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Mean–variance portfolio selection based on a generalized BNS stochastic volatility model

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We study a mean–variance portfolio selection problem via optimal feedback control based on a generalized Barndorff-Nielsen and Shephard stochastic volatility model, where an investor trades in a generalized Black–Scholes market. The random coefficients of the market are driven by non-Gaussian Ornstein–Uhlenbeck processes that are independent of the underlying multi-dimensional Brownian motion. Our contribution is to explicitly compute and justify optimal portfolios over an admissible set that is large enough to cover some important classes of strategies such as the class of feedback controls of Markov type. Concretely, the mean–variance efficient portfolios and efficient frontiers are explicitly calculated through the method of *generalized* linear-quadratic control and explicitly constructed solutions to three integro-partial differential equations under a quite mild condition that only requires one stock whose appreciation-rate process is different from the interest-rate process. Related minimum variance issue is also addressed via our main results.

Keywords: mean–variance portfolio selection; non-Gaussian Ornstein–Uhlenbeck process; generalized Black–Scholes model; optimal feedback control; integro-partial differential equation

2000 AMS Subject Classifications: 45K05; 60H10; 60H30; 60K30; 91B06; 93E20

1. Introduction

This paper studies the mean–variance portfolio selection problem in an incomplete financial market that consists of $d+1$ primitive assets: one bond and d risky assets with price processes described by a generalized Black–Scholes model whose random coefficients are driven by external stochastic risk factors of non-Gaussian Ornstein–Uhlenbeck (NGOU) processes that are independent of the underlying d-dimensional Brownian motion. Our market model includes the Barndorff-Nielsen and Shephard (BNS) volatility model suggested in BNS [3] and further studied in, such as, Benth *et al.* [5], Benth and Meyer-Brandis [4] and Lindberg [24] as a special case, and closely relates to the model considered in Delong and Klüppelberg [12]. The volatility level in these models are allowed to have sudden shifts in the upward direction, while decreasing exponentially between such shifts, and moreover, the empirical investigations on exchange rates

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for real market data in BNS [3] demonstrate that such models fit the empirical auto-correlation and the leptokurtic behaviour of log-return data remarkably well.

With the risk of a portfolio being measured by the variance of its return, mean–variance portfolio selection is concerned with the allocation of wealth among a variety of securities so as to minimize a portfolio's variance subject to the constraint that its expected return equals a prescribed level over a fixed planning horizon. The model was first proposed and solved in the single-period setting by Markowitz in his Nobel-Prize-winning work [26,27]. Since then, there are numerous extensions and applications of the method in both mean–variance portfolio selection (see, e.g. [22,23,36,38–40]) and mean–variance hedging (see, e.g. [2,8,9,13,16,17,21,30,33], and references therein). However, due to the fact that the proper notion of admissibility for the hedging problem in the general case with an unbounded short interest rate has not been worked out yet (see, e.g. [16,21]), we will adopt the GLQ control method to solve our portfolio selection problem. The reason for us to use the terminology of GLQ is because our Riccati-type equation is an integro-partial differential equation (IPDE) that can be viewed as a generalization of the traditional ordinary differential Riccati equation in studying stochastic LQ control problems (see, e.g. [22,23,38–40] for details).

Concretely, we are to explicitly compute and justify optimal portfolios over an admissible set that is large enough to cover some important classes of strategies of practical interests such as the class of feedback controls of Markov type (see, e.g. [15]). In doing so, the mean–variance efficient portfolios and efficient frontiers are explicitly calculated through the method of generalized linear-quadratic (GLQ) control and explicitly constructed solutions to three IPDEs under a quite mild condition that only requires one stock whose appreciation-rate process is different from the unbounded short interest-rate process. Related minimum variance issue is also addressed via our main results.

Here, we remark that portfolio optimization in stochastic factor models has recently gained much attention in the financial literature, and particularly, Merton's classic portfolio optimization problems for the BNS volatility model and its generalized form have been discussed by Benth *et al.* [5], Lindberg [24], Delong and Klüppelberg [12], etc., where power utility functions with exponents between 0 and 1 are considered. So their problems are different from our mean–variance portfolio selection problem (interested readers are also referred to Bielecki *et al.* [7] and Steinbach [35] for discussions on crucial differences between Merton's utility and Markowitz's mean–variance types of models). In addition, our portfolio is defined to be a vector consisting of the money values of different stocks just as studied in Zhou and Yin [40], which is in contrary to the ones as in most of the literature, where a portfolio is defined to be the fractions of stocks by implicitly assuming that the wealth process never takes value of zero.

In the end, we point out that the external factors used in this paper are assumed to be observable just as in all the above-mentioned publications, and certain integrability of the tails of the so-called Lévy measures is imposed as in the existing studies (see, e.g. [5]), which guarantees that all of the related integrals used in this paper are meaningful (finite). This imposed integrability condition can be realized in practice since the total wealth in the world is finite and hence the jump sizes associated with leverage effect, etc. for some Lévy process (subordinator) used in this paper should be bounded by a certain positive constant although it may be huge. A close related concept to the integrability condition is the so-called infinite mean and/or infinite variance (non-integrable) heavy-tail phenomenon, which is widely used in financial engineering, telecommunication systems, statistical analysis, etc. However, the recent study in Dai [10] indicates that, due to the limitations of computer hardware and software, all the random variables used in a computer simulation are the truncated versions of their original ones. In other words, a heavy-tail random variable used in a simulation is not a real heavy-tail one. So many heavy-tail-related simulations in finance and telecommunication need new interpretations. For example, Dai [10] uses the heavy traffic theory developed in [11] and the concept of the truncated random variable to provide certain new and reasonable interpretations to several well-known heavy-tail-related simulations conducted by Bell Labs scientists for the purpose of telecommunication traffic data modelling. Therefore, the imposed integrability condition should be a reasonable assumption for our study. Nevertheless, here we need to make it clear that the purposes of research, the used and developed mathematical theory/methodology in Dai [10,11] are completely different from the study of the current paper.

The rest of the paper is organized as follows. In Section 2, we describe our financial market model. In Section 3, we present our main theorem. In Section 4, we discuss the feasibility of our mean–variance portfolio selection problem under a quite mild condition and provide the proof of our main theorem.

2. The financial market model

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space on which are defined a standard d-dimensional Brownian motion $W \equiv \{W(t), t \in [0,T]\}$ with $W(t) = (W_1(t), \ldots, W_d(t))'$ and h-dimensional subordinator $L \equiv \{L(t), t \in [0,T]\}$ with $L(t) \equiv (L_1(t), \ldots, L_h(t))'$ and càdlàg sample paths for some fixed $T \in [0, \infty)$ (see, e.g. [1,6,32] for more details about subordinators and Lévy processes), where the prime denotes the corresponding transpose of a matrix or a vector. Moreover, W, L and their components are assumed to be independent of each other. Related to the probability space, we suppose that there is a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t \equiv \sigma\{W(s), L(\lambda s) : 0 \leq s \leq t\}$ for each $t \in [0,T]$, $\lambda = (\lambda_1, \ldots \lambda_h)' > 0$, and $L(\lambda s) = (L_1(\lambda_1 s), \ldots, L_h(\lambda_h s))'$.

Our financial market is a multivariate Lévy-driven Ornstein–Uhlenbeck (OU)-type stochastic volatility model with leverage effect, which consists of $d+1$ assets. One of the $d+1$ assets is supposed to be a risk-free account whose price $S_0(t)$ is subject to the ordinary differential equation (ODE),

$$\begin{aligned} \mathrm{d}S_0(t) &= r(Y(t^-))S_0(t)\,\mathrm{d}t, \\ S_0(0) &= s_0 > 0, \end{aligned} \tag{1}$$

and the other d assets are stocks whose vector price process $S(t) = (S_1(t), \ldots, S_d(t))'$ satisfies the following stochastic differential equation (SDE) for each $t \in [0,T]$,

$$\begin{aligned} \mathrm{d}S(t) &= \mathrm{diag}(S(t^-))\{b(Y(t^-))\,\mathrm{d}t + \sigma(Y(t^-))\,\mathrm{d}W(t)\}, \\ S(0) &= s > 0, \end{aligned} \tag{2}$$

where the diag(v) denotes the $d \times d$ diagonal matrix whose entries in the main diagonal are v_i with $i \in \{1, \ldots, d\}$ for a d-dimensional vector $v = (v_1, \ldots, v_d)'$ and all the other entries are zero, and $Y(t)$ is a Lévy-driven OU-type process described by the following SDE:

$$\begin{aligned} \mathrm{d}Y(t) &= -\Lambda Y(t^-)\,\mathrm{d}t + \mathrm{d}L(\lambda t), \\ Y(0) &= y_0, \end{aligned} \tag{3}$$

where $\Lambda = \mathrm{diag}(\lambda)$ and $y_0 = (y_{10}, \ldots, y_{h0})'$. In addition, the coefficients in Equation (2) are defined as follows:

$$\begin{aligned} &r(y) : R_c^h \longrightarrow [0, \infty), \\ &b(y) \equiv (b_1(y), \ldots, b_d(y))' : R_c^h \longrightarrow [0, \infty)^d, \\ &\sigma(y) \equiv (\sigma_{mn}(y))_{d\times d} : R_c^h \longrightarrow (0, \infty)^{dd}, \end{aligned}$$

where $R_c^h \equiv (c_1, \infty) \times \cdots \times (c_h, \infty)$ with $c_i = y_{i0}\mathrm{e}^{-\lambda_i T}$. Furthermore, we suppose that the coefficients in Equation (2) satisfy the following conditions:

C1. The functions $r(y)$, $b(y)$ and $\sigma(y)$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\begin{aligned} 0 \le r(y) &\le A_r + B_r\|y\|, \\ \|b(y)\| &\le A_b + B_b\|y\|, \\ \|\sigma(y)\sigma(y)'\| &\le A_\sigma + B_\sigma\|y\|, \\ \|(\sigma(y)\sigma(y)')^{-1}\| &\le \frac{1}{b_\sigma\|y\|}, \end{aligned}$$

where the norm $\|A\|$ takes the largest absolute value of all components of a vector A or all entries of a matrix A, and $A_r \ge 0$, $A_b \ge 0, A_\sigma \ge 0, B_r \ge 0, B_b \ge 0, B_\sigma \ge 0$, and $b_\sigma > 0$ are constants.

C2. The derivatives $\partial r(y)/\partial y_i$, $\partial b(y)/\partial y_i$, and $\partial(\sigma(y)\sigma(y)')^{-1}/\partial y_i$ for all $i \in \{1, \ldots, h\}$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\begin{aligned} \left|\frac{\partial r(y)}{\partial y_i}\right| &\le \bar{A}_r + \bar{B}_r\|y\|, \\ \left\|\frac{\partial b(y)}{\partial y_i}\right\| &\le \bar{A}_b + \bar{B}_b\|y\|, \\ \left\|\frac{\partial(\sigma(y)\sigma(y)')^{-1}}{\partial y_i}\right\| &\le \bar{A}_\sigma + \bar{B}_\sigma\|y\|, \end{aligned}$$

where $\bar{A}_r$, $\bar{A}_b$, $\bar{A}_\sigma$, $\bar{B}_r$, $\bar{B}_b$ and $\bar{B}_\sigma$ are some nonnegative constants.

We next impose conditions for each subordinator L_i with $i \in \{1, \ldots, h\}$, which can be represented by (see, e.g. [19, Theorem 13.4 and Corollary 13.7])

$$L_i(t) = a_i t + \int_{(0,t]}\int_{z_i>0} z_i N_i(\mathrm{d}s, \mathrm{d}z_i), \quad t \ge 0. \tag{4}$$

for each $i \in \{1, \ldots, h\}$. In Equation (4), $N_i((0,t] \times A) \equiv \sum_{0<s\le t} I_A(L(s) - L_i(s^-))$ denotes a Poisson random measure with a deterministic, time-homogeneous intensity measure $\nu_i(\mathrm{d}z_i)\,\mathrm{d}s$, where $I_A(\cdot)$ is the index function over the set A, the constant a_i is taken to be zero, and ν_i the Lévy measure satisfying

$$\int_{z_i>0} (\mathrm{e}^{Cz_i} - 1)\nu_i(\mathrm{d}z_i) < \infty, \tag{5}$$

where C is supposed to be a reasonable large positive constant to guarantee all of the related integrals in this paper meaningful and it can be concretely recorded through our discussion. Note that the condition in Equation (5) is on the integrability of the tails of the Lévy measures. Due to the total wealth in the world being finite, the jump sizes corresponding to the subordinator $L(t)$ associated with leverage effect etc. should be bounded by some positive constant although it may be huge (see also related discussions in [10,11]).

3. The optimal portfolio

Consider an investor with initial wealth $x_0 > 0$ and let $X(t)$ denote the total wealth of the investor at time $t \in [0, T]$. Suppose that the trading of shares takes place continuously and that transaction cost and consumptions are not considered, then by generalizing the discussion of Yong and Zhou [37, p. 57] that

$$\begin{aligned} \mathrm{d}X(t) &= (r(Y(t^-))X(t^-) + B(Y(t^-))'u(t))\,\mathrm{d}t + u(t)'\sigma(Y(t^-))\,\mathrm{d}W(t), \\ X(0) &= x_0, \end{aligned} \tag{6}$$

where $u(\cdot) = (u_1(\cdot), \ldots, u_d(\cdot))'$ is called a portfolio of the investor, whose ith component $u_i(t)$ ($i \in \{1, \ldots, d\}$) is the total market value of the investor's wealth in the ith asset, and $B(\cdot)$ is defined by

$$B(Y(t^-)) \equiv (b_1(Y(t^-)) - r(Y(t^-)), \ldots, b_d(Y(t^-)) - r(Y(t^-)))'. \tag{7}$$

Note that our analysis will focus only on $u(t)$ since the asset in the bond is determined by $u_0(t) = X(t) - \sum_{i=1}^{d} u_i(t)$. Note that the above way to model the wealth process allows a portfolio to be well defined even if the wealth is zero or negative, which represents extra certified funds available for the investor (readers are referred to [7,40] for more explanations).

Next, let $L^2_{\mathcal{F}}([0,T]; R^d)$ denote the set of all R^d-valued measurable stochastic processes $Z(t)$ adapted to $\{\mathcal{F}_t, t \in [0,T]\}$ such that $E[\int_0^{\mathrm{T}} \|Z(t)\|^2\,\mathrm{d}t] < \infty$, where the norm $\|A\|$ takes the largest absolute value of all components of a vector or all entries of a matrix A, and let $L^2_{\mathcal{F},p}([0,T], R^d)$ denote the corresponding set of predictable processes (see, e.g. Definitions 5.2 and 1.1, respectively, in [18, pp. 21 and 45]). Moreover, let $L^2_{\mathcal{F}_T}(\Omega; R^d)$ denote the set of all R^d-valued, $\mathcal{F}_T$-measurable random variables $\xi \in R^d$ satisfying $E[\|\xi\|^2] < \infty$. Then we can present the following definition.

Definition 3.1 *A portfolio $u(\cdot)$ is said to be admissible if $u(\cdot) \in L^2_{\mathcal{F},p}([0,T]; R^d)$ and the SDE* (6) *has a unique solution $X(\cdot) \in L^2_{\mathcal{F}}([0,T]; R^1)$, which corresponds to $u(\cdot)$. In this case, $(X(\cdot), u(\cdot))$ is called an admissible pair and $\mathcal{A}$ is used to denote the set of all such admissible pairs.*

Note that the way used in Definition 3.1 to define an admissible portfolio has frequently appeared in control theory and related applications (see, e.g., the definition in [15, p. 156], [37, Definition 4.1, p. 63] and [40]). In our case, the set of admissible portfolios stated in Definition 3.1 is big enough to include some important classes of portfolios, e.g. the set of all feedback controls of Markov type, $u(t, X(t^-), Y(t^-))$, such that the corresponding coefficients in Equation (6) satisfy conditions (23)–(26) as stated in Lemma 4.1.

Definition 3.2 *The mean–variance portfolio selection is a constrained stochastic optimization problem that is parameterized by $p \in R$:*

$$\inf_{(X(\cdot),u(\cdot))\in\mathcal{A}} J(x_0, y_0, u(\cdot)) \equiv \mathrm{Var}(X(T)) = E[X(T) - p]^2 \tag{8}$$

subject to

$$E[X(T)] = p. \tag{9}$$

Furthermore, the problem is called feasible if there exists at least one portfolio satisfying all the constraints. The problem is called finite if it is feasible and the infimum of $J(x_0, y_0, u(\cdot))$ is finite. Finally, an optimal portfolio to the above problem, if it ever exists, is called an efficient portfolio corresponding to p, and the corresponding $(\mathrm{Var}(X(T)), p) \in R^2$ *and* $(\sigma_{X(T)}, p) \in R^2$ *are interchangeably called an efficient point, where $\sigma_{X(T)}$ denotes the standard deviation of $X(T)$. The set of all the efficient points is called the efficient frontier.*

Here, we remark that the optimization model given in Equations (8) and (9) is a faithful replication in the form of the original Markowitz single-period model presented in [26,27]. To introduce our main theorem, for each $y \in R_c^h$, we define

$$\rho(y) \equiv B(y)'[\sigma(y)\sigma(y)']^{-1}B(y), \tag{10}$$

$$P(t,y) \equiv E_{t,y}[\mathrm{e}^{-\int_t^{\mathrm{T}}(\rho(Y(s))-2r(Y(s)))\,\mathrm{d}s}] > 0, \tag{11}$$

$$H(t,y) \equiv \frac{1}{P(t,y)}E_{t,y}[\mathrm{e}^{-\int_t^{\mathrm{T}}(\rho(Y(s))-r(Y(s)))\,\mathrm{d}s}] > 0, \tag{12}$$

$$\theta \equiv \sum_{i=1}^{h}\lambda_i E\left[\int_0^{\mathrm{T}}\int_{z_i>0} P(t,Y(t^-)+z_ie_i)(H(t,Y(t^-)+z_ie_i)-H(t,Y(t^-)))^2\nu(\mathrm{d}z_i)\,\mathrm{d}t\right]. \tag{13}$$

In the sequel, we use $|\cdot|$ to denote the Euclidean norm and assume that there is at least one stock whose appreciation-rate process is different from the interest-rate process, i.e.

$$E\left[\int_0^{\mathrm{T}}|B(Y(t^-))|^2\,\mathrm{d}t\right] > 0. \tag{14}$$

THEOREM 3.1 *Under Inequality* (14) *the efficient portfolio corresponding to p is given by*

$$u^*(t,X(t^-),Y(t^-)) = -(\sigma(Y(t^-))\sigma(Y(t^-))')^{-1}B(Y(t^-))(X(t^-)+(\alpha^*-p)H(t,Y(t^-))), \tag{15}$$

where

α

with the corresponding Lagrange multiplier $\alpha^*_{\min} = 0$ *and the associated expected terminal wealth*

$$p_{\min} = \frac{P(0, y_0)H(0, y_0)}{P(0, y_0)H(0, y_0)^2 + \theta} x_0. \tag{20}$$

Moreover, the portfolio that achieves the above minimum variance is given by

$$u^*_{\min}(t, X(t^-), Y(t^-)) = -(\sigma(Y(t^-))\sigma(Y(t^-))')^{-1} B(Y(t^-))(X(t^-) - p_{\min} H(t, Y(t^-))). \tag{21}$$

The proof of the corollary is a direct conclusion of Theorem 3.1. Hence we omit that here. From this theorem, one can see that the parameter p can be restricted to $p \in [p_{\min}, \infty)$ when one defines the efficient frontier for the mean–variance problem (8)–(9).

