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STOCHASTIC OPTIMAL CONTROL OF ATO SYSTEMS WITH BATCH ARRIVALS VIA DIFFUSION APPROXIMATION

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We study the stochastic optimal control for an assemble-to-order system with multiple products and components that arrive at the system in random batches and according to renewal reward processes. Our purpose is to maximize expected infinite-horizon discounted profit by selecting product prices, component production rates, and a dynamic sequencing rule for assembly. We refine the solution of some static planning problem and a discrete review policy to batch arrival environment and develop an asymptotically optimal policy for the system operating under heavy traffic, which indicates that the system can be approximated by a diffusion process and exhibits a state space collapse property.

1. INTRODUCTION

An assemble-to-order (ATO) system is a system to hold inventories of components that can be rapidly assembled into a wide variety of end products in response to customer orders. With the rapid development of global supply chains, such systems become widely accepted models in the manufacturing industry and lead to an active area of research (readers are referred to Song and Zipkin [12] and Plambeck and Ward [8] for a review of the literature on ATO systems). In an ATO system, pricing, capacity management, and dynamic execution are very challenging. Authors such as Kushner [7] and Plambeck and Ward [8] proposed resolving these stochastic control and optimal control problems in an integrated fashion through diffusion approximations for the systems under heavy traffic, which is in contrast to most of the research in this area, by assuming that the inventory of each

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component is managed independently without regard for the inventory positions of other components.

The current article deals with a more practical and general ATO system, which incorporates batch demand and supply into the framework of [8]. The main objective of the article is to show how to set product prices, component production capacities, the dynamically sequencing rule for assembly and to manage component inventory, in order to asymptotically maximize expected infinite-horizon discounted profit in the batch demand and supply environment. Our asymptotically optimal policy is designed by refining a static planning problem and a discrete review scheduling rule developed in [8] to the system with batch arrivals. In showing that our policy is asymptotically optimal, we need to refine functional limit theorems and justify a uniformly integrable property related to certain extreme value processes for a sequence of scaled and centered renewal reward processes. Then, by employing these results, we can establish a state space collapse property, a heavy-traffic limit theorem, and, finally, prove the asymptotical optimality of our proposed policy along the line of [8].

The rest of the article is organized as follows. In Section 2, we describe our model. In Section 3, we design our asymptotically optimal policy and present our main theorem, which are justified in Section 4. Finally, in the Appendix, we provide the proof of a lemma.

2. THE MODEL

In our ATO system, there are J different components that are assembled into K different products. Product $k \in \{1, \ldots, K\}$ requires a positive, integer amount of type $j \in \{1, \ldots, J\}$ components equal to a_{kj} ($a_{kj} > 0$ for at least one j). Assembly is instantaneous, given the necessary components. At time $t = 0$, the product price vector $p = (p_1, \ldots, p_K)'$ and the component production capacity vector $\gamma = (\gamma_1, \ldots, \gamma_J)'$ are chosen. Then, for the given p, orders for product k arrive in the system in random batches and follow a renewal reward process; that is, the cumulative number of orders for product k that arrive before time t can be denoted by

$$O_k(t) = \sum_{i=1}^{N_k^o(t)} \xi_i^k, \quad k = 1, \ldots, K, \tag{2.1}$$

where $N_k^o(t)$ is a renewal process with rate $\lambda_k \equiv \lambda_k(p)$,

$$N_k^o(t) = \max\{m \geq 0, \chi_k(m) \leq \lambda_k t\}, \qquad \chi_k(m) = \sum_{i=1}^{m} x_k(i), \tag{2.2}$$

and $\{x_k(i), i = 1, 2, \ldots\}$ ($k = 1, \ldots, K$) are K independent i.i.d. (independent and identically distributed) sequences of mean 1 nonnegative random variables having $\mathrm{Var}(x_k(1)) = \sigma^2_{N^o,k}$. The distributions of $x_k(1)$ ($k = 1, \ldots, K$) are all assumed to have an increasing failure rate (IFR), which implies that the nth moment of any of the random variables is finite. ξ_i^k in (2.1) denotes the size of the ith arrival batch. For each k, the integer-valued random sequence $\{\xi_i^k, i = 1, 2, \ldots\}$ is i.i.d., with mean e_k^o and variance $v_{o,k}^2$. We further assume that $1 \leq \xi_i^k \leq u_k^o$ and the batch sizes are independent among different products.

For components of type j, there is an associated unit production cost $c_j > 0$ paid upon the delivery of the component and a physical holding cost $h_j > 0$ per unit time (we assume that component inventory incurs a linear physical holding cost). Components of type j arrive in the system also in random batches and obey a renewal reward process; that is, the cumulative number of components of type j that arrive before time t can be denoted by

$$C_j(t) = \sum_{i=1}^{N_j^c(t)} \eta_i^j, \quad j = 1, \ldots, J, \tag{2.3}$$

where $N_j^c(t)$ is a renewal process with rate γ_j; that is,

$$N_j^c(t) = \max\{m \geq 0, \mathcal{Y}_j(m) \leq \gamma_j t\}, \quad \mathcal{Y}_j(m) = \sum_{i=1}^{m} y_j(i), \tag{2.4}$$

and $\{y_j(i), i = 1, 2, \ldots\}$ $(j = 1, \ldots, J)$ are J independent i.i.d. sequences of mean 1 nonnegative random variables having $\mathrm{Var}(y_j(1)) = \sigma_{N^c,j}^2$; the distributions of $y_j(1)$ $(j = 1, \ldots, J)$ are all assumed to have an IFR. η_i^j in (2.3) is the size of the ith arrival batch. For each j, the integer-valued random sequence $\{\eta_i^j, i = 1, 2, \ldots\}$ is i.i.d., with mean e_j^c and variance $v_{c,j}^2$. We further assume that $1 \leq \eta_i^j \leq u_j^c$ and that the batch sizes are independent of the arrivals of orders and other components.

Now, we will dynamically determine when and in what sequence to assemble outstanding product orders (where we assume that orders leave the system not in batch, but in product). For orders of different products, we will adopt some priority sequencing rule for assembly that will be elaborate later, and for orders of the same product, they will be filled on the basis of first-in first-out (FIFO). Thus, if $A_k(t)$ denotes the cumulative number of type k orders assembled in [0, t], the order queue lengths at time t is given by

$$Q_k(t) \equiv O_k(t) - A_k(t), \quad k = 1, \ldots, K, \tag{2.5}$$

and the component inventory levels at time t are

$$I_j(t) \equiv C_j(t) - \sum_{k=1}^{K} a_{kj} A_k(t), \quad j = 1, \ldots, J. \tag{2.6}$$

The objective for the above system is to maximize the below expected infinite-horizon discounted profit by choosing an admissible policy $u = (p_u, \gamma_u, A_u)$ under the condition of high production volume:

$$\Pi \equiv \sum_{k=1}^{K} \int_{0}^{\infty} p_k e^{-\delta t}\, dA_k(t) - \sum_{j=1}^{J} \int_{0}^{\infty} e^{-\delta t}(c_j\, dC_j(t) + h_j I_j(t)\, dt), \qquad \textbf{(2.7)}$$

where an admissible policy $u = (p_u, \gamma_u, A_u)$ specifies the product prices, component production capacity, and the sequencing rule for assembly. We require that p_u and γ_u be nonnegative vectors. Also, the process A_u is integer-valued, nondecreasing, and nonanticipating and has $A_u(-t) = 0$ for all $t > 0$.

