Stochastic integration for fractional Lévy process and stochastic differential equation driven by fractional Lévy noise *

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Abstract In this paper, based on the white noise analysis of square integrable pure-jump Lévy process given by [1], we define the formal derivative of fractional Lévy process defined by the square integrable pure-jump Lévy process as the fractional Lévy noises by considering fractional Lévy process as the generalized functional of Lévy process, and then we define the Skorohod integral with respect to the fractional Lévy process. Moreover, we propose a class of stochastic Volterra equations driven by fractional Lévy noises and investigate the existence and uniqueness of their solutions; In addition, we propose a class of stochastic differential equations driven by fractional Lévy noises and prove that under the Lipschtz and linear conditions there exists unique stochastic distribution-valued solution.

Keywords White noise; fractional Lévy processes; stochastic ordinary linear differential equations; stochastic Volterra equation
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1 Introduction

The study on fractional processes started from the fractional Brownian motion introduced by Kolmogrov [2] and popularized by Mandelbrot and Van Ness [3]. The self-similarity and long-range dependence properties make fractional Brownian motion suitable to model driving noise in different applications such as mathematical finance and network traffic analysis. However, its light tails are often inadequate to model the higher variability phenomena appeared in these practical systems.

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Thus, it is natural to consider the more general fractional processes. Marquardt [4] introduced the fractional Lévy processes, restricted to the case of Lévy processes with zero mean, finite variance and without Brownian components. In [5] and [6] by white noise approach, the authors constructed generalized fractional Lévy processes as Lévy white noise functionals under a simple condition on Lévy measure. In our previous work [7] [8], we defined infinite-dimensional fractional Lévy processes on Gel′fand Triple and investigate its properties of distribution and sample path.

In order to use the fractional Lévy processes to model the higher variability phenomena in real-world systems, it is necessary to investigate stochastic calculus for fractional Lévy processes and stochastic differential equations driven by these processes. In [4], the authors defined stochastic integral for a class of deterministic integrands with respect to real-valued fractional Lévy processes; In [8], we defined stochastic integral for a class of real deterministic functions and deterministic operator-valued processes with respect to fractional Lévy processes on Gel′fand Triple; In [9], by using S-transform, the authors constructed the stochastic calculus for convoluted Lévy processes which are built by convoluting a Volterra-type kernel with a pure jump, zero expectation Lévy process with finite moment of any order. Especially, the authors investigate the Skorohod integral for fractional Lévy processes whose underlying Lévy process has finite moment of any order.

The purpose of this paper is to investigate stochastic calculus of fractional Lévy processes whose underlying Lévy processes are square integrable and investigate stochastic differential equation driven by fractional Lévy processes.

Motivated by the study of Lokka and Proske [1] about the infinite dimensional analysis of the square integrable pure jump Lévy process on the Poisson space, we investigate the stochastic calculus for fractional Lévy processes whose underlying Lévy proceses are only squre integrable. Our results generalize that of [9] which demands the underlying Lévy processes have finite moment of any order. Moreover, we investigate stochastic Volterra equation driven by fractional Lévy noise. In addition, we obtain a unique local continuous solution for the stochastic differential equation driven by fractional Lévy noise with Lipschtz and linear conditions.

This paper is organized as follows: In Section 2, we recall the basic results about the infinite dimensional analysis of the square integrable pure jump Lévy process given by [1], all of our work is done in this framework; Based on the infinite dimensional analysis of Lévy process, in Section 3, we define Skorohod integral with respect to fractional Lévy processes and investigate its S-transform. Moreover, we obtain an integration transformation formula which can be used to transform stochastic differential equations driven by fractional Lévy noises with different parameter into the same kind of stochastic differential equations with the same parameter β; In Section 4, we investigate the condition of existence and uniqueness of the solution for stochastic differential equation of Volterra type driven by fractional Lévy processes. In Section 5, we obtain a unique continuous global solution for a stochastic differential equation driven by fractional Lévy noise with Lipschtz and linear conditions.

2 Infinite dimensional analysis of square integrable pure jump Lévy process

The Lévy process $X = \{X_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ is a stochastic continuous process with stationary and independent increments and the characteristic function of X_1 takes the form:

$$\widehat{\mu}(u) = \exp\{i\gamma u - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} [e^{iux} - 1 - iux1_{|x|\leq 1}] d\nu(x)\}, u \in \mathbb{R}, \tag{2.1}$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and ν is the Lévy measure satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (|x|^2 \wedge 1) d\nu(x) < \infty. \tag{2.2}$$

Then, for any $t \geq 0$, X has the Lévy-Itô decomposition

$$X_t = \gamma t + B_t + \int_0^t \int_{|x|\geq 1} xN(ds, dx) + \int_0^t \int_{|x|\leq 1} x\widetilde{N}(ds, dx), \tag{2.3}$$

where

$$N((0,t] \times A) = \sum_{s \leq t} 1_A(\triangle X_s), A \in \mathscr{B}(\mathbb{R})$$

is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ ($\mathbb{R}_0 =: \mathbb{R} \setminus \{0\}$) and

$$\widetilde{N}((0,t] \times A) = N((0,t] \times A) - t\nu(A) \tag{2.4}$$

is its compensation. $B = \{B_t, t \geq 0\}$ is a Brownian motion with mean 0 and covariance operator σ^2 which is independent of N. The first integral in (2.3) converges in probability (even a.s.), the second one converges in $L^2(\Omega, \mathscr{F}, P)$. If in (2.1), $\sigma = 0$, we call X a Lévy process without Brownian component. In this paper, we always assume that the Lévy process X has no Brownian part. Furthermore, we suppose $\mathbb{E}[X_1] = 0$ and

$$\int_{|x|>1} |x|^2 d\nu(x) < \infty.$$

Thus, (2.1) can be written as

$$\widehat{\mu}(u) = \exp\{\int_{\mathbb{R}} [e^{iux} - 1 - iux] d\nu(x)\}, u \in \mathbb{R},$$

and

$$X_t = \int_0^t \int_{\mathbb{R}} x\widetilde{N}(ds, dx).$$

In this case, X is a martingale and we call it pure-jump Lévy process. We will work with a two-side Lévy process constructed by taking two independent copies $X^{(1)} = \{X_t^{(1)}, t \geq 0\}$ and $X^{(2)} = \{X_t^{(2)}, t \geq 0\}$ of a one-side Lévy process and setting

$$X_t = \begin{cases} X_t^{(1)}, & t \geq 0 \\ -X_{-t_-}^{(2)}, & t < 0 \ . \end{cases}$$

Next we recall the basic results about the infinite dimensional analysis of the square integrable pure jump Lévy process given by [1].

