Brownian Approximations for Queueing Networks with Finite Buffers: Modeling, Heavy Traffic Analysis and Numerical Implementations

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Brownian Approximations for Queueing Networks with Finite Buffers: Modeling, Heavy Traffic Analysis and Numerical Implementations

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Dedication

to

my parents and family

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Summary

This dissertation is concerned with the performance analysis of queueing networks under different blocking schemes (communication blocking or buffer overflow). Brownian models (semimartingale reflecting Brownian motions) are proposed for approximate analysis of the queueing networks. The approximations are justified by heavy traffic limit theorems. A general numerical algorithm via finite element method is implemented to compute the stationary distribution of a semimartingale reflecting Brownian motion in a d-dimensional box. Brownian estimates of the performance measures are presented numerically. Comparisons with known results are given to show the effectiveness of the Brownian models and the algorithm. These performance measures include long-run average throughput rate of the system, long-run average queue length and the long-run average blocking (or loss) rate at each station.

Motivated by applications in communication networks, manufacturing systems and computer architectures, our focus is on modeling of queueing networks with finite buffers. Concretely, we deal with queueing networks of d single server stations. Each station has a finite capacity waiting buffer, and all customers served at a station are homogeneous in terms of service requirements and routings. When a communication blocking scheme is used in the network, we assume that the network has feedforward structure. When a loss scheme is used, the network is allowed to have feedback. We show that the properly normalized d-dimensional queue length process converges weakly to a d-dimensional reflected Brownian motion (RBM) in a rectangular box under a heavy traffic condition. In addition to the usual requirement that the external arrival rate is close to the service rate at each station, the heavy traffic condition requires that the buffer size at each station is in the order of $1/(1-\rho_i)$, where ρ_i is the traffic intensity at station i. The techniques used in existing heavy traffic

limit theorems do *not* apply here because the solution to our Skorohod problem is not unique. Our proof relies heavily on a uniform dominated oscillation result for solutions to the Skorohod problems. We show that any limiting process is an SRBM of the type defined in Taylor and Williams [38]. We also use results of Dai and Williams [16] on existence and uniqueness of semimartingale RBM in a general polyhedral domain. Our theorems provide a solid foundation for using Brownian models to estimate performance measures of the networks.

To make practical use of SRBM's approximate models of queueing networks, we present a general implementation via finite element method to compute the stationary distribution of SRBM. We compare the numerical results from our algorithm with known analytical results for SRBM, and also employ the implementation to estimate the performance measures of several illustrative finite buffer networks. All the numerical comparisons show that our Brownian estimates give reasonably accurate estimates.

CHAPTER 1

Introduction

Queueing network models with *finite buffers* provide powerful and realistic tools for performance evaluation of discrete flow systems such as communication networks, manufacturing systems and computer architectures . Despite of a growing literature on the performance analysis of this type of networks, there is still no viable analytical method for predicting performances of such networks. In this dissertation, we propose Brownian system models for open queueing networks with finite buffers under different blocking schemes. We further justify the Brownian approximations by proving so called *heavy traffic limit theorems*. Finally, we employ the finite element method to implement an algorithm to numerically compute the stationary distribution of the Brownian models.This implementation provides a practical computer tool for performance analysis of queueing networks with finite buffers.

The queueing network under study has d single server stations with first-in-first-out (FIFO) service discipline at each station. Associated with each station there is a waiting buffer that has finite size. We assume that all customers visiting a station are homogeneous in terms of service requirements and routings. Such a network is so called a *single class* network in literature because at each station there is one customer class. The network is of the type described in the pioneering paper of Jackson [28] with the following extensions: (a) the service times at each station are independent, identically distributed (i.i.d.) with a *general* distribution; (b) the interarrival times associated with each arrival stream are i.i.d. with a general distribution; (c) the buffer size at each station is *finite*. Such extensions are important for applications because non-exponential service time distributions and finite waiting rooms are common place in telecommunication networks and manufacturing systems, see Gerla and etc [19], Kroner and etc [30], Nikolaidis and Akyildiz [32].

It is the finite buffer restriction that distinguishes this work from others in literature. When a buffer is full, different blocking schemes (communication blocking or buffer overflow) can be employed. When communication blocking is used, we further assume that the network has a feedforward routing structure. An example of 2-station tandem network under communication blocking is pictured in Figure 1.1. When the buffer at station 2 is full, server at station 1 stops working although a customer may still occupy station 1. When the buffer at station 1 is full, the external arrival stream to station 1 is turned off.

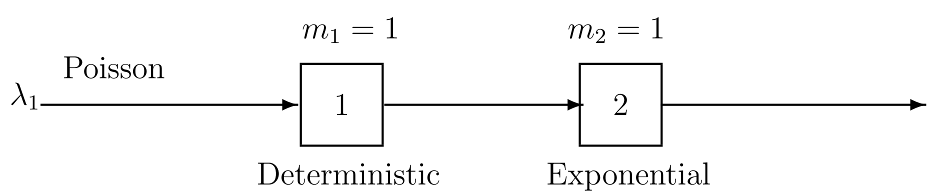

Figure 1.1: A tandem network under communication blocking

For such a type of finite buffer queueing networks, one is interested in the performance measures like the average work-in-process (WIP) level and percentage of time that a station is blocked. Until now, there is no good analytical tool for predicting such performance measures accurately and efficiently. In this dissertation, we propose to use a d-dimensional semimartingale reflected Brownian motion (SRBM) in a rectangular box to approximate the d-dimensional queue length process. We then employ the finite element method to implement a general algorithm of Dai and Harrison [12] for computing the stationary distribution of the SRBM. These computational tools lead to estimates of the performance measures of the queueing network.

Given a $d \times d$ positive definite matrix Γ, a d-dimensional vector θ and $d \times 2d$ matrix R (whose i^{th} column is denoted by v_i). A d-dimensional continuous stochastic process Z is said to be an SRBM in a d-dimensional box $\mathbf{S}$ associated with data (Γ, θ, R) if (see section 2.2 for a more precise definition),

1. $Z(t) = X(t) + \sum_{i=1}^{2d} v_i Y_i(t)$ for all $t \geq 0$;
2. Z has paths in $\mathbf{S}$;
3. $X = \{X(t)\}$ is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ;
4. for $i = 1, ..., 2d$, $Y_i(0) = 0$, Y_i is non-decreasing and $Y_i(\cdot)$ can increase only at times t such that $Z(t)$ is on the boundary of $\mathbf{S}$.

This definition suggests that the SRBM Z behaves like an ordinary Brownian motion with drift vector θ and covariance matrix Γ in the interior of box $\mathbf{S}$. When Z hits the boundary of box $\mathbf{S}$, the process $Y_i(\cdot)$ increases, causing an overall pushing in the direction of v_i. The magnitude of the pushing is the minimal amount required to keep Z inside the box $\mathbf{S}$.

For example, consider the network pictured in Figure 1.1. Let $Q_i(t)$ $(i = 1, 2)$ be the number of customers at time t at station i. The Poisson arrival stream to station 1 has average rate λ_1. Service times at station 1 are deterministic with mean 1. Service times at station 2 are exponentially distributed with mean 1. The buffer size at each station is 25. Then $Q(t)$ can be approximated by an SRBM with data

$\Gamma = \gamma I$, $\theta = (\lambda - 1, 0)$. The reflection matrix R is given by

$$R = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}. \tag{0.1}$$

The i^{th} column v_i of R is the reflection direction on face F_i as shown in Figure 1.2, where γ is the system throughput rate and I is the 2×2 identity matrix. The algorithm described in Chapter 5 will compute the Brownian estimates of long run average queue length and throughput rate.

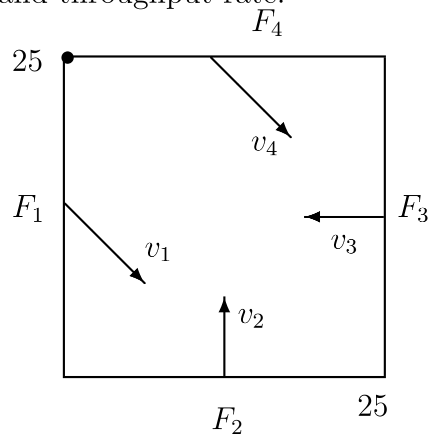

Figure 1.2: Reflection on an 2-dimensional state space

The Brownian approximation described in the previous paragraph is justified by a heavy traffic limit theorem. The theorem says that under a *heavy traffic condition* the properly scaled d-dimensional queue length process will converge in distribution to a d-dimensional SRBM. Let ρ_i be *traffic intensity* at station i. It is the product of mean service time and effective arrival rate (resulting from external arrivals as well as internal transitions) at station i. One can interpret ρ_i as the long-run fraction of time that server i is busy, or server utilization at station i. The heavy traffic condition requires that the traffic intensity ρ_i at station i is close to one. and the buffer size at station i is in the order of $1/(1 - \rho_i)$. As a consequence, the theorem suggests that if the buffer size is in the order of $1/(1 - \rho_i)^2$, one essentially will not "see" finite buffer effects. This insight provides some qualitative description as to when we can assume the network has infinite buffers.

One can trace back heavy traffic analysis of this type to Iglehart and Whitt [26, 27] which treat single station, multi-server queueing systems under FIFO discipline, Harrison [22] that deals with tandem queueing systems. Their heavy traffic limits were given as a complicated function of multidimensional Brownian motion. Harrison [23] again considered tandem queueing networks, and introduced reflected Brownian motion on the nonnegative orthant as the diffusion limit for the first time. These results are extended by Reiman [35], who proves a theorem for networks of Jackson form with the exponential distributions replaced by general ones. Johnson [29] generalized Reiman's result to a network with two customer types, one of which has preemptive-resume priority over the other at all stations. Chen and Shanthikumar [7] extended Reiman's result to networks in which stations may have multiple servers. Peterson [34] proved an analogous result for multiclass network in which the routing is deterministic and feedforward. For multiclass network with feedback, Reiman [36] proved a theorem to justify the approximation of the workload process by a one-dimensional RBM, and the proof due to Reiman was subsequently simplified and generalized by Dai and Kurtz [15]. Similar progress has been made in the area of diffusion approximations for single class closed queueing network with Markovian routing, see Chen and Mandelbaum [5, 6].

All of the works above heavily depend on the uniqueness of the solutions to their Skorohod problems. In other words, their Skorohod problems require more strict constraints on their reflection matrices, see Harrison and Reiman [24], Dupuis and Ishii [17]. However, the uniqueness fails in many Skorohod problems that come up in multiclass queueing networks with feedback and finite buffer queueing networks, including the network pictured in Figure 1.1, see Bernard and El Kharroubi [1], Mandelbaum [31].

Due to the non-uniqueness property of the Skorohod problem, the techniques in our proof of heavy traffic limit theorems differ from the ones used in existing limit theorems (Reiman [35, 36], Peterson [34]). The method involves two novel ideas. The first idea is to prove a *uniform oscillation theorem* and the other is to prove a martingale property for the limiting process. Combining these two results with the existence and uniqueness of an SRBM in a general polyhedron (Dai and williams [16]), we finish the proof.

Roughly speaking, for a d-dimensional box $\mathbf{S}$ and a matrix R, a pair of functions (z, y) is called a solution to a Skorohod problem associated with $(\mathbf{S}, R)$ if for a given d-dimensional function x, we have (an exact definition will be introduced in section 3.2)

1. $z(t) = x(t) + Ry(t) \in \mathbf{S}$ for $t \geq 0$,
2. for each i, y_i is nondecreasing with $y_i(0) = 0$, and y_i can increase only at times t for which $z(t)$ reaches the boundary F_i.

To state the uniform oscillation theorem, let $(z^n(\cdot), y^n(\cdot))$ be a solution to a Skorohod problem associated with $x^n(\cdot)$ on a rectangular state space $\mathcal{S}^n$ with reflection matrix R^n for $n = 1, 2, ...$ Assume that $\mathbf{S}^n$ is a sequence of rectangular boxes and $R^n \to R$ as $n \to \infty$. Suppose that at each corner of $\mathbf{S}^n$, a corresponding $d \times d$ matrix obtained from R is completely-S as defined in Taylor and Williams [38]. Then there is a constant C such that for any $0 \leq t_1 < t_2$,

$$\mathrm{Osc}(z^n(\cdot), [t_1, t_2]) \leq C \max\{\mathrm{Osc}(x^n(\cdot), [t_1, t_2]), \Gamma^n\},$$

$$\mathrm{Osc}(y^n(\cdot), [t_1, t_2]) \leq C \max\{\mathrm{Osc}(x^n(\cdot), [t_1, t_2]), \Gamma^n\},$$

where, for a function $f(\cdot)$ and an interval $[t_1, t_2]$, $\mathrm{Osc}(f(\cdot), [t_1, t_2]) = \sup_{t_1 \leq s \leq t \leq t_2} |f(t) - f(s)|$, and Γ^n is the largest jump size of $y^n(\cdot)$, which is fixed for each n if the station number is fixed.

To state the martingale property for the limiting process, let $Z^n(\cdot)$ be the scaled queue length process and $Y^n(\cdot)$ be the vector of scaled cumulative blocking time and idle time processes, or scaled cumulative loss and idle time processes. Then $(Z^n(\cdot), Y^n(\cdot))$ is a solution to the Skorohod problem corresponding to a process $X^n(\cdot)$. We show that $\{(X^n(\cdot), Z^n(\cdot), Y^n(\cdot))\}$ has a subsequence converging to $(X(\cdot), Z(\cdot), Y(\cdot))$, where $X(\cdot)$ is a Brownian motion with drift vector θ and some covariance matrix. We need to show that $Z(\cdot)$ is an SRBM corresponding to $X(\cdot)$. A key to the proof is to show the following martingale property: $\{X(t) - \theta t, t \geq 0\}$ is a martingale with respect to the filtration generated by $X(\cdot)$ and $Y(\cdot)$.

CHAPTER 2

Network System Models

In this chapter, we describe queueing network models with finite buffers under different blocking schemes. Their corresponding Brownian approximating models are presented. The Brownian models are rooted from heavy traffic limit theorems, which will be presented in Chapter 4.

2.1 Intree-like Network under Communication Blocking

2.1.1 Queueing Network Model

The first type of queueing network under consideration has d single server stations indexed by $i \in J = \{1, ..., d\}$. The size of the buffer associated with each station i is *finite*. Therefore, at each station i there are at most b_i customers, including the one possibly being served. The network is assumed under first-in-first-out (FIFO) service discipline. Customers visiting station i are homogeneous in terms of service time distribution and routing mechanism. All customers eventually leave the network. Namely, the network is open. An example of a 5-station network is pictured in Figure 2.1. For each station i, let $E_i(t)$ be the number of *external* customer arrivals to station i when the arrival process is *turned on* for t units of time and $S_i(t)$ be the number of customer departures from station i in t units of server i *busy* time. If station i has no

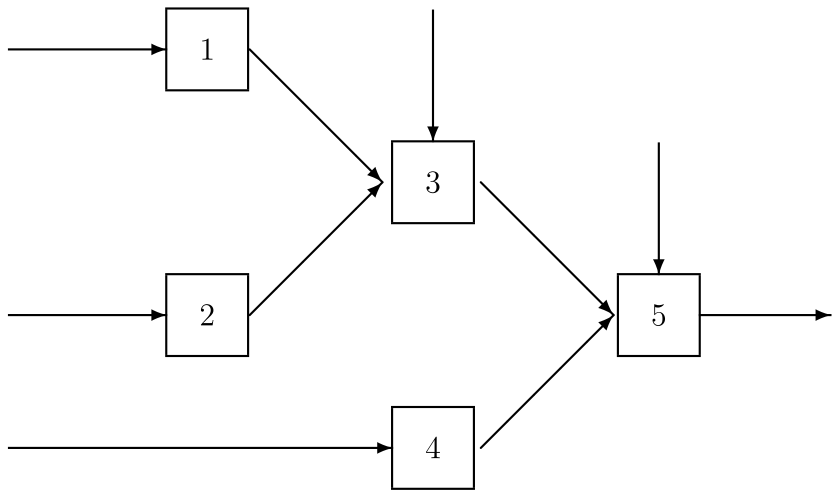

Figure 2.1: A five station intree-like network

external arrivals, $E_i(t) = 0$ for all $t \geq 0$.