A high-volume condition is defined according to a sequence of systems that are indexed by $n \in \{1, 2, \ldots\}$: Batch arrival rates tend to infinity in a manner that preserves the structure of the batch demand functions; that is,

$$\lambda_k^n(p) \equiv n\lambda_k(p), \quad k = 1, \ldots, K.$$

In the sequel, when we wish to refer to any process or other quantity associated with the ATO system having batch arrival rate function λ^n, we superscript the proper symbol by n, such as Π^n. An admissible policy refers to an entire sequence, $u = (p_u^n, \gamma_u^n, A_u^n)$ (with $\lambda_{u,k}^n \equiv \lambda_k^n(p_u^n)$) that specifies an admissible policy for each n. Therefore, our objective is to find such a policy that maximizes the expected Π defined in (2.7) asymptotically in a certain sense as $n \to \infty$.

3. THE ASYMPTOTICALLY OPTIMAL POLICY AND MAIN THEOREM

Our asymptotically optimal policy can be proposed through two stages: solving a static planning problem to yield a first-order approximation to the optimal prices and production capacities, and designing a discrete review policy that minimizes instantaneous financial holding costs at each review point by distributing components to product orders.

3.1. The Static Planning Problem

In our proposed policy, optimal prices and production capacities are determined based on the solution of the following static programming problem:

$$\bar{\pi}(\theta) \equiv \max_{p \geq 0, \gamma \geq 0} \sum_{k=1}^{K} p_k \lambda_k(p) e_k^o - \sum_{j=1}^{J} c_j \gamma_j e_j^c, \qquad \textbf{(3.1)}$$

subject to

$$\sum_{k=1}^{K} a_{kj} \lambda_k(p) e_k^o \leq (\gamma_j + \theta_j) e_j^c, \quad j = 1, \ldots, J, \qquad \textbf{(3.2)}$$

for a fixed $\theta \in R^J$.

To discuss the existence and uniqueness of the solution of the problem described by (3.1) and (3.2), we need a few standard assumptions on the batch demand function λ.

First, $\lambda(p)$ is a continuously differentiable function, and the Jacobian matrix $[\partial\lambda_k(p)/\partial p_m]_{k,m=1,\ldots,K}$ is nonsingular everywhere. Second, batch demand for any one product is strictly decreasing in the price of that product but is nondecreasing in the price of any other product; so, $\partial\lambda_k(p)/\partial p_k < 0$ and $\partial\lambda_k(p)/\partial p_m \geq 0$, $m \neq k$. Third, batch demand for each product decreases when all products' prices increase by the same amount; so $\sum_{m=1}^{K} \partial\lambda_k(p)/\partial p_m < 0$, $k = 1, \ldots, K$. Hence, we can further assume that the revenue rate

$$r(\lambda) \equiv \sum_{m=1}^{K} \lambda_k p_k(\lambda) e_k^o \tag{3.3}$$

is strictly concave, where p is the unique inverse function of λ by Lemma 1 in [8].

Lemma 3.1: *If $\theta = 0$, there is a unique solution (p^*, γ^*) to the static programming problem (3.1)–(3.2), which satisfies*

$$p_k^* > \sum_{j=1}^{J} a_{kj} c_j > 0 \quad \textit{for every}\, k = 1, \ldots, K \,\textit{and}\, \gamma^* > 0.$$

If $\theta_j \leq \gamma_j^*$ for $j = 1, \ldots, J$, the perturbed problem (3.1)–(3.2) has a unique optimal solution $(p^*(\theta), \gamma^*(\theta))$ such that

$$p^*(\theta) = p^*, \quad \gamma^*(\theta) = \gamma^* - \theta, \quad \textit{and} \quad \bar{\pi}(\theta) - \bar{\pi} = \sum_{j=1}^{J} c_j \theta_j e_j^c,$$

where $\bar{\pi} = \bar{\pi}(0)$.

Lemma 3.1 is a generalization of Lemma 2 in [8] and its proof is provided in the Appendix.

3.2. The Discrete-Review Policy for Assembly

The discrete-review policy employed for assembly release orders at review time points l, $2l$, $3l, \ldots$ and does nothing at all other times, which can be described as follows: Given the number of orders assembled by time $(i-1)l$, $A_*((i-1)l)$ with $A_*(0) = 0$, and the shortage of each component $S_j(il)$ with the corresponding shortage process defined as

$$S_j(t) \equiv \sum_{k=1}^{K} a_{kj} O_k(t) - C_j(t) = \sum_{k=1}^{K} a_{kj} Q_k(t) - I_j(t), \quad j = 1, 2, \ldots, J. \tag{3.4}$$

Then at each review time point il for $i \in \{1, 2, \ldots\}$, we allocate available inventory to product orders in order to minimize instantaneous holding costs, where we assume that assembly is instantaneous. Concretely, at time point $t = il$, we solve the following linear program:

$$\min_{Q(t)\geq 0, I(t)\geq 0} \delta \sum_{k=1}^{K} p_k^* Q_k(t) + \sum_{j=1}^{J} h_j I_j(t) \tag{3.5}$$

subject to

$$I_j(t) \equiv \sum_{k=1}^{K} a_{kj} Q_k(t) - S_j(t) \geq 0, \quad j = 1, 2, \ldots, J, \tag{3.6}$$

$$O(t) - Q(t) \geq A_*((i-1)l) \tag{3.7}$$

to get a suitable optimal solution $(Q^*(t), I^*(t))$, as explained in [8]. Then, the assembly policy at time point il can be proposed recursively as follows:

$$A_*(il) = O(il) - Q^*(S(il), O(il), A_*((i-1)l), \quad i = 1, 2, \ldots \tag{3.8}$$

In the sequel, we assume that the vector $(p_1^* + \sum_{j=1}^{J} h_j a_{1j}, \ldots, p_k^* + \sum_{j=1}^{J} h_j a_{Kj})$ is not parallel to the vector $(a_{1j}, \ldots, a_{Kj})$ for any $j = 1, \ldots, J$; hence, there is a unique solution $(q^*(S), i^*(S))$ to the linear program (3.5)–(3.6) for every feasible S (see [8] for more discussions).