Let ξ_n denote denote the n'th Hermite function, the set of Hermite functions $\{\xi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$ and by $\mathscr{S}'(\mathbb{R}^d)$ the space of tempered distributions. The nuclear topology on $\mathscr{S}(\mathbb{R}^d)$ is induced by the pre-Hilbertian norms

$$\|\phi\|_p^2 := \sum_{\alpha=(\alpha_1,\ldots,\alpha_d)\in\mathbb{N}^d} (1+\alpha)^{2p}(\phi, \xi_\alpha)^2_{L^2(\mathbb{R}^d)}, p \in \mathbb{N}_0,$$

where $(1+\alpha)^{2p} = \prod_{i=1}^d (1+\alpha_i)^{2p}$, $\xi_\alpha(x_1, \ldots, x_d) = \prod_{i=1}^d \xi_{\alpha_i}(x_i)$. Let $\mathbb{U} = \mathbb{R} \times \mathbb{R}_0$, define

$$\mathscr{S}(\mathbb{U}) := \{\phi \in \mathscr{S}(\mathbb{R}^2) : \phi(x, 0) = \frac{\partial \phi}{\partial y}(x, 0) = 0\}.$$

$\mathscr{S}(\mathbb{U})$ is a closed subspace of $\mathscr{S}(\mathbb{R}^2)$, thus it is a countably Hilbertian nuclear algebra endowed with the topology induced by the norms $\|\cdot\|_p$, and its dual $\mathscr{S}'(\mathbb{U}) \supset \mathscr{S}'(\mathbb{R}^2)$. For $\phi \in \mathscr{S}(\mathbb{U})$, $\Phi \in \mathscr{S}'(\mathbb{U})$, the action of Φ on ϕ is given by $\langle \Phi, \phi \rangle = \int_{\mathbb{U}} \Phi(x, y)\phi(x, y)dxdy$. Assume that ν is the Lévy measure on $\mathbb{R}_0$ satisfying

$$\int_{\mathbb{R}_0} |x|^2 d\nu(x) < \infty. \tag{2.5}$$

Denote λ the Lebesgue measure on $\mathbb{R}$ and let π denote the measure on $\mathbb{U}$ given by $\pi = \lambda \times \nu$. By Lemma 2.1 of [1], there exists an element denoted by $1 \otimes \dot{\nu}$ in $\mathscr{S}'(\mathbb{U})$ such that

$$\langle 1 \otimes \dot{\nu}, \phi \rangle = \int_{\mathbb{U}} \phi(x)\pi(dx), \phi \in \mathscr{S}(\mathbb{U}). \tag{2.6}$$

Denote $L^2(\mathbb{U}, \pi)$ by the space of all square integrable functions on $\mathbb{U}$ with respect to π, let $(\cdot,\cdot)_\pi$ the inner product on $L^2(\mathbb{U}, \pi)$ and $|\cdot|_\pi$ the corresponding norms on this space. Define $\mathscr{N}_\pi := \{\phi \in \mathscr{S}(\mathbb{U}) : |\phi|_\pi = 0\}$, then $\mathscr{N}_\pi$ is a closed ideal of $\mathscr{S}(\mathbb{U})$. Let $\widetilde{\mathscr{S}}(\mathbb{U})$ be the space $\widetilde{\mathscr{S}}(\mathbb{U}) := \mathscr{S}(\mathbb{U})/\mathscr{N}_\pi$ endowed with the topology induced by the system of norms $\|\widehat{\phi}\|_{p,\pi} := \inf_{\psi\in\mathscr{N}_\pi} \|\phi + \psi\|_p$, then $\widetilde{\mathscr{S}}(\mathbb{U})$ is a nuclear algebra. Let $\widetilde{\mathscr{S}'}(\mathbb{U})$ be the dual of $\widetilde{\mathscr{S}}(\mathbb{U})$, and for $p \in \mathbb{N}$, let $\widetilde{\mathscr{S}_p}(\mathbb{U})$ denote the completion of $\widetilde{\mathscr{S}}(\mathbb{U})$ with respect to the norm $\|\cdot\|_{p,\pi}$, $\widetilde{\mathscr{S}'_{-p}}(\mathbb{U})$ denote the dual of $\widetilde{\mathscr{S}_p}(\mathbb{U})$. $\widetilde{\mathscr{S}}(\mathbb{U})$ is the projective limit of $\{\widetilde{\mathscr{S}_p}(\mathbb{U}), p \geq 0\}$, and $\widetilde{\mathscr{S}'}(\mathbb{U})$ is the inductive limit of $\{\widetilde{\mathscr{S}'_{-p}}(\mathbb{U}), p \geq 0\}$. $\widetilde{\mathscr{S}}(\mathbb{U})$ has similar nice properties as the classical Schwartz space. Thus, Lokka and

Proske [1] introduce it to construct the white noise analysis of Lévy process.
Theorem 2.1 ([1]) (1)There exists a probability measure μ_π on $\widetilde{\mathscr{S}'}(\mathbb{U})$ such that

$$\int_{\widetilde{\mathscr{S}'}(\mathbb{U})} e^{i\langle\omega,\phi\rangle} d\mu_\pi(\omega) = \exp\{\int_{\mathbb{U}}(e^{i\phi(x)}-1)d\pi(x)\}, \forall\phi\in\widetilde{\mathscr{S}}(\mathbb{U}). \qquad (2.7)$$

(2) There exists a $p_0\in\mathbb{N}$ such that $1\otimes\dot{\nu}\in\widetilde{\mathscr{S}'_{-p_0}}(\mathbb{U})$, and a natural number $q_0>p_0$ such that the imbedding operator $\widetilde{\mathscr{S}'_{-p_0}}(\mathbb{U})\hookrightarrow\widetilde{\mathscr{S}'_{-q_0}}(\mathbb{U})$ is Hilbert-Schimidt and $\mu_\pi(\widetilde{\mathscr{S}'_{-q_0}}(\mathbb{U}))=1$.