4. Proof of Theorem 3.1

First of all, we have the following lemma:

LEMMA 4.1 *Consider the following q-dimensional SDE with an external factor, where q is a positive integer,*

$$\mathrm{d}M(t) = \hat{b}(t, M(t), Y(t^-))\,\mathrm{d}t + \hat{\sigma}(t, M(t), Y(t^-))\,\mathrm{d}W(t), \quad M(0) = m_0 \tag{22}$$

whose coefficients $\hat{b}(\cdot,\cdot,\cdot)$ *and* $\hat{\sigma}(\cdot,\cdot,\cdot)$ *satisfy, for any* $x \in R^q$,

$$\|\hat{b}(t, x, Y(t^-))\| \le (D_1 + D_2\|x\|)\mathrm{e}^{\alpha\|L(\lambda t)\|}, \tag{23}$$

$$\|\hat{\sigma}(t, x, Y(t^-))\hat{\sigma}(t, x, Y(t^-))'\| \le (D_1 + D_2\|x\|^2))\mathrm{e}^{\alpha\|L(\lambda t)\|}, \tag{24}$$

where α, D_1 *and* D_2 *are some nonnegative constants,* $L(\lambda t) = (L_1(\lambda_1 t), \ldots, L_h(\lambda_h t))'$. *Moreover, for any* $x_1, x_2 \in R^q$,

$$\|\hat{b}(t, x_1, Y(t^-)) - \hat{b}(t, x_2, Y(t^-))\| \le D_3\|x_1 - x_2\|\mathrm{e}^{\alpha\|L(\lambda t)\|}, \tag{25}$$

$$\begin{aligned} &\|\hat{\sigma}(t, x_1, Y(t^-))\hat{\sigma}(t, x_1, Y(t^-))' - 2\hat{\sigma}(t, x_1, Y(t^-))\hat{\sigma}(t, x_2, Y(t^-)) \\ &\quad + \hat{\sigma}(t, x_2, Y(t^-))\hat{\sigma}(t, x_2, Y(t^-))'\| \le D_3\|x_1 - x_2\|^2\mathrm{e}^{\alpha\|L(\lambda t)\|}, \end{aligned} \tag{26}$$

where D_3 *is a nonnegative constant. Then there exists a unique solution M to Equation* (22), *which is continuous and*

$$M(\cdot) \in L^2_{\mathcal{F}}([0, T], R^q, P). \tag{27}$$

Proof Consider a sequence of increasing positive numbers $k \in \{1, 2, \ldots\}$ and define

$$\tau_k \equiv \inf\{t > 0, \|L(\lambda t)\| > k\}. \tag{28}$$

Then it follows from [31, Theorem 3, p. 4] and condition (5) that $\{\tau_k\}$ is a sequence of nondecreasing $\{\mathcal{F}_t\}$-stopping times and satisfies $\tau_k \to \infty$ a.s. as $k \to \infty$ since $L(\lambda t)$ is a h-dimensional nonnegative and nondecreasing càdlàg process. Moreover, define $\sigma_k \equiv T \wedge \tau_k$. Therefore, we can claim that Equation (22) has a unique solution in the Banach space

$$L^2_{\mathcal{F}}([0, \sigma_k], R^q, P) \equiv \left\{ f(t, \omega) \text{ is } \mathcal{F}_t\text{-adapted } R^q\text{-valued such that } E\left[\int_0^{\sigma_k} \|f(t, \omega)\|^2\,\mathrm{d}t\right] < \infty \right\}$$

endowed with the norm $\|f\| = (E[\int_0^{\sigma_k} \|f(t, \omega)\|^2\,\mathrm{d}t])^{1/2}$ (see, e.g. the definition in Section 2 and the corresponding explanation in the proof of Theorem 1 of [34]).

In fact, define a sequence of processes $\{M_n(t), t \in [0, \sigma_k]\}$ with $M_0(t) = M(0) = m_0$ and $M_n(0) = m_0$ for all $n \in \{0, 1, \ldots\}$ and $t \in [0, \sigma_k]$ as follows:

$$M_{n+1}(t) = M(0) + \int_0^t \hat{b}(s, M_n(s), Y(s^-))\,\mathrm{d}s + \int_0^t \hat{\sigma}(s, M_n(s), Y(s^-))\,\mathrm{d}W(s). \tag{29}$$

Equivalently, for each $t \in [0, T]$, we can rewrite Equation (29) as

$$M_{n+1}(t \wedge \sigma_k) = m_0 + \int_0^{t\wedge\sigma_k} \hat{b}(s, M_n(s), Y(s^-))\,\mathrm{d}s + \int_0^{t\wedge\sigma_k} \hat{\sigma}(s, M_n(s), Y(s^-))\,\mathrm{d}W(s). \tag{30}$$

By Karatzas and Shreve [20, Proposition 2.18, p. 9] and the similar inductive argument as used by Applebaum [1, Proof of Theorem 6.2.3], we can get a process $M = \{M(t) : t \in [0, \sigma_k]\}$ as follows:

$$M(\cdot) = \lim_{n\to\infty} M_n(\cdot) \quad \text{in } L^2_{\mathcal{F}}([0, \sigma_k], R^q, P), \tag{31}$$

which is the unique solution of Equation (29) in $L^2_{\mathcal{F}}([0, \sigma_k], R^q, P)$. In addition, due to Equation (31), we have

$$E\left[\int_0^{\sigma_k} \|M(t)\|^2\,\mathrm{d}t\right] < \infty. \tag{32}$$

Hence, by Øksendal [28, Theorem 3.2.5], we can take $M(t)$ to be a continuous version in terms of t. Moreover, let M^k be the unique solution of Equation (29) corresponding to each k. Then we can conclude that if $l \le k$, $M^k(t) = M^l(t)$ for $t \le \sigma_l$ since $\{\sigma_k\}$ is a sequence of nondecreasing stopping times. Thus, define $M^\infty(t) = M^k(t)$ for all $t \le \sigma_k$, then by Equation (32) and the similar argument as used for Definition 2.4 and its remark on p. 57 of Ikeda and Watanabe [18], we know that $M(t) \equiv M^\infty(t)$ for all $t \le T = \lim_{k\to\infty} \sigma_k$ is the unique solution of Equation (22) in $L^2_{\mathcal{F}}([0, T], R^q, P)$ (readers are also referred to [14, p. 192] and [1, pp. 309–310] for some related discussion). Hence we finish the proof of Lemma 4.1. ■

Then, based on Lemma 4.1, we can show that the portfolio selection problem described by Equations (8) and (9) is feasible under condition (14). To do so, consider a portfolio $u^0(t) = 0$ for all $t \in [0, T]$, which is corresponding to the one that puts all the money in the risk-free account. The associated wealth process $X^0(\cdot)$ satisfies

$$\mathrm{d}X^0(t) = r(Y(t^-))X^0(t)\,\mathrm{d}t, \quad X^0(0) = x_0, \quad Y(0) = y_0 \tag{33}$$

and its expected terminal wealth is given by

$$p^0 \equiv E[X^0(T)] = x_0 E\left[\exp\left(\int_0^{\mathrm{T}} r(Y(s^-)\right)\,\mathrm{d}s\right]. \tag{34}$$

Then we have the following lemma.

LEMMA 4.2 *For each $y = (y_1, \ldots, y_h)' \in R^h_c$, define*

$$\psi(t, y) \equiv E_{t,y}[\mathrm{e}^{\int_t^{\mathrm{T}} r(Y(s))\,\mathrm{d}s}], \tag{35}$$

then the problem (8)–(9) *is feasible for every* $p \in R^1$ *and* $p \neq p^0$ *if*

$$\gamma \equiv E\left[\int_0^T \left|\psi(t, Y(t^-))B(Y(t^-))\right|^2 \, dt\right] > 0 \tag{36}$$

and Equation (36) *holds if and only if Equation* (14) *holds. Moreover, if Equation* (14) *holds, then for any* $p \in R^1$, *an admissible portfolio that satisfies* $E[X(T)] = p$ *is given by*

$$u(t) = \frac{p - p^0}{\gamma} B(Y(t^-))' \psi(t, Y(t^-)). \tag{37}$$

Remark 4.1 If Equation (14) fails, then the problem (8)–(9) is feasible only if $p = p^0$, which is a trivial case.

Proof First, it follows from conditions C1, C2, Equation (5), and the similar proof as used in Benth *et al.* [5] that $\psi(t, y)$ defined in Equation (35) is a solution of the following IPDE:

$$\begin{aligned} &\frac{\partial}{\partial t}\psi(t,y) = -r(y)\psi(t,y) + \sum_{i=1}^{h} \lambda_i y_i \frac{\partial}{\partial y_i}\psi(t,y) - \sum_{i=1}^{h} \lambda_i \int_{z_i>0} (\psi(t, y + z_i e_i) - \psi(t,y))\nu_i(dz_i), \\ &\psi(T, y) = 1, \end{aligned} \tag{38}$$

where e_i is the h-dimensional unit vector with the ith component one and all other components zero. In addition, we have that

$$\psi(t,y) \in C^{1,1}([0,T) \times R_c^h, R^1), \tag{39}$$

$$E\left[\int_0^T |\psi(t, Y(t^-))|^2 \, dt\right] < \infty, \tag{40}$$

$$\sum_{i=1}^{h} E\left[\int_0^T \int_{z_i>0} |\psi(t, Y(t^-) + z_i e_i) - \psi(t, Y(t^-))|^2 \nu(dz_i) \, dt\right] < \infty, \tag{41}$$

where $C^{1,1}([0,T) \times R_c^h, R^1)$ denotes the space of continuous functions $v : [0,T] \times R_c^h \to R^1$ which are of $C^{1,1}$ with respect to $(t,y) \in [0,T) \times R_c^h$. Furthermore, for any $(t,y) \in [0,T] \times R_c^h$, it follows from Equation (35), condition C1, condition (5), and the independent assumption on Y_i with $i \in \{1, \ldots, h\}$ that there is some positive constant K such that

$$\psi(t,y) \leq \exp\left(K(T-t) + B_r \sum_{i=1}^{h} \frac{y_i}{\lambda_i}\right). \tag{42}$$

Second, we show that problem (8)–(9) is feasible for every $p \in R^1$ and $p \neq p^0$ if the condition (36) holds. To do so, we construct a family of portfolios $u^\beta(\cdot) = \beta u(\cdot)$ for $\beta \in R^1$, where

$$u(t) \equiv u(t, Y(t^-)) = B(Y(t^-))\psi(t, Y(t^-)). \tag{43}$$

Then it follows from the following argument that $u^\beta(\cdot)$ is admissible for each $\beta \in R^1$. First, it follows from Equation (39) that $u^\beta(\cdot)$ is $\{\mathcal{F}_t\}$-predictable. Then, by condition C1 and Equation (42), we know that there is some nonnegative constant K_1 such that

$$\|u(t, Y(t^-))\| \leq K_1 e^{(1+B_b+B_r(1+\sum_{i=1}^{h}(1/\lambda_i)))\|L(\lambda t)\|}. \tag{44}$$

Thus, by Equations (44) and (5), we know that $u^\beta(\cdot) \in L^2_{\mathcal{F},p}([0,T]; R^d)$. Now substituting $u^\beta(\cdot)$ into Equation (6), we obtain

$$X^\beta(t) = \hat{b}(t, X^\beta(t^-), Y(t^-))\,\mathrm{d}t + \hat{\sigma}(t, X^\beta(t^-), Y(t^-))\,\mathrm{d}W(t),$$

where the coefficients $\hat{b}(\cdot,\cdot,\cdot)$ and $\hat{\sigma}(\cdot,\cdot,\cdot)$ are given as follows:

$$\hat{b}(t,x,y) = r(y)x + \beta|B(y)|^2\psi(t,y),$$
$$\hat{\sigma}(t,x,y) = \beta\psi(t,y)B(y)'\sigma(y).$$

Then by conditions C1 and (44), one can check that all the conditions stated in Lemma 4.1 are satisfied. Therefore, there is a unique solution $X^\beta(\cdot)$ to Equation (6) corresponding to each $u^\beta(\cdot)$, which has all the properties as stated in Lemma 4.1. Hence, $u^\beta(\cdot)$ is an admissible portfolio for each $\beta \in R^1$.

Now, by the linearity and the uniqueness of solution to Equation (6), we have $X^\beta(t) = X^0(t) + \beta X^1(t)$, where $X^0(\cdot)$ satisfies (33) and $X^1(\cdot)$ is the solution to the equation

$$\begin{aligned} \mathrm{d}X^1(t) &= [r(Y(t^-))X^1(t^-) + B(Y(t^-))'u(t)]\,\mathrm{d}t + u(t)'\sigma(Y(t^-))\,\mathrm{d}W(t), \\ X^1(0) &= 0, \quad Y(0) = y_0. \end{aligned} \tag{45}$$

Thus, the problem (8)–(9) is feasible for every $p \in R^1$ (actually here we allow $p = p^0$), if there exists $\beta \in R^1$ such that

$$p = E[X^\beta(T)] = E[X^0(T)] + \beta E[X^1(T)]. \tag{46}$$

In other words, problem (8)–(9) is feasible for every $p \in R^1$ if $E[X^1(T)] \neq 0$. Moreover, let $\phi(t,x,y) = x\psi(t,y)$, then it follows from Itô's formula in Øksendal and Sulem [29, p. 8] and Equation (38) that

$$\begin{aligned} \mathrm{d}[X^1(t)\psi(t,Y(t^-))] &= \psi(t,Y(t^-))B(Y(t^-))'u(t)\,\mathrm{d}t + \sum_{i=1}^h \int_{z_i>0} F_i(t,z_i)\tilde{N}_i(\mathrm{d}t,\mathrm{d}z_i) \\ &\quad + \psi(t,Y(t^-))u(t)'\sigma(Y(t^-))\,\mathrm{d}W(t), \end{aligned} \tag{47}$$

where, for each $i \in \{1,\dots,h\}$,

$$\tilde{N}_i(\lambda_i t,\, \mathrm{d}z_i) = N_i(\lambda_i t, dz_i) - \lambda_i\,\mathrm{d}t\nu_i(\mathrm{d}z_i). \tag{48}$$

and for each $\omega \in \Omega$ and $z_i \in (0,\infty)$ with $i \in \{1,\dots,h\}$,

$$F_i(t,z_i,\omega) \equiv X^1(t^-,\omega)(\psi(t,Y(t^-,\omega) + z_i e_i)) - \psi(t,Y(t^-,\omega))). \tag{49}$$

Then, by Equation (39) and Lemma 4.1, we know that $F(t,z_i,\omega)$ is $\mathcal{B}((0,\infty)) \times \{\mathcal{F}_t\}$-measurable for each t and is a.s. left-continuous in t, that is, it is $\{\mathcal{F}_t\}$-predictable. Then it follows from the mean-value theorem, Equations (38)–(41), Lemma 4.1 and the similar argument used in Benth *et al.* [5] that

$$E\left[\int_0^{\sigma_k}\int_{z_i>0} |F(t,z_i,\omega)|\nu_i(\mathrm{d}z_i)\,\mathrm{d}t\right] < \infty, \tag{50}$$

where σ_k is defined in the proof of Lemma 4.1. Furthermore, by Equation (50) and the similar argument as explained in Definition 2.4 and its associated remark in Ikeda and Watanabe [18, p. 57], the claim is still true if the stopping time σ_k is replaced by T. In addition, it follows from condition C1, Equations (5) and (43), (42) that

$$E\left[\int_0^{\mathrm{T}} \psi(s,Y(s^-))u(s)'\sigma(Y(s^-))\sigma(Y(s^-))'u(s)\psi(s,Y(s^-))\,\mathrm{d}s\right] < \infty. \tag{51}$$

Therefore, we can see that the second and the third terms on the right-hand side of Equation (47) are $\{\mathcal{F}_t\}$-martingales. Thus, integrating from 0 to T on both sides of Equation (47), then taking expectation and using Equation (43), we have

$$E[X^1(T)] = E\left[\int_0^{\mathrm{T}} |\psi(t, Y(t^-))B(Y(t^-))|^2 \, \mathrm{d}t\right]. \tag{52}$$

Hence, if Equation (36) holds, then $E[X^1(T)] \neq 0$, which implies that the problem (8)–(9) is feasible for every $p \in R^1$ and $p \neq p^0$.

Third, it follows from Equation (35) that $\psi(t, y) \geq 1$ for all $(t, y) \in [0, T] \times R_c^h$. Therefore, Equation (36) holds if and only if Equation (14) is true. Moreover, if Equation (14) holds for $p \in R^1$ and $p \neq p^0$, it follows from the proof of condition (36) that a constant β can be determined according to Equations (46) and (52). Thus, we get an admissible portfolio given by Equation (37), which satisfies $E[X(T)] = p$. In addition, if $p = p^0$, then it is obvious that $u(t) \equiv 0$ is an admissible portfolio with $E[X(T)] = p^0$, which can also be represented by Equation (37). Hence we finish the proof of Theorem 4.2. ■

Next, to solve the constraint optimization problem (8)–(9), we first apply the Lagrange multiplier technique to handle the constraint $E[X(T)] = p$ and obtain the unconstrained problem parameterized by the Lagrange multiplier $\alpha \in R^1$,

$$\inf_{(X(\cdot),u(\cdot))\in\mathcal{A}} J_1(x_0, y_0, u(\cdot), \alpha). \tag{53}$$

where

$$\begin{aligned} J_1(x_0, y_0, u(\cdot), \alpha) &\equiv E[|X(T) - p|^2 + 2\alpha(X(T) - p)] \\ &= E[(X(T) + \alpha - p)^2] - \alpha^2. \end{aligned} \tag{54}$$

Then we have the following lemma:

LEMMA 4.3 *There exists an optimal feedback control for problem* (53), *which is given by*

$$u^*(t, x, Y(t^-)) = -(\sigma(Y(t^-))\sigma(Y(t^-))')^{-1}B(Y(t^-))(x + (\alpha - p)H(t, Y(t^-))) \tag{55}$$

and the corresponding optimal value is given by

$$\begin{aligned} \inf_{(X(\cdot),u(\cdot))\in\mathcal{A}} J_1(x_0, y_0, u(\cdot), \alpha) &= (\alpha - p)^2(P(0, y_0)H(0, y_0)^2 + \theta - 1) \\ &\quad + 2(\alpha - p)(P(0, y_0)H(0, y_0)x_0 - p) + P(0, y_0)x_0^2 - p^2, \end{aligned} \tag{56}$$

where the constant θ is given by Equation (13).