For each i, let $\{u_i(k), k \geq 1\}$ and $\{v_i(k), k \geq 1\}$ be i.i.d. random sequences with mean $Eu_i(1) = Ev_i(1) = 1$. Then we put the following assumptions on the processes $E_i(\cdot)$ and $S_i(\cdot)$. Each $E_i(\cdot)$ is associated with an i.i.d. interarrival times sequence $\{\xi_i(k) = (1/\lambda_i)u_i(k), k \geq 1\}$ with mean value $E\xi_i(1) = 1/\lambda_i < \infty$, variance $\sigma_{a,i}^2$ and squared coefficient of variation SCV $c_{a,i}^2 = \lambda_i^2 \sigma_{a,i}^2$. Similarly, $S_i(\cdot)$ is associated an i.i.d. service time sequence $\{\eta_i(k) = (1/\mu_i)v_i(k), k \geq 1\}$ with mean value $m_i = E\eta_i(1) = 1/\mu_i < \infty$, variance $\sigma_{s,i}^2$ and SCV $c_{s,i}^2 = \mu_i^2 \sigma_i^2$. Therefore, $E_i(\cdot)$ and $S_i(\cdot)$ can be denoted by

$$E_i(t) = \sup\left\{k : \sum_{l=1}^{k} \xi_i(l) \leq t\right\}, \tag{1.1}$$

$$S_i(t) = \sup\left\{k : \sum_{l=1}^{k} \eta_i(l) \leq t\right\}. \tag{1.2}$$

We assume that routing is deterministic. That is, customers leave station i will all go next to station $\sigma(i) \in J \equiv \{1, 2, ..d\}$ or leave the system. Due to this routing requirement, we call the network an *intree-like* network, see Figure 2.1.

As mentioned before, an important new feature in the network is that the sizes of buffers are *finite*. When the buffer at station $\sigma(i)$ is full, server i stops working until the buffer $\sigma(i)$ has free space available although a customer may still occupy station i. In the literature of queueing theory, this is called *communication blocking*. In the following section, we will look at other mechanisms in dealing with buffer overflow problem. The blocking in the network introduces new complications in heavy traffic theory.

Let $Q_i(t)$ be the number of customers at station i at time t, including possibly the one being served. Let $Y_i^b(t)$ be the amount of time that buffer i is full in time interval $[0,t]$ and $Y_i^0(t)$ be the amount of time that server i has been idle while server i is not blocked in $[0,t]$. We are interested in estimating performance measures, including the long-run average buffer size

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Q_i(s)ds, \tag{1.3}$$

the long run average time that buffer i is full,

$$\lim_{t\to\infty} \frac{Y_i^b(t)}{t}, \tag{1.4}$$

and the long-run average server utilization rate

$$1 - \lim_{t\to\infty} \frac{Y_i^0(t)}{t} - \lim_{t\to\infty} \frac{Y_{\sigma(i)}^b(t)}{t}. \tag{1.5}$$

We propose that the d-dimensional queue length process $Q = \{Q(t), t \geq 0\}$ be approximated by a reflecting Brownian motion (SRBM) $\tilde{Q} = \{\tilde{Q}(t), t \geq 0\}$ to be defined in the next subsection, where $Q(\cdot)$ is the vector of queue length processes, namely, $Q(\cdot) = (Q_1(\cdot), Q_2(\cdot), ..., Q_d(\cdot))'$. Performance measures like those in (1.3)-(1.5) can be estimated from their Brownian counterparts. Approximating procedure will be justified by a heavy traffic limit theorem in Chapter 4.

2.1.2 Semimartingale Reflecting Brownian Motion

In this subsection, we introduce some standard terminology in the study of reflecting Brownian motion, see Harrison and Reiman [24], Taylor and Williams [38], Dai and

Williams [16]. More specifically, we consider a class of *semimartingale* reflecting Brownian motion (SRBM). Let $\mathbf{S}$ be a d-dimensional box with $2d$ boundary faces as follows,

$$\mathbf{S} \equiv \left\{ x = (x_1, ..., x_d)' \in R^d : \ 0 \le x_i \le b_i, \text{ for } i \in J \right\}. \tag{1.6}$$

$$F_i = \{x \in S : x_i = 0\}, \ F_{i+d} = \{x \in S : x_i = b_i\} \text{ for } i = 1, ..., d. \tag{1.7}$$

Let Γ a $d \times d$ positive definite matrix, θ be a d-dimensional vector and R be a $d \times 2d$ matrix (whose ith column is denoted by v_i).

A triple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ will be called a filtered space if Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω, and $\{\mathcal{F}_t, t \ge 0\}$ is an increasing family of sub-σ-fields of $\mathcal{F}$, i.e., a filtration. If, in addition, $\mathbf{P}$ is a probability measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ is called a filtered probability space.

Definition 2.1.1 *An SRBM associated with the data $(\mathbf{S}, \theta, \Gamma, R)$ that has initial distribution π is a continuous, $\{\mathcal{F}_t\}$-adapted, d-dimensional process Z defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ such that under $\mathbf{P}$,*

$$Z(t) = X(t) + \sum_{i=1}^{2d} v_i Y_i(t) \ \text{ for all } t \ge 0, \tag{1.8}$$

where

1. *Z has continuous paths in S, P-a.s.,*

2. *X is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ such that $\{X(t) - \theta t, \mathcal{F}_t, t \ge 0\}$ is a martingale and $\mathbf{P}X^{-1}(0) = \pi$,*

3. *Y is an $\{\mathcal{F}_t\}$-adapted, $2d$-dimensional process such that $\mathbf{P}$-a.s., for each $i \in \{1, ..., 2d\}$, the i^{th} component Y_i of Y satisfies*

   (a) $Y_i(0) = 0$,

   (b) Y_i is continuous and non-decreasing,

(c) Y_i can increase only when Z is on the face F_i.

An SRBM Z as defined above behaves like a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ in the interior of state space $\mathbf{S}$. When the boundary face F_i is hit, the process Y_i increases, causing an instantaneous displacement of Z in the direction given by v_i, the magnitude of the displacement is the minimal amount of requirement to keep Z always inside $\mathbf{S}$. Therefore, we call Γ, θ and R the covariance matrix, the drift vector and the reflection matrix of Z, respectively. When explicit dependence on an initial distribution π is needed, we use $\mathbf{P}_\pi$ to denote the probability measure. When the initial distribution π is concentrated on a point $x \in \mathbf{S}$, we use $\mathbf{P}_x$ to denote the probability measure.

One can derive parameters Γ, θ and R for different queueing network models. Once these parameters are given, the corresponding Brownian approximating models are derived.

2.1.3 Brownian System Model

In this section, we present a suitable SRBM as an approximating model for the intree-like queueing network discussed in previous section. Let $B_i^0(t)$ be the cumulative amount of time that buffer i is not full during time interval $[0, t]$. As a matter of definition, we have

$$B_i^0(t) = t - Y_i^b(t),$$

where $Y_i^b(t)$ is the amount of time that buffer i is full in the time interval $[0, t]$ as defined before. We model the external arrival processes in the following way. The arrival process at station i is turned on only when the buffer at the station is not full. Therefore $E_i(B_i^0(t))$ is the number of external arrivals to station i by time t.

Recall that customers leaving station i will go next to station $\sigma(i)$. Because of the communication blocking mechanism used, server i is blocked $Y_{\sigma(i)}^b(t)$ units of time in $[0,t]$. Let $B_i(t)$ be the cumulative amount of time that server i is busy in $[0,t]$. we have

$$B_i(t) = t - (Y_i^0(t) + Y_{\sigma(i)}^b(t)).$$

Therefore $S_i(B_i(t))$ is the number of departures from station i by time t. Moreover we can write down the main equation that governs the dynamics of the queue length processes. Namely,

$$Q_i(t) = Q_i(0) + E_i(B_i^0(t)) + \sum_{j\in J,\sigma(j)=i} S_j(B_j(t)) - S_i(B_i(t)),\ i \in J, \tag{1.9}$$

where $Q_i(0)$ is the initial queue length at station i. To set up a connection between the Brownian system model and the queue length process $Q(t)$, we define the following centered processes $\hat{E}_i$ and $\hat{S}_i$ by

$$\hat{E}_i(t) = E_i(t) - \lambda_i t \tag{1.10}$$

$$\hat{S}_i(t) = S_i(t) - \mu_i t. \tag{1.11}$$

Let

$$\Xi_i(t) = Q_i(0) + \hat{E}_i(B_i^0(t)) + \sum_{j\in J,\sigma(j)=i} \hat{S}_j(B_j(t)) - \hat{S}_i(B_i(t)) \tag{1.12}$$

$$\theta_i = \lambda_i + \sum_{j\in J,\sigma(j)=i} \mu_j - \mu_i. \tag{1.13}$$

Let $Q(t) = (Q_1(t),...,Q_d(t))'$, $\Xi(t) = (\Xi_1(t),...,\Xi_d(t))'$, $Y^0(t) = (Y_1^0(t),...,Y_d^0(t))'$, $Y^b(t) = (Y_1^b(t),...,Y_d^b(t))'$ and $\theta = (\theta_1,...,\theta_d)'$. After going through the standard centering process as in Harrison [21], we have

$$Q(t) = \Xi(t) + \theta t + R^0 Y^0(t) + R^b Y^b(t), \tag{1.14}$$

where R^0 and R^b are $d \times d$ matrix given by

$$R_{ij}^0 = \begin{cases} \mu_i, & \text{if } i = j \\ -\mu_j, & \text{if } j < i \text{ and } \sigma(j) = i, \\ 0, & \text{if } j < i \text{ and } \sigma(j) \neq i \text{ or } j > i, \end{cases} \tag{1.15}$$

$$R^b_{ij} = \begin{cases} -(\lambda_i + \sum_{l<i,\sigma(l)=i} \mu_l), & \text{if } i = j, \\ \mu_i, & \text{if } j > i \text{ and } \sigma(i) = j, \\ 0, & \text{if } j > i \text{ and } \sigma(i) \neq j \text{ or } j < i. \end{cases} \tag{1.16}$$

Furthermore, we have

$$Q(t) \in \mathbf{S},\ t \geq 0, \tag{1.17}$$

$$Y^0_i(0) = 0,\ Y^0_i(\cdot) \text{ is continuous and nondecreasing},\ i \in J, \tag{1.18}$$

$$Y^b_i(0) = 0,\ Y^b_i(\cdot) \text{ is continuous and nondecreasing},\ i \in J, \tag{1.19}$$

$$Y^0(\cdot) \text{ increases only at times t when } Q_i(t) = 0,\ i \in J, \tag{1.20}$$

$$Y^b_i(\cdot) \text{ increases only at times t when } Q_i(t) = b_i,\ i \in J. \tag{1.21}$$

Comparing (1.8) and (1.14), we see that if Ξ is a Brownian motion, then Q will be an SRBM of the type defined in Definition 2.1.1 of previous section. In Chapter 4, we will rigorously justify that Ξ can indeed be approximated by a Brownian motion under a heavy traffic scaling. For the purpose of performance analysis, we just simply replace the queue length process by an $(\mathbf{S}, \theta, \Gamma, R)$-SRBM $Z(t) = X(t) + RY(t)$ with θ given by (1.13), $R = (R^0, R^b)$ given by (1.15)- (1.16) and Γ given by

$$\Gamma = \text{diag}(\lambda_1 c^2_{a,1}\gamma_1, ..., \lambda_d C^2_{a,d}\gamma_d) + (I - P')\text{diag}(\mu_1 c^2_{s,1}\gamma_1, ..., \mu_d c^2_{s,d}\gamma_d)(I - P) \tag{1.22}$$

with $P_{ij} = 1$ if $j = \sigma(i)$ and zero otherwise. γ_i $(i = 1, ..., d)$ is the long-run average rate at which services are completed at station i. That is,

$$\gamma_i = \lim_{t\to} \mu_i B_i(t)/t.$$

They are unknown and can be computed iteratively via the algorithm developed in Chapter 5.

2.1.4 Performance Comparisons for a Tandem Network

In this section, we use our algorithm developed in Chapter 5 to compute some performance measures for a two-station tandem network with finite buffers. The comparisons are given among computed results using our algorithm and existing estimates.

Consider the simple queueing network pictured in Figure 2.2. The network consists of two stations arranged in series under FIFO service discipline. Arriving customers go to station 1 first. After completing service there, they go next to station 2, and after completing service at station 2, they exit the system. The input process to station 1 is a Poisson process with average arrival rate λ_1. Service times at station 1 are deterministic of duration $m_1 = 1$, and service times at station 2 are exponentially distributed with mean $m_2 = 1$. There is a storage buffer in front of station i ($i = 1, 2$) that can hold 24 waiting customers, in addition to the customer occupying the service station. When the buffer in front of station 1 is full, the Poisson input is simply turned off, and in similar fashion, server 1 stops working when the buffer in front of station 2 is full, although a customer may still occupy station 1 when the server is idle because of such blocking. The steady-state performance measures in which we are interested are

- The long-run average queue length q_i at station i ($i = 1, 2$).
- The long-run average throughput rate γ.

The average throughput rate can be equivalently viewed as (a) the average rate at which new arrivals are accepted into the system, or as (b) the average rate at which services are completed at the first station, or as (c) the average rate at which customers departure from the system.

This queueing network model was studied by Dai and Harrison [12]. As an example, we do some numerical comparisons of performance measures such as q_i and

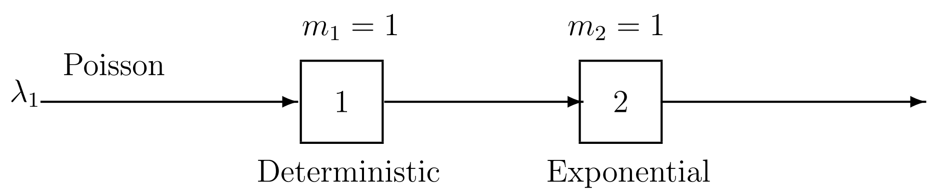

Figure 2.2: A tandem network with finite buffers

γ by employing an algorithm similar to the one developed in Chapter 5. The main equation that governs the dynamics of the queue length process is given by (1.12)-(1.21). Namely,

$$Q_1(t) = \Xi_1(t) + \theta_1 t + Y_1^0(t) - \lambda_1 Y_1^b(t) + Y_2^b(t),$$
$$Q_2(t) = \Xi_2(t) + \theta_2 t - Y_2^0(t) + Y_1^b(t) - Y_2^b(t).$$

To be consistent with the formulation in Dai and Harrison [12], let $Y_1(t) = Y_1^0(t)$, $Y_2(t) = Y_2^0(t)$, $Y_3(t) = \lambda_1 Y_1^b(t)$ and $Y_4(t) = Y_2^b(t)$. Then the queue length process can be approximated by a SRBM as explained in Dai and Harrison [12]. That is,

$$\tilde{Q}(t) = \Xi(t) + \theta t + RY(t),$$

where Ξ is a Brownian motion with drift zero and covariance γI, $\theta = (\lambda_1 - 1, 0)'$ and the reflection matrix R is given by

$$R = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}.$$

The state space $\mathbf{S}$ for the SRBM $Q(t)$ is shown in Figure 1.2, where the boundary size is 25. As discussed in Dai and Harrison [12], γ is unknown and it can be computed iteratively.

Table 2.1: Iterative calculation of throughput rate γ for $\lambda_1 = 0.9$

$\lambda_1 = 0.9, n = 14$				
Iterative number	1	2	3	4
Trial value γ	1.0	0.898354	0.899045	0.899041
Computed q_1	5.208493	4.684678	4.688321	4.688300
Computed q_2	6.632457	6.165969	6.169345	6.169325
Computed γ	0.898354	0.899045	0.899041	0.899041

Table 2.1 shows the computed throughput rate γ obtained by our algorithm and the iterative procedure in Dai and Harrison [12]. The Poisson input rate $\lambda_1 = 0.9$ and the number n of grid points in interval $[0, 25]$ is 14.

In Table 2.2, we give performance estimates derived from the approximate Brownian model with our algorithm, identified in the table as FEM (Finite Element Method) estimates. The estimates are taken from the third iteration. The QNET and SIM estimates are obtained from Dai and Harrison [12].