3.3. The Proposed Asymptotically Optimal Policy and Main Result

Let B_θ be a J-dimensional Brownian motion with drift θ and covariance matrix Γ whose (i, j)th entry is given by

$$\Gamma_{i,j} \equiv \sum_{k=1}^{K} a_{ki} a_{kj} (\sigma_k^o)^2 + (\sigma_j^c)^2 I_{\{i=j\}}, \tag{3.9}$$

where $\sigma_k^o = \sqrt{\lambda_k^* v_{o,k}^2 + (e_k^o)^2 (\lambda_k^*)^3 \sigma_{N^o,k}^2}$, $\sigma_j^o = \sqrt{\gamma_j^* v_{c,j}^2 + (e_j^c)^2 (\gamma_j^*)^3 \sigma_{N^c,j}^2}$, and $\lambda^* = \lambda(p^*)$. Define the limiting cost of queuing and holding inventory as follows:

$$\mathcal{H}(\theta) \equiv E\left[\int_0^\infty e^{-\delta t} \left(\sum_{k=1}^{K} p_k^* \delta q_k^*(B_\theta(t)) + \sum_{j=1}^{J} h_j i_j^*(B_\theta(t)) \right) dt \right] \tag{3.10}$$

and find a maximizer

$$\theta^* = \operatorname{argmax} \left(\delta^{-1} \sum_{j=1}^{J} c_j \theta_j - \mathcal{H}(\theta) \right). \tag{3.11}$$

The existence of such a maximizer is guaranteed due to Lemmas 5 and 6 in [8].

Under the high-volume condition and in the nth system, if prices are exactly p^*, component production capacities are $\gamma^* - n^{-1/2}\theta^*$, and the discrete-review policy

A_*^n corresponds to the review period length

$$l^n = (1/|n\lambda^*|)^{2/3}, \tag{3.12}$$

with $|\cdot|$ being the Euclidean norm, then the policy $* = (p^*,\ \gamma^* - n^{-1/2}\theta^*,\ A_*^n)$ is asymptotically optimal in high volume, which can be described in the following theorem.

THEOREM 3.1: *For a policy u, define*

$$\tilde{\Pi}_u^n \equiv \frac{\Pi_u^n - \delta^{-1} n \bar{\pi}}{\sqrt{n}}. \tag{3.13}$$

Then the policy $*$ *having*

$$E(\tilde{\Pi}_*^n) \to \delta^{-1} \sum_{j=1}^{J} c_j \theta_j^* - \mathcal{H}(\theta^*) \tag{3.14}$$

is asymptotically optimal under the high-volume condition in the sense that

$$\liminf_{n\to\infty} E(\tilde{\Pi}_*^n) \geq \limsup_{n\to\infty} E(\tilde{\Pi}_u^n) \tag{3.15}$$

for any other admissible policy u.

4. DEMONSTRATING THEOREM 3.1

Under the high-volume condition and a policy $u = (p_u^n,\ \gamma_u^n,\ A_u^n)$, we recall the definitions of product and component cumulative arrival processes for the nth system—that is, $O_{u,k}^n(\cdot)$ and $C_{u,j}^n(\cdot)$ with associated renewal processes as follows:

$$\begin{aligned} N_{u,k}^{o,n}(t) &= \max\{m \geq 0, \chi_k(m) \leq \lambda_{u,k}^n t\}, \\ N_{u,j}^{c,n}(t) &= \max\{m \geq 0, y_j(m) \leq n\gamma_{u,j}^n t\}. \end{aligned} \tag{4.1}$$

Moreover, we define the following scaled and centered processes:

$$\tilde{O}_{u,k}^n(t) \equiv \frac{O_{u,k}^n(t) - ne_k^o \lambda_k(p_u^n)t}{\sqrt{n}} = \sqrt{n}(n^{-1} O_{u,k}^n(t) - e_k^o \lambda_k(p_u^n)t), \tag{4.2}$$

$$\tilde{C}_{u,j}^n(t) \equiv \frac{C_{u,j}^n(t) - ne_j^c \gamma_{u,j}^n t}{\sqrt{n}} = \sqrt{n}(n^{-1} C_{u,j}^n(t) - e_j^c \gamma_{u,j}^n t). \tag{4.3}$$

4.1. A Uniformly Integrable Property

For a policy u with $p_u^n \to p^*$, define $\lambda^* \equiv \lim_{n\to\infty} \lambda(p^n)$ since $\lambda(p)$ is continuous in p.

LEMMA 4.1: *For a policy u with $p_u^n \to p^*$ and $\gamma_u^n \to \gamma^*$ as $n \to \infty$, we have*

$$\tilde{O}_{u,k}^n(t) \Rightarrow \sigma_k^o B_k^o, \qquad \tilde{C}_{u,j}^n(t) \Rightarrow \sigma_j^c B_j^c, \tag{4.4}$$

$$\frac{O_{u,k}^n(t)}{n} \longrightarrow e_k^o \lambda_k^* t, \qquad \frac{C_{u,j}^n(t)}{n} \longrightarrow e_j^c \gamma_j^* t, \quad u.o.c.\ a.s. \tag{4.5}$$

where B_k^o $(k = 1, \ldots, K)$ and B_j^o $(j = 1, \ldots, J)$ are $K + J$ independent standard Brownian motions, $\Rightarrow$ denotes weak convergence, and σ_k^o and σ_j^o are defined in (3.9).

PROOF: The weak convergence claimed in (4.4) can be proved by applying the renewal reward Functional Central Limit Theorem (see, e.g., Whitt [13, Thm. 7.4.1]) and the Random Change of Time Theorem (see, e.g., Billingsley [1, Sect. 17]). For the convergence stated in (4.5), due to (4.4), it follows from the Skorohod representation theorem (Whitt [13] or Ethier and Kurtz [5]) that there exists a common supporting probability space such that $\tilde{O}_{u,k}^n(t) \to \sigma_k^o B_k^o$, u.o.c. a.s. Since Brownian motion almost surely has continuous sample paths, we conclude that $O_{u,k}^n(t)/n \to e_k^o \lambda_k^* t$, u.o.c. a.s. The second part of (4.5) can be explained in the same way. Hence, we finish the proof of the lemma. ■

PROPOSITION 4.1: *For a policy u with $p_u^n \to p^*$ and $\gamma_u^n \to \gamma^*$ as $n \to \infty$, we have that*

$$\left\{ \int_0^\infty e^{-\delta t} \sup_{0\le s\le t} |\tilde{O}_{u,k}^n(s)|\, dt, n \ge 1 \right\} \quad and \quad \left\{ \int_0^\infty e^{-\delta t} \sup_{0\le s\le t} |\tilde{C}_{u,j}^n(s)|\, dt,\ n \ge 1 \right\}$$

are uniformly integrable families for each $k = 1, \ldots, K$ and $j = 1, \ldots, J$. Furthermore, as $n \to \infty$, we have

$$E\int_0^\infty e^{-\delta t} \tilde{O}_{u,k}^n(t)\, dt \longrightarrow 0 \quad and \quad E\int_0^\infty e^{-\delta t} \tilde{C}_{u,j}^n(t)\, dt \longrightarrow 0. \tag{4.6}$$