From now on, for all q_0, p_0 are described in the Theorem 2.1. Set $\Omega=\widetilde{\mathscr{S}'}(\mathbb{U})$ and μ_π given by Theorem 2.1 on which [1] give the infinite dimensional calculus for pure jump measure, and all of our following discussion is based on this probability space.

Let $C_n(\cdot)$ be the generalized Charlier polynomials given by [1], for all $m,n\in\mathbb{N}$, $\varphi^{(n)}\in\widetilde{\mathscr{S}}(\mathbb{U})^{\widehat{\otimes}n}$, $\psi^{(m)}\in\widetilde{\mathscr{S}}(\mathbb{U})^{\widehat{\otimes}m}$, ($\widehat{\otimes}$ denotes the symmetrized tensor product), the following orthogonality relation holds,

$$\int_{\widetilde{\mathscr{S}'}(\mathbb{U})}\langle C_n(\omega),\varphi^{(n)}\rangle\langle C_m(\omega),\psi^{(m)}\rangle d\mu_\pi(\omega)=\begin{cases}0, & n\neq m\\ n!(\varphi^{(n)},\psi^{(n)})_\pi, & n=m\end{cases}.$$

Especially, for $n=1$, $C_1(\omega)=\omega-1\otimes\dot{\nu}$. Since $L^2(\mathbb{U})$ is dense in $\widetilde{\mathscr{S}}(\mathbb{U})$, for $f\in L^2(\mathbb{U})$, there exists a sequence of functions $f_n\in\widetilde{\mathscr{S}}(\mathbb{U})$ such that $f_n\to f$ in $\widehat{L}^2(\mathbb{U},\pi)$ as $n\to\infty$. Define $\langle C_1(\omega),f\rangle$ by $\langle C_1(\omega),f\rangle=\lim_{n\to\infty}\langle C_1(\omega),f_n\rangle$ (*limit in* $L^2(\mu_\pi)$), the definition is independent of the choice of approximating sequence and the following isometry holds

$$\int_{\widetilde{\mathscr{S}'}(\mathbb{U})}\langle C_1(\omega),f\rangle^2 d\mu_\pi(\omega)=\int_{\widetilde{\mathscr{S}'}(\mathbb{U})}\langle\omega-1\otimes\dot{\nu},f\rangle^2 d\mu_\pi(\omega)=|f|_\pi^2. \qquad (2.8)$$

For any Borel sets $\Lambda_1\subset\mathbb{R}$ and $\Lambda_2\subset\mathbb{R}_0$ such that the 0 is not in the closure of Λ_2, define the random measure

$$N(\Lambda_1,\Lambda_2):=\langle\omega,1_{\Lambda_1\times\Lambda_2}\rangle, \widetilde{N}(\Lambda_1,\Lambda_2):=\langle\omega-1\otimes\dot{\nu},1_{\Lambda_1\times\Lambda_2}\rangle.$$

From the characterization function of μ_π, it is easy to deduced that N is a Poisson random measure, and $\widetilde{N}$ is the corresponding compensated measure. The compensator of $N(\Lambda_1,\Lambda_2)$ is given by $\langle 1\otimes\dot{\nu},1_{\Lambda_1\times\Lambda_2}\rangle$ which is equal to $\pi(\Lambda_1\times\Lambda_2)$. Moreover,

$$\int_{\mathbb{U}}\phi(s,x)\widetilde{N}(ds,dx)=\langle\omega-1\otimes\dot{\nu},\phi\rangle, \phi\in L^2(\mathbb{U},\pi). \qquad (2.9)$$

Then, the pure jump Lévy process has a representation

$$X_t=\int_0^t\int_{\mathbb{R}}x\widetilde{N}(ds,dx)=\langle\omega-1\otimes\dot{\nu},f_t\rangle, f_t(s,x)=x1_{s\leq t}(s). \qquad (2.10)$$

The Wiener integral of $g \in L^2(\mathbb{R})$ with respect to X has the following representation

$$\int_{\mathbb{R}} g(s)dX_s = \int_{\mathbb{R}}\int_{\mathbb{R}} g(s)x\widetilde{N}(ds,dx) = \langle \omega - 1\otimes\dot{\nu}, \phi\rangle, \tag{2.11}$$

where $\phi(s,x) = xg(s)$.

Define the space $\mathscr{P}(\widetilde{\mathscr{S}'}(\mathbb{U})) = \{f : \widetilde{\mathscr{S}'}(\mathbb{U}) \to C, f(\omega) = \sum_{n=0}^{N}\langle \omega^{\otimes n}, \phi^{(n)}\rangle$, $\omega \in \widetilde{\mathscr{S}'}(\mathbb{U}), \phi^{(n)} \in \widetilde{\mathscr{S}}(\mathbb{U})^{\widehat{\otimes} n}, N \in \mathbb{N}\}$, f is called a continuous polynomial function if $f \in \mathscr{P}(\widetilde{\mathscr{S}'}(\mathbb{U}))$ and it admits a unique representation of the form

$$f(\omega) = \sum_{n=0}^{\infty}\langle C_n(\omega), f_n\rangle, f_n \in \widetilde{\mathscr{S}}(\mathbb{U})^{\widehat{\otimes} n}.$$

For any number $p \geq q_0$, define the Hilbert space $(\mathscr{S})_p^1$ as the completion of $\mathscr{P}(\widetilde{\mathscr{S}'}(\mathbb{U}))$ with respect to the norm

$$\|f\|_{p,1}^2 = \sum_{n=0}^{\infty}(n!)^2\|f_n\|_{p,\pi}^2.$$

The corresponding inner product is

$$((f,g))_{p,1} = \sum_{n=0}^{\infty}(n!)^2((f_n,g_n))_{p,\pi}.$$

where $((\cdot,\cdot))_{p,\pi}$ denotes the inner product on $\widetilde{\mathscr{S}_p}(\mathbb{U})^{\widehat{\otimes} n}$. Obviously, $(\mathscr{S})_{p+1}^1 \subset (\mathscr{S})_p^1$. In [1], the authors define $(\mathscr{S})^1$ as the projective limit of $\{(\mathscr{S})_p^1, p \geq q_0\}$, and it is a nuclear Fréchet space which can be densely imbedding in $L^2(\mu_\pi)$. Denote $(\mathscr{S})_{-p}^{-1}$ as the dual of $(\mathscr{S})_p^1$, $(\mathscr{S})^{-1}$ as the inductive limit of $\{(\mathscr{S})_{-p}^{-1}, p \geq q_0\}$ which is equal to the dual of $(\mathscr{S})^1$. $F \in (\mathscr{S})^{-1}$ if and only if F admits an expansion