Proof First, it follows from conditions C1, C2, Equation (5) and the related discussion in Benth *et al.* [5] that $P(t, y)$ defined in Equation (11) is a solution of the IPDE

$$\begin{aligned} \frac{\partial}{\partial t}P(t, y) &= (\rho(y) - 2r(y))P(t, y) + \sum_{i=1}^{h} \lambda_i y_i \frac{\partial}{\partial y_i}P(t, y) \\ &\quad - \sum_{i=1}^{h} \lambda_i \int_{z_i>0} (P(t, y + z_i e_i) - P(t, y))\nu_i(\mathrm{d}z_i), \\ P(T, y) &= 1 \end{aligned} \tag{57}$$

which can be considered as an *extended* Riccati-type equation since it is a general formation of an ordinary differential Riccati-type equation. Moreover, $P(t, y)$ has the similar properties with

$\psi(t,y)$ as stated in Equation (39)–(41). In addition, note that

$$\bar{H}(t,y) = P(t,y)H(t,y) = E_{t,y}[\mathrm{e}^{\int_t^{\mathrm{T}}(\rho(Y(s))-r(Y(s)))\,\mathrm{d}s}],$$

so we can conclude that

$$\begin{aligned}\frac{\partial}{\partial t}H(t,y) &= \left(r(y) + \frac{1}{P(t,y)}\sum_{i=1}^{h}\lambda_i\int_{z_i>0}(P(t,y+z_ie_i)-P(t,y))\nu_i(\mathrm{d}z_i)\right)H(t,y)\\ &\quad+\sum_{i=1}^{h}\lambda_i y_i\frac{\partial}{\partial y_i}H(t,y) - \frac{1}{P(t,y)}\sum_{i=1}^{h}\lambda_i\int_{z_i>0}(P(t,y+z_ie_i)H(t,y+z_ie_i)\\ &\quad-P(t,y)H(t,y))\nu_i(\mathrm{d}z_i)H(T,y)=1. \qquad (58)\end{aligned}$$

That is, $H(t,y)$ defined in Equation (12) is a solution to the IPDE (58). Moreover, it follows from the definition of $H(t,y)$ that $0 < H(t,y) < 1$ for $t \in [0,T)$ and $H(T,y) = 1$ since $r(y) \geq 0$. In addition, due to $P(t,y), \bar{H}(t,y) \in C^{1,1}([0,T) \times R_c^h, R^1)$ and $P(t,y) > 0$, we know that

$$H(t,y) \in C^{1,1}([0,T) \times R_c^h, R^1)). \qquad (59)$$

Second, we show that $u^*(\cdot)$ in Equation (55) is indeed an admissible portfolio. As a matter of fact, due to Equation (59), we know that $u^*(\cdot)$ is $\{\mathcal{F}_t\}$-predictable, and then substituting $u^*(\cdot)$ into Equation (6), we get the following SDE:

$$\mathrm{d}X(t) = \hat{b}(t,X(t^-),Y(t^-))\,\mathrm{d}t + \hat{\sigma}(t,X(t^-),Y(t^-))\,\mathrm{d}W(t) \qquad (60)$$

whose coefficients are given as follows, for each $t \in [0,T]$, $x \in R^1$ and $y \in R_c^h$,

$$\hat{b}(t,x,y) = r(y)x + B(y)'u^*(t), \hat{\sigma}(t,x,y) = u^*(t)'\sigma(y),$$

which satisfy the conditions as stated in Lemma 4.1 due to conditions C1 and C2. Hence there is a unique adapted solution $X(\cdot)$ to Equation (60), which implies that $u^*(\cdot)$ taken in the form of Equation (55) is an admissible portfolio.

Third, let $u(\cdot)$ be any admissible control and $X(\cdot)$ be the corresponding state trajectory of Equation (6). Moreover, define

$$\phi(t,x,y) = P(t,y)[x + (\alpha - p)H(t,y)]^2.$$

Then it follows from the It$\hat{o}$'s formula (see, e.g. Øksendal and Sulem [29, Theorem 1.16, p. 8]) that

$$\begin{aligned}\mathrm{d}\phi(t,X(t),Y(t^-)) &= P(t,Y(t^-))\{(u(t)-u^*(t,X(t),Y(t^-)))'\\ &\quad\times[\sigma(Y(t^-))\sigma(Y(t^-))'](u(t)-u^*(t,X(t),Y(t^-)))\}\,\mathrm{d}t + M(t), \qquad (61)\end{aligned}$$

where $M(t)$ is a $\{\mathcal{F}_t\}$-martingale due to the similar argument as used for Equation (47).

Finally, as explained in the proof of Lemma 4.2, integrating both sides of Equation (61) from 0 to T and taking expectations, we have

$$\begin{aligned} J_1(x_0, y_0, u(\cdot), \alpha) = &(\alpha - p)^2 (P(0, y_0)H(0, y_0)^2 + \theta - 1) \\ &+ 2(\alpha - p)(P(0, y_0)H(0, y_0)x_0 - p) + P(0, y_0)x_0^2 - p^2 \\ &+ E\int_0^{\mathrm{T}} P(t, Y(t^-))\{(u(t) - u^*(t, X(t), Y(t^-)))' \\ &\times [\sigma(Y(t^-))\sigma(Y(t^-))'](u(t) - u^*(t, X(t), Y(t^-)))\}\,\mathrm{d}t. \end{aligned} \tag{62}$$

Moreover, by the definition in Equation (11), we know that $P(t, Y(t^-)) > 0$. Thus, it follows from Equation (62) that the optimal feedback control and the optimal value are given by Equations (55) and (56), respectively. Hence we finish the proof of Lemma 4.3. ■

Proof of Theorem 3.1 Since the condition (14) is true, then it follows from Lemma 4.2 that the optimization problem (8)–(9) is feasible for any $p \in R^1$. Thus, there exists an admissible portfolio $u(\cdot)$ that satisfies Equation (37) such that the wealth process $X(\cdot)$ is uniquely determined by the SDE (6) and satisfies $E[|X(t)|^2] < \infty$ for each $t \in [0, T]$. Hence, we can conclude that the problem (8)–(9) is finite for any $p \in R^1$. Now, note that $J(x_0, y_0, u(\cdot))$ is strictly convex in $u(\cdot)$ and the constraint function $EX(T) - p$ is affine in $u(\cdot)$. Hence it follows from Luenberger [25, Duality theorem 1, p. 224] and [40, Remark in the proof of Theorem 5.1] that the optimal value of problem (8)–(9) for any $p \in R^1$ is given by

$$\inf_{(X(\cdot),u(\cdot))\in\mathcal{A}} J(x_0, y_0, u(\cdot)) = \sup_{\alpha\in R^1} \inf_{(X(\cdot),u(\cdot))\in\mathcal{A}} J_1(x_0, y_0, u(\cdot), \alpha) > -\infty, \tag{63}$$

where J_1 is defined in Equation (54). By Equation (56) in Lemma 4.3 and Equation (62), we know that

$$\inf_{(X(\cdot),u(\cdot))\in\mathcal{A}} J_1(x_0, y_0, u(\cdot), \alpha)$$

is a quadratic function in terms of $(\alpha - p)$. Due to the finiteness as shown in Equation (63), we can conclude that

$$P(0, y_0)H(0, y_0)^2 + \theta - 1 \le 0. \tag{64}$$

However, if the equality holds in Equation (64), then it follows from Equation (63) and Lemma 4.3 that $P(0, y_0)H(0, y_0)x_0 - p = 0$ must hold for every $p \in R^1$. This is a contradiction. Thus, we know that Equation (17) is true. Now, due to Equation (63), the maximizer (16) can be obtained by maximizing the quadratic function (56) in terms of $(\alpha - p)$ and therefore its corresponding optimal value can be obtained as given in Equation (18). In the end, let $\alpha = \alpha^*$ in Equation (55), then we get the optimal feedback control (15). Hence, we finish the proof of the theorem. ■

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Mean-variance hedging based on an incomplete market with external risk factors of non-Gaussian OU processes[1]

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Abstract

In this paper, we prove the global risk optimality of the hedging strategy of contingent claim, which is explicitly (or called semi-explicitly) constructed for an incomplete financial market with external risk factors of non-Gaussian Ornstein-Uhlenbeck (NGOU) processes. Analytical and numerical examples are both presented to illustrate the effectiveness of our optimal strategy. Our study establishes the connection between our financial system and existing general semimartingale based discussions by justifying required conditions. More precisely, there are three steps involved. First, we firmly prove the no-arbitrage condition to be true for our financial market, which is used as an assumption in existing discussions. In doing so, we explicitly construct the square-integrable density process of the variance-optimal martingale measure (VOMM). Second, we derive a backward stochastic differential equation (BSDE) with jumps for the mean-value process of a given contingent claim. The unique existence of adapted strong solution to the BSDE is proved under suitable terminal conditions including both European call and put options as special cases. Third, by combining the solution of the BSDE and the VOMM, we reach the justification of the global risk optimality for our hedging strategy.

Key words: Mean-variance hedging, Global risk minimization, Non-Gaussian Ornstein-Uhlenbeck process, Generalized Black-Scholes model, Variance-optimal martingale measure, Backward stochastic differential equation with jumps, Integral-partial differential equation

1 Introduction

In this paper, we justify the global risk optimality of the hedging strategy of contingent claim, which is explicitly constructed for an incomplete market defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The financial market has $d+1$ primitive assets: one bond with constant interest rate and d risky assets. The price processes of the assets are described by a generalized Black-Scholes model with coefficients driven by the market regime caused by

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leverage effect, etc. The financial market model includes the Barndorff-Nielsen & Shephard (BNS) volatility model proposed by Barndorff-Nielsen and Shephard [3] and further studied in Benth **et al.** [4], Benth and Meyer-Brandis [5], Lindberg [36], etc. as a particular case. Our model is closely related to the one considered in Delong and Klüppelberg [17]. As pointed out in Barndorff-Nielsen and Shephard [3], these models fit real market data quite well. Nevertheless, such models also induce incompleteness of the financial markets, which means that it is impossible to replicate perfectly contingent claims based on the bond and the d primitive risky assets. A rule for designing a good hedging strategy is to minimize the mean squared hedging error over the set $\bar{\Theta}$ of all reasonable trading strategy processes,

$$\inf_{u\in\bar{\Theta}} E\left[(v+(u\cdot D)(T)-H)^2\right], \tag{1.1}$$

where H is a random variable representing the discounted payoff of the claim, D is the discounted price process of d risky assets, v is the initial endowment and T is the time horizon. Mathematically speaking, one seeks to compute the orthogonal projection of $H-v$ on the space $\bar{\Theta}$ of stochastic integrals.

To solve the mean-variance hedging problem (1.1), we explicitly construct a trading strategy for the financial market and justify it to be the global risk-minimizing hedging strategy by using the following procedure.

First, we explicitly construct the square-integrable density process of a variance-optimal martingale measure (VOMM) Q^*. As a result, the set of equivalent (local) martingale measures with square-integrable densities, i.e.,

$$\mathcal{U}_2^e(D) \equiv \left\{Q\sim P : \frac{dQ}{dP}\in L^2(P), D \text{ is a } Q\text{-local martingale}\right\} \tag{1.2}$$

is nonempty. Hence, our market is arbitrage-free (e.g, Delbaen and Schachermayer [16]). Second, we derive an BSDE with jumps and external random factors of non-Gaussian Ornstein-Uhlenbeck (NGOU) type for the mean value process of the option H (i.e., $E_{Q^*}[H|\mathcal{F}_t]$). The unique existence of adapted solution to the BSDE is proved under suitable terminal conditions including both European call and put options as special cases. Third, by combining the solution to the BSDE and the VOMM, we get the optimal hedging strategy for our market.

A closely related (local) risk minimizing problem was initially introduced by Föllmer and Sondermann [20] under complete information, who also suggested an approach for the computation of a minimizing strategy in an incomplete market by extending the martingale approach of Harrison and Kreps [24]. The basic idea of the approach was to introduce a measure of riskiness in terms of a conditional mean square error process where the discounted price process is a square-integrable martingale. Furthermore, the answer to the hedging problem is provided by the **Galtchouk-Kunita-Watanabe decomposition** of the claim. Then, this concept of local-risk minimization was further extended for the semimartingale case by Föllmer and Schweizer [21], and Schweizer [45, 46], where the minimal martingale measure and Föllmer-Schweizer (F-S) decomposition play a central role. Interested readers are referred to Föllmer and Schweizer [22], Schweizer [48] for more recent surveys about (local) risk minimization and mean-variance hedging.

Owing to the fact that one cares about the total hedging error and not the daily profit-loss ratios, the solution with respect to global-risk minimization of the unconditional expected squared hedging error presented in (1.1) was considered (e.g., surveys in Pham [40] and Schweizer [48]). Then, the study on global-risk minimization was further developed by Cěrný and Kallsen [7], who showed that the hedging model (1.1) admits a solution in a very general class of arbitrage-free semimartingale markets where local-risk minimization may fail to be well defined. The key point of their approach is the introduction of the opportunity-neutral measure P^* that turns the dynamic asset allocation problem into a myopic one. Furthermore, the minimal martingale measure relative to P^* coincides with the variance-optimal martingale measure relative to the original probability measure P. Recently, to overcome the difficulties appeared in Cěrný and Kallsen [7] (i.e., a process N appeared in Definition 3.12 is very hard to find and the VOMM Q^* in Proposition 3.13 is notoriously difficult to determine), the authors in Jeanblanc **et al.** [31] developed a method via stochastic control and backward stochastic differential equations (BSDEs) to handle the mean-variance hedging problem for general semimartingales. Furthermore, the authors in Kallsen and Vierthauer [33] derived semi-explicit formulas for the optimal hedging strategy and the minimal hedging error by applying general structural results and Laplace transform techniques. In addition to these works, some related studies in both general theory and concrete results in specific setups for the mean-variance hedging problem can be found in, such as, Arai [2], Chan **et al.** [9], Duffie and Richardson [18], Gourieroux **et al.** [23], Heath **et al.** [25], Laurent and Pham [37], and references therein.

Comparing with the above studies, our contribution of the current research is threefold. First, we firmly prove the no-arbitrage condition to be true for our financial market, i.e., the set defined in (1.2) is nonempty. This condition is used as an assumption for the existence of the VOMM in existing discussions (e.g., Arai [2], Cěrný and Kallsen [7], Chan **et al.** [9], Jeanblanc **et al.** [31], Kallsen and Vierthauer [33]). In doing so, we explicitly (or called semi-explicitly) construct a measure through identifying its explicit density by the general structure presented in Cěrný and Kallsen [7]. Then, we justify it to be the VOMM for our market model by proving the equivalent conditions given in Cěrný and Kallsen [8]. Second, in applying our VOMM to obtain the optimal hedging strategy, we derive an BSDE with jumps for the mean value process of the option H. Here, we lift the requirements that the contingent claims are bounded (e.g., Heath and Schweizer [26], Cěrný and Kallsen [8]) or satisfy Lipschitz condition (e.g., Roch [42], Chan **et al.** [9]) to guarantee the corresponding integral-partial differential equation (IPDE) to have a classic or viscosity solution. Furthermore, the unique existence of an adapted solution to our derived BSDE is firmly proved under certain conditions while in the recent study of Jeanblanc **et al.** [31], such existence of an adapted solution to their constructed BSDE is only showed as an equivalent condition to guarantee the existence of an optimal strategy. More importantly, our BSDE can be solved by developing related numerical algorithms through the given terminal option H (see, e.g., Dai [15]). Third, from the purpose of easy applications, our discussion is based on a multivariate financial market model, which is in contrast to existing studies (e.g., Cěrný and Kallsen [7], Chan **et al.** [9], Jeanblanc **et al.** [31], Kallsen and Vierthauer [33]). Therefore, unlike the studies in Hubalek **et al.** [27] and Kallsen and Vierthauer [33], our option H is generally related to a multivariate terminal function and hence a BSDE involved approach is employed. Actually, whether one can extend the Laplace transform related method developed in Hubalek **et al.** [27] and Kallsen and Vierthauer [33] for single-variate terminal function to our general multivariate case is still an open problem.

Note that our study in this paper establishes the connection between our financial system and existing general semimartingale based study in Cěrný and Kallsen [7] since we can overcome the difficulties in Cěrný and Kallsen [7] by explicitly constructing the process N and the VOMM Q^* as mentioned earlier. Furthermore, our objective and discussion in this paper are different from the recent study of Jeanblanc **et al.** [31] since the authors in Jeanblanc **et al.** [31] did not aim to derive any concrete expression. Nevertheless, interested readers may make an attempt to extend the study in Jeanblanc **et al.** [31] and apply it to our financial market model to construct the corresponding explicit results.

Finally, when the random variable H in (1.1) is taken to be a constant (e.g., a prescribed daily expected return), the associated hedging problem reduces to a mean-variance portfolio selection problem as studied in Dai [10] by an alternative feedback control method. In this case, the optimal policies can be explicitly obtained by both the feedback control method in Dai [10] and the martingale method presented in the current paper. In the late method, the related BSDE is a degenerate one. From this constant option case, we can construct two insightful examples to provide the effective comparisons between the two methods. More precisely, our newly constructed hedging strategy can slightly outperform the feedback control based policy. However, the performance between the two methods is consistent in certain sense.

The remainder of the paper is organized as follows. We formulate our financial market model in Section 2 and present our main theorem Section 3. Analytical and numerical examples are given in Section 4. Our main theorem is proven in Section 5. Finally, in Section 6, we conclude this paper with remarks.

2 The Financial Market

2.1 The Model

We use $(\Omega, \mathcal{F}, P)$ to denote a fixed complete probability space on which are defined a standard d-dimensional Brownian motion $W \equiv \{W(t), t \in [0,T]\}$ with $W(t) = (W_1(t), ..., W_d(t))'$ and h-dimensional subordinator $L \equiv \{L(t), t \in [0,T]\}$ with $L(t) \equiv (L_1(t), ..., L_h(t))'$ and càdlàg sample paths for some fixed $T \in [0, \infty)$ (e.g., Applebaum [1], Bertoin [6], and Sato [44] for more details about subordinators and Lévy processes). The prime denotes the corresponding transpose of a matrix or a vector. Furthermore, W, L, and their components are assumed to be independent of each other. For each given $\lambda = (\lambda_1, ...\lambda_h)' > 0$, we let $L(\lambda s) =$

$(L_1(\lambda_1 s), ..., L_h(\lambda_h s))'$. Then, we suppose that there is a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ related to the probability space, where $\mathcal{F}_t \equiv \sigma\{W(s), L(\lambda s) : 0 \leq s \leq t\}$ for each $t \in [0, T]$.