PROOF: We only establish that the conclusions for $\tilde{O}_k^n(t)$ and the arguments for $\tilde{C}_j^n(t)$ are similar. To simplify, we suppress the subscript indicating the policy u. For any $k = 1, \ldots, K$ and $t \ge 0$, let

$$I_{1,k}^n(N_k^{o,n}(t)) \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{N_k^{o,n}(t)} (\xi_i^k - e_k^o), \qquad I_{2,k}^n(t) \equiv \frac{1}{\sqrt{n}}\, e_k^o(N_k^{o,n}(t) - n\lambda_k^n t).$$

Then we have

$$\tilde{O}_k^n(t) = I_{1,k}^n(N_k^{o,n}(t)) + I_{2,k}^n(t). \tag{4.7}$$

To prove the uniform integrability, it suffices to prove (see, e.g., the explanation in Billingsley [1, p. 1])

$$\sup_{n\geq 1} E\left(\int_0^\infty e^{-\delta t} \sup_{0\leq s\leq t} |\tilde{O}_k^n(s)|\, dt\right)^2 < \infty. \tag{4.8}$$

Moreover, note that

$$E\left(\int_0^\infty e^{-\delta t} \sup_{0\leq s\leq t} |\tilde{O}_k^n(s)|\, dt\right)^2 \leq 2E\left(\int_0^\infty e^{-\delta t} \sup_{0\leq s\leq t} |I_{1,k}^n(N_k^{o,n}(s))|\, dt\right)^2 + 2E\left(\int_0^\infty e^{-\delta t} \sup_{0\leq s\leq t} |I_{2,k}^n(s)|\, dt\right)^2.$$

Therefore, we only need to prove that

$$E\left(\int_0^\infty e^{-\delta t} \sup_{0\leq s\leq t} |I_{1,k}^n(N_k^{o,n}(s))\, dt\right)^2 \quad \text{and} \quad E\left(\int_0^\infty e^{-\delta t} \sup_{0\leq s\leq t} |I_{2,k}^n(s)|\, dt\right)^2 \tag{4.9}$$

are uniformly bounded in n. It follows from Lemma 4 in [8] that the claim for the second part in (4.9) is true. For the first part in (4.9), note that

$$\sup_{0\leq s\leq t} |I_{1,k}^n(N_k^{o,n}(s))| \leq 3 \sup_{0\leq s\leq t} |I_{1,k}^n(N_k^{o,n}(s) + 1)|. \tag{4.10}$$

Since $\Xi_k(m) \equiv \sum_{i=1}^m (\xi_i^k - e_k^o)$ is a square integrable martingale in terms of the multiparameter filtration $\mathcal{F} = \{\mathcal{F}_m = \sigma(\xi_1^k, \ldots, \xi_m^k;\ x_k(1), \ldots, x_k(m)),\ m = 1,\ 2, \ldots\}$ and $N_k^{o,n}(t) + 1$ is a $\mathcal{F}_m$-stopping time (see, e.g., Dai and Dai [4] or Williams [14]), it follows from the L^p maximum inequality (obtained from Doob's Submartingale Inequality (Theorem 70.1) and Lemma 52.3 in Volume 1 of

Rogers and Williams [10]) that

$$E\left(\sup_{0\le s\le t} |I^n_{1,k}(N^{o,n}_k(s)+1)|\right)^2 \tag{4.11}$$

$$\le \frac{4}{n} E|\Xi_k(N^{o,n}_k(t)+1)|^2 \tag{4.12}$$

$$= \frac{4}{n} E(\xi^k_1 - e^o_k)^2 E(N^{o,n}_k(t)+1) \tag{4.13}$$

$$= \frac{4}{n} E(\xi^k_1 - e^o_k)^2 E(\chi_k(N^{o,n}_k(t)+1)) \tag{4.14}$$

$$= \frac{4}{n} E(\xi^k_1 - e^o_k)^2 E(n\lambda_k(p^n)t + \chi_k(N^{o,n}_k(t)+1) - n\lambda_k(p^n)t)$$

$$\le \frac{4}{n} E(\xi^k_1 - e^o_k)^2 (n\lambda_k(p^n)t + E(x_k(1))^2), \tag{4.15}$$

where we have used the L^p maximal inequality in (4.12), Wald's second moment identity in (4.13) (see, e.g., Chow and Teicher [3, Thm. 3]), Wald's first moment identity in (4.14), and Lorden's inequality in (4.15) (see, e.g., Gut [6, pp. 99–100]). Notice that $\lambda(p^n) \to \lambda(p^*) = \lambda^*$ as $n \to \infty$; thus, by (4.10)–(4.15), we conclude that

$$E\left(\sup_{0\le s\le t} |I^n_{1,k}(s)|\right)^2 \le c_1 t + c_2 \tag{4.16}$$

for large enough n, where c_1 and c_2 are some positive finite constants that are independent of n and t.

Now by the Functional Law of the Iterative Logarithm (see, e.g., Chen and Yao [2, Thm. 5.13]), we can conclude that, for each n and almost surely,

$$\sup_{0\le s\le t} |I^n_{1,k}(s)| \le c_3(\omega, n)t$$

for large enough t, where c_3 is some positive constant that is independent of t but might depend on n and sample path ω. Thus, for each n, we almost surely have

$$\int_0^\infty e^{-\delta t} \sup_{0\le s\le t} |I^n_{1,k}(s)|\, dt < \infty. \tag{4.17}$$

Hence, by (4.16) and (4.17), it follows from Jensen's inequality and the Tonelli theorem (see, e.g., Royden [9]) that

$$E\left(\int_0^\infty e^{-\delta t} \sup_{0\le s\le t} |I^n_{1,k}(s)|\, dt\right)^2 \le \int_0^\infty e^{-2\delta}(c_1 t + c_2)\, dt$$

is uniformly bounded in n. Therefore, the uniformly integrable property claimed in the lemma is true.

Finally, as to the convergence property (4.6) stated in Proposition 4.1, it can be proved by employing the previous Lemma 4.1 and a similar proof used in [8, Lemma 4.1]. Hence, we finish the proof of the proposition. ∎

4.2. State Space Collapse

In this subsection, we show that the employed assembly policy exhibits a certain state space collapse property that reduces the problem dimension from $K+J$ to J since queue lengths and inventory levels are, with a very high probability, deterministic functions of the shortage process. Here we remark that the assumption having bounded batch sizes is only required by the following lemma.