$$F(\omega) = \sum_{n=0}^{\infty}\langle C_n(\omega), F_n\rangle, F_n \in \widetilde{\mathscr{S}'}(\mathbb{U})^{\widehat{\otimes} n},$$

and there exists a $p \geq q_0$ such that

$$\|F\|_{-p,-1}^2 = \sum_{n=0}^{\infty}\|F_n\|_{-p,\pi}^2 < \infty.$$

For $F \in (\mathscr{S})^{-1}$, $f \in (\mathscr{S})^1$,

$$\langle\langle F, g\rangle\rangle = \sum_{n=0}^{\infty} n!\langle F_n, f_n\rangle_\pi,$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ is an extension of the inner product on $L^2(\mu_\pi)$. $(\mathscr{S})^1$ is called space of stochastic test functions, $(\mathscr{S})^{-1}$ is called space of stochastic distribution functions, they are pairs of dual spaces, $(\mathscr{S})^1 \subset L^2(\mu_\pi) \subset (\mathscr{S})^{-1}$.

Next we recall the S-transform given by [1] which can transform stochastic distribution functions to deterministic functionals. Let

$$\widetilde{e}(\phi,\omega) := \exp(\langle\omega, \ln(1+\phi)\rangle - \langle 1\otimes\dot{\nu}, \phi\rangle),$$

it is analytic as a function of $\phi \in \widetilde{\mathscr{S}_{q_0}}$ for $\phi \in \widetilde{\mathscr{S}_{q_0}}$ satisfying $\phi(x) > -1$ for all $x \in \mathbb{U}$. Moreover, it has the following chaos expansion,

$$\widetilde{e}(\phi,\omega) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle C_n(\omega), \phi^{\otimes n}\rangle.$$

Denote $U_p := \{\phi \in \widetilde{\mathscr{S}(\mathbb{U})} : \|\phi\|_{p,\pi} < 1\}$, by the chaos expansion of $\widetilde{e}(\phi,\omega)$ $\widetilde{e}(\phi,\omega) \in (\mathscr{S})^1_p$ if and only if $\phi \in U_p$.

Definition 2.2([1]) Let $F \in (\mathscr{S})^{-1}_{-p}$, $\xi \in U_p$, the the S-transform of F is defined by

$$S(F)(\xi) := \langle\langle F, \widetilde{e}(\xi,\omega)\rangle\rangle.$$

For example, if $F = \sum_{n=0}^{\infty}\langle C_n(\omega), F_n\rangle \in (\mathscr{S})^{-1}_{-p}$, $\xi \in U_p$, then $S(F)(\xi) = \sum_{n=0}^{\infty}\langle F_n, \xi^{\otimes n}\rangle_\pi$.

Denote $\mathscr{U} = Hol(0)$ the algebra of germs of functions that are holomorphic in a neighborhood of 0. The S-transform is isomorphic between $(\mathscr{S})^{-1}$ and $\mathscr{U}$. Since $f, g \in \mathscr{U}$, then $fg \in \mathscr{U}$, then the following definition of Wick product is well-defined.

Definition 2.3([1]) (Wick product) Let $F, G \in (\mathscr{S})^{-1}$, define the Wick product $F \diamond G$ of F and G by

$$F \diamond G = S^{-1}(S(F)S(G)).$$

Let $F : \mathbb{U} \to (\mathscr{S})^{-1}$ be the random fields with chaos expansion

$$F(x) = \sum_{n=0}^{\infty}\langle C_n(\omega), F_n(\cdot, x)\rangle,$$

where $F_n(\cdot,x) \in \mathscr{S}'(\mathbb{U})^{\widehat{\otimes} n}$ and $\|F(x)\|_{-p,-1} < \infty$, for some $p > 0$. Let $\mathbb{L}$ denote the set of all $F : \mathbb{U} \to (\mathscr{S})^{-1}$ such that $\widetilde{F_n} \in \mathscr{S}'(\mathbb{U})^{\widehat{\otimes}(n+1)}$($\widetilde{F_n}$ is the symmetrization of F) and $\sum_{n=0}^{\infty}|\widetilde{F_n}|^2_{-p,\pi} < \infty$ for some $p > 0$.

Definition 2.4([1])(Skorohod integral) For $F \in \mathbb{L}$, define the Skorohod integral $\delta(F)$ by

$$\delta(F) := \sum_{n=0}^{\infty}\langle C_{n+1}(\omega), \widetilde{F_n}\rangle.$$

From the assumption on $\mathbb{L}$, we see that $\delta(F) \in (\mathscr{S})^{-1}$. For the predictable integrands, the Skorohod integral coincides with the usual Ito-type integral with respect to the compensated Poisson random measure.

Proposition 2.5([1]) If $F \in \mathbb{L}$, then $\delta(F) \in (\mathscr{S})^{-1}_{-p}$ for some $p > 0$ and

$$S\delta(F)(\xi) = \int_{\mathbb{U}} SF(x)(\xi)\xi(x)\pi(dx), \xi \in U_p. \tag{2.12}$$

3 S-transform and Skorohod integral for fractional Lévy processes

In this section, we give the S-transform and the Skorohod integral for fractional Lévy processes based on Section 2.1.

First, we recall the definition of fractional Lévy processes (for more details, see [4], [7], [8]). The β-fractional Lévy process $\{X_t^\beta, t \geq 0\}$ $(0 < \beta < \frac{1}{2})$ is defined by:

$$X_t^\beta = \int_{\mathbb{R}} I_-^\beta \chi_{[0,t]}(s) dX_s = \frac{1}{\Gamma(\beta+1)} \int_{-\infty}^{\infty} ((t-s)_+^\beta - (-s)_+^\beta) dX_s, \tag{3.1}$$

where X is a two-side Lévy process satisfying all the assumptions in Section 2,

$$\chi_{[0,t]}(s) = \begin{cases} 1, & 0 < s < t \\ -1, & t < s < 0 \\ 0, & \text{else.} \end{cases}$$

Furthermore, I_-^β is the Riemann-Liouville fractional integral operator defined by

$$(I_-^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_t^{+\infty} (s-t)^{\beta-1} f(s) ds \ , f \in \mathscr{S}(\mathbb{R}),$$

where $x_+ = x \vee 0$, and $\Gamma(\cdot)$ is the Gamma function.