The financial market under consideration is a multivariate Lévy-driven OU type stochastic volatility model, which consists of $d+1$ assets. One of the $d+1$ assets is risk-free, whose price $S_0(t)$ is subject to the ordinary differential equation (ODE) with constant interest rate $r \geq 0$,

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = s_0 > 0. \tag{2.1}$$

The other d assets are stocks whose vector price process $S(t) = (S_1(t), ..., S_d(t))'$ satisfies the following stochastic differential equation (SDE) for each $t \in [0, T]$,

$$\begin{cases} dS(t) = \mathrm{diag}(S(t^-))\{b(Y(t^-))dt + \sigma(Y(t^-))dW(t)\}, \\ S(0) = s > 0. \end{cases} \tag{2.2}$$

Here and in the sequel, the $\mathrm{diag}(v)$ denotes the $d \times d$ diagonal matrix whose entries in the main diagonal are v_i with $i \in \{1, ..., d\}$ for a d-dimensional vector $v = (v_1, ..., v_d)'$ and all the other entries are zero. $Y(t)$ is a Lévy-driven OU type process described by the following SDE,

$$\begin{cases} dY(t) = -\Lambda Y(t^-)dt + dL(\lambda t), \\ Y(0) = y_0, \end{cases} \tag{2.3}$$

where $\Lambda = \mathrm{diag}(\lambda)$ and $y_0 = (y_{10}, ..., y_{h0})'$. Now, define

$$\begin{aligned} b(y) &\equiv (b_1(y), ..., b_d(y))' : R_c^h \to [0, \infty)^d, \\ \sigma(y) &\equiv (\sigma_{mn}(y))_{d\times d} : \quad R_c^h \to (0, \infty)^{dd}, \end{aligned}$$

where $R_c^h \equiv (c_1, \infty) \times ... \times (c_h, \infty)$ with $c_i = y_{i0}e^{-\lambda_i T}$. Thus, we can impose the following conditions related to the coefficients in (2.2)-(2.3):

C1. The functions $b(y)$ and $\sigma(y)$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\|b(y)\| \leq A_b + B_b\|y\|, \tag{2.4}$$

$$\|\sigma(y)\sigma(y)'\| \leq A_\sigma + B_\sigma\|y\|, \tag{2.5}$$

$$\left\|\left(\sigma(y)\sigma(y)'\right)^{-1}\right\| \leq \frac{1}{b_\sigma\|y\|}, \tag{2.6}$$

where the norm $\|A\|$ takes the largest absolute value of all components of a vector A or all entries of a matrix A, and $A_b \geq 0, A_\sigma \geq 0, B_b \geq 0, B_\sigma \geq 0, b_\sigma > 0$ are constants.

C2. The derivatives $\frac{\partial b(y)}{\partial y_i}$ and $\frac{\partial(\sigma(y)\sigma(y)')^{-1}}{\partial y_i}$ for all $i \in \{1, ..., h\}$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\left\|\frac{\partial b(y)}{\partial y_i}\right\| \leq \bar{A}_b + \bar{B}_b\|y\|, \tag{2.7}$$

$$\left\|\frac{\partial(\sigma(y)\sigma(y)')^{-1}}{\partial y_i}\right\| \leq \bar{A}_\sigma + \bar{B}_\sigma\|y\|, \tag{2.8}$$

where $\bar{A}_b$, $\bar{A}_\sigma$, $\bar{B}_b$ and $\bar{B}_\sigma$ are some nonnegative constants.

We now introduce the conditions for each subordinator L_i with $i \in \{1, ..., h\}$, which can be represented by (e.g., Theorem 13.4 and Corollary 13.7 in Kallenberg [32])

$$L_i(t) = \int_{(0,t]} \int_{z_i > 0} z_i N_i(ds, dz_i), \; t \geq 0. \tag{2.9}$$

Here and in the sequel, $N_i((0,t] \times A) \equiv \sum_{0<s\leq t} I_A(L_i(s) - L_i(s^-))$ denotes a Poisson random measure with deterministic, time-homogeneous intensity measure $\nu_i(dz_i)ds$. $I_A(\cdot)$ is the index function over the set A. ν_i is the Lévy measure satisfying

$$\int_{z_i > 0} \left(e^{Cz_i} - 1\right) \nu_i(dz_i) < \infty \tag{2.10}$$

with C taken to be a sufficiently large positive constant to guarantee all of the related integrals in this paper meaningful. Note that the condition in (2.10) is on the integrability of the tails of the Lévy measures (readers are referred to Dai ([10, 11, 12, 13, 14]) for the justification of its reasonability).

2.2 Admissible Strategies

First, we use $D(t) = (D_1(t), ..., D_d(t))'$ to denote the associated d-dimensional discounted price process, i.e., for each $m \in \{1, ..., d\}$,

$$D_m(t) = \frac{S_m(t)}{S_0(t)} = e^{-rt} S_m(t). \tag{2.11}$$

Furthermore, we define $L^2_{\mathcal{F}}\left([0,T], R^d, P\right)$ to be the set of all R^d-valued measurable stochastic processes $Z(t)$ adapted to $\{\mathcal{F}_t, t \in [0,T]\}$ such that $E\left[\int_0^T \|Z(t)\|^2 dt\right] < \infty$. Thus, it follows from Lemma 5.1 that $D(\cdot)$ is a continuous $\{\mathcal{F}_t\}$-semimartingale. In addition, $D(\cdot)$ is locally in $L^2_{\mathcal{F}}([0,T], R^d, P)$, i.e., there is a localizing sequence of stopping times $\{\sigma_n\}$ with $n \in \mathcal{N} \equiv \{0, 1, 2, ...\}$ such that, for any $n \in \mathcal{N}$,

$$\sup\{E\left[D^2(\tau)\right] : \text{all stopping } \; \tau \; \text{ time satisfying } \tau \leq \sigma_n\} < \infty. \tag{2.12}$$

Second, let $L(D)$ denote the set of D-integrable and predictable processes in the sense of Definition 6.17 in page 207 of Jacod and Shiryaev [30]. Furthermore, let $u_i(t)$ denote the number of shares invested in stock $i \in \{1, ..., d\}$ at time t and define $u(t) \equiv (u_1(t), ..., u_d(t))'$. Then, we have the following definitions concerning admissible strategies.

Definition 2.1 **An R^d-valued trading strategy u is called simple if it is a linear combination of strategies $ZI_{(\tau_1,\tau_2]}$ where $\tau_1 \leq \tau_2$ are stopping times dominated by σ_n for some $n \in \mathcal{N}$ and Z is a bounded $\mathcal{F}_{\tau_1}$-measurable random variable. Furthermore, the set of all such simple trading strategies is denoted by $\Theta(D)$.**

Definition 2.2 **A trading strategy $u \in L(D)$ is called admissible if there is a sequence $\{u^n, n \in \mathcal{N}\}$ of simple strategies such that: $(u^n \cdot D)(t) \to (u \cdot D)(t)$ in probability as $n \to \infty$ for any $t \in [0,T]$ and $(u^n \cdot D)(T) \to (u \cdot D)(T)$ in $L^2(P)$ as $n \to \infty$. Furthermore, the set of all such admissible strategies is denoted by $\bar{\Theta}(D)$.**

3 Main Theorem

First, for each $y \in R_c^h$, define

$$
\begin{aligned}
&(3.1) \qquad & B(y) &\equiv (b_1(y) - r, ..., b_d(y) - r)', \\
&(3.2) \qquad & \rho(y) &\equiv B(y)' \left[\sigma(y)\sigma(y)'\right]^{-1} B(y), \\
&(3.3) \qquad & P(t,y) &\equiv E_{t,y}\left[e^{-\int_t^T \rho(Y(s))ds}\right] > 0, \\
&(3.4) \qquad & O(t) &\equiv P(t, Y(t)), \\
&(3.5) \qquad & a(t) &\equiv (\mathrm{diag}(D(t)))^{-1}\left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)^{-1} B(t, Y(t^-)), \\
&(3.6) \qquad & \hat{Z}(t) &\equiv \frac{O(t)\mathcal{E}(-a \cdot D)(t)}{O_0}, \quad O_0 = O(0).
\end{aligned}
$$

Note that the process $a(\cdot)$ presented in (3.5) is corresponding to the adjustment process defined in Lemma 3.7 of Cerny and Kallsen [7]. Furthermore, the process $\hat{Z}(\cdot)$ presented in (3.6) is associated with the density process defined in Proposition 3.13 of Cerny and Kallsen [7]. In addition, here and in the sequel, $\mathcal{E}(N) = \{\mathcal{E}(N)(t), t \in [0,T]\}$ denotes the stochastic exponential for a univariant continuous semimartingale $N = \{N(t), t \in [0,T]\}$ (e.g., pages 84-85 of Protter [41]) with

$$
(3.7) \qquad \mathcal{E}(N)(t) = \exp\left\{N(t) - \frac{1}{2}[N,N](t)\right\}
$$

where $[\cdot,\cdot]$ denotes the quadratic variation process of N.

Second, let $L^2_{\mathcal{F},p}([0,T], R^d, P)$ denote the set of all R^d-valued predictable processes (see, e.g., Definition 5.2 in page 21 of Ikeda and Watanabe [28]) and let $L^2_p([0,T],\ R^h, P)$ be the set of all R^h-valued predictable processes $\tilde{Z}(t,z) = (\tilde{Z}_1(t,z),\ ...,\ \tilde{Z}_h(t,z))'$ satisfying

$$
E\left[\sum_{i=1}^h \int_0^T \int_{z_i > 0} \left|\tilde{Z}_i(t,z)\right|^2 \nu_i(dz_i)dt\right] < \infty.
$$

Furthermore, let

$$
\begin{aligned}
&(3.8) \qquad & \bar{Z}(t) &\equiv \frac{\hat{Z}(t^-)}{\hat{Z}(t)}, \\
&(3.9) \qquad & \bar{B}_i(Y(t^-)) &\equiv \sum_{j=1}^d \left(\left(B(Y(t^-))'\left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)^{-1}\right)\right)_j \sigma_{ji}(Y(t^-)), \\
&(3.10) \qquad & F(t, z_i)) &\equiv \frac{P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))}{P(t, Y(t^-))},
\end{aligned}
$$

where, e_i is the h-dimensional unit vector with the ith component one. Then, we define

$$
\begin{aligned}
(3.11)\qquad & g\left(t, V(t^-), \bar{V}(t), \tilde{V}(t,\cdot), Y(t^-)\right) \\
\equiv\ & -\sum_{i=1}^{d} \bar{V}_i(t)\bar{B}_i(Y(t^-)) \\
& +\sum_{i=1}^{h}\int_{z_i>0}\left(\tilde{V}_i(t,z_i)F(t,z_i)\bar{Z}(t) + V(t^-)\left(F(t,z_i)\bar{Z}(t)\right)^2\right)\lambda_i\nu_i(dz_i).
\end{aligned}
$$

Definition 3.1 **For a given random variable H, a 3-tuple $(V, \bar{V}, \tilde{V})$ is called a $\{\mathcal{F}_t\}$-adapted strong solution of the BSDE**

$$
\begin{aligned}
(3.12)\qquad V(t) &= H - \int_t^T g(s, V(s^-), \bar{V}(s), \tilde{V}(s,\cdot), Y(s^-))ds \\
&\quad - \int_t^T \sum_{i=1}^{d} \bar{V}_i(s)dW_i(s) - \int_t^T \sum_{i=1}^{h}\int_{z_i>0}\tilde{V}_i(s,z_i)\tilde{N}_i(\lambda_i ds, dz_i)
\end{aligned}
$$

if $V \in L^2_{\mathcal{F}}([0,T], R, P)$ is a càdlàg process, $\bar{V} = (\bar{V}_1, ..., \bar{V}_d) \in L^2_{\mathcal{F},p}([0,T], R^d, P)$, $\tilde{V} = (\tilde{V}_1, ..., \tilde{V}_h) \in L^2_p([0,T], R^h, P)$, and (3.12) holds a.s., where

$$
(3.13)\qquad \tilde{N}_i(\lambda_i dt, dz_i) \equiv N_i(\lambda_i dz_i, dt) - \lambda_i\nu_i(dz_i)dt \quad \textbf{for each } i \in \{1, ..., h\}.
$$

To impose suitable condition on the option H, we use $L^{\gamma}_{\mathcal{F}_T}(\Omega, R^d, P)$ for a positive integer γ to denote the set of all R^d-valued, $\mathcal{F}_T$-measurable random variables $\xi \in R^d$ satisfying $E\left[\|\xi\|^{\gamma}\right] < \infty$.

Assumption 3.1 $H \in L^4_{\mathcal{F}_T}(\Omega, R, P)$ **and there exists a sequence of random variables $H_{\tau_n} \in L^2_{\mathcal{F}_{T\wedge\tau_n}}(\Omega, R, P)$ satisfying $H_{\tau_n} \to H$ in L^2 as $n \to \infty$ and $H_{\tau_n}(\omega) = H(\omega)$ for all $\omega \in \{\omega, \tau_n(\omega) \geq T\}$, where $\{\tau_n\}$ is a sequence of nondecreasing $\{\mathcal{F}_t\}$-stopping times satisfying $\tau_n \to \infty$ a.s. as $n \to \infty$.**

As pointed out in Dai [12], under conditions **C1**, **C2**, and (2.10), the discounted European call and put options satisfy Assumption 3.1. Now, we can state our main theorem of the paper as follows.

Theorem 3.1 **Under conditions C1, C2, (2.10), and Assumption 3.1, let $(V, \bar{V}, \tilde{V})$ be the unique $\{\mathcal{F}_t\}$-adapted strong solution of the BSDE in (3.12). Then, the optimal hedging strategy $\phi \in \bar{\Theta}(D)$ for (1.1) is given by**

$$
(3.14)\qquad \phi(t) = \xi(t) - (v + \Psi(t^-) - V(t^-))a(t),
$$

where, the pure hedge coe□cient ξ is given by

$$
\begin{aligned}
(3.15)\qquad \xi(t) &= \left(\tilde{c}^{D^*}(t)\right)^{-1}\left(\tilde{c}^{DV^*}(t)\right), \\
(3.16)\qquad \tilde{c}^{D^*}(t) &= \textbf{diag}(D(t))\left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)\textbf{diag}(D(t)), \\
(3.17)\qquad \tilde{c}^{DV^*}(t) &= \left(\sum_{i=1}^{d} D_1(t)\sigma_{1i}(Y(t^-))\bar{V}_i(t), ..., \sum_{i=1}^{d} D_d(t)\sigma_{di}(Y(t^-))\bar{V}_i(t)\right)'.
\end{aligned}
$$

In addition, Ψ is the unique solution of the SDE

$$\Psi(t) = ((\xi - (v - V_{-})a) \cdot D)(t) - (\Psi_{-} \cdot (a \cdot D))(t). \tag{3.18}$$

Remark 3.1 **The process $V(\cdot)$ appeared in Theorem 3.1 is actually the conditional mean value process,**

$$V(t) = E_{Q^*}\left[H\left|\mathcal{F}_t\right.\right] \quad \textbf{with} \quad dQ^* \equiv \hat{Z}(T)dP. \tag{3.19}$$

Since it is not easy to be computed directly as the Markovian based conditional process $O(t, Y(t))$, we turn to use the BSDE in (3.12) to evaluate it, which is convenient for us to design the optimal hedging policy as explained in Introduction of the paper.

The proof of Theorem 3.1 will be provided in Section 5.

4 Performance Comparisons

The material in this section is partially reported in the short conference version of the current paper (see, Dai [12]). To be convenient and clear for readers, we refine it here. Note that the interest rate r in (2.1) here is taken to be zero. Furthermore, the financial market is assumed to be self-financing, which implies that $X(t) = v + (u \cdot D)(t)$. In addition, the terminal option H is taken to be a constant p, i.e., $H = p$. In this case, the optimal policies can be explicitly obtained by the feedback control method studied in Dai [10] and the martingale method presented in the current paper. In the late method, the related BSDE is a degenerate one, which can be easily observed from (3.19) in Remark 3.1. However, from this constant option $H = p$, we can construct two insightful examples to provide the effective comparisons between the two methods.

More precisely, by (18) in Theorem 3.1 of Dai [10], we know that the terminal variance under the optimal policy stated in (15) of Theorem 3.1 of Dai [10] is given by

$$Var(X^*(T)) = \frac{P(0, y_0)}{1 - P(0, y_0)} (p - v)^2. \tag{4.1}$$

In addition, by using Theorem 3.1 in the current paper and Theorem 4.12 in Cěrný and Kallsen [7], we know that the hedging error under the optimal policy in (3.14) is given by

$$Herr = P(0, y_0) (p - v)^2. \tag{4.2}$$

For the purpose of performance comparisons, we calculate the differences between the optimal terminal variances in (4.1) and the optimal hedging errors in (4.2), i.e.,

$$\begin{aligned} Error &= Var(X^*(T)) - Herr \\ &= \frac{(P(0, y_0))^2}{1 - P(0, y_0)} (p - v)^2 \\ &> 0. \end{aligned} \tag{4.3}$$

The result shown in the last inequality of (4.3) is intuitively right since the optimal strategy in (3.14) is taken over a general decision set given in Definition 2.2 and the one in (15) of Theorem 3.1 of Dai [10] is taken in an **ad-hoc** approach. Nevertheless, the errors are very small as displayed in the following numerical examples.

Example 4.1 Here, we suppose that the financial market is given by the Black-Scholes model

$$dD(t) = D(t)(\alpha dt + \beta dB(t)), \tag{4.4}$$

where α and β are given constants. Owing to Definition 2.1.4(b) in pages 273-274 of ∅ksendal [39], the option $H = p$ (a positive constant) is not attainable and hence the associated hedging error can not be zero if the initial endowment $v \neq p$. However, by the simulated results displayed in Figures 1 and 2, we see that the absolute error between the optimal variance based on the policy in (15) of Theorem 3.1 of Dai [10] and the optimal hedging error based on the strategy in (3.14) approaches zero as the terminal time increases. The rate of convergence is heavily dependent on the volatility β. If β is relatively large, the di□erence requires more time to reach zero. Nevertheless, if the millisecond is employed to represent the time unit in a supercomputer based trading system, the required time for the convergence makes sense in practice.

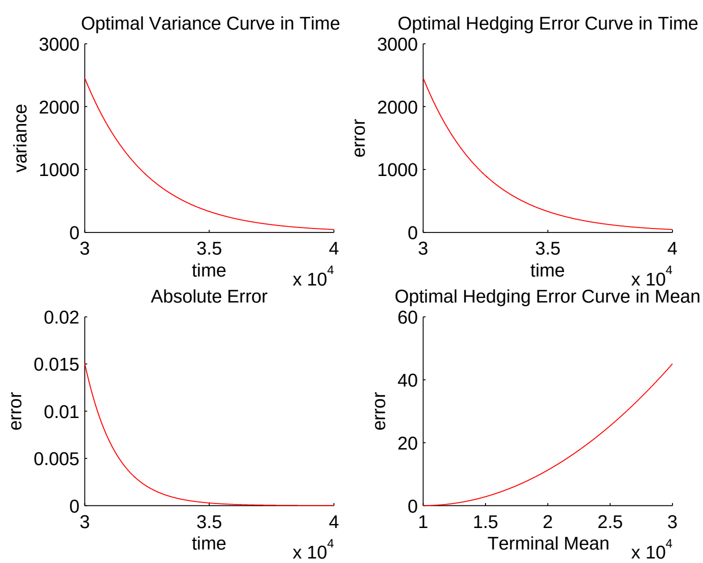

Figure 1: Errors using Black-Scholes model with $r = 0$, $v = 10000$, $p = 30000$, $T = 40000$, $\alpha = 2$, $\beta = 100$.