LEMMA 4.2: *Under any policy u with $p_u^n \to p^*$ and $\gamma_u^n \to \gamma^*$ as $n \to \infty$, and for any finite constant α, there exists a constant β such that*

$$P\left(\max_{i=0,\dots,I^n-1}\max_{k=1,\dots,K}|O_k^n((i+1)l^n) - O_k^n(il^n) - e_k^o\lambda_k(p_u^n)nl^n| < \alpha n^{1/3}\right) \geq 1-\beta n^{-1/6} \tag{4.18}$$

$$P\left(\max_{i=0,\dots,I^n-1}\max_{j=1,\dots,J}|C_j^n((i+1)l^n) - C_j^n(il^n) - e_j^c\gamma_{u,j}^n nl^n| < \alpha n^{1/3}\right) \geq 1-\beta n^{-1/6}, \tag{4.19}$$

where $I^n = \lfloor 1/l^n \rfloor$.

PROOF: It is sufficient only to establish (4.18), similar arguments yield (4.19). First, notice that for every n,

$$\begin{aligned} &P\left(\max_{i=0,\dots,I^n-1}\max_{k=1,\dots,K}|O_k^n((i+1)l^n) - O_k^n(il^n) - e_k^o\lambda_k(p_u^n)nl^n| > \alpha n^{1/3}\right) \\ &\quad \leq \sum_{k=1}^{K}\sum_{i=0}^{I^n-1} P\left(O_k^n((i+1)l^n) - O_k^n(il^n) \geq \Delta_k^{n,+}\right) \\ &\quad\quad + \sum_{k=1}^{K}\sum_{i=0}^{I^n-1} P\left(O_k^n((i+1)l^n) - O_k^n(il^n) < \Delta_k^{n,-}\right), \end{aligned}$$

where $\Delta_k^{n,+} = \lfloor e_k^o\lambda_k(p_u^n)nl^n + \alpha n^{1/3}\rfloor$ and $\Delta_k^{n,-} = \lceil e_k^o\lambda_k(p_u^n)nl^n - \alpha n^{1/3}\rceil$. Moreover, let $\bar{\Delta}_k^{n,+} \equiv \lfloor \Delta_k^{n,+}/u_k^o \rfloor$; then we have

$$\begin{aligned}
P\big(O_k^n((i+1)l^n) - O_k^n(il^n) \ge \Delta_k^{n,+}\big) &= P\left(\sum_{m=N_k^{o,n}(il^n)+1}^{N_k^{o,n}((i+1)l^n)} \xi_m^k \ge \Delta_k^{n,+}\right) \\
&\le P\big(N_k^{o,n}((i+1)l^n) - N_k^{o,n}(il^n) \ge \Delta_k^{n,+}/u_k^o\big) \\
&\le P\big(N_k^{o,n}((i+1)l^n) - N_k^{o,n}(il^n) \ge \bar{\Delta}_k^{n,+}\big) \\
&= P\big(\chi_k(\bar{\Delta}_k^{n,+}) \le n\lambda_k(p_u^n)l^n\big). \qquad (4.20)
\end{aligned}$$

The last equation in (4.20) is obtained from basic renewal theory (more explanations can be found in [8]). Note that $e_k^o \ge 1$; then the right-hand side of (4.20) is bounded by

$$\begin{aligned}
& P(\chi_k(\bar{\Delta}_k^{n,+})/u_k^o - \bar{\Delta}_k^{n,+} \le n\lambda_k(p_u^n)l^n/u_k^o - \bar{\Delta}_k^{n,+}) \\
&\le P\Big(\chi_k(\bar{\Delta}_k^{n,+})/u_k^o - \bar{\Delta}_k^{n,+} \le 2 - \alpha n^{1/3}/u_k^o\Big) \\
&\le P\Big(|\chi_k(\bar{\Delta}_k^{n,+})/u_k^o - \bar{\Delta}_k^{n,+}| > \alpha n^{1/3}/u_k^o - 2\Big) \\
&\le \frac{E\left|\sum_{m=1}^{\bar{\Delta}_k^{n,+}} (x_k(m)/u_k^o - 1)\right|^5}{\big(\alpha n^{1/3}/u_k^o - 2\big)^5} \\
&\le \left(90\sqrt{\frac{5}{4}}\right)^5 \frac{E\left|\sum_{m=1}^{\bar{\Delta}_k^{n,+}} (x_k(m)/u_k^o - 1)^2\right|^{5/2}}{(\alpha n^{1/3}/u_k^o - 2)^5} \\
&\le \left(90\sqrt{\frac{5}{4}}\right)^5 \frac{(e_k^o\lambda_k(p_u^n)/(u_k^o|\lambda_k^*|^{2/3} + \alpha/u_k^o)^{5/2} E|x_k(1)/u_k^o - 1|^5 n^{-5/6}}{(\alpha/u_k^o - 2n^{-1/3})^5}, \qquad (4.21)
\end{aligned}$$

where we used Markov's inequality to get the third inequality in (4.21), used Burkholder's inequality to get the fourth inequality, and used the following fact for the fifth inequality: for $z_k(m) \equiv (x_k(m)/u_k^o) - 1$, any $\varepsilon_1 > 0$, and any integer $M > 0$, $E|\sum_{m=1}^{M} z_k^2(m)|^{(1+\varepsilon_1)} \le M^{1+\varepsilon_1}(E|z_k(1)|^{(2+2\varepsilon_1)})$, which is proved in Lemma 3 of [8].

Similarly, due to the assumption that the interarrival time distribution is IFR, we have

$$\begin{aligned}
&P\big(O_k^n((i+1)l^n) - O_k^n(il^n) < \Delta_k^{n,-}\big)\\
&\le P\big(N_k^{o,n}((i+1)l^n) - N_k^{o,n}(il^n) < \Delta_k^{n,-}\big)\\
&= P\big(e_k^o \chi_k(\Delta_k^{n,-}+1) > n e_k^o \lambda_k(p_u^n) l^n\big)\\
&\le \left(90\sqrt{\frac{5}{4}}\right)^5 \frac{(e_k^o\lambda_k(p_u^n)/|\lambda_k^*|^{2/3} - \alpha)^{5/2} E|e_k^o x_k(1) - 1|^5 n^{-5/6}}{(\alpha - 2n^{-1/3})^5}
\end{aligned} \tag{4.22}$$

Then, it follows from (4.20)–(4.22) that

$$\begin{aligned}
&P\left(\max_{i=0,\ldots,I^n-1}\ \max_{k=1,\ldots,K} |O_k^n((i+1)l^n) - O_k^n(il^n) - e_k^o\lambda_k(p_u^n) n l^n| > \alpha n^{1/3}\right)\\
&\le I^n \sum_{k=1}^K \big\{P(\chi_k(\bar{\Delta}_k^{n,+}) \le n\lambda_k(p_u^n)l^n) + P(\chi_k(\Delta_k^{n,-}+1) > n\lambda_k(p_u^n)l^n)\big\}\\
&\le \beta n^{-1/6},
\end{aligned}$$

where we used the fact that $I^n = O(n^{2/3})$ and β is a positive constant that depends only on $\lambda_1^*, \ldots, \lambda_1^*$, α, $E|x_k(1)/u_k^o - 1|^5$, and $E|e_k^o x_k(1) - 1|^5$. Hence, we finish the proof. ■