2.1.5 Numerical Prediction for a Three-Station Network

Consider the intree-like queueing network pictured in Figure 2.3. The input processes to station i ($i = 1, 2$) are Poisson processes with arrival rate λ_i. Service times at station 1 are deterministic of duration $m_1 = 1$. Service times at station 2 and 3 are exponentially distributed with mean $m_2 = 1$ and $m_3 = 0.5$. There is a storage buffer in front of station i that can hold 24 waiting customers ($i = 1, 2, 3$), in addition to the customer being served. When the buffer in front of station i is full, the Poisson input process is simply turned off, and in similar fashion, servers 1 and 2 stop working when the buffer in front of station 3 is full, although a customer may still occupy station 1

Table 2.2: Performance comparisons for a tandem network

$\lambda_1 = 0.9, n = 14$			
	γ	q_1	q_2
FEM	0.8990	4.6883	6.1694
SIM	0.8991	5.1291	6.2691
QNET	0.8995	4.8490	6.3184
$\lambda_1 = 1.0, n = 14$			
	γ	q_1	q_2
FEM	0.9688	13.7865	11.2135
SIM	0.9690	13.87	11.07
QNET	0.9688	13.75	11.25
$\lambda_1 = 1.1, n = 14$			
	γ	q_1	q_2
FEM	0.9801	20.6679	12.4572
SIM	0.9801	20.4801	12.3801
QNET	0.9801	20.5239	12.4445
$\lambda_1 = 1.2, n = 14$			
	γ	q_1	q_2
FEM	0.9804	20.6671	12.4572
SIM	0.9804	20.4804	12.4804
QNET	0.9807	20.2688	12.4676

or 2 when the server is idle because of such blocking. The steady-state performance measures on which we focus are

- The long-run average queue length q_i at station i $(i = 1, 2, 3)$.
- the long-run average rate at which new arrivals are accepted into the system, or as the long-run average rate γ_i at which services are completed at station i $(i = 1, 2)$.
- The long-run average throughput rate γ_3 at station 3, or the long-run average rate at which services are completed.

In these terminologies, "queue length" means that the number of customers at the station, either waiting or being served. Obviously, we have $\gamma_3 = \gamma_1 + \gamma_2$.

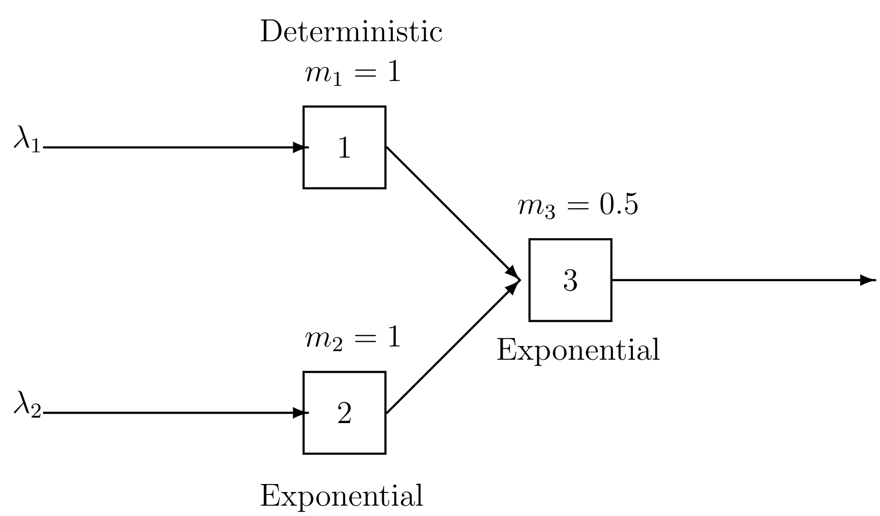

Figure 2.3: A three station intree-like network

As an alternative to simulation, we compute the approximated performance measures above via SRBM and our algorithm developed in Chapter 5. The SRBM can be written as

$$Q_1(t) = \Xi_1(t) + \theta_1 t + Y_1^0(t) - \lambda_1 Y_1^b(t) + Y_3^b(t),$$

$$Q_2(t) = \Xi_2(t) + \theta_2 t + Y_2^0(t) - \lambda_2 Y_2^b(t) + Y_3^b(t),$$
$$Q_3(t) = \Xi_3(t) - Y_1^0(t) - Y_2^0(t) + 2Y_3^0(t) - 2Y_3^b(t).$$

Let $Y_i(t) = Y_i^0(t)$ $(i = 1, 2, 3)$, $Y_4(t) = \lambda_1 Y_1^b(t)$, $Y_5(t) = \lambda_2 Y_2^b(t)$ and $Y_6(t) = Y_3^b(t)$, then we have

$$Q(t) = \Xi(t) + \theta t + RY(t),$$

where $\xi(\cdot)$ is a 3-dimensional approximate Brownian motion with covariance matrix Γ and drift vector 0. The three dimensional vector $\theta = (\lambda_1 - 1, \lambda_2 - 1, 0)'$, and the reflection matrix R is given by

$$R = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 & 0 & -2 \end{pmatrix}.$$

Notice that B_i^0 $(i = 1, 2)$ and B_i $(i = 1, 2, 3)$ are continuous and nondecreasing processes, then there exist constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\lambda_1 B_1^0(t) \sim \gamma_1 t, \quad B_1(t) \sim \gamma_1 t, \tag{1.23}$$
$$\lambda_2 B_2^0(t) \sim \gamma_2 t, \quad B_2(t) \sim \gamma_2 t, \tag{1.24}$$
$$2B_3(t) \sim (\gamma_1 + \gamma_2)t. \tag{1.25}$$

Therefore by (1.22), the covariance matrix Γ can be denoted by

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & 2\gamma_2 & -\gamma_2 \\ 0 & -\gamma_2 & \gamma_1 + 2\gamma_2 \end{pmatrix}. \tag{1.26}$$

The two constants γ_1 and γ_2 will be calculated iteratively by employing our algorithm developed in Chapter 5. To see this point, let δ_i represent the long-run average amount of pushing per unit of time needed on boundary F_i in order to keep the SRBM Q

inside the box **S**. Then by the basic adjoint relationship introduced in Chapter 5, we have

$$(\lambda_1 - 1) + \delta_1 - \delta_4 + \delta_6 = 0, \tag{1.27}$$

$$(\lambda_2 - 1) + \delta_2 - \delta_5 + \delta_6 = 0, \tag{1.28}$$

$$-\delta_1 - \delta_2 + 2\delta_3 - 2\delta_6 = 0. \tag{1.29}$$

Next notice that

$$B_i^0(t) = t - \frac{1}{\lambda_i} Y_i(t),\ i = 1, 2,$$

$$B_1(t) = t - Y_1(t) - Y_6(t),$$

$$B_2(t) = t - Y_2(t) - Y_6(t),$$

$$2B_3(t) = t - Y_3(t).$$

Then by (1.23) to (1.25), we have that

$$\gamma_1 = \lambda_1 - \delta_4, \tag{1.30}$$

$$\gamma_1 = 1 - \delta_1 - \delta_6, \tag{1.31}$$

$$\gamma_2 = \lambda_2 - \delta_5, \tag{1.32}$$

$$\gamma_2 = 1 - \delta_2 - \delta_6, \tag{1.33}$$

$$\gamma_1 + \gamma_2 = 2 - 2\delta_3. \tag{1.34}$$

From (1.27) to (1.29), we see that (1.30) and (1.31) are equivalent, (1.32) and (1.33) are equivalent, (1.34) is equivalent to the summation of (1.30) and (1.32). All of these relationships hold as we expect. Then the following iterative procedure naturally suggests itself:

1. start with trial values of γ_1 and γ_2 (say, $\gamma_1 = \gamma_2 = 1$),

2. set the covariance matrix Γ in (1.26) to compute the data set of the SRBM,

3. compute the steady-state performance characteristic δ_4 and δ_5,

4. use (1.30) and (1.32) to determine new values of γ_1 and γ_2,

5. repeat the procedure 1-4 until convergence is obtained.

By the above procedure and the algorithm developed in Chapter 5, the computed performance measures for $\lambda_1 = 0.9$, $\lambda_2 = 1.0$ and $n = 5$ are shown in Table 2.3, including the long-run average throughput rates γ_i $(i = 1, 2)$, the long-run average queue length q_i $(i = 1, 2, 3)$, and δ_i $(i = 1, 2, ..., 6)$, the long-run average amount of pushing per unit of time on boundary F_i. An important feature is that δ_i $(i = 1, 2, ..., 6)$ satisfies (1.27)-(1.29) with very small error as we expect.

Table 2.4 shows the average throughput rates and average queue lengths for different arrival rates at station 1 and station 2 with our algorithm and the above iterative procedure. All results are taken from the forth iteration and $n = 5$. An unusual observation in the table should be pointed out here. When we increase the Poisson input rate λ_2 to 1.2 at station 2, the long-run average rate γ_2 at station 2 does not increase.

2.2 Tree-like Queueing Network under Communication Blocking

2.2.1 Queueing Network Model

The second type of queueing network discussed in this part has the tree-like structure under a communication blocking scheme. Similar to intree-like network, here we consider a queueing network which consists of d single server stations indexed by $i \in J = \{1, ..., d\}$. The size of the buffer associated with each station i is *finite*. Therefore, at each station i there are at most b_i customers, including the one possibly

Table 2.3: Iterative calculation of throughput rate γ_1 and γ_2

$\lambda_1 = 0.9, \lambda_2 = 1.0, n = 5$				
Iterative number	1	2	3	4
Trial value γ_1	1.0	0.895180	0.896106	0.896081
Trial value γ_2	1.0	0.958692	0.961853	0.961695
Computed q_1	6.534646	6.217643	6.223954	6.223704
Computed q_2	12.341952	12.165035	12.168961	12.168808
Computed q_3	9.198588	9.032063	9.036501	9.036329
Computed γ_1	0.895180	0.896106	0.896081	0.896082
Computed γ_2	0.958692	0.961853	0.961695	0.961703
Computed δ_1	0.093603	0.094088	0.094054	0.094056
Computed δ_2	0.030091	0.028341	0.028440	0.028435
Computed δ_3	0.069861	0.067822	0.067905	0.067901
Computed δ_4	0.003306	0.002516	0.002531	0.002530
Computed δ_5	0.035553	0.032451	0.032593	0.032586
Computed δ_6	0.011217	0.009806	0.009864	0.009862

Table 2.4: Performance estimates for the three station network

$\lambda_1 = 0.9, \lambda_2 = 1.0, n = 5$					
	γ_1	γ_2	q_1	q_2	q_3
FEM	0.896082	0.961703	6.223704	12.168808	9.036329
$\lambda_1 = 1.0, \lambda_2 = 1.1$					
	γ_1	γ_2	q_1	q_2	q_3
FEM	0.960058	0.967873	14.768292	18.397146	11.950783
$\lambda_1 = 1.1, \lambda_2 = 1.2$					
	γ_1	γ_2	q_1	q_2	q_3
FEM	0.977809	0.951475	18.896440	21.870421	11.894797

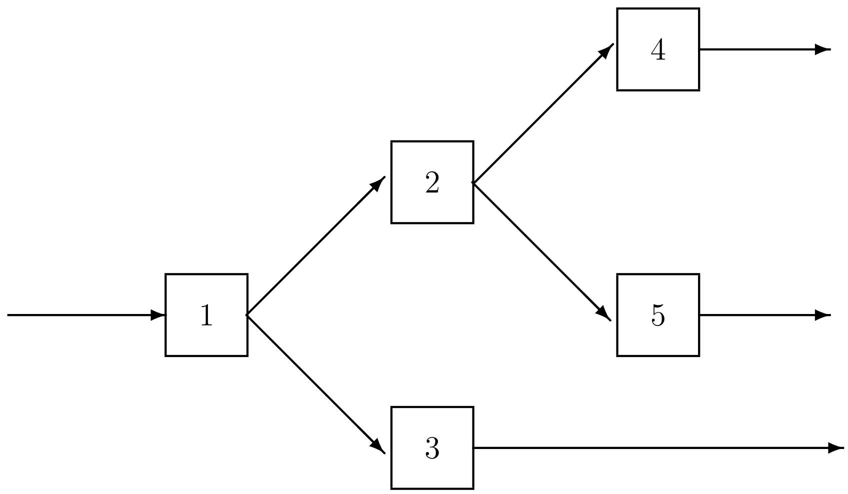

Figure 2.4: A five station tree-like network

being served. The network is assumed under FIFO service discipline. Customers visiting station i are homogeneous in terms of service time distribution and routing mechanism. All customers eventually leave the network. Namely, the network is open.

Arrivals and services are the same as in the intree-like network. The routing structure is tree-like, for example, see Figure 2.4. Stations are numbered in an increasing order, and only at station 1 there is an external arrival stream. Upon completion of service at station i, a customer goes next to a station $j \in \sigma(i)$ with probability P_{ij}. $\sigma(i)$ indexes the stations where a customer will visit after he finishes service at station i. It can be denoted by

$$\sigma(i) \equiv \{j \in J, P_{ij} > 0, j > i\}.$$

We assume that $\sigma(i) \cap \sigma(j) = \emptyset$ for $i \neq j$. That is, each station has at most one predecessor. When the buffer at one of stations in $\sigma(i)$ is full, server i stops working although a customer may still occupy station i when the server is idle because of such blocking.

Finally, we suppose that the routing of a customer in the network is independent of all previous history. To be more precise for the statement, let $\phi^i(k)$ be the routing vector for the k^{th} customer who finishes service at station i, that is,

$$\phi^i = \{\phi_{ij}(k), k \geq 1\}, \; for \; i, j \in J. \tag{2.1}$$

If $\phi_{ij}(k) = 1$, the k^{th} customer at station i becomes a customer at station j. If $\phi_{ij}(k) = 0$ for any $j \in J$, the k^{th} customer at station i leaves the system. Therefore $\phi^i(k)$ is a d-dimensional "Bernoulli Random Variable" with parameter P_i', where P_i denotes the i^{th} row of $P = \{P_{ij}\}$ with the spectral radius less than unity, the prime denote the transpose. We assume that $\phi = \{\phi^i(k), k \geq 1\}$ is i.i.d. and $\phi^1, \phi^2, ..., \phi^d$ are independent and independent of the arrival processes and service processes. Furthermore, let

$$\Phi^i(k) \equiv \phi^i(1) + ... + \phi^i(k), \tag{2.2}$$

or, in component form,

$$\Phi_{ij}(k) \equiv \sum_{l=1}^{k} \phi_{ij}(l) \; for \; i, j \in J, \tag{2.3}$$

where $\Phi_{ij}(k)$ is the cumulative number of customers to station j for the first k customers leaving station i.