Define $(K^\beta f)(s,x) := x I_-^\beta f(s)$, the Riemann-Liouville fractional differential operator are applied only to the time variable s. Since $I_-^\beta \chi_{[0,t]} \in L^2$, by (2.11), X_t^β has the following representation

$$X_t^\beta = \langle C_1, K^\beta \chi_{[0,t]} \rangle = \delta(K^\beta \chi_{[0,t]}). \tag{3.2}$$

Thus, by (2.11), we get the S-transform of fractional Lévy process

$$SX_t^\beta(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} I_-^\beta \chi_{[0,t]}(s) y \eta(s,y) \nu(dy) ds, \eta \in U_p, p > q_0. \tag{3.3}$$

For g satisfying $I_-^\beta g \in L^2$, the Wiener integral with respect to X^β can be written as

$$\int_{\mathbb{R}} g(t) dX_t^\beta = \int_{\mathbb{R}} I_-^\beta g(t) dX_t = \langle C_1, K^\beta g \rangle = \delta(K^\beta g). \tag{3.4}$$

Since $\int_{\mathbb{R}} g(t) dX_t^\beta \in L^2(\Omega)$, its S-transform is given by

$$S(\int_{\mathbb{R}} g(t) dX_t^\beta)(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} I_-^\beta g(t) y \eta(t,y) \nu(dy) dt, \eta \in U_p, p > q_0. \tag{3.5}$$

By the following fractional integral by parts formula of operator $I_\pm^\beta$:

$$\int_{-\infty}^{\infty} f(s) I_+^\beta g(s) ds = \int_{-\infty}^{\infty} g(s) I_-^\beta f(s) ds, \ \ f, g \in \mathscr{S}(\mathbb{R})$$

which can be extended to $f \in L^p(\mathbb{R})$, $g \in L^r(\mathbb{R})$ with $p > 1$, $r > 1$ and $\frac{1}{p} + \frac{1}{r} = 1 + \beta$, where

$$(I_+^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^t (t-s)^{\beta-1} f(s) ds \ ,$$

(3.3) can be written as

$$SX_t^\beta(\eta) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} 1_{[0,t]}(s) y I_+^\beta \eta(\cdot, y)(s) \nu(dy) ds = \int_0^t \int_{\mathbb{R}_0} y I_+^\beta \eta(\cdot, y)(s) \nu(dy) ds, t \geq 0$$

Hence,

$$\frac{d}{dt} SX_t^\beta(\eta) = \int_{\mathbb{R}_0} I_+^\beta \eta(\cdot, y)(t) y \nu(dy), \eta \in U_p, p > q_0, t \geq 0. \tag{3.6}$$

Note that (3.6) also holds for $t < 0$. We denote $\dot{X}_t^\beta$ the fractional Lévy noise in the following sense:

$$S\dot{X}_t^\beta(\eta) = \frac{d}{dt} SX_t^\beta(\eta), \eta \in U_p, p > q_0.$$

Then we can prove that $\dot{X}_t^\beta$ is a generalized stochastic distribution function and it has a chaos expansion.
Theorem 3.1 $\dot{X}_t^\beta \in (\mathscr{S})_{-p}^{-1}$ for all $p > \max\{1, q_0\}$ and

$$\dot{X}_t^\beta = \langle C_1, \lambda_t \rangle, \tag{3.7}$$

where

$$\lambda_t(u, y) = y I_-^\beta \delta_t(u) = y(t-u)_+^{\beta-1} / \Gamma(\beta)$$

Proof: We first show that $\langle C_1, \lambda_t \rangle \in (\mathscr{S})_{-p}^{-1}$ for all $p > \max\{1, q_0\}$. By the estimate

$$|\int_{\mathbb{R}} (t-u)_+^{\beta-1} \xi_n(u) du| \leq C n^{\frac{2}{3} - \frac{\beta}{2}},$$

from section 4 of [12] , where C is a certain constant independent of t,

$$\begin{aligned} \|\langle C_1, \lambda_t \rangle\|_{-1,-p}^2 &= \frac{\int_{\mathbb{R}} |y|^2 d\nu(y)}{\Gamma(\beta)} \sum_{n=1}^\infty (n+1)^{-2p} \langle (t-\cdot)_+^{\beta-1}, \xi_n \rangle_{L^2(\mathbb{R})}^2 \\ &= A \sum_{n=1}^\infty (n+1)^{-2p} (\int_{\mathbb{R}} (t-u)_+^{\beta-1} \xi_n(u) du)^2 \\ &\leq AC \sum_{n=1}^\infty (n+1)^{-2p + \frac{4}{3} - \beta} \\ &< +\infty, for\ p > \max\{1, q_0\}, \end{aligned} \tag{3.8}$$

where $A=\frac{\int_{\mathbb{R}}|y|^2 d\nu(y)}{\Gamma(\beta)}$ is a positive constant. Thus, $\langle C_1,\lambda_t\rangle\in(\mathscr{S})^{-1}_{-p}$ for all $p>\max\{1,q_0\}$. Next, we prove (3.7) holds. In fact,

$$(I^\beta_-\delta_t)(s)=\frac{1}{\Gamma(\beta)}\int_{\mathbb{R}_+}\frac{\delta_t(s+u)du}{u^{1-\beta}}=\frac{(t-s)^{\beta-1}_+}{\Gamma(\beta)}.$$

Taking S-transform of $\langle C_1,\lambda_t\rangle$, we have

$$\begin{aligned}S\langle C_1,\lambda_t\rangle(\eta)&=\int_{\mathbb{R}}\int_{\mathbb{R}_0}\frac{y(t-s)^{\beta-1}_+}{\Gamma(\beta)}\eta(s,y)\nu(dy)ds\\&=\int_{\mathbb{R}}\int_{\mathbb{R}_0}yI^\beta_-\delta_t(s)\eta(s,y)\nu(dy)ds\\&=\int_{\mathbb{R}}\int_{\mathbb{R}_0}y\delta_t(s)I^\beta_+\eta(\cdot,y)(s)\nu(dy)ds\\&=\int_{\mathbb{R}_0}yI^\beta_+\eta(\cdot,y)(t)\nu(dy),\eta\in U_p,p>q_0.\end{aligned}$$

Hence, it follows from (3.6) that (3.7) holds. □

By (3.5) for g satisfying $I^\beta_-g\in L^2$, we have

$$\int_{\mathbb{R}}g(t)dX^\beta_t=\int_{\mathbb{R}}g(t)\diamond\dot{X}^\beta_t dt. \tag{3.9}$$

Based on (3.9) we can define Skorohod integral for $(\mathscr{S})^{-1}$-valued processes with respect to the fractional Lévy process X^β as follows.