Example 4.2 Here, we assume that the financial market is presented by the BNS model

$$dD(t) = D(t)((\alpha + \beta Y(t^-))dt + \sqrt{Y(t^-)}dB(t)), \tag{4.5}$$

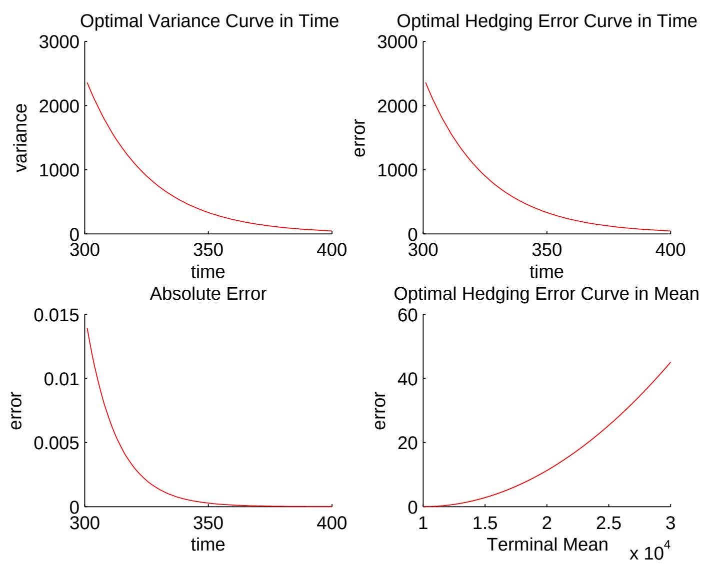

Figure 2: Errors using Black-Scholes model with $y_0 = 10$, $r = 0$, $v = 10000$, $p = 30000$, $T = 400$, $\alpha = 2$, $\beta = 10$.

where α and β are given constants. Furthermore, owing to the remarks to the condition in (2.10) and owing to the discussions in Dai [11], we suppose that the driving subordinator $L(\lambda\cdot)$ with $\lambda = 1$ to the SDE in (2.3) is a compound Poisson process. The interarrival times of the process are exponentially distributed with mean $1/\mu$ and the jump sizes of the process are also exponentially distributed with mean $1/\mu_1$. By the simulated results displayed in Figure 3, we see that the similar illustration displayed in Example 4.1 also makes sense for the current example, where δ appeared in Figure 3 is the length of equally divided subintervals of $[0, T]$. In addition, by the simulated results, we also see that, by perfect hedging is impossible in an incomplete market, the mean-variance hedging errors can be very small in many cases when terminal time increases.

5 Proof of Theorem 3.1

The proof consists of four parts presented in the subsequent four subsections: the justification of a proposition related to the discounted price process, the demonstration of a proposition related to the VOMM, the illustration of unique existence of solution to a type of BSDEs with jumps, and the remaining proof of Theorem 3.1.

5.1 The Proposition Related to the Discounted Price Process

Proposition 5.1 **Under conditions C1, C2, and (2.10), we have that $D(\cdot)$ is a continuous $\{\mathcal{F}_t\}$-semimartingale, i.e.,**

$$D(\cdot) = D_0 + M^D(\cdot) + B^D(\cdot), \tag{5.1}$$

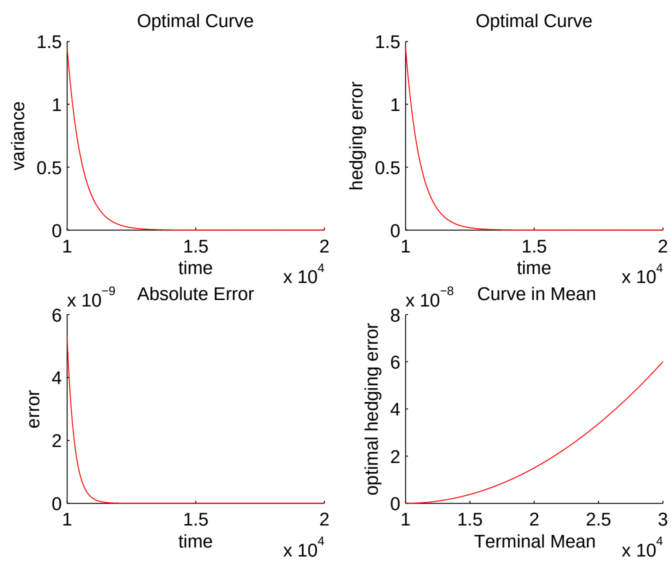

Figure 3: Errors using BNS model with $y_0 = 10$, $r = 0$, $v = 10000$, $p = 30000$, $T = 200$, $\delta = 0.01$, $\alpha = 0.5$, $\beta = 0.02$, $\mu = 10$, $\mu_1 = 8$.

where $M^D(\cdot)$ and $B^D(\cdot)$ are an $\{\mathcal{F}_t\}$-martingale and a predictable process of finite variation respectively. Furthermore, $D(\cdot)$ is locally in $L^2_{\mathcal{F}}([0,T], R^d, P)$ in the sense as stated in (2.12).

We divide the proof of the proposition into two parts. First, we have the following lemma.

Lemma 5.1 **Under (2.10), the unique adapted solution to the SDE in (2.3) for each $\hat{t} > t$, $i \in \{1, ..., h\}$, and $y \in (0, \infty)^h$ is given by**

$$Y_i(\hat{t}) = y_i e^{-\lambda_i(\hat{t}-t)} + \int_t^{\hat{t}} e^{-\lambda_i(s-t)} dL_i(\lambda_i s) \geq y_i e^{-\lambda_i \hat{t}}, \qquad Y_i(t) = y_i. \tag{5.2}$$

Furthermore, under conditions C1, C2, and (2.10), there is a unique solution $(S_0(t), S(t)')$ for (2.2)-(2.3), which is an $\{\mathcal{F}_t\}$-adapted and continuous semimartingale with

$$S(\cdot) \in L^2_{\mathcal{F}}\left([0,T], R^d, P\right). \tag{5.3}$$

In addition, for each $m \in \{1, ..., d\}$,

$$\begin{aligned} S_m(t) &= S_m(0) \exp\left\{ \int_0^t \left[b_m(Y(s^-)) - \frac{1}{2}\sum_{n=1}^d \sigma_{mn}^2(Y(s^-)) \right] ds \right. \\ &\quad \left. + \int_0^t \sum_{n=1}^d \sigma_{mn}(Y(s^-)) dW_n(s) \right\}. \end{aligned} \tag{5.4}$$

PROOF. The claim concerning (5.2) directly follows from pages 316-317 in Applebaum [1]. Furthermore, owing to conditions **C1** and **C2**, we know that our market given by (2.2)-(2.3) satisfies the conditions as required by Lemma 4.1 in Dai [10]. Thus, our market has a unique solution, which is $\{\mathcal{F}_t\}$-adapted, continuous, and mean-square integrable as stated in Lemma 5.1. In order to prove (5.4), let

$$X_m(t) = \int_0^t \alpha_m(Y(s^-))ds + \int_0^t \beta_m(Y(s^-))' dW(s), \tag{5.5}$$

where, for any $s \in [0,T]$,

$$\begin{aligned}\alpha_m(Y(s^-)) &= b_m(Y(s^-)) - \frac{1}{2}\sum_{n=1}^{d}\sigma_{mn}^2(Y(s^-)),\\ \beta_m(Y(s^-)) &= (\sigma_{m1}(Y(s^-)),...,\sigma_{md}(Y(s^-)))'.\end{aligned}$$

Then, by condition **C1**, there exists some nonnegative constant D_1 such that

$$\begin{aligned}E\left[\int_0^T \left|\alpha_m(Y(s^-))\right| ds\right] &\leq D_1 T + \left(B_b + \frac{1}{2}B_\sigma\right) T e^{\sum_{i=1}^h y_{i0}} \prod_{i=1}^{h} E\left[e^{L_i(\lambda_i T))}\right]\\ &< \infty,\end{aligned} \tag{5.6}$$

where we have used the facts that $L(\lambda t)$ is nonnegative and nondecreasing in t, the independence assumption among $L_i(\lambda_i \cdot)$ for $i \in \{1,...,h\}$, and

$$a + b\|L(\lambda t)\| \leq \left(\frac{1}{\epsilon} \vee a\right) e^{b\epsilon\|L(\lambda t)\|} \text{ for any } a \geq 0,\ b \geq 0,\ \epsilon > 0, \tag{5.7}$$

$$Y_i^{t,y_i}(\hat{t}) \leq y_i + L_i(\lambda_i \hat{t}) - L_i(\lambda_i t) \quad \text{for any } \hat{t} \geq t, \tag{5.8}$$

$$E\left[e^{CL_i(\lambda_i t)}\right] = \exp\left(\lambda_i t \int_{z_i > 0} \left(e^{Cz_i} - 1\right) \nu_i(dz_i)\right) < \infty. \tag{5.9}$$

Similarly, we can show that

$$E\left[\int_0^T \beta_m^2(Y(s^-))ds\right] < \infty. \tag{5.10}$$

Note that $W(\cdot)$ and $L_i(\lambda_i \cdot)$ for $i \in \{1,...,h\}$ are independent; W is $\{\mathcal{F}_t, t \in [0,T]\}$-martingale; $\alpha_m(Y(t^-))$ and $\beta_m(Y(t^-))$ are $\mathcal{F}_t$-adapted. Then, it follows from Definition 4.1.1 in Øksendal [39] and the associated Itô's formula (e.g., Theorem 4.1.2 in Øksendal [39]) that $S_m(t)$ given in (5.4) for each m is the unique solution of (2.2).

Now, we show that $S_m(\cdot)$ for each $m \in \{1,...,d\}$ is a square-integrable $\{\mathcal{F}_t\}$-semimartingale. To do so, we rewrite (2.2) in its integral form

$$S_m(t) = S_m(0) + \int_0^t S_m(s) b_m(Y(s^-))ds + \int_0^t S_m(s) \sum_{n=1}^{d} \sigma_{mn}(Y(s^-)) dW_n(s). \tag{5.11}$$

Then, the third term on the right-hand side of (5.11) is a square-integrable $\{\mathcal{F}_t\}$-martingale. In fact, it follows from (5.2) that, for each $i \in \{1, ..., h\}$ and $\hat{t} > t$,

$$\begin{aligned}
\lambda_i \int_t^{\hat{t}} Y_i^{(t,y_i)}(s)ds &= y_i + L_i(\lambda_i \hat{t}) - L_i(\lambda_i t) - Y_i^{(t,y_i)}(\hat{t}) \\
&\leq y_i + L_i(\lambda_i \hat{t}) - L_i(\lambda_i t) \\
&= y_i + L_i(\lambda_i(\hat{t} - t)),
\end{aligned} \tag{5.12}$$

where the last equality in (5.12) holds in distribution. Thus, it follows from Condition **C1** and (5.4) in Lemma 5.1 that

$$\begin{aligned}
E\left[\int_0^T \left(S_m(s)\sum_{n=1}^d \sigma_{mn}(Y(s^-))\right)^2 ds\right] &\leq ds_m^2 C T^{\frac{1}{2}} \left(E\left[e^{C\|L(\lambda T)\|}\right]\right)^{\frac{1}{2}} \\
&< \infty,
\end{aligned} \tag{5.13}$$

where C is some positive constant and we have used Theorem 39 in page 138 of Protter [41] and the condition (2.10). Therefore, by Theorem 4.40(b) in page 48 of Jacod and Shiryaev [30], we know that the third term in (5.11) is a square-integrable $\{\mathcal{F}_t\}$-martingale.

Furthermore, by the same method, we can show that the second term on the right-hand side of (5.11) is of finite variation a.s. and is square-integrable over $[0, T]$. Therefore, we conclude that $S_m(\cdot)$ for each $m \in \{1, ..., d\}$ is a square-integrable $\{\mathcal{F}_t\}$-semimartingale. Hence, we complete the proof of Lemma 5.1. □

Proof of Proposition 5.1

It follows from Lemma 5.1 and the Ito's formula that, for each $m \in \{1, ..., d\}$,

$$B_m^D(t) = \int_0^t D_m(s)(b_m(Y(s^-)) - r)ds, \tag{5.14}$$

$$M_m^D(t) = \int_0^t D_m(s)\sum_{n=1}^d \sigma_{mn}(Y(s^-))dW_n(s). \tag{5.15}$$

Note that, by the similar calculation as in (5.13), we have

$$E\left[\int_0^t \left(D_m(s)\sum_{n=1}^d \sigma_{mn}(Y(s^-))\right)^2 ds\right] < \infty \tag{5.16}$$

for all $t \in [0, T]$. Thus, it follows from Theorem 4.40(b) in page 48 of Jacod and Shiryaev [30] that M^D is an $\{\mathcal{F}_t\}$-martingale. Furthermore, it follows from a similar explanation with the end of the proof for Lemma 5.1 that B^D is a predictable process of finite variation and square-integrable. Thus, we know that D is a continuous $\{\mathcal{F}_t\}$-semimartingale. Moreover, it is locally in $L^2(P)$ since we may take $\sigma_n \equiv \inf\{\tau : D^2(\tau) \geq n\}$ as the sequence of localizing times. Hence, we complete the proof of Proposition 5.1. □

5.2 A Proposition Related to the VOMM

First of all, we use $\mathcal{P}_D(\bar{\Theta})(D)$ to denote the set of all signed $\bar{\Theta}$-martingale measures in the sense that $Q(\Omega) = 1$ and $Q \ll P$ with

$$\frac{dQ}{dP} \in L^2(P) \text{ and } E\left[\frac{dQ}{dP}(u \cdot D)(T)\right] = 0$$

for a signed measure Q on $(\Omega, \mathcal{F})$ and all $u \in \bar{\Theta}(D)$. Then, we have the following proposition.

Proposition 5.2 **Under conditions C1, C2, and (2.10), the following claims are true:**

1. $\hat{Z}$ **is a** $\{\mathcal{F}_t\}$**-martingale, where** $\hat{Z}(\cdot)$ **is given in (3.6);**

2. **The measure** Q^* **defined in (3.19) is an equivalent martingale measure (EMM), and** $Q^* \in \mathcal{U}_2^e(D)$ **that is defined in (1.2);**

3. **The measure** Q^* **is the VOMM in the sense that**

$$Var\left(\frac{dQ^*}{dP}\right) = \min_{Q \in \mathcal{P}_D(\bar{\Theta})} Var\left(\frac{dQ}{dP}\right).$$

We divide the proof of the proposition into demonstrating six lemmas as follows.

Lemma 5.2 **Under conditions C1, C2, and (2.10),** $P(t, y)$ **defined in (3.3) is a solution of the following IPDE**

$$\begin{cases} \frac{\partial}{\partial t}P(t,y) = \rho(y)P(t,y) + \sum_{i=1}^{h} \lambda_i y_i \frac{\partial}{\partial y_i} P(t,y) \\ \qquad\qquad - \sum_{i=1}^{h} \lambda_i \int_{z_i>0} (P(t, y + z_i e_i) - P(t,y))\nu_i(dz_i), \\ P(T,y) = 1. \end{cases} \tag{5.17}$$

for $y \in R_c^h$**. Furthermore, we have**

$$P(t,y) \in C^{1,1}([0,T) \times R_c^h, R^1), \tag{5.18}$$

$$E\left[\int_0^T |P(t, Y(t^-))|^2 dt\right] < \infty, \tag{5.19}$$

$$\sum_{i=1}^{h} E\left[\int_0^T \int_{z_i>0} |P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))|^2 \nu(dz_i) dt\right] < \infty. \tag{5.20}$$

Proof. It follows from conditions **C1**, **C2**, and (5.2) that, for each $i \in \{1, ..., h\}$,

$$\|\rho(Y(t))\| \le A_\rho + B_\rho \|Y(t)\|, \tag{5.21}$$

$$\left\|\frac{\partial \rho(Y(t))}{\partial y_i}\right\| \le \bar{A}_1 + \bar{A}_2\|Y(t)\| + \bar{A}_3\|Y(t)\|^2 + \bar{A}_4\|Y(t)\|^3, \tag{5.22}$$

where $\bar{A}_i$ for $i \in \{1,2,3,4\}$ are some nonnegative constants, A_ρ and B_ρ are given by

$$A_\rho = \frac{2(A_b + r)B_b}{b_\sigma} + \frac{(A_b + r)^2}{b_\sigma K}, \qquad B_\rho = \frac{B_\sigma^2}{b_\sigma}$$

with $K = \min\{y_{i0}e^{-\lambda_i T}, i = 1, ..., h\}$. Then, based on an idea as used in Benth **at al.** [4], we can prove Lemma 5.2 by the following four steps.

First, by direct calculation, we know that $P(t, y)$ is finite for any $(t, y) \in [0, T] \times R_c^h$, i.e.,

$$P(t,y) \le \exp\left(K_1(T-t) + B_\rho \sum_{i=1}^{h} \frac{y_i}{\lambda_i}\right) < \infty, \tag{5.23}$$

where the nonnegative constant K_1 is given by

$$K_1 = A_\rho + \sum_{i=1}^{h} \lambda_i \int_{z_i > 0} \left(e^{\frac{B_\rho z_i}{\lambda_i}} - 1\right) \nu_i(dz_i).$$

Second, we prove that $P \in C^{0,1}\left([0,T] \times R_c^h, R^1\right)$ and the mapping $(t, y) \to \frac{\partial P}{\partial y_i}(t, y)$ for each $i \in \{1, ..., h\}$ is continuous.. The continuity of $P(\cdot, y)$ for each $y \in R_c^h$ can be shown as follows. Owing to the condition (2.4) and the fact (5.12), we know that

$$\exp\left(\int_t^T \rho(Y^{t,y}(s))ds\right) \le \exp\left(A_\rho T + \sum_{i=1}^{h} \frac{B_\rho}{\lambda_i}(y_i + L_i(\lambda_i T))\right). \tag{5.24}$$

By (2.10) and (5.9), we know that the function on the right-hand side of (5.24) is integrable for each fixed $y \in R_c^h$. Then, it follows from the Lebesgue's dominated convergence theorem that $P(t, y)$ for each y is continuous in terms of $t \in [0, T]$.

Next, we show that $\frac{\partial P}{\partial y_i}(t, \cdot)$ with $i \in \{1, ..., h\}$ for all $t \in [0, T]$ exist and are continuous. In fact, consider an arbitrary but fixed point y and take a compact set $U \subset R_c^h$ such that y is in the interior of U. Note that all points in U can be assumed to be bounded by some positive constant M. Thus, by (5.22), (5.2), (5.8) and (5.7), we have, for all $s \ge t$,

$$\left|\frac{\partial}{\partial y_i}\rho(Y^{t,y}(s))\right| \le \left(\sum_{i=1}^{4} \bar{A}_i\right) e^{3hM + 3\sum_{i=1}^{h} L_i(\lambda_i T)}, \tag{5.25}$$

where $Y^{t,y}(s)$ denotes the process with the initial value y at time t. Owing to (2.10) and (5.9), the function on the right-hand side of (5.25) is integrable. Thus, it follows from Theorem 2.27(b) in Folland [19] that the partial derivative of $\int_t^T \rho(Y^{t,y}(s))ds$ in terms of y_i for each $i \in \{1, ..., h\}$ exists. Hence, we have

$$\left|\frac{\partial}{\partial y_i}\left(e^{\int_t^T \rho(Y^{t,y}(s))ds}\right)\right| \le T\left(\left(\sum_{i=1}^{4} \bar{A}_i\right) e^{\left(A_\rho T + 3hM + B_\rho \sum_{i=1}^{h} \frac{1}{\lambda_i}\right) + \sum_{i=1}^{h}\left(3 + \frac{B_\rho}{\lambda_i}\right) L_i(\lambda_i T)}\right). \tag{5.26}$$

Again, by (2.10) and (5.9), we know that the function on the right-hand side of (5.26) is integrable. Therefore, by Theorem 2.27(b) in Folland [19], we can conclude that $P(t, y)$ is differentiable with respect to $y \in R_c^h$. Furthermore, by (5.2), (5.26) and the Lebesgue's dominated convergence theorem, we obtain that the mapping $(t, y) \to \frac{\partial P}{\partial y_i}(t, y)$ for each $i \in \{1, ..., h\}$ is continuous. Hence, $P(t, y) \in C^{0,1}\left([0, T] \times R_c^h, R^1\right)$.