Finally, let $Q_i(t)$ be the number of customers at station i, including possibly the one being served. Let $Y_i^0(t)$ be the cumulative time that station i is empty and all of stations $j \in \sigma(i)$ are not full in $[0, t]$. Let $Y_j^b(t)$ be the cumulative amount of time that station $j \in \sigma(i)$ is full and every station l ($l \in \sigma(i)$, $l > j$) is not full for some $i \in J \cup \{0\}$ with $\sigma(0) = 1$. Then $\sum_{j \in \sigma(i)} Y_j^b(t)$ denotes the total amount of time that station i is blocked by time t. Similar to intree-like network, we are interested in quantities, including long-run average buffer size

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t Q_i(t)(s) ds,$$

and the long-run average server utilization rate

$$1 - \lim_{t\to\infty} \frac{Y_i^0(t)}{t} - \lim_{t\to\infty} \frac{1}{t} \sum_{j\in\sigma(i)} Y_j^b(t).$$

2.2.2 Brownian System Model

In this section, we present a suitable SRBM as an approximating model for the tree-like queueing network described in previous section. Let $B_i(t)$ be the cumulative amount of time that the server i is busy in $[0, t]$. Then we have

$$B_i(t) = t - Y_i^0(t) - \sum_{j\in\sigma(i)} Y_j^b(t).$$

Furthermore let $B_1^0(t)$ be the cumulative amount of time that buffer 1 is not full during time interval $[0, t]$. As a matter of definition, we have

$$B_1^0(t) = t - Y_1^b(t).$$

Then the queue length process at station i can be represented by

$$Q_1(t) = Q_1(0) + E_1(B_0^1(t)) - S_1(B_1(t)),$$
$$Q_i(t) = Q_i(0) + \Phi_{ji}(S_j(B_j(t))) - S_i(B_i(t)),\ (i > 1 \text{ and some } j < i, i \in \sigma(j)),$$

where $Q_i(0)$ is the initial queue length at station i. Similar to previous discussion in intree-like network, we go through the standard centering process as in Harrison [21]. Let

$$\hat{E}_1(t) = E_1(t) - \lambda_1 t, \tag{2.4}$$

$$\hat{S}_i(t) = S_i(t) - \mu_i t, \tag{2.5}$$

$$\hat{\Phi}_{ji}(k) = \Phi_{ji}(k) - P_{ji} k, \tag{2.6}$$

where k takes values in nonnegative integer set. Let

$$\Xi_1(t) \ \ = Q_1(0) + \hat{E}_1(B_0^1(t)) - \hat{S}_1(B_1(t)),$$

$$\begin{aligned}
\Xi_i(t) &= Q_i(0) + \hat{\Phi}_{ji}(S_j(B_j(t))) + P_{ji}\hat{S}_j(B_j(t)) - \hat{S}_i(B_i(t)) \ (i > 1),\\
\theta_1 &= \lambda_1 - \mu_1,\\
\theta_i &= P_{ji}\mu_j - \mu_i \qquad\qquad (i > 1).
\end{aligned}$$

Let $Q(t) = (Q_1(t), ..., Q_d(t))'$, $\Xi(t) = (\Xi_1(t), ..., \Xi_d(t))'$, $Y^0(t) = (Y_1^0(t), ..., Y_d^0(t))'$, $Y^b(t) = (Y_1^b(t), ..., Y_d^b(t))'$, $\theta = (\theta_1, ..., \theta_d)$ and P be the $d \times d$ routing matrix. Then we have

$$Q(t) = \Xi(t) + \theta t + R^0 Y^0(t) + R^b Y^b(t), \tag{2.7}$$

where R^0 and R^b are $d \times d$ matrix given by

$$R^0_{ij} = \begin{cases} \mu_i, & \text{if } i = j,\\ -\mu_j P_{ji}, & \text{if } j < i \text{ and } P_{ji} > 0,\\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

$$R^b_{ij} = \begin{cases} -\lambda_1, & \text{if } i = j = 1,\\ \mu_i, & \text{if } j > i \text{ and } j \in \sigma(i),\\ -\mu_l P_{li}, & \text{if } j \in \sigma(l),\ P_{li} > 0 \text{ for some } l < i,\\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

Furthermore, we have

$$Q(t) \in \mathbf{S},\ t \geq 0, \tag{2.10}$$

$$Y_i^0(0) = 0,\ Y_i^0(\cdot) \text{ is continuous and nondecreasing},\ i \in J, \tag{2.11}$$

$$Y_i^b(0) = 0,\ Y_i^b(\cdot) \text{ is continuous and nondecreasing},\ i \in J, \tag{2.12}$$

$$Y^0(\cdot) \text{ increases only at times t when } Q_i(t) = 0,\ i \in J, \tag{2.13}$$

$$Y_i^b(\cdot) \text{ increases only at times t when } Q_i(t) = b_i,\ i \in J. \tag{2.14}$$

Therefore, similar to the discussion in intree-like network, we can simply replace the queue length process by an $(\mathbf{S}, \theta, \Gamma, R)$-SRBM with R given by (2.8)-(2.9) and

$$\Gamma = \text{diag}(\lambda_1 c_{a,1}^2\gamma_1, ..., \lambda_d c_{a,d}^2\gamma_d) + (I - P')\text{diag}(\mu_1 c_{s,1}^2\gamma_1, ..., \mu_d c_{s,d}^2\gamma_d)(I - P) + \sum_{j=1}^{d} \mu_j\gamma_j\Gamma^j,$$

where γ_i $(i = 1, ..., d)$ is the long-run average rate at which services are completed at station i, and

$$\Gamma_{lk}^{j} = \begin{cases} P_{jl}(1 - P_{jl}), & \text{if } l = k, \\ -P_{jl}P_{jk}, & \text{if } l \neq k. \end{cases}$$

2.3 Queueing Network with Feedback and Loss

2.3.1 Queueing Network Model

In this section, we will look at some new mechanism for a network in dealing with buffer overflow problem. The queueing network under consideration consists of d single server stations indexed by $i \in J = \{1, ..., d\}$. The size of the buffer associated with each station i is *finite*. Therefore, at each station i there are at most b_i customers, including the one possibly being served. The network is assumed under FIFO service discipline. Customers visiting station i are homogeneous in terms of service time distribution and routing mechanism. All customers eventually leave the network, namely the network is open. An example of the network is pictured in Figured 2.5.

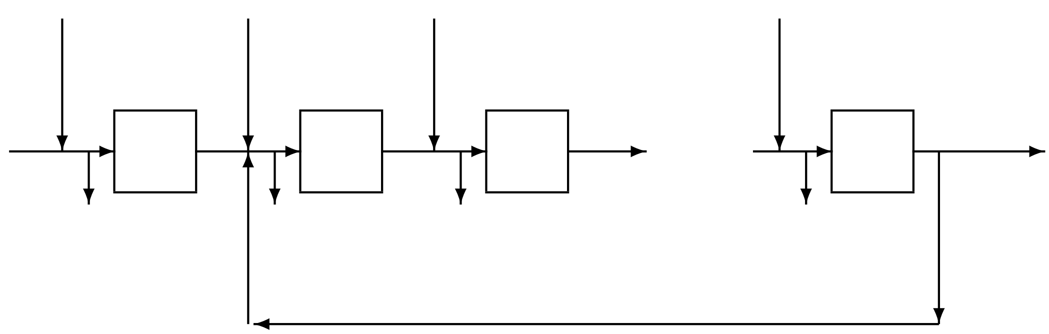

Figure 2.5: A network with feedback and loss

Again the arrivals and services we specified are the same as before. The routing is different from the previous two models. Upon completion of service at station i, a customer goes next to a station $j \in J$ with probability P_{ij} and exits the network with probability $1 - \sum_j P_{ij}$, independent of all previous history. To be more precise for the statement, let $\phi^i(k)$ be the routing vector for the k^{th} customer who finishes service at station i. That is,

$$\phi^i = \{\phi_{ij}(k), k \geq 1\}\,,\ for\ i, j \in J. \tag{3.1}$$

If $\phi_{ij}(k) = 1$, the k^{th} customer at station i becomes a customer at station j. If $\phi_{ij}(k) = 0$ for any $j \in J$, the k^{th} customer at station i leaves the system. Therefore $\phi^i(k)$ is a d-dimensional "Bernoulli Random Variable" with parameter P_i', where P_i denotes the i^{th} row of $P = \{P_{ij}\}$ with the spectral radius less than unity, the prime denote the transpose. We assume that $\phi = \{\phi^i(k), k \geq 1\}$ is i.i.d. and $\phi^1, \phi^2, ..., \phi^d$ are independent and independent of the arrival processes and service processes. Furthermore, let

$$\Phi^i(k) \equiv \phi^i(1) + ... + \phi^i(k), \tag{3.2}$$

or, in component form,

$$\Phi_{ij}(k) \equiv \sum_{l=1}^{k} \phi_{ij}(l)\ for\ i, j \in J, \tag{3.3}$$

where $\Phi_{ij}(k)$ is the cumulative number of customers to station j for the first k customers leaving station i.

A customer arrives at a full buffer station i. Instead of going into the station, it either goes next to station j with probability $\bar{P}_{ij}$, or is lost with probability $1 - \sum_j \bar{P}_{ij}$, independent of all previous history. Similar to the discussion before, let $\bar{\phi}_i(k)$ be the routing vector for the k^{th} deflected customer at station i, that is,

$$\{\bar{\phi}_{ij}(k), k \geq 1\}. \tag{3.4}$$

If $\bar{\phi}_{ij} = 1$, the k^{th} customer is deflected from station i to some station j. If $\bar{\phi}_{ij} = 0$ for all $j \in J$, the k^{th} customer leaves the network and is lost. Therefore $\bar{\phi}_i(k)$ is a d-dimensional "Bernoulli random variable" with parameter $\bar{P}_i'$ where $\bar{P}_i$ denotes the i^{th} row of $\bar{P} = (\bar{P}_{ij})$ with the spectral radius less than unity. We assume that $\bar{\phi} = \{\bar{\phi}_i(k), k \geq 1\}$ is i.i.d., $\bar{\phi}_1, ..., \bar{\phi}_d$ are independent of the arrival processes and service processes. Furthermore, let

$$\bar{\Phi}^i(k) \equiv \bar{\phi}_i(1) + ... + \bar{\phi}_i(k), \tag{3.5}$$

or, in component-wise form,

$$\bar{\Phi}_{ij}(k) \equiv \sum_{l=1}^{k} \bar{\phi}_{ij}(l) \; for \; i, j \in J, \tag{3.6}$$

where $\bar{\Phi}_{ij}(k)$ is the total number of customers to station j for the first k customers leaving station i due to its full buffer.

Finally, let $Q_i(t)$ be the number of customers at station i, including the one being served. Let $Y_i^0(t)$ be the cumulative amount of time that station i is empty in $[0, t]$ and $Y_i^b(t)$ be the cumulative number of customers lost at station i due to the full buffer by time t. The quantities are of interests, including the average buffer size

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t Q_i(s) ds,$$

and the long-run average number of customers that are lost at station i

$$\lim_{t \to \infty} \frac{Y_i^b(t)}{t},$$

and the server utilization rate

$$1 - \lim_{t \to \infty} \frac{Y_i^0(t)}{t}.$$

All of these performance measures can be estimated from their Brownian counterparts.

2.3.2 Brownian System Model

In this section, we present a suitable SRBM as an approximating model for the queueing network described in the previous section. Let $B_i(t)$ be the cumulative amount of time that server i is busy in $[0,t]$. As a matter of definition, we have

$$B_i(t) = t - Y_i^0(t).$$

Then we can write down the main equation that governs the dynamics of the queue length process, that is,

$$Q_i(t) = Q_i(0) + E_i(t) + \sum_{j \neq i} (\Phi_{ji}(S_j(B_j(t)) + \bar{\Phi}_{ji}(Y_j^b(t))) - Y_i^b(t) - S_i(B_i(t)),$$

where the third term on the right hand side denotes the cumulative number of customers routing to station i from other stations. It includes customers either deflected or finished service at other stations.

Similar to the discussion before, we go through the standard centering process as in Harrison [21]. Let

$$\begin{aligned} \hat{E}_i(t) &= E_i(t) - \lambda_i t, \\ \hat{S}_i(t) &= S_i(t) - \mu_i t, \\ \hat{\Phi}_{ji}(k) &= \Phi_{ji}(k) - P_{ji} k, \\ \hat{\bar{\Phi}}_{ji}(k) &= \bar{\Phi}_{ji}(k) - \bar{P}_{ji} k, \end{aligned}$$

where k takes values in nonnegative integer set. Let

$$\begin{aligned} \Xi_i(t) &= Q_i(0) + \hat{E}_i(t) + \sum_{j \neq i} P_{ji} \hat{S}_j(B_j(t)) \\ &\quad + \sum_{j \neq i} \{\hat{\Phi}_{ji}(S_j(B_j(t)) + \hat{\bar{\Phi}}_{ji}(Y_j^b(t))\} - \hat{S}_i(B_i(t)), \\ \theta_i &= \lambda_i + \sum_{j \neq i} \mu_j P_{ji} - \mu_i. \end{aligned}$$

Let $Q(t) = (Q_1(t), ..., Q_d(t))'$, $\Xi(t) = (\Xi_1(t), ..., \Xi_d(t))'$, $Y^0(t) = (Y_1^0(t), ..., Y_d^0(t))'$, $Y^b(t) = (Y_1^b(t), ..., Y_d^b(t))'$, $\theta = (\theta_1, ..., \theta_d)' R^0 = (I - P')$ and $R^b = -(I - \bar{P}')$. Then,

$$Q(t) = \Xi(t) + \theta t + R^0 \text{diag}(\mu) Y^0(t) + R^b Y^b(t), \tag{3.7}$$

$$Q(t) \in \mathbf{S},\ t \geq 0, \tag{3.8}$$

$$Y_i^0(0) = 0,\ Y_i^0(\cdot) \text{ is continuous and nondecreasing},\ i \in J, \tag{3.9}$$

$$Y_i^b(0) = 0,\ Y_i^b(\cdot) \text{ is nondecreasing},\ i \in J, \tag{3.10}$$

$$Y^0(\cdot) \text{ increases only at times t when } Q_i(t) = 0,\ i \in J, \tag{3.11}$$

$$Y_i^b(\cdot) \text{ increases only at times t when } Q_i(t) = b_i,\ i \in J. \tag{3.12}$$

Similar to the discussion before, the queue length process in (3.7) can be replaced by an $(\mathbf{S}, \theta, \Gamma, R)$-SRBM with covariance matrix given by

$$\begin{aligned} \Gamma &= \text{diag}(\lambda_1 c_{a,1}^2 \gamma_1, ..., \lambda_d c_{a,d}^2 \gamma_d) + (I - P')\text{diag}(\mu_1 c_{s,1}^2 \gamma_1, ..., \mu_d c_{s,d}^2 \gamma_d)(I - P) \\ &\quad + \sum_{j=1}^{d} \mu_j \gamma_j \Gamma^j + \sum_{j=1}^{d} (1 - \gamma_j) \bar{\Gamma}^j \end{aligned}$$

where γ_i $(i = 1, ..., d)$ is the long-run average rate at which services are completed at station i, and

$$\Gamma_{lk}^j = \begin{cases} P_{jl}(1 - P_{jl}), & \text{if } l = k, \\ -P_{jl} P_{jk}, & \text{if } l \neq k, \end{cases}$$

$$\bar{\Gamma}_{lk}^j = \begin{cases} \bar{P}_{jl}(1 - \bar{P}_{jl}), & \text{if } l = k, \\ -\bar{P}_{jl} \bar{P}_{jk}, & \text{if } l \neq k. \end{cases}$$

CHAPTER 3

Oscillation, Compactness and Convergence

3.1 Convex Polyhedron and SRBM

In this section, we give some general background on convex polyhedron and SRBM. A polyhedron is defined in terms of m ($m \geq 1$) d-dimensional unit vectors $\{n_i, i \in J\}$, $J \equiv \{1, ..., m\}$ and an m-dimensional vector $(b_1 ... b_m)'$, where the prime denotes transpose. The state space $\mathbf{S}$ is defined by

$$\mathbf{S} \equiv \left\{ x \in R^d : n_i \cdot x \geq b_i \text{ for all } i \in J \right\} \tag{1.1}$$

where $n_i \cdot x = n_i' x$ represents the inner product of the vectors n_i and x. It is assumed that the interior of $\mathbf{S}$ is non-empty and that the set $((n_1, b_1), ..., (n_m, b_m))$ is minimal in the sense that no proper subset defines $\mathbf{S}$, that is, for any strict subset $K \subset J$, the set $\{x \in R^d : n_i \cdot x \geq b_i \text{ for any } i \in K\}$ is strictly larger than $\mathbf{S}$. This minimal property is equivalent to the following assumption that each of the faces

$$F_i \equiv \{x \in \mathbf{S} : n_i \cdot x = b_i\} \text{ for } i \in J \tag{1.2}$$

where F_i is a $(d-1)$-dimension superplane, see Theorem 8.2 in Brondsted [2]. As a consequence, n_i is the unit normal to face F_i that points into the interior of $\mathbf{S}$.

Definition 3.1.1 *For each* $\emptyset \neq K \subset J$, *define* $F_K = \cap_{i \in K} F_i$ *and let* $F_\emptyset = \mathbf{S}$. *A* *set* $K \subset J$ *is maximal if* $K \neq \emptyset$, $F_K \neq \emptyset$ *and* $F_K \neq F_{\bar{K}}$ *for any* $\bar{K} \supset K$ *such that* $\bar{K} \neq K$.

Definition 3.1.2 *A convex polyhedron* $\mathbf{S}$ *is simple if for each* $K \subset J$ *such that* $K \neq \emptyset$ *and* $F_K \neq \emptyset$, *exactly* $\mid K \mid$ *distinct faces contain* F_K.

This definition is equivalent to one of the following two conditions,

1. Every nonempty subset of a maximal set is maximal,
2. $K \subset J$ is maximal whenever $K \neq \emptyset$ and $F_K \neq \emptyset$.

A point $x_0 \in \mathbf{S}$ is a vertex of $\mathbf{S}$ if $F_K = \{x_0\}$ for some $K \subset J$. If $\mathbf{S}$ is simple, precisely d faces meet at any vertex of $\mathbf{S}$.

Let θ be a vector in R^d, Γ be a $d \times d$ symmetric, positive definite matrix, and R be a $d \times m$ matrix. A triple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ will be called a filtered space if Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω, and $\{\mathcal{F}_t, t \geq 0\}$ is an increasing family of sub-σ-fields of $\mathcal{F}$, i.e., a filtration. If, in addition, $\mathbf{P}$ is a probability measure on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is called a filtered probability space. Now we present the definition of a semimartingale reflected Brownian motion (SRBM) on a general convex polyhedron.