Definition 3.2(1) $F:\mathbb{R}\to(\mathscr{S})^{-1}$ is differentiable, if $\forall t\in\mathbb{R}$,

$$\lim_{\triangle t\to 0}\frac{F(t+\triangle t)-F(t)}{\triangle t}$$

exists in $(\mathscr{S})^{-1}$.

(2) Suppose $F:\mathbb{R}\longrightarrow(\mathscr{S})^{-1}$ is a given function such that $\langle\langle F(s),f\rangle\rangle\in L^1(\mathbb{R},ds)$ for all $f\in\mathscr{S}$, then $\int_{\mathbb{R}}F(s)ds$ is defined to be the unique element of $(\mathscr{S})^{-1}$ such that

$$\langle\langle\int_{\mathbb{R}}F(s)ds,f\rangle\rangle=\int_{\mathbb{R}}\langle\langle F(s),f\rangle\rangle ds.$$

Definition 3.3 Suppose that $F:\mathbb{R}\longrightarrow(\mathscr{S})^{-1}$ such that $F(s)\diamond\dot{X}^\beta(s)$ is $ds-$ integrable in $(\mathscr{S})^{-1}$. Then, we define the Skorohod integral of F with respect to X^β by

$$\delta^\beta(F):=\int_{\mathbb{R}}F(s)\delta X^\beta(s):=\int_{\mathbb{R}}F(s)\diamond\dot{X}^\beta_s ds. \tag{3.10}$$

In particular, if $A\subset\mathbb{R}$ is a Borel set, then

$$\int_A F(s)\delta X^\beta(s):=\int_{\mathbb{R}}1_A(s)F(s)\diamond\dot{X}^\beta_s ds. \tag{3.11}$$

By Definition 3.3, we get
Proposition 3.4 Let $F : \mathbb{R} \longrightarrow (\mathscr{S})^{-1}$ be Skorohod integrable with respect to X^β, $Y \in (\mathscr{S})^{-1}$, then

$$Y \diamond \delta^\beta(F) = \delta^\beta(Y \diamond F),$$

the equation holds whenever one side exists.
Proposition 3.5 Let $F : \mathbb{R} \longrightarrow (\mathscr{S})^{-1}$ be Skorohod integrable with respect to X^β and $K^\beta F \in \mathbb{L}$, then

$$\delta^\beta(F) = \delta(K^\beta F).$$

Proof: The S-transform of $\delta^\beta(F)$ is given by

$$S(\int_{\mathbb{R}} F(t)\delta X_t^\beta)(\eta) = \int_{\mathbb{R}}\int_{\mathbb{R}_0} S(F(t))(\eta) y I_+^\beta \eta(\cdot, y)(t)\nu(dy)dt, \eta \in U_p.$$

On the other hand, by Proposition 2.8, the S-transform of $\delta(K^\beta F)$ is given by

$$\begin{aligned} S(\delta(K^\beta F))(\eta) &= \int_{\mathbb{R}}\int_{\mathbb{R}_0} S(I_-^\beta F(t)y)(\eta)\eta(t,y)\nu(dy)dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}_0} S(I_-^\beta F(t)\eta(t,y))(\eta)y\nu(dy)dt \\ &= \int_{\mathbb{R}_0} [S(\int_{\mathbb{R}} I_-^\beta F(t)\eta(t,y)dt)(\eta)]y\nu(dy) \\ &= \int_{\mathbb{R}_0} [S(\int_{\mathbb{R}} F(t) I_+^\beta \eta(\cdot, y)(t)dt)(\eta)]y\nu(dy) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}_0} S(F(t))(\eta) y I_+^\beta \eta(\cdot, y)(t)\nu(dy)dt, \eta \in U_p. \end{aligned}$$

Then,

$$S(\int_{\mathbb{R}} F(t)\delta X_t^\beta)(\eta) = S(\delta(K^\beta F))(\eta),$$

Hence, we get the desired result. □

Especially, for $F \in L^p(\Omega \times \mathbb{R})$ with $1 < p < \frac{1}{\beta}$ and $I_-^\beta F \in L^2(\Omega \times \mathbb{R})$, we have

$$\int_{\mathbb{R}} F(t)\delta X_t^\beta = \int_{\mathbb{R}} I_-^\beta F(t) dX_t, \tag{3.12}$$

which holds whenever one side exists.

Let $0 < \alpha < \beta < \frac{1}{2}$, X^α is a fractional Lévy processes defined by the same underlying zero mean square integrable Lévy process X of X^β , that is

$$X_t^\alpha = \int_{\mathbb{R}} I_-^\alpha \chi_{[0,t]}(s) dX_s = \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^t ((t-s)^\alpha - (-s)_+^\alpha) dX_s.$$