Third, we prove the square-integrable property (5.20) to be true. In fact, it follows from condition (2.10) that $\nu_i(\cdot)$ $(i \in \{1, ..., h\})$ is a σ-finite measure since $\nu_i([\epsilon, \infty)) < \infty$ for any $\epsilon > 0$. In addition, it is easy to see that the nonnegative function $|P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))|^2$ is a measurable one on the product space $[0, T] \times R_c^h \times \Omega$. Hence, by the mean value theorem, (5.25), (5.26), the Jensen's inequality, and the differentiability of $P(t, y)$ in y, we have

$$
\begin{aligned}
(5.27)\quad & E\left[\int_0^T \int_{z_i>0} |P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))|^2 \nu_i(dz_i)dt\right] \\
\leq\ & K_3 K_4 \left(e^{(6+\frac{2B_\rho}{\lambda_i})} \int_{0<z_i<1} z_i^2 \nu_i(dz_i) + \int_{z_i \geq 1} \left(e^{(8+\frac{2B_\rho}{\lambda_i})z_i} - 1\right) \nu_i(dz_i) + \int_{z_i \geq 1} \nu_i(dz_i)\right) \\
<\ & \infty,
\end{aligned}
$$

where K_3 and K_4 are some positive constants. Furthermore, it follows from (5.23), (5.8), and (2.10) that (5.19) is true.

Fourth, we prove that $P(t, y)$ satisfies the IPDE (5.17). In fact, for each $t \in [0, T)$, it follows from the time-homogeneity of Y that

$$
(5.28)\qquad g(T-t, y) \equiv E_{0,y}\left[e^{-\int_0^{T-t} \rho(Y(s))ds}\right] = E_{t,y}\left[e^{-\int_t^T \rho(Y(s))ds}\right] = P(t, y).
$$

Since $P(t, y) \in C^{0,1}\left([0, T] \times R_c^h\right)$, it follows from the Itô's formula (see, e.g., Theorem 1.14 and Theorem 1.16 in pages 6-9 of ∅ksendal and Sulem [38]) that, for each fixed t,

$$
\begin{aligned}
(5.29)\qquad & g(T-t, Y^{0,y}(l)) \\
=\ & g(T-t, y) - \sum_{i=1}^h \lambda_i \int_0^l Y_i^{0,y_i}(s^-) \frac{\partial g}{\partial y_i}(T-t, Y^{0,y}(s^-))ds \\
& + \sum_{i=1}^h \int_0^l \int_{z_i>0} (g(T-t, Y^{0,y}(s^-) + z_i e_i) - g(T-t, Y^{0,y}(s^-)))N_i(\lambda_i ds, dz_i).
\end{aligned}
$$

Furthermore, let $\hat{g}(t, z_i, \omega) \equiv g(T-t, Y^{0,y}(s^-, \omega) + z_i e_i) - g(T-t, Y^{0,y}(s^-), \omega))$ for each $z_i \in (0, \infty)$, $i \in \{1, ..., h\}$ and $\omega \in \Omega$. Then, $\hat{g}$ is $\{\mathcal{F}_t\}$-predictable. Thus, owing to (5.20) (here we need to use an arbitrary but fixed y to replace y_0), it follows from Theorem 4.2.3 in Applebaum [1] (or the explanation in page 61-62 of Ikeda and Watanabe [28]) that the last term in (5.29) is a semimartingale. Thus, taking expectations on both sides of (5.29), we get

$$
\begin{aligned}
& \frac{E[g(T-t, Y^{0,y}(l))] - g(T-t, y)}{l} \\
=\ & \sum_{i=1}^h \frac{\lambda_i}{l} \int_0^l E\left[Y_i^{0,y_i}(s^-) \frac{\partial g}{\partial y_i}(T-t, Y^{0,y}(s^-))\right] ds \\
& - \sum_{i=1}^h \frac{\lambda_i}{l} \int_0^l \int_{z_i>0} E[g(T-t, Y^{0,y}(s^-) + z_i e_i) - g(T-t, Y^{0,y}(s^-))]\nu_i(dz_i)ds.
\end{aligned}
$$

Then, by letting $l \downarrow 0$, we know that $P(t,\cdot)$ is in the domain of the infinitesimal generator of Y, which is denoted by $\mathcal{A}$, that is,

$$\begin{aligned}(5.30) \qquad \mathcal{A}g(T-t,y) &= \sum_{i=1}^{h} \lambda_i y_i \frac{\partial}{\partial y_i} g(T-t,y) \\ &\quad - \sum_{i=1}^{h} \lambda_i \int_{z_i>0} (g(T-t,y+z_i e_i) - g(T-t,y))\nu_i(dz_i).\end{aligned}$$

Now, by (5.23), we see that $g(T-t,y) = P(t,Y^{0,y}(l)) \in L^2(\Omega,P)$ for each $t \in [0,T)$ and all l in a neighborhood of zero such that $t-l \leq T$. Thus, we have

$$\begin{aligned}(5.31) \qquad E_{0,y}[g(T-t,Y(l))] &= E_{0,y}\left[E_{0,Y(l)}\left[e^{-\int_0^{T-t}\rho(Y(s))ds}\right]\right] \\ &= E_{0,y}\left[E_{0,y}\left[e^{-\int_0^{T-t}\rho(Y(s+l))ds}\middle|\mathcal{F}_l\right]\right] \\ &= E_{0,y}\left[e^{-\int_l^{T-t+l}\rho(Y(s))ds)}\right] \\ &= E_{0,y}\left[e^{-\int_0^{T-t+l}\rho(Y(s))ds}e^{\int_0^l \rho(Y(s))ds)}\right],\end{aligned}$$

where the second equality in (5.31) follows from the Markov property of Y (e.g., Proposition 7.9 in Kallenberg [32]). Then, we have

$$\begin{aligned}(5.32) \qquad & \frac{E_{0,y}[g(T-t,Y(l))] - g(T-t,y)}{l} \\ &= \frac{1}{l}E_{0,y}\left[e^{-\int_0^{T-t+l}\rho(Y(s))ds}\left(e^{\int_0^l \rho(Y(s))ds} - 1\right)\right] + \frac{g(T-t+l,y) - g(T-t,y)}{l}.\end{aligned}$$

Now, by the fundamental theorem of calculus, as $l \downarrow 0$ and a.s., we have

$$(5.33) \qquad e^{-\int_0^{T-t+l}\rho(Y^{0,y}(s))ds}\left\{\frac{1}{l}\left(e^{\int_0^l \rho(Y^{0,y}(s))ds} - 1\right)\right\} \to \rho(y)e^{-\int_0^{T-t}\rho(Y^{0,y}(s))ds}.$$

Furthermore, by the mean-value theorem, we have

$$(5.34) \qquad \frac{1}{l}\left|e^{\int_0^l \rho(Y^{0,y}(s))ds} - 1\right| \leq \sup_{l\in[0,T]}\left|\rho(Y^{0,y}(l))e^{\int_0^l \rho(Y^{0,y}(s))ds}\right|.$$

Since the function in the left-hand side of (5.33) is uniformly bounded by an integrable function, it follows from the dominated convergence theorem that the right-derivative of $g(T-\cdot,y)$ at t exists and satisfies

$$(5.35) \qquad \mathcal{A}g(T-t,y) = \rho(y)g(T-t) + \frac{\partial g}{\partial t}(T-t,y).$$

Hence, by (5.28) and (5.35), we know that $P(t,y)$ satisfies (5.17). In addition, we have

$$|P(t,y+z_i\delta_{ij}) - P(t,y)| \leq K_5 E\left[e^{3\sum_{j=1}^h (2L_j(\lambda_j T)+z_i\delta_{ij})}\left(e^{\sum_{j=1}^h\left(\frac{B_\rho}{\lambda_j}(2L_j(\lambda_j T)+z_i\delta_{ij})\right)}\right)\right] z_i.$$

where K_5 is some positive constant. Thus, by the Lebesgue's dominated convergence theorem, we can conclude that

$$\int_{z_i>0} |P(t, y + z_i e_i) - P(t, y)| \nu_i(dz_i)$$

is continuous in t. Therefore, it follows from (5.17) that $\frac{\partial P}{\partial t}(t, y)$ is continuous in $t \in [0, T)$, which implies that $P \in C^{1,1}\left([0, T) \times R_c^h, R^1\right)$. Hence, we complete the proof of Lemma 5.2. $\square$

Lemma 5.3 Let $O(t) \equiv P(t, Y(t))$ defined in (3.4). Then, under conditions C1, C2, and (2.10), O is a $(0, 1]$-valued semimartingale with $O(T) = 1$. Furthermore, define

$$K \equiv \mathcal{L}(O) \equiv \left(\frac{1}{O_-}\right) \cdot O \ \text{ with } \ K(0) = 0 \ \text{ and } \ O_-(t) \equiv O(t^-). \tag{5.36}$$

Then, K is an $\{\mathcal{F}_t\}$-semimartingale and has the following canonical decomposition

$$dK(t) \equiv d\mathcal{L}(O)(t) = \rho(Y(t^-))dt + \sum_{i=1}^{h} \int_{z_i>0} F(t, z_i)\tilde{N}_i(\lambda_i dt, dz_i), \tag{5.37}$$

where, $F(t, z_i, \omega))$ is defined in (3.10).

Proof. First, we show that O is an $\{\mathcal{F}_t\}$-semimartingale. In fact, it follows from the Ito's formula (see, e.g., Theorem 1.14 and Theorem 1.16 in pages 6-9 of Øksendal and Sulem [38]) and Lemma 5.2 that

$$\begin{aligned} O(t) &= P(0, y_0) + \int_0^t \rho(Y(s^-))P(s, Y(s^-))ds \\ &\quad + \sum_{i=1}^{h} \int_0^t \int_{z_i>0} (P(s, Y(s^-) + z_i e_i) - P(s, Y(s^-)))\tilde{N}_i(\lambda_i ds, dz_i). \end{aligned} \tag{5.38}$$

Then, by Lemma 5.2 and the claim in pages 61-62 of Ikeda and Watanabe [28], we know that the third term in the right-hand side of (5.38) is an $\{\mathcal{F}_t\}$-martingale. Furthermore, by (5.21) and the similar proof as used for Lemma 5.1, we know that the second term on the right-hand side of (5.38) is of finite variation a.s. Hence, we get that O is an $\{\mathcal{F}_t\}$-semimartingale. Thus, it follows from (5.38) and the definition of $K(t)$ that (5.37) is true.

Second, M^K defined as follows is an $\{\mathcal{F}_t\}$-martingale,

$$M^K(t) = \sum_{i=1}^{h} \int_0^t \int_{z_i>0} F(s, z_i)\tilde{N}_i(\lambda_i ds, dz_i). \tag{5.39}$$

In fact, by the mean-value theorem, (5.21), (5.2) (2.10), and the fact that $\nu_i(\cdot)$ ($i \in \{1, ..., h\}$) is a σ-finite measure since $\nu_i([\epsilon, \infty)) < \infty$ for any $\epsilon > 0$, we have

$$E\left[\int_0^T \int_{z_i>0} |F(t, z_i)|^2 \nu(dz_i)dt\right] < \infty. \tag{5.40}$$

Thus, it follows from (5.40) and the claims in pages 61-62 of Ikeda and Watanable [28] that M^K is an $\{\mathcal{F}_t\}$-martingale. Therefore, we can conclude that K is an $\{\mathcal{F}_t\}$-semimartingale. Hence, Lemma 5.3 is true. □

Lemma 5.4 **Let b^D and c^D be the drift and the covariance matrix processes associated with D, b^K is the drift process associated with K. Then, under conditions C1, C2, and (2.10), we have**

$$b^K = \left(b^D\right)' \left(c^D\right)^{-1} b^D. \tag{5.41}$$

Furthermore, the process a defined in (3.5) satisfies the following relationship,

$$a \equiv \left(c^D\right)^{-1} b^D. \tag{5.42}$$

PROOF. First of all, it follows from Lemma 5.1 and Lemma 5.3 that

$$b^D(t) = (D_1(t)(b_1(Y(t^-)) - r), ..., D_d(t)(b_d(Y(t^-)) - r))', \tag{5.43}$$
$$c^D(t) = \text{diag}(D(t)) \left(\sigma(Y(t^-))\sigma(Y(t^-))'\right) \text{diag}(D(t)), \tag{5.44}$$
$$b^K(t) = O^{-1}(t)b^O(t) = \rho(Y(t^-)). \tag{5.45}$$

Then, by simple calculations, we know that (5.41) and (5.42) are true. Hence, we complete the proof of Lemma 5.4. □

For convenience, we will use $C^D_{ij} \equiv [D_i, D_j]$ to denote the co-quadratic variation processes with $i, j \in \{1, ..., d\}$ for the process D and write interchangeably $c^{D_i D_j} \equiv c^D_{ij}$ and $c^{D_i} = c^D_{ii}$. Furthermore, similar notations are also used for other processes related in the following discussions.

Lemma 5.5 **Under conditions C1, C2, and (2.10), $\hat{Z}$ is an $\{\mathcal{F}_t\}$-and P-martingale.**

PROOF. First, we show that $a \in L(D)$. In fact, it follows from the condition **C1**, (5.2), and (5.1) that $\|Y(t^-))\| \geq \min\{y_{i0}e^{-\lambda_i T}, i = 1, ..., h\} > 0$ for any $t \in [0, T]$. Then, for $m, n \in \{1, ..., d\}$, we have

$$\begin{aligned} \bar{\rho}(Y(t^-)) &\equiv \sum_{m=1}^{d} \left(B(Y(t^-))' \left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)^{-1}\right)^2_m \sum_{n=1}^{d} \sigma^2_{mn}(Y(t^-)) \\ &\leq C_{\bar{\rho}} + \frac{B_b^2 B_\sigma}{b_\sigma^2} \|Y(t^-)\|, \end{aligned} \tag{5.46}$$

where $C_{\bar{\rho}}$ is some positive constant. Thus, it follows from the Kunita-Watanable inequality (e.g., Theorem 25 in page 69 of Protter [41]) that

$$\begin{aligned} E\left[\sum_{m=1}^{d}\sum_{n=1}^{d}\left|\int_0^T a_m(t)a_n(t)d\left[M^D_m, M^D_n\right](t)\right|\right] &\leq d^2 E\left[\int_0^T \bar{\rho}(Y(t^-))dt\right] \\ &< \infty, \end{aligned} \tag{5.47}$$

where a_m and M_m^D with $m \in \{1, ..., d\}$ are the mth components of a and M^D respectively. Furthermore, it follows from (5.1) that

$$E\left[\sum_{m=1}^{d}\int_0^T a_m(t)D_m(t)B_m(Y(t^-))dt\right] = E\left[\int_0^T \rho(Y(t^-))dt\right] < \infty. \tag{5.48}$$

Then, by (5.47)-(5.48), Definition 6.17 of page 207, Definition 4.3 of page 180, Definition 6.12 of page 206, and Definition 2.6 of page 76 in Jacod and Shiryaev [30], we know that $a \in L(D)$. Thus, $(a \cdot D)(T)$ is well defined.

In addition, it follows from Theorem 4.5(a) in page 180 of Jacod and Shiryaev [30] that, for each $u \in L(D)$, we have,

$$(u \cdot D)(t) = \lim_{k\to\infty}\sum_{i=1}^{d}\int_0^t u_i(s)I_{\{\|u(s)\|\leq k\}}dM_i^D(s) + \sum_{i=1}^{d}\int_0^t u_i(s)dB_i^D(s), \tag{5.49}$$

where the limit in the first term on the right-hand side of (5.49) corresponds to the convergence in probability uniformly on every compact set of $[0, T]$. Therefore, by (5.1), (2.10), (5.14)-(5.15), (5.49), and the Lebesgue dominated convergence theorem, we know that

$$(a \cdot D)(T) = \sum_{m=1}^{d}\int_0^T a_m(t)dD_m(t). \tag{5.50}$$

Now, it follows from Lemma 5.3 that O is a semimartingale. Thus, it follows from Conditions **C1**, **C2** and (5.50) that $(a \cdot D)$ is also a semimartingale. Then, by Corollary 8.7(b) and equation 8.19 in pages 135-138 of Jacod and Shiryaev [30], we have that

$$\begin{aligned} \hat{Z}(t) &= \mathcal{E}(K - (a \cdot D) - [K, (a \cdot D)])(t) \\ &= \mathcal{E}\left(M^K - (a \cdot M^D) + (b^K - a'b^D) \cdot A\right)(t) \\ &= \mathcal{E}(G)(t), \end{aligned} \tag{5.51}$$

where the second equality follows from the facts that $A(t) = t$, $K(0) = 0$ and the independence among driving Brownian motions and Lévy processes. The third equality follows from Lemma 5.4. Furthermore, M^K and M^D are given by (5.39) and (5.15), which are $\{\mathcal{F}_t\}$-martingales. Hence,

$$G \equiv M^K - \left(a \cdot M^D\right) \tag{5.52}$$

is also an $\{\mathcal{F}_t\}$-martingale. Thus, it follows from Theorem 4.61 in page 59 of Jacod and Shiryaev [30] that $\hat{Z}$ is an $\{\mathcal{F}_t\}$-local martingale.

Second, we prove that $\hat{Z}$ is of class (D), i.e., the set of random variables

$$\{\hat{Z}(\tau), \tau \text{ is finite valued } \{\mathcal{F}_t\} - \text{stopping times}\}$$

is uniformly integrable (e.g., Definition 1.46 in page 11 of Jacod and Shiryaev [30]).