PROPOSITION 4.2: *Under any policy u with $p_u^n \to p^*$ and $\gamma_u^n \to \gamma^*$ as $n \to \infty$, there exists a constant β such that under the proposed assembly policy,*

$$P(Q^n(il^n) = q^*(S^n(il^n)) \quad \textit{for all } i = 1, 2, \ldots, I^n) \ge 1 - \beta n^{-1/6}.$$

PROOF: Note that q^* is a Lipschitz continuous function (see, e.g., Schrijver [11, Thm. 10.5] and more explanations in [8]) and there exists some positive constant κ such that for any $S^1, S^2 \in R^J$,

$$\max_{k=1,\ldots,K} |q_k^*(S^1) - q_k^*(S^2)| \le \kappa \max_{j=1,\ldots,J} |S_j^1 - S_j^2|.$$

Next, for each fixed n, define

$$\alpha \equiv \min_{k=1,\ldots,K} \frac{e_k^o \lambda_k^*}{2|\lambda^*|^{2/3}\left(1 + \kappa \max_{j=1,\ldots,J}\left(1 + \sum_{k=1}^K a_{kj}\right)\right)};$$

furthermore, let $\mathcal{A}$ and $\mathcal{B}$ denote the sets

$$\mathcal{A} \equiv \left\{ \omega\colon \max_{i=0,\ldots,I^n-1} \max_{k=1,\ldots,K} |O_k^n((i+1)l^n) - O_k(il^n) - e_k^o \lambda_k(p_u^n) n l^n| < a n^{1/3} \right\},$$

$$\mathcal{B} \equiv \left\{ \omega\colon \max_{i=0,\ldots,I^n-1} \max_{j=1,\ldots,J} |C_j^n((i+1)l^n) - C_j(il^n) - e_j^c \gamma_{u,j}^n n l^n| < a n^{1/3} \right\}.$$

Then, for a sample path $\omega \in \mathcal{A} \cup \mathcal{B}$, each $k = 1, \ldots, K$, and each $i = 1, \ldots, I^n$, we have

$$O_k^n(il^n) - q_k^*(S^n(il^n)) \geq A_k^n((i-1)l^n),$$

which can be proved by mathematical induction similar to the procedure used in [8, Prop. 1], and only need to replace $\lambda_k(p_u^n)/|\lambda^*|^{2/3}$ by $e_k^o \lambda_k(p_u^n)/|\lambda^*|^{2/3}$ in related steps of their proof. Thus, constraint (3.7) is not violated at $q_k^*(S^n(il^n))$. Then we have

$$Q^n(il^n) = Q^*(S^n(il^n)) = q^*(S^n(il^n)).$$

Therefore, it follows from Lemma 4.2 that the claim in the proposition is true. Hence, we finish the proof. ∎

4.3. A Heavy-Traffic Limit Theorem

Define the capacity imbalance of the nth system under a given policy u as follows:

$$\theta_{u,j}^n \equiv \sum_{k=1}^{K} a_{kj} \lambda_k(p_u^n) e_k^o - \gamma_{u,j}^n e_j^c, \qquad j = 1, \ldots, J.$$

Proposition 4.3: *Let B_θ be a Brownian motion with drift θ and covariance matrix Γ defined in (3.9). Under a policy $u = (p_u^n, \gamma_u^n, A_*^n)$ satisfying*

$$\sqrt{n}\theta_{u,j}^n \longrightarrow \theta_j, \qquad j = 1, \ldots, J, \tag{4.23}$$

as $n \to \infty$, and $p_u^n = p^*(\theta_u^n)$ and $\gamma_u^n = \gamma^*(\theta_u^n)$ so that p_u^n and γ_u^n solve the perturbed static planning problem in (3.1)–(3.2), and A_*^n is defined in (3.12), then as $n \to \infty$,

$$\frac{(S^n, Q^n, I^n)}{\sqrt{n}} \Longrightarrow (B_\theta, q^*(B_\theta), i^*(B_\theta)), \tag{4.24}$$

$$E(\tilde{\Pi}_u^n) \longrightarrow \delta^{-1} \sum_{j=1}^{J} c_j \theta_j - \mathcal{H}(\theta), \tag{4.25}$$

where $\tilde{\Pi}_u^n$ is defined in (3.13) and $\mathcal{H}(\theta)$ is given in (3.10).

PROOF: We first prove (4.24). Note that

$$\frac{S^n_{u,j}(t)}{\sqrt{n}} = \sum_{k=1}^{K} a_{kj}\tilde{O}^n_k(t) - \tilde{C}^n_j(t) + \sqrt{n}\theta^n_{u,j}t. \tag{4.26}$$

By condition (4.23), $\theta^n_{u,j} < \gamma^*_j$ when n is large enough. It follows from Lemma 3.1 that, for large enough n, $p^*(\theta^n_{u,j}) = p^*$ and $\gamma^*(\theta^n_{u,j}) = \gamma^* - \theta^n_{u,\ j}$. Therefore,

$$p^n_u \longrightarrow p^*, \qquad \gamma^n \longrightarrow \gamma^*, \qquad \lambda(p^n_u) \longrightarrow \lambda^*. \tag{4.27}$$

Thus, by Lemma 4.1, the continuous mapping theorem, and the Cramer–Wold device (Billingsley [1]), we have

$$\frac{S^n}{\sqrt{n}} \Rightarrow B_\theta \tag{4.28}$$

as $n \to \infty$, where B_θ is a J-dimensional Brownian motion with drift θ and covariance matrix Γ defined in (3.9).

Due to the discrete-review policy, we adopt arguments analogous to those in the proof of Proposition 2 in [8]; we have that

$$\begin{aligned}
&\sup_{0\le t\le 1}\left|n^{-1/2}Q^n_k(t) - q^*_k(S^n(t)/\sqrt{n})\right| \\
&\quad \le \max_{i=0,\dots,\lfloor 1/l^n\rfloor}\left|n^{-1/2}Q^n_k(il^n) - q^*_k\left(\frac{S^n(il^n)}{\sqrt{n}}\right)\right| \\
&\qquad + \sup_{0\le t\le 1}\left|q^*_k\left(\frac{S^n([(t/l^n)l^n])}{\sqrt{n}}\right) - q^*_k\left(\frac{S^n(t)}{\sqrt{n}}\right)\right| \\
&\qquad + \frac{1}{\sqrt{n}}\max_{i=0,\dots,\lfloor 1/l^n\rfloor}\left|O^n_k((i+1)l^n) - O^n_k(il^n) - e^o_k n\lambda^*_k l^n\right| + \frac{1}{n^{1/6}}\frac{\lambda^*_k e^o_k}{|\lambda_k|^{2/3}}
\end{aligned}$$

converges to zero in probability as $n \to \infty$ by Lemma 4.2, Proposition 4.2, the observation that for any $S \in R^J$, $q^*(n^{-1/2}S) = n^{-1/2}q^*(S)$, and the fact that il^n, $i = 0, 1, \dots, (l^n)^{-1}$, becomes dense in (0, 1). Then it follows from the continuous mapping theorem that

$$\frac{Q^n}{\sqrt{n}} \Rightarrow q^*(B_\theta).$$

Moreover, by (3.6) and the continuous mapping theorem, we get

$$\frac{I^n}{\sqrt{n}} \Rightarrow i^*(B_\theta).$$

Since q^* and i^* are deterministic functions, we get $(S^n, Q^n, I^n)/\sqrt{n} \Rightarrow (B_\theta, q^*(B_\theta), i^*(B_\theta))$.