Then, by the semigroup property of Riemann-Liouvile fractional integration operator, i.e. $I_-^{\alpha} I_-^{\beta} = I_-^{\alpha+\beta}$, $0 < \alpha, \beta, \alpha+\beta < 1$, we get the following relation between two different fractional Lévy processes:

$$X_t^{\beta} = \int_{\mathbb{R}} I_-^{\alpha} I_-^{\beta-\alpha} 1_{[0,t]}(s) dX_s = \frac{1}{\Gamma(\beta-\alpha)} \int_{-\infty}^{\infty} ((t-s)_+^{\beta-\alpha} - (-s)_+^{\beta-\alpha}) dX_s^{\alpha} \tag{3.13}$$

which holds in L^2-sense.
More generally, by Definition 3.3 and the semigroup property of Riemann-Liouvile fractional integration operator, we get the following integral transformation formula between two different fractional Lévy processes:
Proposition 3.6 Let $0 < \alpha < \beta < \frac{1}{2}$, X^{α}, X^{β} are fractional Lévy processes given above. $F : B \times [0,T] \times \Omega \longrightarrow (\mathscr{S})^{-1}$, $B \subset \mathbb{R}$ is a Borel set, F is Skorohod integrable with respect to X^{β}, and $K^{\beta} F \in \mathbb{L}$. Then $K^{\alpha} I_-^{\beta-\alpha} F \in \mathbb{L}$ and

$$\delta^{\beta}(F) = \delta^{\alpha}(I_-^{\beta-\alpha} F). \tag{3.14}$$

Proof: In fact, by the semigroup property of Riemann-Liouvile fractional integration operator and Proposition 3.4, we have the following relations

$$\delta^{\beta}(F) = \delta(K^{\beta} F) = \delta(I_-^{\alpha} K^{\beta-\alpha} F) = \delta^{\alpha}(I_-^{\beta-\alpha} F).$$

Remark 3.7 Proposition 3.6 provides a powerful tool in solving stochastic differential equations driven by fractional Lévy noises, by which one can transform stochastic differential equations driven by fractional Lévy noises with different parameter β into the same kind of stochastic differential equations with the same parameter.

4 The stochastic Volterra equation driven by fractional Lévy process

In this section, we consider the following Skorohod stochastic integral equation of Volterra-type driven by fractional Lévy process

$$U(t) = a(t) + \int_0^t b(t,s) \diamond U(s) ds + \int_0^t \sigma(t,s) \diamond U(s) \delta X_s^{\beta}, 0 \le t \le T. \tag{4.1}$$

Since (4.1) can be written as

$$U(t) = a(t) + \int_0^t b(t,s) \diamond U(s) ds + \int_0^t \sigma(t,s) \diamond U(s) \diamond \dot{X}_s^{\beta} ds, 0 \le t \le T.$$

Thus, (4.1) can be regarded as a special case of the following linear stochastic Volterra equation:

$$U(t) = J(t) + \int_0^t K(t,s) \diamond U(s) ds, 0 \le t \le T \tag{4.2}$$

where $T > 0$ is a given number and $J : [0,T] \to (\mathscr{S})^{-1}$, $K : [0,T] \times [0,T] \to (\mathscr{S})^{-1}$ are given stochastic distribution processes. Then, we first consider

the solution of (4.2) in $(\mathscr{S})^{-1}$.
Lemma 4.1 Let $J : [0,T] \to (\mathscr{S})^{-1}$, $K : [0,T]\times[0,T] \to (\mathscr{S})^{-1}$ are continuous stochastic distribution processes. If there exists a q satisfying $p_0 < q < \infty$ and $M < \infty$ such that

$$\|K(t,s)\|_{-1,-q} < M, 0 \le s \le t \le T,$$

then there exists a unique continuous stochastic distribution process solves (4.2) which is given by

$$U(t) = J(t) + \int_0^t H(t,s) \diamond U(s)ds, \tag{4.3}$$

where

$$\begin{aligned} H(t,s) &= \sum_{n=1}^{\infty} K_n(t,s), \\ K_{n+1}(t,s) &= \int_s^t K_n(t,u) \diamond K(u,s)du, n \ge 1, \\ K_1(t,s) &= K(t,s). \end{aligned}$$

Proof: The iteration method of Theorem 3.4.2 of [13] is also valid in our context. Thus, we omit it here.
Theorem 4.2 Assume that $a : [0,T] \to (\mathscr{S})^{-1}$ is continuous, $b : [0,T] \times [0,T] \to (\mathscr{S})^{-1}$ and $\sigma : [0,T]\times[0,T] \to (\mathscr{S})^{-1}$ are bounded continuous function. Then the equation (4.1) has a unique solution in $(\mathscr{S})^{-1}$.
Proof: Let $J(t) = a(t)$, $K(t,s) = b(t,s) + \sigma(t,s) \diamond \dot{X}_s^{\beta}$, $J(t)$ is clearly continuous. Since b and σ is bounded and continuous, it suffices to investigate $\dot{X}_t^{\beta}$. By Theorem 3.1, $\dot{X}_t^{\beta} \in (\mathscr{S})_{-p}^{-1}$ all $p > \max\{1, q_0\}$ and

$$\dot{X}_t^{\beta} = \langle C_1, \lambda_t \rangle,$$

where

$$\lambda_t(u,y) = yI_-^{\beta}\delta_t(u) = y(t-u)_+^{\beta-1}/\Gamma(\beta).$$

From [14], Hermite functions $\{\xi_n\}$ is the orthogonal basis in $L^2(\mathbb{R})$ and

$$|\xi_n(x)| \le \begin{cases} Cn^{-\frac{1}{12}}, & |x| \le 2\sqrt{n}, \\ Ce^{-\gamma x^2}, & |x| \ge 2\sqrt{n}\ , \end{cases}$$

where C and γ are certain positive constants independent of n. Let $t > s$, then we have

$$
\begin{aligned}
&\langle (t-u)_+^{\beta-1} - (s-u)_+^{\beta-1}, \xi_n \rangle_{L^2(\mathbb{R})} \\
&= \int_{\mathbb{R}} [(t-u)_+^{\beta-1} - (s-u)_+^{\beta-1}] \xi_n(u) du \\
&= \int_{\mathbb{R}} [(t-s-u)_+^{\beta-1} - (-u)_+^{\beta-1}] \xi_n(s+u) du \\
&= (t-s)^\beta \int_{\mathbb{R}} [(1-u)_+^{\beta-1} - (-u)_+^{\beta-1}] \xi_n(s+(t-s)u) du \\
&\leq (t-s)^\beta \{ \int_{|s+(t-s)u| \leq 2\sqrt{n}} C n^{-\frac{1}{12}} [(1-u)_+^{\beta-1} - (-u)_+^{\beta-1}] du \\
&\quad + \int_{|s+(t-s)u| > 2\sqrt{n}} C e^{-\gamma u^2} [(1-u)_+^{\beta-1} - (-u)_+^{\beta-1}] du \} \\
&\leq C (t-s)^\beta n^{\frac{1}{12}} \int_{\mathbb{R}} [(1-u)_+^{\beta-1} - (-u)_+^{\beta-1}] du \\
&\leq \widetilde{C} n^{\frac{1}{12}} (t-s)^\beta .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\dot{X}_t^\beta - \dot{X}_s^\beta\|_{-1,-p}^2 = \|\lambda_t(u,y) - \lambda_s(u,y)\|_{-p,\pi}^2 \\
&= \left(\int_{\mathbb{R}} |y|^2 d\nu(y)\right) \sum_{n=1}^{\infty} (n+1)^{-2p} \frac{1}{\Gamma(\beta)} \langle (t-u)_+^{\beta-1} - (s-u)_+^{\beta-1}, \xi_n \rangle_{L^2(\mathbb{R})}^2 \\
&\leq \widetilde{\widetilde{C}} |s-t|^{2\beta} \sum_{n=1}^{\infty} (n+1)^{-2p+\frac{1}{6}} \\
&\leq C' |s-t|^{2\beta} < \infty, for\ p > \max\{1, q_0\},
\end{aligned}
$$

