following are equivalent for a ring R*:

(1) *r.* σ-$\dim(R) = 0$;
(2) *Every flat right R-module is* 1*-flat*;
(3) *Every flat right R-module is projective relative to each epimorphism $B \to C$, where B and C are cotorsion*;
(4) *For any short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if B and C are cotorsion, then A is cotorsion*;
(5) *For any non-negative integer m, every m-cotorsion right R-module is cotorsion*;
(6) *Every flat right R-module is m-flat for any integer $m \geq 1$.*

Remark 4.9. (1) Let R be a ring and n a non-negative integer. We note that, if every right R-module is n-cotorsion, then $r.\ \sigma$-$\dim(R) \leq n$. Indeed, by Theorem 4.1, every n-flat right R-module is projective, and hence $(n+1)$-flat. Thus $r.\ \sigma$-$\dim(R) \leq n$ by Theorem 4.7.

(2) Let $R = \mathbb{Z}$, the ring of integers. Then every R-module is 1-cotorsion, and so σ-$\dim(R) \leq 1$. However, in the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are cotorsion, but $\mathbb{Z}$ is not cotorsion, so σ-$\dim(R) \neq 0$ by Corollary 4.8(4). Thus σ-$\dim(R) = 1$.

We end this article with the following theorem.

Theorem 4.10. *Let R be a ring with $rD(R) < \infty$ and n a fixed non-negative integer. Then the following are equivalent*:

(1) *r.* σ-dim$(R) \leq n$;
(2) *Every right R-module is n-cotorsion*;
(3) $pd(M) \leq n$ *for every flat cotorsion right R-module M*;
(4) $pd(\mathscr{C}_0(M)) \leq n$ *for every flat right R-module M*;
(5) $pd(\mathscr{F}_0(M)) \leq n$ *for every cotorsion right R-module M*;
(6) *Every projective right R-module is n-cotorsion.*

Proof. (2) $\Rightarrow$ (1) holds by Remark 4.9(1). (2) $\Rightarrow$ (6) is clear.

(2) $\Rightarrow$ (3) follows from Theorem 4.1.

(4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) hold since cotorsion envelopes of flat modules are always flat and flat covers of cotorsion modules are always cotorsion.

(1) $\Rightarrow$ (2) Let N be any right R-module. We may assume $id(N) = m < \infty$ by hypothesis. For any n-flat right R-module M, we have $\sigma_R(M) = \infty$ by (1), and so $\mathrm{Ext}^1(M, N) = 0$ since N is m-cotorsion, which implies that M is projective. Therefore every right R-module is n-cotorsion by Theorem 4.1.

(3) $\Rightarrow$ (2) Let M be any flat right R-module, we only need to show that $pd(M) \leq n$ by Theorem 4.1. Note that M is m-cotorsion for some non-negative integer m since $rD(R) < \infty$. Consider a cotorsion resolution of M

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^{m-1} \to C^m \to \cdots,$$

where each C^i is cotorsion, $L^i = \mathrm{coker}(C^{i-2} \to C^{i-1}) \to C^i$ is a cotorsion envelope of L^i, $i = 0, 1, \ldots, C^{-2} = 0$, $C^{-1} = M$. Since M is flat, C^i is flat cotorsion and L^i is flat, $i = 0, 1, \ldots$. Note that $0 \to M \to C^0 \to C^1 \to \cdots \to C^{m-1} \to L^m \to 0$ is exact, so L^m is cotorsion by Proposition 3.3. Since $pd(L^m) \leq n$ and $pd(C^i) \leq n$ by (3), $i = 0, 1, \ldots, m-1$, $pd(M) \leq n$.

(6) $\Rightarrow$ (2) For any flat right R-module M, $pd(M) = m < \infty$ since $rD(R) < \infty$. Thus there exists an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is projective, $i = 0, 1, \ldots, m$. Thus M is n-cotorsion by Remark 3.4 since each P_i is n-cotorsion. $\square$

Remark 4.11. By Theorem 4.10, if $r.$ σ-dim$(R) = 0$, then R is either right perfect or $rD(R) = \infty$.

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