Definition 3.1.3 *An SRBM associated with the data* $(\mathbf{S}, \theta, \Gamma, R)$ *that has initial distribution* π *is a continuous,* $\{\mathcal{F}_t\}$*-adapted, d-dimensional process* Z *defined on some filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ *such that under* $\mathbf{P}$,

$$Z(t) = X(t) + RY(t) \;\; for \; all \; t \geq 0, \tag{1.3}$$

where

1. Z *has continuous paths in* $\mathbf{S}$, $\mathbf{P}$*-a.s.,*
2. *under* $\mathbf{P}$, X *is a d-dimensional Brownian motion with drift vector* θ *and covariance matrix* Γ *such that* $\{X(t)-\theta t, \mathcal{F}_t, t \geq 0\}$ *is a martingale and* $PX^{-1}(0) = \pi$,

3. Y is an $\{\mathcal{F}_t\}$-adapted, m-dimensional process such that $\mathbf{P}$-a.s., for each $i \in 1, ..., m$, the i^{th} component Y_i of Y satisfies

(a) $Y_i(0) = 0$,

(b) Y_i is continuous and non-decreasing,

(c) Y_i can increase only when Z is on the face F_i, i.e.

$$\int_0^t I_{F_i}(Z(s))dY_i(s) = Y_i(t) \text{ for all } t \geq 0. \tag{1.4}$$

Definition 3.1.4 *A square matrix A is called an $\mathcal{S}$-matrix if there is a vector $x \geq 0$ such that $Ax > 0$. The matrix A is completely-$\mathcal{S}$ if and only if each principal submatrix of A is an $\mathcal{S}$-matrix.*

From Definition 3.1.4, we have the following geometric interpretation for a 2×2 completely-$\mathcal{S}$ matrix $A = (v_1, v_2)$. The 2-dimensional vectors v_1 and v_2 are the inward vectors on the boundaries F_1 and F_2 of the nonnegative othant. At the corner O, there exists a positive linear combination $x_1v_1 + x_2v_2$, $x_1 > 0$ and $x_2 > 0$ such that $x_1v_1 + x_2v_2$ points to the interior of the nonnegative othant.

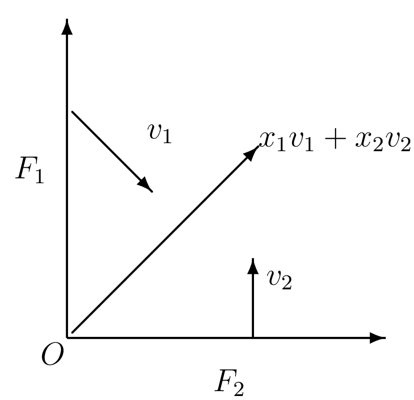

Figure 3.1: Geometric interpretation of completely-$\mathcal{S}$ matrix

In order to apply the completely-$\mathcal{S}$ condition to a general polyhedron, we do the following extension. Let N denote the $m \times d$ matrix whose i^{th} row is given by the row vector n_i' for each $i \in J$. For an $m \times m$ matrix A and $K \subset J$, let A_K denote the $\mid K \mid \times \mid K \mid$ matrix obtained from A by deleting those rows and columns with indices in $J \setminus K$. Concerning the matrix N and the reflection matrix R, the following assumptions $(A.1)$ and $(A.2)$ are employed throughout our analysis, that is,

- $(A.1)$: the matrix $(NR)_K$ is an $\mathcal{S}$-matrix for each maximal $K \subset J$;
- $(A.2)$: the matrix $(NR)'_K$ is an $\mathcal{S}$-matrix for each maximal $K \subset J$.

Let v_i denote the i^{th} column of the matrix R for each $i \in J$, then conditions $(A.1)$ and $(A.2)$ are equivalent to $(A.1)'$ and $(A.2)'$ respectively.

- $(A.1)'$: For each maximal $K \subset J$, there is a positive linear combination $v = \sum_{i \in K} a_i v_i$, $a_i > 0$ for $i \in K$, such that $n_i \cdot v > 0$ for all $i \in K$;
- $(A.2)'$: For each maximal $K \subset J$, there is a positive linear combination $\eta = \sum_{i \in K} c_i n_i$, $c_i > 0$ for $i \in K$, such that $\eta \cdot v_i > 0$ for all $i \in K$.

Conditions $(A.1)$ and $(A.2)$ will be the key assumptions in proving the oscillation theorem introduced in next section. They also take an important role in the proof of existence and uniqueness in law of an SRBM starting from each point $x \in \mathbf{S}$, see Theorem 1.3 in Dai and Williams [16].

Let $C_{\mathbf{S}} = C([0,\infty), R^d_{\mathbf{S}} \times R^m_+ \times \mathbf{S}) = \{(x, y, z) : x, y, z$ are continuous functions from $[0,\infty)$ into $R^d_{\mathbf{S}}, R^m_+, \mathbf{S}$ with $x(0) \in \mathbf{S}$, respectively$\}$, $\mathcal{M} = \sigma\{(y,z)(s) : 0 \le s < \infty, (y,z) \in C([0,\infty), R^m_+ \times \mathbf{S})\}$, and for each $t \ge 0$, $\mathcal{M}_t = \sigma\{(y,z)(s) : 0 \le s \le t, (y,z) \in C([0,\infty), R^m_+ \times \mathbf{S})\}$, where R^m_+ is m-dimensional nonnegative vector space. Then the following proposition gives the existence and uniqueness in law of an SRBM associated with $(\mathbf{S}, \theta, \Gamma, R)$ and initial distribution π.

Proposition 3.1 *Suppose that Assumptions $(A.1)$ and $(A.2)$ hold. Then there exists a unique probability measure Q_π on the filtered probability space $(C_{\mathbf{S}}, \mathcal{M}, \{\mathcal{M}_t\})$ such that Z together with Q_π is an (S, θ, Γ, R)-SRBM.*

Proof. This is a direct generalization of Theorem 1.3 of Dai and Williams [16], see Dai and Kurtz [14], or by using the same argument as the well-posed Martingale problem, see page 182 and Problems 49, 50 in Ethier and Kurtz [18]. □

3.2 $(\mathbf{S}, R)$-Regulation Problem: Oscillation

With $(\mathbf{S}, R)$ referred to previous section, we study the oscillation property of an $(\mathbf{S}, R)$-regulation. First, we state the definition of an $(\mathbf{S}, R)$-Regulation problem in $D_{R^d}[0,T]$ which is the path space of all functions $f : [0,T] \to R^d$ which are right continuous and have left limits. The space $D_{R^d}[0,T]$ is endowed with Skorohod topology

Definition 3.2.1 *Given $T > 0$ and $x \in D_{R^d}[0,T]$ with $x(0) \in \mathbf{S}$, an $(\mathbf{S}, R)$- Regulation of x over [0,T] is a pair $(z, y) \in D_{\mathbf{S}}[0,T] \times D_{R_+^m}[0,T]$ such that*

1. *$z(t) = x(t) + Ry(t) \in \mathbf{S}$ for all $t \in [0,T]$,*
2. *for each $i \in J$, y_i is nondecreasing, $y_i(0) = 0$, and y_i can increase only at times $t \in [0,T]$ for which $z(t) \in F_i$.*

The following lemma shows that there exists a unique solution for one dimensional $(\mathbf{S}, R)$-regulation problem with $\mathbf{S} = R_+$.

Lemma 3.1 *Let $\mathbf{S} = R_+$ and $x \in D_R[0,T]$ with $x(0) \in R_+$ for any given $T > 0$. Then for any $\alpha > 0$, there exists a unique solution given by (z, y) for (R_+, α)-regulation problem*

$$z(t) = x(t) + \alpha\, y(t), \tag{2.1}$$

$$y(t) = \alpha^{-1} \sup_{0 \le s \le t} x^-(s) \ for\ each\ t \in [0,T]. \tag{2.2}$$

Proof. It is easy to show that (z, y) defined by (2.1) and (2.2) is a solution of (R_+, α)-regulation problem. In fact, clearly, $z = x + \alpha y$ and $z(t) = x(t) + \alpha y(t) \geq x(t) + x^-(t) = x^+(t) \geq 0$. Thus $z(t) \in R_+$. Since $x(0) \geq 0,\ y(0) = 0$. Obviously y is non-decreasing. Moreover z and y are right continuous and have left limit at any point $t_0 \in [0, T]$. To show this, notice that $x \in D_R[0, T]$, then

$$\lim_{t \to t_0^-} x^-(t) = x^-(t_0^-).$$

Thus for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$| x^-(t) - x^-(t_0^-) | < \epsilon$$

for all $t \in [t_0 - \delta, t_0)$. Therefore for any $t_1, t_2 \in [t_0 - \delta, t_0)$ and $t_1 \leq t_2$, we have

$$| x^-(t_1) - x^-(t_2) | < 2\epsilon.$$

Hence

$$0 \leq y(t_2) - y(t_1) \leq 2\epsilon.$$

Therefore y has left limit at t_0. Similarly, we can prove that y is right continuous at t_0. Thus $z, y \in D_{R^+}[0, T]$.

Now we show that 2 is true in Definition 3.2.1. Here we only study the case when y increases to the left of $t_0 > 0$. Then for each $\delta > 0,\ 0 \leq y(t_0 - \delta) < y(t_0)$. (If $t_0 = 0$, it only has the right-side case). We divide this into two cases since the left limits exist for $y(\cdot)$ and $x(\cdot)$; case (a) $y(t_0^-) = y(t_0)$, case (b) $y(t_o^-) < y(t_0)$.

For case (a), $z(t_0) = 0$ follows from Lemma 8.1 in K.L.Chung and R.J.Williams [10] since $y(\cdot)$ is right continuous.

For case (b), if $x^-(t_0) < y(t_0)$, then

$$\begin{aligned} y(t_0) &= \sup_{0 \leq s \leq t_0} x^-(s) \\ &= \max\left\{ \sup_{0 \leq s < t_0} x^-(s),\ x^-(t_0) \right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{ y(t_0^-),\ x^-(t_0)\right\} \\
&< y(t_0).
\end{aligned}
$$

That is a contradiction. So $x^-(t_0) = y(t_0) > y(t_0 - \delta) \geq 0$ and hence $x(t_0) < 0$. Finally, $z(t_0) = x(t_0) + y(t_0) = 0$.

To prove the uniqueness, suppose that $\langle \bar{z}, \bar{y}\rangle$ is another solution for (R_+, α)-regulation problem. Then by (1) in definition 3.2.1, we have

$$z(t) - \bar{z}(t) = y(t) - \bar{y}(t).$$

Since $z, y, \bar{z}, \bar{y} \in D_{R_+}[0, \infty)$, they are bounded in finite interval $[0, t]$, see page 110 in Billingsley [3]. Hence by Fubini's Theorem or Proposition 10 on page 68 of Wang [40], we have the following formula of integral by parts for Lebesgue Stieltjes-integral.

$$
\begin{aligned}
0 &\leq (y(t) - \bar{y}(t))^2 + \sum_{0<s\leq t} ((y(s) - \bar{y}(s)) - (y(s^-) - \bar{y}(s^-)))^2 \\
&= 2\int_0^t (y(t) - \bar{y}(s)) d(y(s) - \bar{y}(s)) \\
&= 2\int_0^t (z(s) - \bar{z}(s)) d(y(s) - \bar{y}(s)) \\
&= (-2)\int_0^t z(s) d\bar{y}(s) - 2\int_0^t \bar{z}(s) dy(s) \leq 0.
\end{aligned}
$$

where the third equality follows from 2 in Definition 3.2.1. Thus $y(t) = \bar{y}(t),\ z(t) = \bar{z}(t)$ for all t. □

Before we move to the discussion of the oscillation property for general $(\mathbf{S}, R)$-regulation problem, we introduce some notations and results on the decomposition of state space $\mathbf{S}$. For convenience, we restate Lemma B.1 of Dai and Williams (1994), which will be used several times.

Lemma 3.2 *There is a constant $C \geq 1$ which depends only on $\{n_i, i \in J\}$ such that for each $K : \emptyset \neq K \subset J$ and each $F_K \neq \emptyset$, and each $x \in \mathbf{S}$,*

$$d(x, F_K) \leq C \sum_{i\in K} (n_i \cdot x - b_i). \tag{2.3}$$

□

Then for each $\varepsilon \geq 0$ and $K \subset J$ (including the empty set) , define

$$F_K^{\varepsilon} = \left\{ x \in R^d : 0 \leq n_i \cdot x - b_i \leq C_{\varepsilon} \text{ for all } i \in K,\ n_i \cdot x - b_i > \varepsilon \text{ for all } i \in J \setminus K \right\} \quad (2.4)$$

where $C_{\varepsilon} = Cm\varepsilon$ and C is given by Lemma 3.2. Then by Lemma 4.1 and Lemma 4.2 in Dai and Williams [16] for each $\varepsilon \geq 0$, we have

$$\mathbf{S} = \cup_{K \in \Xi} F_K^{\varepsilon}, \quad (2.5)$$

where Ξ denotes the collection of subsets of J consisting of all maximal sets in J together with the empty set. If $K \subset J$ is maximal, the conditions $(A.1)$ and $(A.2)$ hold for $(\mathbf{S}_K, R_K)$, where

$$\mathbf{S}_K = \{x \in R^d : n_i \cdot x \geq b_i \text{ for all } i \in K\},$$

and R_K is the $d\times \mid K \mid$ matrix whose columns are given by i^{th} column of matrix R. Finally, for a function f defined from $[t_1,\ t_2] \subset [0,\ \infty)$ into R^k for some $k \geq 1$, let

$$\text{Osc}(f, [t_1, t_2]) = \sup_{t_1 \leq s \leq t \leq t_2} |f(t) - f(s)|. \quad (2.6)$$

Then we have the following oscillation result for a sequence of $(\mathbf{S}^n, R^n)$-regulation problems.

Theorem 3.1 *For any $T > 0$, given a sequence of $\{x^n\}_{n=1}^{\infty} \in D_{R^d}[0, T]$ with the initial values $x^n(0) \in \mathbf{S}^n$. Let (z^n, y^n) be an $(\mathbf{S}^n, R^n)$-regulation of x^n over $[0.T]$, where $(z^n, y^n) \in D_{R^d}[0, T] \times D_{R^m}[0, T]$. Assuming that all $\mathbf{S}^n$ have the same shape,i.e., the only difference is the corresponding boundary size b_i^n. Assuming that $\{b_i^n\}$ belongs to some bounded set, and the jump sizes of y^n are bounded by Γ^n for each n. Then if (N, R) satisfies $(A.1)$, $(A.2)$ and $R^n \to R$ as $n \to \infty$, we have*

$$\text{Osc}(z^n, [t_1, t_2]) \leq C\ \max\left\{\text{Osc}(x^n, [t_1, t_2]), \Gamma^n\right\}, \quad (2.7)$$

$$\text{Osc}(y^n, [t_1, t_2]) \leq C\ \max\left\{\text{Osc}(x^n, [t_1, t_2]), \Gamma^n\right\}. \quad (2.8)$$

Where C depends only on $(N, R, \mid K \mid)$ for all $K \in \Xi$.

Proof. Since $R^n \to R$ and (N, R) satisfies conditions $(A.1)$ and $(A.2)$, without loss of generality, we suppose that there is a common η such that $(A.1)'$ and $(A.2)'$ are true for each maximal set $K \in \Xi$ for all $n \geq 1$. We prove this theorem via an induction on the size of J, which is the common index set for faces of $\mathbf{S}^n$ $(n \geq 1)$.

First consider the case $\mid J \mid= 1$. Then $R^n = v_1^n$ is a vector in R^d for each n. From condition $(A.1)$, we have that $n_i \cdot v_1^n > 0$. Then from Lemma 3.1, we can see that y^n is uniquely determined by the 1-dimensional regulator mapping for $n_1 \cdot x^n - b_1^n$,

$$n_1 z^n(t) = n_1 x^n(t) + n_1 v_1^n y^n(t) \tag{2.9}$$

$$y^n(t) = \sup_{0 \leq s \leq t} (n_1 x^n(s) - b_1^n)^- / n_1 v_1^n \tag{2.10}$$

It is clear that $y^n(0) = 0$ and $y^n(\cdot)$ is non-decreasing, and (2.9), (2.10) defines a $([b_1^n, \infty), n_1 \cdot v_1^n)$-regulation of $n_1 \cdot v_1^n$ over $[0, T]$. Then we have

$$\mathrm{Osc}(y^n, [t_1, t_2]) \leq \frac{1}{n_1 v_1^n} \max \left\{ \mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n \right\}, \tag{2.11}$$

$$\mathrm{Osc}(z^n, [t_1, t_2)) \leq (1 + \frac{\| v_1^n \|}{n_1 v_1^n}) \max \left\{ \mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n \right\}. \tag{2.12}$$

Finally, let $C = \sup_n \max\{1 + \frac{\|v_1^n\|}{n_1 v_1^n}, \frac{1}{n_1 v_1^n}\}$, then $1 \leq C < \infty$ and C depends only on $(N, R, \mid K \mid)$ for $K \in \Xi$ because $v_1^n \to v_1$. Thus we have the theorem is true for $\mid J \mid= 1$.