where $\widetilde{\widetilde{C}}, \widetilde{C}, C'$ are positive constants. Thus, $\dot{X}_t^\beta$ is continuous in $(\mathscr{S})^{-1}$. Hence, by Lemma 4.1, the equation (4.1) has a unique solution in $(\mathscr{S})^{-1}$. $\square$

5 A general existence and uniqueness theorem

In this section, we consider a existence and uniqueness results for the stochastic equation satisfying linear growth and Lipschtz condition. First we consider a general stochastic differential equation with Lipschtz and linear growth conditions in the generalized distribution space, and we obtain a unique continuous global solution in $(\mathscr{S})_{-p}^{-1}$ with $p > q_0$ a natural number. Then, we consider a stochastic differential equation driven by fractional Lévy noise with Lipschtz condition and linear conditions, because of the boundedness of fractional Lévy noise in $(\mathscr{S})_{-p}^{-1}$ for $p > \max\{1, q_0\}$, we obtain a global solution.

Lemma 5.1 Let $p > q_0$ be a natural number and suppose that $F : [0, +\infty) \times (\mathscr{S})_{-p}^{-1} \to (\mathscr{S})_{-p}^{-1}$ satisfies the following two conditions:

$$\|F(t, Y)\|_{-1,-p} \leq C(1 + \|Y\|_{-1,-p}) \tag{5.1}$$

$$\|F(t,Y)-F(t,Z)\|_{-1,-p} \le C\|Y-Z\|_{-1,-p} \tag{5.2}$$

for all $t \in [0,+\infty)$, $Y,Z \in (\mathscr{S})^{-1}_{-p}$, C is a constant independent of t, Y, Z. Then the differential equation

$$\frac{dU(t)}{dt} = F(t,U(t)), U(0) = U_0 \in (\mathscr{S})^{-1}_{-p} \tag{5.3}$$

has a unique t-continuous global solution $U : [0,+\infty) \to (\mathscr{S})^{-1}_{-p}$.
Proof: We can use the classical iteration methods of the linear differential equation to verify the lemma, we omit it here. □

Based on the above results, we consider the following stochastic equation driven by fractional Lévy noise:

$$U(t) = U_0 + \int_0^t b(U(s))ds + \int_0^t \sigma(U(s))X^{\beta}_s, t \ge 0. \tag{5.4}$$

which can be written as

$$U(t) = U_0 + \int_0^t b(U(s))ds + \int_0^t \sigma(U(s)) \diamond \dot{X}^{\beta}_s ds, t \ge 0. \tag{5.5}$$

or

$$\frac{dU(t)}{dt} = b(U(t)) + \sigma(U(t)) \diamond \dot{X}^{\beta}_t, U(0) = U_0 \in (\mathscr{S})^{-1}_{-p}, t \ge 0. \tag{5.6}$$

Theorem 5.2 Let $p > \max\{1, q_0\}$ be a natural number, suppose that $b : (\mathscr{S})^{-1}_{-p} \to (\mathscr{S})^{-1}_{-p}$ and $\sigma : (\mathscr{S})^{-1}_{-p} \to (\mathscr{S})^{-1}_{-p}$ satisfies the following conditions:

$$\begin{aligned}
&\|b(Y)\|_{-1,-p} \le C(1+\|Y\|_{-1,-p}),\\
&\|b(Y)-b(Z)\|_{-1,-p} \le C\|Y-Z\|_{-1,-p},\\
&\|\sigma(Y)\|_{-1,-p} \le C(1+\|Y\|_{-1,-p}),\\
&\|\sigma(Y)-\sigma(Z)\|_{-1,-p} \le C\|Y-Z\|_{-1,-p},
\end{aligned}$$

for all $Y,Z \in (\mathscr{S})^{-1}_{-p}$, with C independent of Y, Z. Then the differential equation (5.5) has a unique continuous solution $U : [0,\infty) \to (\mathscr{S})^{-1}_{-p}$.
Proof: Let $F(t,Y) = b(Y) + \sigma(Y) \diamond \dot{X}^{\beta}_t$, by (3.8) the deduction in Theorem 4.2, we can get for $p > \max\{1, q_0\}$, $\forall t \ge 0, \|\dot{X}^{\beta}(t)\|_{-1,p} \le M$, then

$$\begin{aligned}
\|F(t,Y)\|_{-1,-p} &= \|b(Y) + \sigma(Y) \diamond \dot{X}^{\beta}_t\|_{-1,-p}\\
&\le \|b(Y)\|_{-1,-p} + \|\sigma(Y) \diamond \dot{X}^{\beta}_t\|_{-1,-p}\\
&= \|b(Y)\|_{-1,-p} + \|\sigma(Y)\|_{-1,-p}\|\dot{X}^{\beta}(t)\|_{-1,-p}\\
&\le C(1+M)(1+\|Y\|_{-1,-p})
\end{aligned}$$

Similarly,

$$\|F(t,Y)-F(t,Z)\|_{-1,-p} \le C(1+M)\|Y-Z\|_{-1,-p}.$$

That is, F satisfies Lipschtz condition and linear growth condition. Hence, by Lemma 5.1, we deduce that the stochastic equation (5.5) has a unique global solution $U : [0, \infty) \to (\mathscr{S})_{-p}^{-1}$. $\square$

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