In fact, consider an arbitrary finite-valued $\{\mathcal{F}_t\}$-stopping time $\tau \leq T$ and an arbitrary constant $\gamma > 0$. Then, we have

$$E\left[\left|\hat{Z}(\tau)\right| I_{\{|\hat{Z}(\tau)|\geq\gamma\}}\right] \leq \frac{1}{P(0,y_0)}\left(E\left[(\mathcal{E}(-a\cdot D)(\tau))^2\right]\right)^{1/2}\left(P\{|\hat{Z}(\tau)|\geq\gamma\}\right)^{1/2}, \tag{5.53}$$

where we have used the facts that $0 < O(\cdot) \leq 1$ and D is continuous. Furthermore, let

$$U_1(t) = \int_0^t \rho(Y(s^-))ds, \tag{5.54}$$

$$U_2(t) = \sum_{n=1}^{d}\int_0^t \bar{B}_n(Y(s^-))dW_n(s), \tag{5.55}$$

where $\bar{B}(Y(s^-))$ is defined in (3.8). Hence, $U_2(t)$ is a continuous $\{\mathcal{F}_t\}$-martingale. Thus,

$$\begin{aligned} E\left[\left(\mathcal{E}\left(-a\cdot D\right)(\tau)\right)^2\right] &= E\left[e^{(-2(U_1(\tau)+U_2(\tau))-[U_1+U_2,U_1+U_2](\tau))}\right] \\ &\leq E\left[e^{-2U_2(\tau)}\right] \\ &\leq \left(E\left[e^{8[U_2,U_2](T)}\right]\right)^{\frac{1}{2}} \\ &< \infty, \end{aligned} \tag{5.56}$$

where the third inequality follows from the optional sampling theorem, the fact that $e^{-2U_2(t)}$ is a submartingale by the Jensen's inequality, and Theorem 39 in page 138 of Protter [41]. The last inequality follows from conditions **C1-C2**. Therefore, it follows from (5.56) that $\sup_\tau E\left[\left|\hat{Z}(\tau)\right|\right] \leq K_1$, where K_1 is some positive constant. Thus, by the Markov's inequality, we have that

$$P\{|\hat{Z}(\tau)|\geq\gamma\} \leq \frac{K_1}{\gamma} \to 0 \ \text{ as } \gamma\to\infty \tag{5.57}$$

for all stopping time $\tau \leq T$. Therefore, it follows from (5.53)-(5.57) that $\hat{Z}$ is of class (D). Hence, it follows from (5.51) and Proposition 1.47(c) in page 12 of Jacod and Shiryaev [30] that $\hat{Z}$ is a uniformly integrable $\{\mathcal{F}_t\}$- and P-martingale. □

Lemma 5.6 **Under conditions C1, C2, and (2.10), Q^* is an equivalent martingale measure.**

PROOF. First, we use P_t to denote the restriction of P to $\mathcal{F}_t$ for each $t \in [0,T]$. Then, we define $dQ_t^* \equiv \hat{Z}(t)dP_t$ and $dQ^* \equiv \hat{Z}(T)dP$. Owing to (3.3)-(3.6), we know that $\hat{Z}(t) > 0$ for each $t \in [0,T]$. Furthermore, note that $\hat{Z}$ is a $\{\mathcal{F}_t\}$- and P-martingale. Hence, it follows from the discussion in page 166 of Jacod and Shiryaev [30] that Q^* is equivalent to P with the density process $\hat{Z}$.

Next, we show that D is an Q^*-martingale. In fact, since D is an P-semimartingale with the decomposition given in (5.1), it follows from Girsanov-Meyer Theorem (e.g., Theorem

35 in page 132 of Protter [41]) that D is also an Q^*-semimartingale with the decomposition $D = \tilde{D} + \bar{D}$. The process $\bar{D}$ is an Q^*-finite variation process. For each $m \in \{1, ..., d\}$,

$$
\begin{aligned}
(5.58) \quad \tilde{D}_m(t) &= M_m^D(t) - \int_0^t \frac{1}{\hat{Z}(s)} d\left[\hat{Z}, M_m^D\right](s) \\
&= M_m^D(t) - \sum_{n=1}^{d} \int_0^t \frac{D_m(s)\sigma_{mn}(Y(s^-))}{\hat{Z}(s)O_0} d\left[O\mathcal{E}(-a \cdot D), W_n\right]^c(s) \\
&= M_m^D(t) - \sum_{n=1}^{d} \int_0^t \frac{D_m(s)\sigma_{mn}(Y(s^-))}{\hat{Z}(s)O_0} \mathcal{E}(-a \cdot D(s)) \Big(d\left[O, W_n\right]^c(s) \\
&\quad + O(s) d\left[U, W_n\right]^c(s) + \frac{1}{2} d[[O, U]^c, W_n]^c(s)\Big) \\
&= M_m^D(t) - \sum_{n=1}^{d} \int_0^t D_m(s)\sigma_{mn}(Y(s^-)) d\left[U, W_n\right]^c(s) \\
&= M_m^D(t) + \sum_{r=1}^{d}\sum_{n=1}^{d} \int_0^t D_m(s)\sigma_{mn}(Y(s^-)) a_r(s) d\left[D_r, W_n\right]^c(s) \\
&= M_m^D(t) + \int_0^t \sum_{n=1}^{d} D_m(s)\sigma_{mn}(Y(s^-)) \bar{B}_n(Y(s^-)) ds,
\end{aligned}
$$

where $\bar{B}_n(Y(s^-))$ is defined in (3.9). The second equality in (5.58) follows from Theorem 29 in page 75 of Protter [41], the proof of Corollary in page 83 of Protter [41], the fact that W is continuous, Theorem 4.52 in page 55 of Jacod and Shiryaev [30], and the explanation in page 70 of Protter [41]. The third equality in (5.58) follows from the Ito's formula for multi-dimensional semimartingales (e.g., Theorem 33 in pages 81-82 of Protter [41]), and the associated function f is taken to be $f(O, U) = Oe^U$. Furthermore, a_r is the rth component of a, and U is defined by $U(t) \equiv -a \cdot D(t) - \frac{1}{2}[a \cdot D, a \cdot D](t)$. Thus, we have

$$
(5.59) \qquad \bar{D}(t) = D(t) - \tilde{D}(t) = \bar{s} \equiv (s_1, ..., s_d)' \text{ or } D(t) = \tilde{D}(t) + \bar{s},
$$

where s_i for each $i \in \{1, ..., d\}$ is the initial price as given in (2.2).

Therefore, to show that D is an Q^*-martingale, it suffices to show that $\tilde{D}$ is an Q^*-martingale. More precisely, by the last equation in the proof of Theorem 35 in pages 132-133 of Protter [41], we have that

$$
(5.60) \qquad \tilde{D}_m(t) = \left(M_m^D(t) - \frac{1}{\hat{Z}(t)}\left[\hat{Z}, M_m^D\right](t)\right) + \int_0^t \left[\hat{Z}, M_m^D\right](s^-) d\left[\frac{1}{\hat{Z}}\right](s).
$$

Then, we can show that the both terms on the right-hand side of (5.60) are Q^*-martingales.

For the first term on the right-hand side of (5.60), it follows from integration by parts (e.g., equations (*) and (**) in page 132 of Protter [41]), the Ito's formula (e.g., Theorem

1.14 and Theorem 1.16 in pages 6-9 of Øksendal and Sulem [38]), and Lemma 5.2 that

$$
\begin{aligned}
(5.61)\quad & \left(M_m^D(t) - \frac{1}{\hat{Z}(t)}\left[\hat{Z}, M_m^D\right](t)\right)\hat{Z}(t) \\
= \; & \int_0^t \hat{Z}(s^-)dM_m^D(s) + \int_0^t M_m^D(s)d\hat{Z}(s) \\
= \; & \int_0^t \hat{Z}(s^-)dM_m^D(s) - \sum_{n=1}^{d}\int_0^t M_m^D(s)\hat{Z}(s^-)\bar{B}_n(Y(s^-))dW_n(s) \\
& + \sum_{i=1}^{h}\int_0^t\int_{z_i>0} \frac{M_m^D(s)\mathcal{E}((-a\cdot D)(s))}{O_0}(P(s,Y(s^-)+z_ie_i) - P(s,Y(s^-)))\tilde{N}_i(\lambda_i ds, dz_i),
\end{aligned}
$$

where $\bar{B}_n(Y(s^-))$ is defined in (3.9). The second equality follows from (5.39)-(5.15) and the fact that

$$
d\hat{Z}(t) = \hat{Z}(t^-)dG(t) \tag{5.62}
$$

owing to (5.51)-(5.52), the definition of Doléans-Dade exponential, and Theorem 37 in pages 84-85 of Protter [41].

Then, we can show that each of the three terms on the right-hand side of (5.61) is an Q^*-martingale.

The claim that the first term on the right-hand side of (5.61) is a Q^*-martingale can be proved as follows. First, it follows from the similar argument as used in (5.64) that M^D is a square integrable P-martingale. Second, by the Tonelli's Theorem (e.g., Theorem 20 in page 309 of Royden [43]) and the Hölder's inequality, we have

$$
\begin{aligned}
(5.63)\quad & E\left[\int_0^T \hat{Z}^2(s)d\left[M_m^D, M_m^D\right](s)\right] \\
\leq \; & \bar{K}\int_0^T \left(E\left[O^8(s)\right]\right)^{1/2}\left(E\left[(\mathcal{E}(-a\cdot D)(s))^{16}\right]\right)^{1/4}\left(E\left[D_m^{16}(s)\right]\right)^{1/4} ds \\
< \; & \infty,
\end{aligned}
$$

where $\bar{K}$ is some positive constant. The last inequality in (5.63) follows from the similar arguments as in (5.56) and (5.13). Thus, it follows from Theorem 4.40(b) in page 48 of Jacod and Shiryaev [30] that the first term on the right-hand side of (5.61) is an $\{\mathcal{F}_t\}$- and P-martingale.

The claim that the second term on the right-hand side of (5.61) is an Q^*-martingale can be proved as follows. It follows from (5.13) and Exercise 3.25 in page 163 of Karatzas and Shreve [34] that

$$
E\left[\int_0^t \left(M_m^D(s)\right)^{16} ds\right] < \infty. \tag{5.64}
$$

Then, by (5.64), the Hölder's inequality and the similar method as used in (5.63), we know that the second term on the right-hand side of (5.61) is an $\{\mathcal{F}_t\}$- and P-martingale.

The claim that the third term on the right-hand side of (5.61) is an Q^*-martingale can be proved as follows. It follows from the Tonelli's Theorem (e.g., Theorem 20 in page 309 of Royden [43]) that

$$
\begin{aligned}
(5.65)\quad & E\left[\int_0^T \int_{z_i>0} \frac{|M_m^D(t)|}{O_0} \left|(P(t,Y(t^-)+z_i e_i) - P(t,Y(t^-)))\mathcal{E}((-a\cdot D)(t))\right| \nu_i(dz_i)dt\right] \\
\leq\ & K_1 \left(E\left[\int_0^T \int_{z_i>0} \sup_{\xi(Y(t^-))\in[0,z_i]} \left|\frac{\partial P(t,Y(t^-)+\xi(Y(t^-))e_i)}{\partial y_i}\right|^2 z_i \nu_i(dz_i)dt\right]\right)^{\frac{1}{2}} \\
<\ & \infty,
\end{aligned}
$$

where K_1 is some positive constant. The inequalities in (5.65) follow from the similar proofs as used in (5.56), (5.64), the Hölder's inequality, the proof of (5.27), and the fact that

$$
\begin{aligned}
\int_{z_i>0} z_i \nu(dz_i) &\leq \int_{0<z_i<1} z_i \nu_i(dz_i) + \int_{z_i\geq 1} z_i \nu_i(dz_i) < \infty. \\
&= \int_{0<z_i<1} z_i \nu_i(dz_i) + \int_{z_i\geq 1} (e^{z_i}-1) + \int_{z_i\geq 1} \nu_i(dz_i) \\
&< \infty.
\end{aligned}
$$

Then, it follows from (5.65) and the argument in pages 61-62 in Ikeda and Watanable [28] that the third term on the right-hand side of (5.61) is also an $\{\mathcal{F}_t\}$- and P-martingale.

Therefore, by summarizing the discussions for the three terms on the right-hand side of (5.61), we know that the process given by (5.61), is an $\{\mathcal{F}_t\}$- and P-martingale. Moreover, by applying Proposition 3.8(a) in page 168 of Jacod and Shiryaev [30], we can conclude that the first term on the right-hand side of (5.60) is an Q^*-martingale.

For the second term on the right-hand side of (5.60), we can show that it is also an $\{\mathcal{F}_t\}$- and Q^*-martingale. In fact, since $\hat{Z}$ is a density process of Q^* in terms of P and $\left(\frac{1}{\hat{Z}}\right)\hat{Z} = 1$ (that is an P-martingale), it follows from Proposition 3.8(a) in page 168 of Jacod and Shiryaev [30] that $\frac{1}{\hat{Z}}$ is an Q^*-martingale. Furthermore, it follows from the Ito's formula (e.g., Theorem 32 in page 78 of Protter [41]), (5.62) and the calculation of $d\hat{Z}(t)$ in the last equality in (5.61) that

$$
\begin{aligned}
(5.66)\quad d\left(\frac{1}{\hat{Z}(t)}\right) &= \frac{1}{\hat{Z}(t^-)} \sum_{n=1}^d \left(\bar{B}_n(Y(t^-))\right)^2 dt - \frac{1}{\hat{Z}(t^-)} \sum_{n=1}^d \bar{B}_n(Y(t^-)) dW_n(t) \\
&\quad - \sum_{i=1}^h \int_{z_i>0} \frac{F(t,z_i)}{\hat{Z}(t)} \tilde{N}_i(\lambda_i dz_i, dt),
\end{aligned}
$$

where $\bar{B}(Y(t^-))$ is defined in (3.9). Thus, it follows from (5.66) that $\frac{1}{\hat{Z}}$ is squarely integrable under Q^*, i.e.,

$$
(5.67)\, E_{Q^*}\left[\left(\frac{1}{\hat{Z}(t)}\right)^2\right] \leq E_{Q^*}\left[\sup_{0\leq s\leq T} \frac{1}{\hat{Z}^2(s)}\right]
$$

$$
\begin{aligned}
&\leq 4E_{Q^*}\left[\frac{1}{\hat{Z}^2(T)}\right] \\
&\leq 4\left(E\left[\hat{Z}^2(T)\right]\right)^{\frac{1}{2}}\left(E\left[\frac{1}{\hat{Z}^4(T)}\right]\right)^{\frac{1}{2}} \\
&= \frac{4}{O_0}\left(E\left[(\mathcal{E}((-a\cdot D)(T)))^2\right]\right)^{\frac{1}{2}}\left(E\left[\frac{1}{(\mathcal{E}((-a\cdot D)(T)))^4}\right]\right)^{\frac{1}{2}} \\
&< \infty,
\end{aligned}
$$

where the second inequality in (5.67) follows from the Doob's martingale inequality (e.g., Theorem 2.1.5 in page 74 of Applebaum [1]) since $\frac{1}{\hat{Z}}$ is an Q^*-martingale. The last inequality of (5.67) follows from the similar argument as in (5.56).

Therefore, to show that the second term on the right-hand side of (5.60) is an Q^*-martingale, it suffices to show that the following expectation under Q^* is finite owing to (5.67) and Theorem 4.40(b) in page 48 of Jacod and Shiryaev [30],

$$
\begin{aligned}
&E_{Q^*}\left[\int_0^T \left(\left[\hat{Z}, M_m^D\right](s^-)\right)^2 d\left[\frac{1}{\hat{Z}}, \frac{1}{\hat{Z}}\right](s)\right] \\
&= E_{Q^*}\left[\int_0^T \left(\left[\hat{Z}, M_m^D\right]^c(s^-)\frac{1}{\hat{Z}(s^-)}\right)^2 \sum_{n=1}^d \left(\bar{B}_n(Y(s^-))\right)^2 ds\right] \\
&\quad + \sum_{i=1}^h E_{Q^*}\left[\int_0^T \int_{z_i>0} \left(\left[\hat{Z}, M_m^D\right]^c(s^-)\frac{F(s,z_i)}{\hat{Z}(s)}\right)^2 \lambda_i\nu_i(dz_i)ds\right].
\end{aligned} \tag{5.68}
$$

The first term on the right-hand side of (5.68) is finite since

$$
\begin{aligned}
&E_{Q^*}\left[\int_0^T \frac{1}{\hat{Z}^2(s^-)}\left(\left[\hat{Z}, M_m^D\right]^c(s^-)\right)^2 \bar{\rho}(Y(s^-))ds\right] \\
&\leq K_1\left(\frac{4}{3}E\left[\left(\frac{1}{\mathcal{E}(-a\cdot D)(T)}\right)^3\right]\right)^{\frac{1}{2}}\left(\frac{20}{19}E\left[(\mathcal{E}(-a\cdot D)(T))^{20}\right]\right)^{\frac{1}{8}}\left(\int_0^T E\left[D_m^8(s)\right]ds\right)^{\frac{1}{8}} \\
&< \infty,
\end{aligned} \tag{5.69}
$$

where K_1 is some positive constant. The first inequality in (5.69) follows from the Doob's martingale inequality (e.g., page 74 of Applebaum [1]). The second inequality in (5.69) follows from the similar arguments as in (5.53) and (5.13). Similarly, the second term on the right-hand side of (5.68) is also finite, which can be proved along the line of the discussion as in (5.69).

Thus, it follows from the finiteness of (5.68) that the second term on the right-hand side of (5.60) is an Q^*-martingale. Therefore, by combining this fact with (5.60) and (5.61), we know that $D = \tilde{D} + \bar{s}$ displayed in (5.59) is an Q^*-martingale (i.e. Q^* is an equivalent martingale measure). Finally, by applying the similar discussion as used in (5.63), we conclude that $\frac{dQ^*}{dP} \in L^2(P)$, which implies that $Q^* \in \mathcal{U}_2^e(D)$. $\square$

Lemma 5.7 **Under conditions C1, C2, and (2.10), Q^* is the VOMM.**

PROOF. It suffices to justify that all conditions stated in Theorem 3.25 of Cěrný and Kallsen [7] are satisfied. First of all, for any stopping time τ, we can show that

$$u^{\tau}(t) \equiv a(t)I_{(\tau,T]}(t)\mathcal{E}\left((-aI_{(\tau,T])})\cdot D\right)(t^-) \in \bar{\Theta}(D). \tag{5.70}$$

In fact, it follows from the proof of Lemma 5.6 that $\mathcal{U}_2^e(D)$ is nonempty. Furthermore, since D is a continuous P-semimartingale, it is sufficient to prove that the three equivalent conditions stated in Theorem 2.1 of Cěrný and Kallsen [8] are satisfied for (5.70), which can be done by tedious computations similarly as before. In addition, we can show that $O\mathcal{E}((-aI_{(\tau,T]})\cdot D)$ is of class (D). Therefore, by combining this claim with Lemma 5.4, (5.70), and Theorem 3.25 in Cěrný and Kallsen [7], we know that O and a are the opportunity and adjustment processes in the sense defined Section 3 of [7]. Thus, it follows from Proposition 3.13 in Cěrný and Kallsen [7] that Q^* is the VOMM. Hence, we complete the proof of Proposition 5.2. $\square$

5.3 The Unique Existence of Solution to A Type of BSDEs

Consider the following q-dimensional BSDE with jumps and a terminal condition H

$$\begin{aligned} V(t) &= H-\int_t^T g\left(s,V(s^-),\bar{V}(s),\tilde{V}(s,\cdot),Y(s^-)\right)ds-\int_t^T\sum_{i=1}^{d}\bar{V}_i(s)dW_i(s) \\ &\quad -\int_t^T\sum_{i=1}^{h}\int_{z_i>0}\tilde{V}_i(s,z_i)\tilde{N}_i(\lambda_i ds,dz_i), \end{aligned} \tag{5.71}$$

where $H\in L^2_{\mathcal{F}_T}(\Omega,R^q,P)$, $\bar{V}=\left(\bar{V}_1,...,\bar{V}_d\right)\in R^{q\times d}$, $\tilde{V}=\left(\tilde{V}_1,...,\tilde{V}_h\right)\in R^{q\times h}$, g is a random function: $[0,T]\times R^q\times R^{q\times d}\times L^2_\nu(R^h_+,R^{q\times h})\times R^h\times\Omega\to R^h$ and

$$L^2_\nu(R^h_+,R^{q\times h})\equiv\left\{\tilde{v}:R^h_+\to R^{q\times h},\sum_{i=1}^{h}\int_{z_i>0}\left|\tilde{v}_i(z_i)\right|^2\nu_i(dz_i)<\infty\right\}. \tag{5.72}$$