Secondly suppose that the results (2.7) and (2.8) are true for $1 \leq\mid J \mid< m$. Then consider the case with $\mid J \mid= m$. The proof of the induction steps is divided into the following several parts.

Part (a): Here we claim that there exists a constant $C_1 \geq 1$ that depends only on $(N, R, \mid K \mid)$ for $K \in \Xi$, such that for each $K \in \Xi \setminus \{J\}$, if $y^n_{J \setminus K}$ does not increase on $[t_1, t_2)$ for each $n \geq 1$, then one has

$$\mathrm{Osc}(y^n, [t_1, t_2]) \leq C_1 \max \left\{ \mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n \right\}, \tag{2.13}$$

$$\mathrm{Osc}(z^n, [t_1, t_2]) \leq C_1 \max \left\{ \mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n \right\}. \tag{2.14}$$

In fact, for each $t \in [0, t_2 - t_1)$, we have

$$z^n(t+t_1) = z^n(t_1) + (x^n(t+t_1) - x^n(t_1)) + \sum_{i \in K} v_i^n(y_i^n(t+t_1) - y_i^n(t_1)). \qquad (2.15)$$

It follows that $(z^n(\cdot + t_1), y_K^n(\cdot + t_1) - y_K^n(t_1))$ is an $(\mathbf{S}_K^n, R_K^n)$-regulator of $z^n(t_1)+$ $x^n(\cdot + t_1)$-$x^n(t_1)$ during $[0, t_2 - t_1)$. If $K = \emptyset$, then y^n does not increase on $[t_1, t_2)$, then (2.7) and (2.8) trivially hold with $C_1 = 1$. If $K \neq \emptyset$, then K is maximal and conditions $(A.1)$ and $(A.2)$ hold for $(\mathbf{S}_K^n, R_K^n)$ by Lemma 4.2 in Dai and William [16]. So, by the induction assumption and $\mid K \mid < m$, there exists a constant $C_K \geq 1$ that depends only on $(N_K, R_K, \mid K' \mid)$ for all $K' \in \Xi$, $K' \in K$ and $N_K = \{n_i, i \in K\}$ such that for any $t_2' < t_2$,

$$\begin{aligned} \mathrm{Osc}(y^n, [t_1, t_2']) &= \mathrm{Osc}(y^n(\cdot + t_1), [0, t_2' - t_1]) \\ &\leq C_K \max\left\{\mathrm{Osc}(x^n(\cdot + t_1) - x^n(t_1) + z(t_1), \Gamma^n\right\} \\ &= C_K \max\left\{\mathrm{Osc}(x^n, [t_1, t_2']), \Gamma^n\right\} \\ &\leq C_K \max\left\{\mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n\right\}. \end{aligned} \qquad (2.16)$$

Since the jump sizes of y^n are bounded by Γ^n and $z^n(t) = x^n(t) + R^n y^n(t)$, we have

$$\mathrm{Osc}(y^n, [t_1, t_2]) \leq C_K \max\{\mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n\},$$
$$\mathrm{Osc}(z^n, [t_1, t_2]) \leq C_K' \max\{\mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n\},$$

where $C_K' = \sup_n (1 + \parallel R^n \parallel) C_K$. Taking C_1 to be the maximum of the $C_K's$ for K running through $\Xi \setminus \{J\}$, we have (2.7) and (2.8) are true.

For parts (b) and (c), let ε^n=max$\{\mathrm{Osc}(x^n, [t_1, t_2]), \Gamma^n\}$ for each $n \geq 1$. Without loss of generality, we suppose that $\varepsilon^n > 0$. By lemma 4.1 in J.Dai and R.Williams [16], $z^n(t_1) \in F_K^{n, C_1 \varepsilon^n}$ for some $K \in \Xi$.

Part (b): Suppose that the K found above is not J. Then, for all $i \in J \setminus K$, $d(z^n(t_1), F_i^n) \geq n_i z^n(t_1) - b_i^n > C_1 \varepsilon^n$. Applying the result in part (a) to intervals $[t_1, t_2')$ with $t_2' \leq t_2$ shows that $z^n(t)$ does not reach F_i^n for any $i \in J \setminus K$ during

$[t_1, t_2)$ and therefore $y^n_{J\setminus K}$ does not increase on $[t_1, t_2)$. Thus, by part (a), we have that (2.7) and (2.8) hold in this case. In fact, if there exists such a t_2', $t_1 < t_2' < t_2$ such that $n_i z^n(t) - b^n_i$ does not reach F^n_i during $[t_1, t_2')$ and hits F^n_i at t_2'. Since $n_i z^n(t) - b^n_i$ is right continuous and greater than zero, then $t_2' > t_1$ can be guaranteed. By part (a), we have

$$\begin{aligned} n_i z^n(t_2') - b^n_i &= n_i(z^n(t_2') - z^n(t_1)) + n_i z^n(t_1) - b^n_i \\ &> (-1)C_1 \max\{\text{Osc}(x^n, [t_1, t_2']), \Gamma^n\} + C_1 \max\{\text{Osc}(x^n, [t_1, t_2']), \Gamma^n\} \\ &\geq 0. \end{aligned}$$

This is a contradiction. Therefore, $t_2' = t_2$ and thus part (b) is true.

Part (c): Suppose that the K described before part (b) is equal to J. Since $z^n(t_1) \in F^{n,C_1\varepsilon^n}_J$, then by Lemma $B.1$ in Dai and Williams [16], we have $d(z^n(t_1), F^n_i) \leq C_2\varepsilon^n$ where $C_2 = C_1(Cm)$ and C depends only on N. Then one of the following two cases holds:

(i) $d(z^n(t), F^n_i) \leq 2C_2\varepsilon^n$ for all $t \in [t_1, t_2]$ and $i \in J$. Then we have

$$0 \leq n_i z^n(t) - b^n_i \leq d(z^n(t), F^n_i) \leq 2C_2\varepsilon^n \text{ for all } t \in [t_1, t_2]. \tag{2.17}$$

Furthermore, we get

$$\text{Osc}(n_i z^n, [t_1, t_2]) \leq 4C_2\varepsilon^n. \tag{2.18}$$

Now, since $K = J$ is maximal, then by condition $(A.1)$ and the explanation at the beginning of this proof, there exists a positive linear combination $\eta = \sum_{i\in J}\gamma_i n_i$ ($\gamma_i > 0$, for all i) of the $\{n_i, i \in J\}$ such that $\eta v^n_i > 0$ for all $i \in J$. Then we have

$$\eta z^n(t) = \eta x^n(t) + \sum_{i\in J}(\eta v^n_i) y^n_i(t) \text{ for all } t \in [0, T]. \tag{2.19}$$

By (2.18), (2.19) and the fact which the y^n_i are non- decreasing, we have that there exists a constant C_3' depending only on $(N, R, \mid K \mid)$ for all $K \in \Xi$ such that

$$\min_{i\in J}(\eta v^n_i)\text{Osc}(y^n_1 + ... + y^n_m, [t_1, t_2]) \tag{2.20}$$

$$
\begin{aligned}
&\leq \text{Osc}(\eta z^n, [t_1, t_2]) + \text{Osc}(\eta x^n, [t_1, t_2]) \\
&\leq \sum_{i \in J} \gamma_i \text{Osc}(n_i z^n, [t_1, t_2]) + \text{Osc}(n_i x^n, [t_1, t_2]) \\
&\leq C_3' \varepsilon^n.
\end{aligned}
$$

Then by (2.20) and $z^n = x^n + R^n y^n$, we have

$$
\begin{aligned}
\text{Osc}(y_i^n, [t_1, t_2] &\leq \text{Osc}(y_1^n + ... + y_m^n, [t_1, t_2]) \\
&= \frac{C_3' \varepsilon^n}{\min_{i \in J}(\eta v_i^n)}
\end{aligned}
$$

$$
\text{Osc}(z^n, [t_1, t_2]) \leq (1 + \frac{C_3' \parallel R^n \parallel}{\min_{i \in J}(\eta v_i^n)}) \varepsilon^n.
$$

Finally, let $C_3 = \sup_n (1 + \frac{C_3' \| R^n \|}{\min_{i \in J}(\eta v_i^n)})$. Since $R^n \to R$, then $C_3 < \infty$ and it depends only on $(N, R, \mid K \mid)$.

(ii) There is $i \in J$ and $t_3 \in [t_1, t_2]$ such that$d(z^n(t_3), F_i^n) > 2C_2 \varepsilon^n$. Define

$$
t_1' = inf \, \{t > t_1 : d(z^n(t), F_i^n) > 2C_2 \varepsilon^n \text{ for some } i \in J\} \tag{2.21}
$$

By the existence of left limit of $z^n(t)$, for any small enough $\delta > 0$ and any $i \in J$, we have $d(z^n(t), F_i^n) \leq 2C_2 \varepsilon^n \;\; for \; t \in [t_1, t_1' - \delta]$. Thus by use of part $c(i)$, we see that

$$
\text{Osc}(y^n, [t_1, t_1' - \delta]) \leq C_3 \varepsilon^n, \; \text{Osc}(z^n, [t_1, t_1' - \delta]) \leq C_3 \varepsilon^n. \tag{2.22}
$$

Let $\delta \to 0$ in the above formulas, the following facts are true since y^n and z^n have left limits,

$$
\text{Osc}(y^n, [t_1, t_1')) \leq C_3 \varepsilon^n, \; \text{Osc}(z^n, [t_1, t_1')) \leq C_3 \varepsilon^n. \tag{2.23}
$$

Over $[t_1', t_2]$, by lemma 4.1 in Dai and Williams [16], we have $z^n(t_1') \in F_K^{n, C_1 \varepsilon^n} \; for \; K \in \Xi \setminus \{J\}$, and therefore we have the case in part (b) over $[t_1', t_2]$. Thus,

$$
\text{Osc}(z^n, [t_1', t_2]) \leq C_1 \varepsilon^n, \; \text{Osc}(y^n, [t_1', t_2]) \leq C_1 \varepsilon^n. \tag{2.24}
$$

By (2.23) and (2.24), we have

$$\begin{aligned}\mathrm{Osc}(z^n,[t_1,t_2]) &\leq (C_1+C_3)\varepsilon^n+\Gamma^n\\ &\leq (1+C_1+C_3)\varepsilon^n\\ \mathrm{Osc}(y^n,[t_1,t_2]) &\leq (1+C_1+C_3)\varepsilon^n.\end{aligned}$$

Take $C_4 = 1 + C_1 + C_3$ which depends only on $(N, R, \mid K \mid)$ for all $K \in \Xi$. Then we finish the proof. □

3.3 Weakly Relative Compactness and Convergence.

In this section, we discuss weakly relative compactness and convergence properties of stochastic processes come up in $(\mathbf{S}, R)$-regulation problems. As a preliminary, we present the following lemma which is an extension of lemma 2.4 in Dai and Williams [16].

Lemma 3.3 . *Suppose z^n converges to z in $D_{R^d}[0,\infty)$, y^n converges to y in $D_{R^+}[0,\infty)$ and $y \in C_{R^+}[0,R^+)$. If y^n, y are non-decreasing. Then, for any $f \in C_b(R^d)$, we have*

$$\int_0^t f(z^n(s))dy^n(s) \to \int_0^t f(z(s))dy(s) \text{ as } n\to\infty \tag{3.1}$$

uniformly for t in any compact subset of $[0,\infty)$.

Proof. Notice that $z^n \to z$ in $D_{R^d}[0,\infty)$, then by Proposition 3.5.3 and Remark 3.5.4 in Ethier and Kurtz [18] or page 112 in Billingsley [3], there exists a sequence $\{\gamma_n\}$ of continuous, strictly increasing functions mapping $[0,\infty)$ onto $[0,\infty)$ such that, as

$n \to \infty$, we have

$$z^n(\gamma_n(t)) \to z(t) \text{ u.o.c. in } t \text{ and } \gamma_n(t) \to t. \tag{3.2}$$

Now, fix $t > 0$ and observe that for all $u \in [0, t]$,

$$\begin{aligned} \textstyle\int_0^u \quad & f(z^n(s))dy^n(s) - \int_0^u f(z(s))dy(s) \\ = \quad & \int_0^{\gamma_n^{-1}(u)} (f(z^n(\gamma_n(s))) - f(z(s)))dy^n(\gamma_n(s)) \\ & + \int_u^{\gamma_n^{-1}(u)} f(z(s))dy^n(\gamma_n(s)) \\ & + \int_0^u f(z(s))d(y^n(\gamma_n) - y)(s). \end{aligned} \tag{3.3}$$

The first term on the right hand side of (3.3) converges to zero as $n \to \infty$ uniformly on $u \in$ [0,t] since it is bounded by

$$\max_{0 \le s \le \gamma_n^{-1}(t)} \mid f(z^n(\gamma_n(s)) - f(z(s)) \mid y^n(t)$$

and since $f \in C_b(R^d)$, $y(t)$ is continuous, $y^n(t) \to y(t)$.

The second term tends to zero since it is dominated by

$$\begin{aligned} \| \quad & f \|_\infty \sup_{0 \le u \le t} \mid y^n(u) - y^n(\gamma_n(u)) \mid \\ \le \quad & \| f \|_\infty (\sup_{0 \le u \le t} \mid y^n(u) - y(u) \mid + \sup_{0 \le u \le t} \mid y(u) - y(\gamma_n(u)) \mid \\ & + \sup_{0 \le u \le t} \mid y(\gamma_n(u)) - y^n(\gamma_n(u)) \mid) \end{aligned}$$

and since $y(t)$ is continuous, which implies that $y^n(t) \to y(t)$ u.o.c..

Finally, we claim that the third term tends to zero. In fact, since $f(z(\cdot)) \in D_R[0, \infty)$, by Theorem 3.5.6, Proposition 3.5.3 and Remark 3.5.4 of Ethier and Kurtz [18], there is a sequence of step functions $\{g^k(\cdot)\}_{k=1}^\infty$ of the form:

$$g^k(\cdot) = \sum_{i=1}^{l_k} g^k(t_i^k) I_{[t_i^k, t_{i+1}^k)}(\cdot) \tag{3.4}$$

where $0 = t_1^k < t_2^k < ... < t_{l_{k+1}}^k < \infty$ and $\sup_{0 \le s \le t} \mid f(z(s)) - g^k(s) \mid \to 0$ as $k \to \infty$. Then, we have

$$\mid \quad \int_0^u f(z(s))d(y^n(\gamma_n) - y)(s) \mid \tag{3.5}$$

$$\leq \ \mid \int_0^u (f(z(s)) - g^k(s))d(y^n(\gamma_n) - y)(s) \mid + \mid \int_0^u g^k(s)d(y^n(\gamma_n) - y)(s) \mid$$
$$\leq \sup_{0\leq s\leq t} \mid f(z(s)) - g^k(s) \mid (y^n(\gamma_n)(t) + y(t))$$
$$+ \sup_{0\leq u\leq t} \sum_{i=1}^{l_k} \mid g^k(t_i^k \bigwedge u) \mid\mid (y^n(\gamma_n) - y)(t_{i+1}^k \bigwedge u) - (y^n(\gamma_n) - y)(t_i^k \bigwedge u) \mid .$$

Notice that $y^n(\cdot) \to y(\cdot) \in C[0,\infty)$ u.o.c., then we have $y^n(\cdot)$ is uniformly bounded on any compact subset of $[0,\infty)$. Furthermore, for fixed k, the last term of (3.5) tends to zero as $n \to \infty$, the desired result then follows , that is,

$$\lim_{n\to\infty} \mid \int_0^u f(z(s))d(y^n(\gamma_n) - y)(s) \mid \leq M \sup_{0\leq s\leq t} \mid f(z(s)) - g^k(s) \mid . \tag{3.6}$$

Let $k \to \infty$, we have

$$\lim_{n\to\infty} \mid \int_0^u f(z(s))d(y^n(\gamma_n) - y)(s) \mid = 0 \tag{3.7}$$

uniformly in $u \in [0,t]$. Thus we complete the proof. □

Now we introduce more notations in the space $D_{R^d}[0,\infty)$. For $T > 0$ and $\delta > 0$, let

$$W_x(T_0) = \mathrm{Osc}(x, T_0) = \sup\{\mid x(s) - x(t) \mid, s, t \in T_0\},\ T_0 \subset [0,T], \tag{3.8}$$
$$W'_x(\delta, T) = \inf_{\{t_j\}} \max_{0<j\leq r} W_x[t_{j-1}, t_j) \tag{3.9}$$

where the infimum extends over the finite sets $\{t_i\}$ of points satisfying $0 = t_0 < t_1 < ... < t_r = T$ and $t_j - t_{j-1} > \delta$ for $j = 1, ..., r$. Define

$$\parallel x \parallel_T = \sup_{0\leq t\leq T} \mid x(t) \mid . \tag{3.10}$$

Then we have the following theorem concerning the relative compactness of a sequence of stochastic processes.