Now we proceed to prove (4.25). It follows from the integration by parts theorem for the Riemann–Stieltjes integral that

$$\begin{aligned}\tilde{\Pi}_u^n &= \int_0^\infty \delta e^{-\delta t}\left(\sum_{k=1}^K p_{u,k}^n \tilde{O}_{u,k}^n(t) - \sum_{j=1}^J c_j \tilde{C}_{u,j}^n(t)\right)dt \\ &\quad - \int_0^\infty e^{-\delta t}\left(\delta \sum_{k=1}^K p_{u,k}^n \frac{Q_{u,k}^n(t)}{\sqrt{n}} + \sum_{j=1}^J h_j \frac{I_{u,j}^n(t)}{\sqrt{n}}\right)dt \\ &\quad + \delta^{-1}\sqrt{n}\left(\sum_{k=1}^K p_{u,k}^n \lambda_k(p_u^n)e_k^o - \sum_{j=1}^J c_j \gamma_{u,j}^n e_j^c - \bar{\pi}\right). \qquad \textbf{(4.29)}\end{aligned}$$

By (4.23) and Lemma 3.1 (one can always decompose $\theta_{u,j}^n$ into $(\theta_{u,j}^n/e_j^c)e_j^c$ to be consistent with the lemma), as $n \to \infty$,

$$\sqrt{n}\left(\sum_{k=1}^K p_{u,k}^n \lambda_k(p_u^n)e_k^o - \sum_{j=1}^J c_j \gamma_{u,j}^n e_j^c - \bar{\pi}\right) = \sqrt{n}\left(\sum_{j=1}^J c_j \theta_{u,j}^n\right) \to \sum_{j=1}^J c_j \theta_j. \quad \textbf{(4.30)}$$

Due to the discrete-review policy, one can use a procedure similar to that in the proof of Proposition 2 in [8] to get

$$\begin{aligned}Q_k^n(t) &\le \sum_{j=1}^J \left\{\sum_{k=1}^K a_{kj} \sup_{0\le s\le t} |O_k^n(s) - ne_k^o \lambda_k(p_u^n)s| + |C_j^n(s) - ne_j^c \gamma_{u,j}^n s|\right\} \\ &\quad + nt\sum_{j=1}^J \theta_{u,j}^n + 2\sup_{0\le s\le t}|O_k^n(s) - ne_k^o\lambda_k(p_u^n)s| + n^{1/3}\frac{e_k^o\lambda_k(p_u^n)}{|\lambda^*|^{2/3}}.\end{aligned}$$

Thus, it follows from Proposition 4.1 and (4.23) that $\{\int_0^\infty e^{-\delta t} n^{-1/2} Q_k^n(t)\, dt,\ n \ge 1\}$ forms a uniformly integrable family for each $j = 1, \ldots, J$. Also, because

$$\begin{aligned}I_j^n(t) &\le \sum_{k=1}^K a_{kj} Q_k^n\left(\left\lfloor \frac{t}{l^n}\right\rfloor l^n\right) - S_j^n\left(\left\lfloor \frac{t}{l^n}\right\rfloor l^n\right) \\ &\quad + 2\sup_{0\le s\le t}|C_j^n(s) - ne_j^c\gamma_j^* s| + n^{1/3}\frac{e_j^c\gamma_j^n}{|\lambda^*|^{2/3}}, \\ S_j^n(t) &= \sqrt{n}\left(\sum_{k=1}^K a_{kj}\tilde{Q}_k^n(t) - \tilde{C}_j^n(t)\right) + nt\theta_{u,j}^n,\end{aligned}$$

we conclude that $\{\int_0^\infty e^{-\delta t} n^{-1/2} I_j^n(t)\, dt,\ n \ge 1\}$ is a uniformly integrable family for each $j = 1, \ldots, J$. Thus, it follows from (4.24), the continuous mapping theorem, and the now proved interchange of mean and limit that

$$E\left[\int_0^\infty e^{-\delta t}\frac{Q_k^n(t)}{\sqrt{n}}\,dt\right]\longrightarrow E\left[\int_0^\infty e^{-\delta t}q_k^*(B_\theta(t))\,dt\right],\qquad k=1,\ldots,K, \tag{4.31}$$

$$E\left[\int_0^\infty e^{-\delta t}\frac{I_j^n(t)}{\sqrt{n}}\,dt\right]\longrightarrow E\left[\int_0^\infty e^{-\delta t}i_j^*(B_\theta(t))\,dt\right],\qquad i=1,\ldots,J, \tag{4.32}$$

as $n\to\infty$. Hence, by Proposition 4.1, (4.27), and (4.30)–(4.32), we have

$$E(\tilde{\Pi}_u^n)\to\delta^{-1}\sum_{j=1}^J c_j\theta_j-\mathcal{H}(\theta).$$

Thus, the proof of the proposition is completed. ∎

4.4. Proof of Theorem 3.1

There are two stages involved: obtaining an upper bound on limiting expected infinite-horizon discounted profit, centered and scaled, under any admissible policy u, and showing that the policy $*$ achieves this upper bound. For an admissible policy, note that $\tilde{\Pi}_u^n$ can be expressed in the form of (4.29), and by Lemma 3.1, the following inequality is true:

$$\sum_{k=1}^K p_{u,k}^n\lambda_k(p_u^n)e_k^o-\sum_{j=1}^J c_j\gamma_{u,j}^n e_j^c-\bar{\pi}\le\bar{\pi}(\tilde{\theta}_u^n)-\bar{\pi}=\sum_{j=1}^J c_j\theta_{u,j}^n, \tag{4.33}$$

where the vector $\tilde{\theta}_u^n$ has component $\tilde{\theta}_{u,j}^n=\theta_{u,j}^n/e_j^c$. Then it follows from (4.29), (4.33), and Proposition 4.1 that

$$\limsup_{n\to\infty}E\left[\tilde{\Pi}_u^n\right]\le\limsup_{n\to\infty}\left(\frac{\sqrt{n}}{\delta}\sum_{j=1}^J c_j\theta_{u,j}^n-E\right.$$
$$\left.\times\left[\int_0^\infty e^{-\delta t}\left(\delta\sum_{k=1}^K p_{u,k}^n\frac{Q_{u,k}^n}{\sqrt{n}}+\sum_{j=1}^J h_j\frac{I_{u,j}^n}{\sqrt{n}}\right)dt\right]\right).$$