Furthermore, for any $\tilde{v}\in L^2_\nu(R^h_+,R^{q\times h})$, the associated norm is defined by

$$\|\tilde{v}\|_\nu\equiv\left(\sum_{i=1}^{h}\int_{z_i>0}\left|\tilde{v}_i(z_i)\right|^2\lambda_i\nu_i(dz_i)\right)^{\frac{1}{2}}. \tag{5.73}$$

Proposition 5.3 **Replacing $H\in L^4_{\mathcal{F}_T}(\Omega,R,P)$ by $H\in L^2_{\mathcal{F}_T}(\Omega,R,P)$ in Assumption 3.1. Supposing that $g(t,v,\bar{v},\tilde{v},Y(t^-))$ is $\{\mathcal{F}_t\}$-adapted for any given $(v,\bar{v},\tilde{v})\in R^q\times R^{q\times d}\times L^2_\nu(R^h_+,R^{q\times h})$ with**

$$g(\cdot,0,0,,0,Y(\cdot^-))\in L^2_{\mathcal{F}}\left([0,T],R^q\right) \tag{5.74}$$

such that

$$\left\|\left(g\left(t, v, \bar{v}, \tilde{v}, Y(t^-)\right) - g\left(t, u, \bar{u}, \tilde{u}, Y(t^-)\right)\right) I_{\{t\leq\tau_n\}}\right\| \leq K_n\left(\|u-v\| + \|\bar{u}-\bar{v}\| + \|\tilde{u}-\tilde{v}\|_\nu\right) \tag{5.75}$$

for any $(u, \bar{u}, \tilde{u})$ **and** $(v, \bar{v}, \tilde{v}) \in R^q \times R^{q\times d} \times L^2_\nu(R^h_+, R^{q\times h})$**, where** K_n **depending on** n **are positive constants. Then, the BSDE in (5.71) has a unique solution**

$$(V, \bar{V}, \tilde{V}) \in L^2_{\mathcal{F}}([0,T], R^q, P) \times L^2_{\mathcal{F},p}([0,T], R^{q\times d}, P) \times L^2_p([0,T], R^{q\times h}, P), \tag{5.76}$$

where V **is a càdlàg process. The uniqueness is in the sense: if there exists another solution** $(U, \bar{U}, \tilde{U})$ **as required, then,**

$$E\left[\int_0^T \left(\|U(t)-V(t)\|^2 + \|\bar{U}(t)-\bar{V}(t)\|^2 + \|\tilde{U}(t,\cdot)-\tilde{V}(t,\cdot)\|^2_\nu\right) dt\right] = 0. \tag{5.77}$$

Proof. First, for each $n \in \{1, 2, ...\}$, we define

$$\tau_n \equiv \inf\{t > 0, \|L(\lambda t)\| > n\}. \tag{5.78}$$

Then, it follows from Theorem 3 in page 4 of Protter [41] and condition (2.10) that $\{\tau_n\}$ is a sequence of nondecreasing $\{\mathcal{F}_t\}$-stopping times and satisfies $\tau_n \to \infty$ a.s. as $n \to \infty$ since

$$P\{\tau_n \leq t\} = P\{\|L(\lambda t)\| > n\} \leq \frac{E\left[\|L(\lambda t)\|^2\right]}{n^2} \to 0$$

as $n \to \infty$ for any given $t \in [0, \infty)$, where we have used (2.10), (5.9), (5.7), and the fact that $L(\lambda t)$ is a h-dimensional nonnegative and nondecreasing càdlàg process.

Second, for each n, consider the following BSDE with a random terminal time $\sigma_n \equiv T \wedge \tau_n$ and a terminal condition H_{τ_n},

$$\begin{aligned} V(t) &= H_{\tau_n} - \int_{t\wedge\sigma_n}^{\sigma_n} g\left(s, V(s^-), \bar{V}(s), \tilde{V}(s,\cdot), Y(s^-)\right) ds \\ &\quad - \int_{t\wedge\sigma_n}^{\sigma_n} \sum_{i=1}^d \bar{V}_i(s) dW_i(s) - \int_{t\wedge\sigma_n}^{\sigma_n} \sum_{i=1}^h \int_{z_i>0} \tilde{V}_i(s, z_i)\tilde{N}_i(\lambda_i ds, dz_i). \end{aligned} \tag{5.79}$$

Then, by slightly generalizing the discussion as in Yong and Zhou [53] and Tang and Li [51] (see also El Karoui **et al.** [35], Situ [49], Yin and Mao [52] for related discussions), we know that (5.79) has a unique adapted solution as required over $[0, \sigma_n]$.

Third, for each $n \in \{1, 2, ...\}$, let $\Omega_n = \{\omega \in \Omega : \sigma_n(\omega) = T\}$. Since σ_n is a sequence of nondecreasing stopping times and $\sigma_n \to T$ a.s. as $n \to \infty$, we have that $\Omega = \cup_{n=1}^\infty \Omega_n$ and $\Omega_l \subseteq \Omega_n$ whenever $l \leq n$. Now, we use $\Pi^n(t, z) \equiv (V^n(t), \bar{V}^n(t), \tilde{V}^n(t, z))$ for $t \leq \sigma_n$ and $z \in R^h_+$ to denote the unique solution to (5.79) for each n. Since $H_{\tau_n}(\omega) = H(\omega)$ for all $\omega \in \{\omega : \tau_n(\omega) \geq T\}$, we know that $\Pi^n(t, z) = \Pi^{n-1}(t, z) = ... = \Pi^l(t, z)$ for all $t \leq \sigma_l(\omega)$, a.s. $\omega \in \Omega_l$ and any $z \in R^h_+$. By the continuity of probability, we know that, for any given

$\epsilon > 0$, there exists a sufficiently large $n_0 > 0$ such that $P\{\Omega_n\} > 1 - \epsilon$ when $n > n_0$. Thus, for any given $\delta > 0$ and for all $n, l > n_0$, we have

$$P\left\{\sup_{0\leq t\leq T, z\in R_+^h}\left\|\Pi^n(t\wedge\sigma_n, z) - \Pi^l(t\wedge\sigma_l, z)\right\| > \delta\right\} < \epsilon,$$

that is, $\{\Pi^n(\cdot\wedge\sigma_n,\cdot), n \in \{1, 2, ...\}\}$ is uniformly Cauchy in probability. Thus, it is uniformly convergent in probability to a process $\Pi = \{\Pi(t, z), t \in [0, T], z \in R_+^h\}$. Therefore, we can extract a subsequence from $\{\Pi^n(\cdot\wedge\sigma_n,\cdot), n \in \{1, 2, ...\}\}$ such that the convergence holds uniformly a.s. Hence, we can conclude that Π is a solution to (5.71) and have all the properties as stated in the proposition. Furthermore, assume that $\Pi' = \{\Pi'(t, z), t \in [0, T], z \in R_+^h\}$ is another solution to (5.71). Then, we can conclude that, for all $n \geq l$, $\Pi'(t, z, \omega) = \Pi^l(t, z, \omega)$ for all $t \in [0, T]$, $z \in R_+^h$, and almost all $\omega \in \Omega_l$. In fact, if the claim fails to be true for some $n \geq l$, define $\Pi_n''(t, z, \omega) = \Pi'(t, z, \omega)$ for $\omega \in \Omega_l$ and $\Pi_n''(t, z, \omega) = \Pi^n(t, z, \omega)$ for $\omega \in \Omega_l^c$. Then, Π_n'' and Π^n are distinct solutions to (5.79) with the same terminal condition H_{τ_n}, which contradicts the uniqueness of solution to (5.79). Then, $P\{\Pi(t, z) = \Pi'(t, z)$ for all $t \in [0, T], z \in R_+^h\} = 1$ follows from a straightforward limiting argument as above. Furthermore, by applying the similar argument as used for Definition 2.4 and its associated remark in page 57 of Ikeda and watanabe [28], we know that Π is the unique solution to (5.71) (interested readers are also referred to pages 309-310 of Applebaum [1] for some related discussion). Hence, we complete the proof of Proposition 5.3. □

5.4 Remaining Proof of Theorem 3.1

First of all, by the Hölder's inequality and the similar calculation as for (5.56), we have that

$$\begin{aligned} E\left[(H\mathcal{E}(-a\cdot D)(T))^2\right] &\leq \left(E\left[H^4\right]\right)^{\frac{1}{2}}\left(E\left[(\mathcal{E}(-a\cdot D)(T))^4\right]\right)^{\frac{1}{2}} \\ &< \infty. \end{aligned} \tag{5.80}$$

Thus, it follows from the Jensen's inequality that the process $X = \{X(t), t \in [0, T]\}$ with

$$X(t) \equiv E\left[H\mathcal{E}(-a\cdot D)(T)|\mathcal{F}_t\right] \tag{5.81}$$

is a square-integrable martingale. Thus, by the Martingale representation theorem (e.g., Lemma 2.3 in Tang and Li [51]), we have

$$X(t) = X(0) + \sum_{j=1}^{d}\int_0^t \bar{X}_j(s)dW_j(s) + \sum_{i=1}^{h}\int_0^t\int_{z_i>0}\tilde{X}_i(s, z_i)\tilde{N}_i(\lambda_i ds, dz_i) \tag{5.82}$$

with $\bar{X} = (\bar{X}_1, ..., \bar{X}_d)' \in L^2_{\mathcal{F},p}([0, T], R^d, P)$ and $\tilde{X} = (\tilde{X}_1, ..., \tilde{X}_h)' \in L^2_p([0, T], R^h, P)$. Furthermore, it follows from the Bayes' rule (e.g., Lemma 8.6.2 in page 160 of ∅ksendal [39]) and Proposition 5.2 that

$$X(t) = O_0 E\left[H\hat{Z}(T)|\mathcal{F}_t\right] = O_0\hat{Z}(t)V(t), \tag{5.83}$$

where $V(t)$ is defined in (3.19). Thus, by the integration by parts formula (e.g., Corollary 2 in page 68 of Protter [41]), and (5.82)-(5.83), we have

$$
\begin{aligned}
& dV(t) \\
= \;& \frac{1}{O_0}\left(X(t^-)d\left(\frac{1}{\hat{Z}(t)}\right)+\frac{1}{\hat{Z}(t^-)}dX(t)+d\left[X,\frac{1}{\hat{Z}}\right](t)\right) \\
= \;& g(t,V(t^-),\bar{V}(t),\tilde{V}(t,\cdot),Y(t^-)))dt \\
& +\sum_{i=1}^{d}\bar{V}_i(t)dW_i(t)+\sum_{i=1}^{h}\int_{z_i>0}\tilde{V}_i(t,z_i)\tilde{N}_i(\lambda_i dz_i,dt),
\end{aligned} \tag{5.84}
$$

where g is defined in (3.11) and

$$
\begin{aligned}
\bar{V}_i(t) &= -V(t^-)\bar{B}_i(Y(t^-))+\frac{\bar{X}_i(t)}{O_0\hat{Z}(t^-)}, && \text{for } i=1,...,d,\\
\tilde{V}_i(t,z_i) &= -V(t^-)F(t,z_i)\bar{Z}(t)+\frac{\tilde{X}_i(t,z_i)}{O_0\hat{Z}(t^-)}, && \text{for } i=1,...,h
\end{aligned}
$$

with $\bar{Z}$ given by (3.8). Hence, by (5.84), we know that V satisfies the BSDE (3.12).

Next, we check that $g(t,v,\bar{v},\tilde{v},Y(t^-))$ defined in (3.11) satisfies the conditions as stated in Proposition 5.3. In fact, from (3.11), we see that $g(t,v,\bar{v},\tilde{v},Y(t^-))$ is $\mathcal{F}_t$-adapted for any given $(v,\bar{v},\tilde{v})\in R\times R^{1\times d}\times L^2_{\nu}(R^h_+,R^{1\times h})$ with $g(t,0,0,0,Y(t^-))\equiv 0\in L^2_{\mathcal{F}}([0,T],R,P)$, Furthermore, for the sequence of nondecreasing stopping times $\{\tau_n, n=1,2,...\}$ as defined in (5.78), we have

$$
\left|\bar{Z}(t)\right| I_{\{t\leq\tau_n\}}\leq \bar{K}_n e^{\sum_{i=1}^{h}\frac{2B_\rho}{\lambda_i}\|L(\lambda t)\|}I_{\{t\leq\tau_n\}}\leq\tilde{K}_n,
$$

where $\bar{K}_n$ and $\tilde{K}_n$ are positive constants depending on n. In addition, it follows from the proof of (5.40) that

$$
\left(\int_{z_i>0}(F(t,z_i))^2\nu_i(dz_i)\right)I_{\{t\leq\tau_n\}}\leq\bar{L}e^{\sum_{i=1}^{h}(6+\frac{4B_\rho}{\lambda_i})\|L(\lambda t)\|}I_{\{t\leq\sigma_n\}}\leq\tilde{L}_n,
$$

where $\bar{L}$ is some positive constant and $\tilde{L}_n$ is a positive constant depending on n. Therefore, for any $(u,\bar{u},\tilde{u})$, $(v,\bar{v},\tilde{v})\in R\times R^{1\times d}\times L^2_{\nu}(R^h_+,R^{1\times h})$, we have

$$
\begin{aligned}
& \left\|(g(t,u,\bar{u},\tilde{u},Y(t^-))-g(t,v,\bar{v},\tilde{v},Y(t^-)))I_{\{t\leq\tau_n\}}\right\| \\
\leq\;& h\tilde{K}_n^2\tilde{L}_n\|u-v\|+\|\bar{u}-\bar{v}\|\left(\frac{1}{2}(\rho(Y(t^-))+d)\right)I_{\{t\leq\tau_n\}}+h\lambda_i\tilde{K}_n\left(\tilde{L}_n\right)^{\frac{1}{2}}\|\tilde{u}-\tilde{v}\|_\nu \\
\leq\;& K_n\left(\|u-v\|+\|\bar{u}-\bar{v}\|+\|\tilde{u}-\tilde{v}\|_\nu\right),
\end{aligned}
$$

where K_n is some positive constant depending on n and in the last inequality, we have used (5.21). Thus, all conditions stated in Proposition 5.3 are satisfied, which implies that (3.12) has a unique adapted solution.

Now, for each $t \in [0,T]$ and $B^K(t) = \int_0^t \rho(Y(s^-))ds$, we define the density process

$$Z^{P^*}(t) \equiv \frac{O(t)}{O_0 \mathcal{E}(B^K)(t)}. \tag{5.85}$$

Then, the corresponding probability $P^* \sim P$. Thus, it is the opportunity-neutral probability measure in the sense of Definition 3.16 in Cěrný and Kallsen [7]. Furthermore, by Corollary 8.7(b) and equation (8.19) in pages 135-138 of Jacod and Shiryaev [30], we can rewrite Z^{P^*} in (5.85) as

$$Z^{P^*}(t) \;=\; \mathcal{E}(K)(t)\mathcal{E}\left(-B^K\right)(t) = \mathcal{E}\left(M^K\right)(t) \tag{5.86}$$

for each $t \in [0,T]$, where K is defined in (5.36) and M^K is defined in (5.39). Then, by a similar method as used in the proof of Proposition 5.2(2), we know that Z^{P^*} is a bounded positive martingale. Thus, for each pair of $i,j \in \{1,...,d\}$ and $t \in [0,T]$, we have

$$\langle D_i, D_j\rangle^{P^*}(t) = [D_i, D_j]^{P^*}(t) = [D_i, D_j](t) = \int_0^t \tilde{c}_{ij}^{D^*}(s)ds, \tag{5.87}$$

where the first equality in (5.87) is owing to the continuity of D, Theorem 5.52 in page 55 of Jacod and Shiryaev [30], Theorem 4.47(c) in page 52 of Jacod and Shiryaev [30], the equivalence between P^* and P, and Girsanov-Meyer Theorem in page 132 of Protter [41]. The second equality follows from Theorem 4.47(a) in page 52 of Jacod and Shiryaev [30] since Z^{P^*} is bounded and Girsanov-Meyer Theorem in page 132 of Protter [41]. Furthermore, $\tilde{c}_{ij}^{D^*}$ in the last equality is defined in (3.16).

Now, note that D is continuous. Then, by Theorem 4.52 in page 55 of Jacod and Shiryaev [30] (or the proof of Corollary in page 83 of Protter [41]), we know that $[D_i, V](t)$ and $[D_i, V]^c(t)$ for each $i \in \{1,...,d\}$ under P or P^* have the same compensator. Hence, we have

$$\langle D_i, V\rangle^{P^*}(t) = \left(\langle D_i, V\rangle^c\right)^{P^*}(t) = \left([D_i, V]^c\right)^{P^*}(t) = [D_i, V]^c(t) = \int_0^t \tilde{c}_i^{DV^*}(s)ds, \tag{5.88}$$

where $\tilde{c}_i^{DV^*}$ is defined in (3.17). The last equality of (5.88) follows from Theorem 4.47(a) in page 52 of Jacod and Shiryaev [30] and the fact that

$$\begin{aligned} V(t) &= V(0) + \int_0^t g(s, V(s^-), \bar{V}(s), \tilde{V}(s,\cdot), Y(s^-))ds \\ &\quad + \int_0^t \sum_{i=1}^d \bar{V}_i(s)dW_i(s) + \int_0^t \sum_{i=1}^h \int_{z_i>0} \tilde{V}_i(s,z_i)\tilde{N}_i(\lambda_i dz_i, ds). \end{aligned}$$

Then, it follows from (5.87)-(5.88), Definition 4.6, and equation (4.8) in Cěrný and Kallsen [7] that (3.15) is true.

Finally, the unique existence of solution to (3.18) is owing to Theorem 6.8 in Jacod [29] and the proofs of Lemma 4.9 and Theorem 4.10 in Cěrný and Kallsen [7]. Thus, by Theorem 4.10 in Cěrný and Kallsen [7], we know that the mean-variance hedge strategy is given by (3.14). Hence, we complete the proof of Theorem 3.1. □

6 Conclusion

In this paper, we prove the global risk optimality of the hedging strategy explicitly constructed for an incomplete financial market. Owing to the discussions in Pigorsch and Stelzer [50] and references therein, our discussion in this paper can be extended to the cases that the external risk factors in (2.3) are correlated in certain manners. For the simplicity of notation, we keep the presentation of the paper in the current way. Furthermore, our study in this paper establishes the connection between our financial system and existing general semimartingale based study in Cĕrný and Kallsen [7] since we can overcome the difficulties in Cĕrný and Kallsen [7] by explicitly constructing the process N and the VOMM Q^*. In addition, our objective and discussion in this paper are different from the recent study of Jeanblanc **et al.** [31] since the authors in Jeanblanc **et al.** [31] did not aim to derive any concrete expression. Nevertheless, interested readers may make an attempt to extend the study in Jeanblanc **et al.** [31] and apply it to our financial market model to construct the corresponding explicit results. Finally, unlike the studies in Hubalek **et al.** [27] and Kallsen and Vierthauer [33], our option H is generally related to a multivariate terminal function and hence a BSDE involved approach is employed. Interested readers may take an attempt to study whether the Laplace transform related method developed in Hubalek **et al.** [27] and Kallsen and Vierthauer [33] for single-variate terminal function can be extended to our general multivariate case.

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