Theorem 3.2 *Let $\{X^n(\cdot)\}$ be a sequence of stochastic processes with sample paths in $D_{R^d}[0,\infty)$ and $X^n(0) \in \mathbf{S}^n$ and $\{Y^n(\cdot), Z^n(\cdot)\}$ be a corresponding $(\mathbf{S}^n, R^n)$-regulation* *processes. Then* $\{X^n(\cdot), Y^n(\cdot), Z^n(\cdot)\}$ *is relatively compact if* $X^n(\cdot) \Rightarrow X(\cdot) \subset D_{R^d}[0,\infty)$ *with* $X(0) \in \mathbf{S}$, $\mathbf{S}^n \to \mathbf{S}$, $\Gamma^n \to 0$, $R^n \to R$ *and* (N, R) *satisfying* (*A*.1) *and* (*A*.2).

where $\Rightarrow$ denotes convergence in distribution and $\{X^n(\cdot), Y^n(\cdot), Z^n(\cdot)\}$ have sample paths in the product space $\Omega_{\mathbf{S}^n} = D^{S^n}_{R^d}[0,\infty) \times D^+_{R^m}[0,\infty) \times D_{\mathbf{S}^n}[0,\infty)$.

Proof. The main tools of the proof are Theorem 7.2 and Corollary 7.4 in Chapter 3 in Ethier and Kurtz [18]. Since $X_n(\cdot) \Rightarrow X(\cdot)$, then by Remark 7.3 in Chapter 3 in Ethier and Kurtz [18], the following compact containment condition holds. Namely, for every $\eta > 0$ and $T > 0$, there is a positive constant M such that

$$\inf_n P\{\mid X_n(t) \mid \leq M,\ 0 \leq t \leq T\} \geq 1 - \eta. \tag{3.11}$$

By Theorem 3.1, the following oscillation result holds

$$\mathrm{Osc}((X^n, Y^n, Z^n), [t_1, t_2]) \leq C\ \max \mathrm{Osc}(X^n, [t_1, t_2]), \Gamma^n\} \tag{3.12}$$

where $[t_1, t_2] \subset [0, T]$ and C is a constant which depends only on $(N, R, \mid K \mid)$ and $C \geq 1$. Thus we have

$$\begin{aligned} &\mid (X^n(t), Y^n(t), Z^n(t)) \mid \\ &\leq \mid (X^n(0), Y^n(0), Z^n(0) \mid + C\ \max \mathrm{Osc}(X^n, [0, T]), \Gamma^n\} \\ &\leq 2 \mid X^n(0) \mid + C\ [2\ \|X^n\|_T + (2d+1)] \end{aligned} \tag{3.13}$$

where we used the assumption $\Gamma^n < 2d+1$ without loss of generality since $\Gamma^n \to 0$ and $0 \leq t \leq T$. Combining (3.11) and (3.13) together, the following compact containment condition holds for $\{(X^n, Y^n, Z^n)\}$.

$$\begin{aligned} &\inf_n P\{\mid (X^n(t), Y^n(t), Z^n(t)) \mid \leq 2M + (2M + 2d + 1)C, 0 \leq t \leq T\} \\ &\geq \inf_n P\{\mid X^n(t) \mid \leq M, 0 \leq t \leq T\} \\ &\geq 1 - \eta. \end{aligned} \tag{3.14}$$

It follows from (3.14) that condition (a) in Corollary 7.4 in Ethier and Kurtz [18] is true. For the condition (b) in that corollary , noting (3.8), (3.9) and (3.11), we have, for $\eta > 0$, $T > 0$ and $\delta > 0$,

$$W'_{(X^n,Y^n,Z^n)}(\delta,T) \leq C\, W'_{X^n}(\delta,T) + C\,\Gamma^n. \tag{3.15}$$

Since $\Gamma^n \to 0$, then $C\,\Gamma^n \leq \frac{1}{2}\eta$ for n large enough. So, there exists some $\delta > 0$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\ P\{W'_{(X^n,Y^n.Z^n)}(\delta,T) \geq \eta\} \\
&\leq \limsup_{n\to\infty}\ P\{C\,W'_{X^n}(\delta,T) \geq \frac{1}{2}\eta\} \\
&\leq \frac{\eta}{2C} \\
&\leq \eta.
\end{aligned}$$

Here because $X^n \Rightarrow X$ and Corollary 7.4(b) in Ethier and Kurtz [18], the second inequality is true. Thus condition (b) is true in Corollary 7.4. Therefore, $\{(X^n(\cdot), Y^n(\cdot), Z^n(\cdot))\}$ is relatively compact. Thus we finish the proof. □

Next we present more concrete properties about the regulation processes.

Theorem 3.3 *Assuming the jump sizes of $Z^n(\cdot)$ and $Y^n(\cdot)$ are bounded by Γ^n. Then under the conditions of the previous theorem, any weak limit (X,Y,Z) of (X^n,Y^n,Z^n) results in an SRBM Z defined on the filtered probability space $(C_{\mathbf{S}}, \mathcal{M}, \mathcal{M}_t, Q_\pi)$ with $Q_\pi = P(X,Y,Z)^{-1}$ if under Q_π, X is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ such that $\{X(t) - \theta t, \mathcal{M}_t, t \geq 0\}$ is a martingale and $Q_\pi X^{-1}(0) = \pi$.*

Proof. From Theorem 3.1, we know that (X^n, Y^n, Z^n) is weakly relatively compact. Let (X,Y,Z) be a weak limit of the sequence. So there is a subsequence of (X^n,Y^n,Z^n) that converges to (X,Y,Z). For notational convenience, we assume the sequence itself converges, that is,

$$(X^n, Y^n, Z^n) \Rightarrow (X, Y, Z). \tag{3.16}$$

Let

$$Q_n \equiv P(X^n, Y^n, Z^n)^{-1}, \tag{3.17}$$

$$Q_\pi \equiv P(X, Y, Z)^{-1}. \tag{3.18}$$

Then we have

$$Q_n \Rightarrow Q_\pi. \tag{3.19}$$

Since $\Gamma^n \to 0$, Z and Y are continuous. Moreover, by the weak convergence and Skorohod representation theorem, we can find a common supporting probability space such that

$$Z^n(\cdot) \to Z(\cdot), \text{ u.o.c., } a.s.$$

$$n_i \cdot Z^n - b_i^n \geq 0, \ a.s.$$

Thus $n_i Z(\cdot) - b_i \geq 0$ $a.s.$ and therefore $Z(\cdot) \in \mathbf{S}$ almost surely. The remaining properties in $(3)(a),(b)$ of Definition 3.1.3 follows from the corresponding properties of Y^n and weak convergence.

Finally, we show that (1.3) and $(3)(c)$ in Definition 3.1.3 are true. Notice that

$$(X^n(\cdot), Y^n(\cdot), Z^n(\cdot)) \ \Rightarrow \ (X(\cdot), Y(\cdot), Z(\cdot)) \in C_{\mathbf{S}}. \tag{3.20}$$

Then,the followings are true on some common supporting probability space by Skorohod representation theorem,

$$(X^n(\cdot), Y^n(\cdot), Z^n(\cdot)) \to (X(\cdot), Y(\cdot), Z(\cdot)), \text{ u.o.c.., } a.s.$$

$$Z^n(\cdot) = X^n(\cdot) + R^n Y^n(\cdot), \ a.s.$$

Therefore (1.3) is true since $R^n \to R$.

Now we turn to prove that $3(c)$ in Definition 3.1.3 is true. Since $n_i Z(s) - b_i \geq 0$ Q_π-a.s. for all $s \geq 0$, where equality holds only if $Z(s) \in F_i$, and Y_i is almost surely non-decreasing, then it suffices to prove that, for each $i \in J$,

$$\int_0^{\cdot} ((n_i Z(s) - b_i) \bigwedge 1)\, dY_i(s) = 0 \qquad Q_\pi\text{−}a.s. \tag{3.21}$$

In fact, notice the weak convergence, Lemma 3.3 and $b_i^n \to b_i$, we have the integral process in (3.21) under Q_π is the weak limit point of the sequence

$$\left\{ \left(\int_0^{\cdot} ((n_i Z^n(s) - b_i^n) \bigwedge 1)\, dY_i^n(s); Q_n \right) \right\}. \tag{3.22}$$

Now all of the integral processes in (3.22) are zero almost surely under Q_n. Then we know that (3.21) is true. Therefore we complete the proof of the theorem. □

CHAPTER 4

Heavy Traffic Limit Theorems

In this chapter, our analysis will mainly focus on intree-like queueing network under communication blocking. Other types of networks can be analyzed by employing the similar procedure.

4.1 System Representation

In this section, we first give some review about the intree-like queueing network model introduced in Chapter 2. Then we derive the main equation that governs the dynamics of the queue length process. Finally, a completely-$\mathcal{S}$ property on the reflection matrix is presented.

Recall that $Q_i(t)$ is the number of customers at station i, including possibly the one being served, $Y_i^b(t)$ is the amount of time that buffer i is full in time interval $[0,t]$ and $Y_i^0(t)$ is the amount of time that server i has been idle while server i is not blocked in $[0,t]$, $B_i(t)$ is the cumulative amount of time that server i is busy in $[0,t]$, and $B_i^0(t)$ is the cumulative amount of time that buffer i is not full during time interval $[0,t]$. As a matter of definition, we have

$$B_i^0(t) = t - Y_i^b(t), \tag{1.1}$$

$$B_i(t) = t - (Y_i^0(t) + Y_{\sigma(i)}^b(t)). \tag{1.2}$$

We model the external arrival processes in the following way. The arrival process at station i is turned on only when the buffer at the station is not full. Therefore $E_i(B_i^0(t)$ is the number of external arrivals to station i by time t and $S_i(B_i(t))$ is the number of departures from station i by time t. The routing is deterministic. That is, customers leave station i will all go next to station $\sigma(i) \in J \equiv \{1, 2, ..d\}$ or leave the system. When the buffer at station $\sigma(i)$ is full, server i stops working although a customer may still occupy station i.

Then the main equation that governs the dynamics of the queue length process can be written as

$$Q_i(t) = Q_i(0) + E_i(B_i^0(t)) + \sum_{j \in J, \sigma(j)=i} S_j(B_j(t)) - S_i(B_i(t)), \; i \in J, \tag{1.3}$$

where $Q_i(0)$ is the initial queue length at station i. Let

$$\hat{E}_i(t) = E_i(t) - \lambda_i t, \tag{1.4}$$

$$\hat{S}_i(t) = S_i(t) - \mu_i t. \tag{1.5}$$

It follows from (1.3) that for $i \in J$,

$$\begin{aligned}
Q_i(t) &= Q_i(0) + E_i(B_i^0(t)) + \sum_{j \in J, \sigma(j)=i} S_j(B_j(t)) - S_i(B_i(t)), \\
&= Q_i(0) + \hat{E}_i(B_i^0(t)) + \lambda_i B_i^0(t) + \sum_{j \in J, \sigma(j)=i} (\hat{S}_j(B_j(t)) + \mu_j B_j(t)) \\
&\quad -\hat{S}_i(B_i(t)) - \mu_i B_i(t), \\
&= Q_i(0) + \hat{E}_i(B_i^0(t)) + \sum_{j \in J, \sigma(j)=i} \hat{S}_j(B_j(t)) - \hat{S}_i(B_i(t)) \\
&\quad +\lambda_i B_i^0(t) + \sum_{j \in J, \sigma(j)=i} \mu_j(t - Y_j^0(t) - Y_{\sigma(j)}^b(t)) - \mu_i(t - Y_i^0(t) - Y_{\sigma(i)}^b(t)) \\
&= Q_i(0) + \hat{E}_i(B_i^0(t)) + \sum_{j \in J, \sigma(j)=i} \hat{S}_j(B_j(t)) - \hat{S}_i(B_i(t)) \\
&\quad + \left(\lambda_i + \sum_{j \in J, \sigma(j)=i} \mu_j - \mu_i\right) t + \mu_i Y_i^0(t) + \mu_i Y_{\sigma(i)}^b(t) \\
&\quad - \sum_{j \in J, \sigma(j)=i} \mu_j Y_j^0(t) - \left(\lambda_i + \sum_{j \in J, \sigma(j)=i} \mu_j\right) Y_i^b(t)
\end{aligned}$$

$$\begin{aligned} &= X_i(t) + \mu_i Y_i^0(t) + \mu_i Y_{\sigma(i)}^b(t) \\ &\quad - \sum_{j \in J, \sigma(j)=i} \mu_j Y_j^0(t) - \left(\lambda_i + \sum_{j \in J, \sigma(j)=i} \mu_j \right) Y_i^b(t), \end{aligned} \tag{1.6}$$

where

$$X_i(t) = \xi_i(t) + \theta_i t, \tag{1.7}$$

$$\xi_i(t) = Q_i(0) + \hat{E}_i(B_i^0(t)) + \sum_{j \in J, \sigma(j)=i} \hat{S}_j(B_j(t)) - \hat{S}_i(B_i(t)), \tag{1.8}$$

$$\theta_i = \left(\lambda_i + \sum_{j \in J, \sigma(j)=i} \mu_j - \mu_i \right) t. \tag{1.9}$$

Let $Q(t) = (Q_1(t), ..., Q_d(t))'$, $X(t) = (X_1(t), ..., X_d(t))'$, $Y^0(t) = (Y_1^0(t), ..., Y_d^0(t))'$, $Y^b(t) = (Y_1^b(t), ..., Y_d^b(t))'$, R^0 and R^b be $d \times d$ matrix given by

$$R_{ij}^0 = \begin{cases} \mu_i, & \text{if } i = j, \\ -\mu_j, & \text{if } j < i \text{ and } \sigma(j) = i, \\ 0 & \text{if } j < i \text{ and } \sigma(j) \neq i \text{ or } j > i, \end{cases} \tag{1.10}$$

$$R_{ij}^b = \begin{cases} -(\lambda_i + \sum_{l<i, \sigma(l)=i} \mu_l), & \text{if } i = j, \\ \mu_i, & \text{if } j > i \text{ and } \sigma(i) = j, \\ 0, & \text{if } j > i \text{ and } \sigma(i) \neq j \text{ or } j < i. \end{cases} \tag{1.11}$$

Then we have the following

$$Q(t) = X(t) + R^0 Y^0(t) + R^b Y^b(t),\ t \geq 0, \tag{1.12}$$

$$0 \leq Q_i(t) \leq b_i,\ t \geq 0, \tag{1.13}$$

$$Y_i^0(0) = 0,\ Y_i^0(\cdot) \text{ is continuous and nondecreasing},\ i \in J, \tag{1.14}$$

$$Y_i^b(0) = 0,\ Y_i^b(\cdot) \text{ is continuous and nondecreasing},\ i \in J, \tag{1.15}$$

$$Y^0(\cdot) \text{ increases only at times t when } Q_i(t) = 0,\ i \in J, \tag{1.16}$$

$$Y_i^b(\cdot) \text{ increases only at times t when } Q_i(t) = b_i,\ i \in J. \tag{1.17}$$

Let P be the $d \times d$ matrix with $P_{ij} = 1$ if $j = \sigma(i)$ and zero otherwise, that is, P is the routing matrix. Then R^0 and $X(t)$ can be rewritten as

$$R^0 = (I - P')\text{diag}(\mu_1, ..., \mu_d) \tag{1.18}$$

$$X(t) = Q(0) + \hat{E}(B^0(t)) - (I - P')\hat{S}(B(t)) + \theta t, \tag{1.19}$$

where $\theta = (\theta_1, ..., \theta_d)'$. Recall that the d-dimensional state space with $2d$ boundary faces are given as follows

$$\mathbf{S} \equiv \{x = (x_1, ..., x_d)' \in R^d : 0 \le x_i \le b_i, i \in J\},$$

$$F_i \equiv \{x \in S : x_i = 0\},\ F_{i+d} = \{x \in S : x_i = b_i\} \text{ for } i = 1, ..., d.$$

Let N denote the $2d \times d$ matrix whose i^{th} row is given by the row vector n_i' of the unit normal to face F_i. That is,

$$N = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & 1 \\ -- & --- & ------ & -- \\ -1 & 0 & ... & 0 \\ 0 & -1 & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & -1 \end{pmatrix}. \tag{1.20}$$

Let $R = (R^0, R^b)$, where R^0 and R^b are defined in (1.10) and (1.11). Then matrices N and R satisfy conditions $(A.1)$ and $(A.2)$ introduced in Chapter 3.