Then, by applying Lemma 3.1 and the same procedure as in the proof of Theorem 1 in [8], we conclude that

$$\limsup_{n\to\infty}E\left[\tilde{\Pi}_u^n\right]\le\sup_{\theta\in\Re^J}\left(\delta^{-1}\sum_{j=1}^J c_j\theta_j-\mathcal{H}(\theta)\right)$$
$$=\delta^{-1}\sum_{j=1}^J c_j\theta_j^*-\mathcal{H}(\theta^*).$$

Finally, it follows from Proposition 4.3 that the policy $*$ achieves the upper bound in (4.34); therefore, $*$ is an asymptotically optimal policy. Hence, we finish the proof. ■

APPENDIX
Proof of Lemma 3.1

Note that the static problem (3.1)–(3.2) is equivalent to the following maximization problem:

$$\overline{\pi}(\theta) = \max_{\lambda>0,\gamma>0} r(\lambda) - \sum_{j=1}^{J} \gamma_j c_j e_j^c \tag{A.1}$$

subject to the constraint (3.2). Since $r(\lambda)$ is assumed to be strictly concave, the above maximization problem is equivalent to a minimization convex program and, hence, has a unique optimal solution $(p^*(\theta), \gamma^*(\theta))$ for all $\theta \in R^J$. The Lagrangian function for the above problem, when equivalently considered as a minimization problem, is

$$\mathcal{L}(\lambda, \gamma, u) = -r(\lambda) + \sum_{j=1}^{J} c_j \gamma_j e_j^c - \sum_{j=1}^{J} u_j \left(e_j^c(\gamma_j + \theta_j) - \sum_{k=1}^{K} a_{kj} \lambda_k e_k^o \right). \tag{A.2}$$

Then it follows from the Karush–Kuhn–Tucker (KKT) conditions that

$$\frac{\partial \mathcal{L}}{\partial \lambda_m} = -\frac{\partial_r(\lambda)}{\partial \lambda_m} + \sum_{j=1}^{J} a_{mj} u_j e_m^o = 0, \qquad m = 1, \ldots, K, \tag{A.3}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma_j} = (c_j - u_j) e_j^c = 0, \qquad j = 1, \ldots, J, \tag{A.4}$$

$$\sum_{k=1}^{K} a_{kj} \lambda_k e_k^o - (\gamma_j + \theta_j) e_j^c \leq 0, \qquad j = 1, \ldots, J, \tag{A.5}$$

$$u_j \left(\sum_{k=1}^{K} a_{kj} \lambda_k e_k^o - (\gamma_j + \theta_j) e_j^c \right) = 0, \qquad j = 1, \ldots, J, \tag{A.6}$$

$$u \geq 0. \tag{A.7}$$

From (A.4), we have that $u_j = c_j > 0$. Then it follows from (A.3) that

$$0 < \sum_{j=1}^{J} a_{mj} c_j e_m^o = \frac{\partial r(\lambda)}{\partial \lambda_m} = \sum_{k=1}^{K} \lambda_k \frac{\partial p_k(\lambda)}{\partial \lambda_m} e_k^o + p_m(\lambda) e_m^o,$$

and, moreover, by Lemma 1 in [8], $(\partial p_k(\lambda)/\partial \lambda_m) \leq 0$. Hence, $p_m(\lambda) > \sum_{j=1}^{J} a_{mj} c_j > 0$

$(m = 1, \ldots, K)$. If $\theta = 0$, it follows from (A.5) that

$$\gamma_j^* = \left(\sum_{k=1}^{K} a_{kj} \lambda_k^* e_k^o \right) \Big/ e_j^c > 0,$$

because $p(\lambda) > 0$ implies $\lambda(p) > 0$. If $\theta > 0$, conditions (A.3)–(A.7) are satisfied when $\lambda^*(\theta) = \lambda^*$ and $\gamma_j^*(\theta) = \gamma_j^* - \theta_j$, $j = 1, \ldots, J$. Thus, it follows from the KKT theorem that $(\lambda^*(\theta), \gamma^*(\theta) = (\lambda^*, \gamma^* - \theta))$ is a global maximizing point for (A.1) and (3.2), which implies that $(p^*(\theta) = p(\lambda^*(\theta)), \gamma^*(\theta))$ is a global maximizing point for (3.1)–(3.2) and that

$$\bar{\pi}(\theta) - \bar{\pi} = \sum_{j=1}^{J} c_j \theta_j e_j^c.$$

Thus, we complete the proof.

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References

1. Billingsley, P. (1968). *Convergence of probability measures*. New York: Wiley.
2. Chen, H. & Yao, D.D. (2001). *Fundamentals of queueing networks*. New York: Springer-Verlag.
3. Chow, Y.S. & Teicher, H. (1988). *Probability theory: Independence, interchangeability and martingales*, 2nd ed. New York: Springer-Verlag.
4. Dai, J.G. & Dai, W. (1999). A heavy traffic limit theorem for a class of open queueing networks with finite buffers. *Queueing Systems* 32: 5–40.
5. Ethier, S.N. & Kurtz, T.G. (1986). *Markov processes: Characterization and convergence*. New York: Wiley.
6. Gut, A. (1988). *Stopped random walks: Limit theorems and applications*. New York: Springer-Verlag.
7. Kushner, H.J. (1999). Control and optimal control of assemble-to-order manufacturing systems under heavy traffic. *Stochastics and Stochastics Reports* 66: 233–272.
8. Plambeck, E.L. & Ward, A.R. (2006). Optimal control of a high-volume assemble-to-order system. *Mathematics of Operations Research* 31: 453–477.
9. Royden, H.L. (1988). *Real analysis*, 3rd ed. New York: Macmillan.
10. Rogers, L.C.G. & Williams, D. (2000). *Diffusions, Markov processes and martingales*, 2nd ed. Cambridge: Cambridge University Press.
11. Schrijver, A. (1986). *Theory of linear and integer programming*. New York: Chichester.
12. Song, J.S. & Zipkin, P. (2003). Supply chain operations: Assemble-to-order and configure-to-order systems. In *Handbooks in Operations Research and Management Science*. Supply Chain Management. S. Graves and T. De Kok, eds., Amsterdam: North-Holland.
13. Whitt, W. (2001). *Stochastic-process limits*. New York: Springer-Verlag.
14. Williams, R.J. (1998). Diffusion approximations for open multiclass queueing networks: Sufficient conditions involving state space collapse. *Queueing Systems* 30: 27–88.