Lemma 4.1 *NR satisfies condition $(A.1)$ and $(NR)'$ satisfies condition $(A.2)$,*

Proof. Since $\mathbf{S}$ is an d-dimensional box, it is a simple polyhedron. Therefore conditions $(A.1)$ and $(A.2)$ introduced in section 3.1 are equivalent, see Dai and Williams [16]. So, we only need to prove that NR satisfies condition $(A.1)$. It is equivalent to prove that any $d \times d$ subprincipal matrix M obtained from NR is completely-S. Where we exclude those subprincipal matrices which contain i^{th} row (column) and $(i+d)^{th}$ row (column) of NR simultaneously. This is due to the fact that faces F_i and F_{i+d} parallel each other. Notice that the $(2d) \times (2d)$ matrix NR can be written as,

$$NR = \begin{pmatrix} R^0 & | & R^b \\ ------- & | & ------- \\ -R^0 & | & -R^b \end{pmatrix} \tag{1.21}$$

Then,any $d \times d$ subprincipal matrix M described above has the following decomposition form,

$$M = \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} + \begin{pmatrix} 0 & A_2 \\ A_3 & 0 \end{pmatrix}. \tag{1.22}$$

Where A_1 and A_4 are subprincipal matrices of R^0 and $-R^b$ respectively. A_2 and A_3 are nonnegative matrices. Noticing the structures of R^0 and $-R^b$, we know that both of them are completely-S matrices, then we have that M is completely-$\mathcal{S}$. Thus we finish the proof. $\square$

Remark. Consider the 2-station tandem network pictured in Figure 1.2. By deleting the 2^{th} row and column, the 3^{th} row and column from the corresponding matrix NR. We see that the reflection matrix around the corner formed by faces F_1 and F_4 is $R_{14} = \begin{pmatrix} \mu_1 & \mu_1 \\ \mu_1 & \mu_1 \end{pmatrix}$ which is completely-$\mathcal{S}$. However, the uniqueness of solution for the corresponding $(\mathbf{S}, R_{14})$- regulation problem fails around the corner, see Mandelbaum [31]. Thus, by the localization method, we can only get the existence of solutions for $(\mathbf{S}, R_{14})$-regulation problem and cannot guarantee the uniqueness. Therefore the mappings $x(\cdot) \to y(\cdot)$ and $x(\cdot) \to z(\cdot)$ associated with the $(\mathbf{S}, R_{14})$-regulation problem are not Lipschitz continuous. The continuity property has played a key role in proving heavy traffic diffusion approximations to queueing networks, see Reiman [35],Johnson [29], Peterson [34], H.Chen and Mandelbaum [6], and etc.

4.2 A Heavy Traffic Limit Theorem

In order to prove a heavy traffic limit theorem, we consider a sequence of queueing networks indexed by $n \geq 1$. In the nth network, the external arrival process is $E^n(t) = \{(E_1^n(t), ..., E_d^n(t))', t \geq 0\}$. Each $E_i^n(t)$ $(i = 1, ..., d)$ associates an i.i.d interarrival time sequence $\{(1/\lambda_i^n)u_i(k), k \geq 1\}$ with mean value $1/\lambda_i^n$. The service process is $S^n(t) = \{(S_1^n(t), ..., S_d^n(t))', t \geq 0\}$. Each $S_i^n(t)$ associates an i.i.d service time sequence $\{(1/\mu_i^n)v_i(k), k \geq 1\}$ with mean value $1/\mu_i^n$. The buffer size vector is $b^n = (b_1^n, ..., b_d^n)'$. However, the routing does not depend on n. Let $Q^n = \{(Q_1^n(t), ..., Q_d^n(t))', t \geq 0\}$ be the queue length process associated with the nth network. Let $Y_i^{b,n}(t)$ be the cumulative time that station i is full. Therefore, the cumulative time that station i is blocked is $Y_{\sigma(i)}^{b,n}(t)$ in the time interval $[0, t]$. Let $Y_i^{0,n}(t)$ be the cumulative time that station i is empty while station i is not blocked in $[0, t]$. Put $Y^{0,n} = \{(Y_1^{0,n}(t), ..., Y_d^{0,n}(t))', t \geq 0\}$ and $Y^{b,n} = \{(Y_1^{b,n}(t), ..., Y_d^{b,n}(t))', t \geq 0\}$ and $Y^n(t) = (Y^{0,n}(t), Y^{0,n}(t))'$. Let $B_i^{0,n}(t)$ be the cumulative amount of time that buffer i is not full and $B_i^n(t)$ be the cumulative amount of time that server i is busy in time interval $[0, t]$. We assume

$$Q^n(0), E_1^n, ..., E_d^n, S_1^n, ..., S_d^n \text{ are independent for each } n. \tag{2.1}$$

Moreover suppose, as $n \to \infty$,

$$\lambda_i^n \to \lambda_i, \ , \ \mu_i^n \to \mu_i > 0 \tag{2.2}$$

$$\sqrt{n}(\lambda_i^n - \sum_{j \in J, \sigma(j)=i} \mu_j^n - \mu_i^n), \to \theta_i \tag{2.3}$$

$$\frac{1}{\sqrt{n}} b_i^n \to b_i > 0 \text{ for } i = 1, ..., d \tag{2.4}$$

where λ_i, μ_i and θ_i are some constants. The assumptions (2.3) and (2.4) are called heavy traffic conditions. For the initial state $Q^n(0)$, we assume that as $n \to \infty$

$$\tilde{Q}^n(0) \equiv \frac{1}{\sqrt{n}} Q^n(0) \Rightarrow \tilde{Q}(0) \tag{2.5}$$

for some random vector $\tilde{Q}(0)$, where $'' \Rightarrow''$ denotes convergence in distribution. Furthermore, due to Functional Central Limit Theorem, we have the following facts that

$$\tilde{E}^n(\cdot) \equiv \frac{1}{\sqrt{n}}\hat{E}^n(n\cdot) = \sqrt{n}(\frac{E^n(n\cdot)}{n} - \lambda^n\cdot) \Rightarrow \tilde{E}(\cdot), \tag{2.6}$$

$$\tilde{S}^n(\cdot) \equiv \frac{1}{\sqrt{n}}\hat{S}^n(n\cdot) = \sqrt{n}(\frac{S^n(n\cdot)}{n} - \mu^n\cdot) \Rightarrow \tilde{S}(\cdot) \tag{2.7}$$

where $\lambda^n = (\lambda_1^n, ..., \lambda_d^n)'$ and $\mu^n = (\mu_1^n, ..., \mu_d^n)$. By the independent assumption (2.1), we have

$$(\tilde{Q}^n(0), \tilde{E}^n(\cdot), \tilde{S}^n(\cdot)) \Rightarrow (\tilde{Q}(0), \tilde{E}(\cdot), \tilde{S}(\cdot)).$$

Where $\tilde{E}(\cdot)$ and $\tilde{S}(\cdot)$ are independent d-dimensional zero-drift Brownian motion with covariance matrices Γ^a and Γ^s as follows,

$$\Gamma^a = \text{diag}(\lambda_1 c_{a,1}^2, ..., \lambda_d c_{a,d}^2) = \text{diag}(\lambda)\text{diag}(c_a^2),$$

$$\Gamma^s = \text{diag}(\mu_1 c_{s,1}^2, ..., \mu_d c_{s,d}^2) = \text{diag}(\mu)\text{diag}(c_s^2),$$

Now we are ready to state the heavy traffic limit theorem.

Theorem 4.1 *Under assumptions (2.1)-(2.5), we have*

$$\left(\frac{1}{\sqrt{n}}Q^n(n\cdot), \frac{1}{\sqrt{n}}Y^{0,n}(\cdot), \frac{1}{\sqrt{n}}Y^{b,n}(n\cdot)\right) \Rightarrow \left(\tilde{Q}(\cdot), \tilde{Y}^0(\cdot), \tilde{Y}^b(\cdot)\right) \text{ as } n \to \infty, \tag{2.8}$$

where $\tilde{Q}(\cdot)$, together with $\tilde{Y}^0(\cdot)$ and $\tilde{Y}^b(\cdot)$ is an $(\mathbf{S}, \theta, \Gamma, R)$-semimartingale reflecting Brownian motion with the initial distribution $\mathbf{P}\tilde{Q}^{-1}(0) = \pi$ on filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. $\{\mathcal{F}_t\}$ is the filtration generated by $\tilde{X}$, $\tilde{Y}^0$ and $\tilde{Y}^b$, augmented with all $\mathbf{P}$-null sets. The process $(\tilde{Q}, \tilde{Y}^0, \tilde{Y}^b)$ is uniquely determined in distribution from the following equations.

$$\tilde{Q}_i(t) = \tilde{X}_i(t) + \sum_{j\in J} R_{ij}^0 \tilde{Y}_j^0(t) + \sum_{j\in J} R_{ij}^b \tilde{Y}_j^b(t), \ \mathbf{P}-a.s., \ t \geq 0, \tag{2.9}$$

$$0 \leq \tilde{Q}_i(t) \leq b_i, \ t \geq 0, \ i \in J, \tag{2.10}$$

$$\tilde{Q}(\cdot), \ \tilde{Y}^0(\cdot) \textit{ and } \tilde{Y}^b \textit{ are } \{\mathcal{F}_t\} - \textit{adapted}, \tag{2.11}$$

$$\tilde{Y}_i^0(0) = 0,\ \tilde{Y}_i^0(\cdot)\ \textit{is continuous and nondecreasing},\ i \in J, \tag{2.12}$$

$$\tilde{Y}_i^b(0) = 0,\ \tilde{Y}_i^b(\cdot)\ \textit{is continuous and nondecreasing},\ i \in J, \tag{2.13}$$

$$\tilde{Y}^0(\cdot)\ \textit{increases only at times}\ t\ \textit{when}\ \tilde{Q}_i(t) = 0,\ i \in J, \tag{2.14}$$

$$\tilde{Y}_i^b(\cdot)\ \textit{increases only at times}\ t\ \textit{when}\ \tilde{Q}_i(t) = b_i,\ i \in J, \tag{2.15}$$

where

$$\tilde{X}_i(t) = \tilde{Q}(0) + \tilde{E}_i(t) + \sum_{j \in J, \sigma(j)=i} \tilde{S}_j(t) - \tilde{S}_i(t) + \theta_i t,\ t \geq 0,\ i \in J, \tag{2.16}$$

R^0 *and* R^b *are* $d \times d$ *matrices given by (1.10) and (1.11),* Γ *is the covariance matrix given by*

$$\Gamma = \Gamma^a + (I - P')\Gamma^s(I - P)$$

with $P_{ij} = 1$ *if* $j = \sigma(i)$ *and zero otherwise.* □

The proof of the above heavy traffic limit theorem is divided into the following several steps: justify a fluid limit theorem and a martingale convergence theorem.

4.2.1 Fluid Limit Theorem

Theorem 4.2 *Under assumptions (2.1)-(2.5), as* $n \to \infty$*, we have*

$$\begin{aligned}
\bar{B}_i^{0,n}(t) &= \frac{1}{n} B_i^{0,n}(nt) \Rightarrow t, \\
\bar{B}_i^{n}(t) &= \frac{1}{n} B_i^{n}(nt) \Rightarrow t, \\
\bar{Y}_i^{0,n}(t) &= \frac{1}{n} Y_i^{0,n}(nt) \Rightarrow 0, \\
\bar{Y}_i^{b,n}(t) &= \frac{1}{n} Y_i^{b,n}(nt) \Rightarrow 0.
\end{aligned}$$

Proof. First we rescale (1.12) as follows,

$$\bar{Q}^n(t) = \bar{Q}^n(0) + \bar{X}^n(t) + R^{0,n}\bar{Y}^{0,n}(t) + R^{b,n}\bar{Y}^{b,n}(t)$$

where $\bar{Q}^n(t) = \frac{1}{n}Q^n(nt)$, $\bar{X}^n(t) = \frac{1}{n}X^n(nt)$. For each n, $(\bar{Q}^n(t), \bar{Y}^{0,n}(t), \bar{Y}^{b,n}(t))$ has the properties (1.13) to (1.17) with the state space $\mathbf{S^n}$ given by

$$\mathbf{S^n} = \left\{ x \in R^d_+ : x_i \leq \bar{b}^n_i = \frac{b^n_i}{n} \right\}.$$

By (1.7)-(1.9), we have

$$\bar{X}^n_i(t) \equiv \frac{1}{n}Q^n(0) + \frac{1}{n}\hat{E}^n_i(n\bar{B}^{0,n}_i(t)) + \frac{1}{n}\sum_{j<i,\sigma(j)=i} \hat{S}^n_j(n\bar{B}^n_i(t)) - \frac{1}{n}\hat{S}^n_i(n\bar{B}^n_i(t)) + \theta^n_i t$$

Noticing that $\bar{B}^n_0(t) \leq t$, $\bar{B}^n(t) \leq t$, then by (2.2)-(2.5) and Skorohod representation theorem, we have

$$\bar{X}^n(t) \to 0, \text{ u.o.c., as } n \to \infty \tag{2.17}$$

where u.o.c. means that the convergence is uniformly on compact set.

Now since $\mathbf{S^n}$ are "boxes" of the same shape in d-dimensional space, (N, R) satisfies conditions $(A.1)$ and $(A.2)$ introduced in Chapter 3, $R^n \to R$. Then, by Theorem 3.1, we have

$$\text{Osc}(\bar{Y}^n, [s,t] \subseteq [0,T]) \leq C \text{ Osc}(\bar{X}^n, [s,t] \subseteq [0,T])$$

for any $T \geq 0$, where C depends only on R and N for n large enough.

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} \text{ Osc}(\bar{Y}^n, [s,t] \subseteq [0,T]) \\ &\leq \limsup_{n\to\infty} \text{ Osc}(\bar{Y}^n, [s,t] \subseteq [0,T]) \\ &\leq C \lim_{n\to\infty} \text{ Osc}(\bar{X}^n, [s,t] \subseteq [0,T]) \\ &= 0, \ a.s., \end{aligned} \tag{2.18}$$

where $\bar{Y}^n(t) = (\bar{Y}^{0,n}(t)', \bar{Y}^{0,n}(t)')'$. Notice that $Y^n(0) = 0$ for all n, we have

$$\lim_{n\to\infty} \bar{Y}^n(t) = 0, \text{ u.o.c., } a.s. \tag{2.19}$$

Thus we complete the proof. □

Remark. Since the weak limits in Theorem 4, 2 are constants, the weak convergence is equivalent to convergence in probability, see problem 4 in Chapter 3 of Chung [9].

4.2.2 Martingale Convergence Theorem

Here we prove an adaptedness property on a weak limit process $(\tilde{Q}, \tilde{X}, \tilde{Y})$ of $(\tilde{Q}^n, \tilde{X}^n, \tilde{Y}^n)$, where $\tilde{Q}^n(\cdot) = \frac{1}{\sqrt{n}} Q^n(n\cdot)$, $\tilde{X}^n(\cdot) = \frac{1}{\sqrt{n}} X^n(n\cdot)$ and $\tilde{Y}^n(\cdot) = \frac{1}{\sqrt{n}} Y^n(n\cdot) = \frac{1}{\sqrt{n}}(Y^{0,n}(n\cdot)', Y^{b,n}(n\cdot)')'$. Define

$$\mathcal{G}_t^n = \sigma\left\{\tilde{Q}^n(0), \tilde{E}^n(s), \tilde{S}^n(s), \tilde{Y}^n(s), s \leq t\right\}, \tag{2.20}$$

where $\tilde{Q}^n(0)$, $\tilde{E}^n(s)$ and $\tilde{S}^n(s)$ are defined in (2.5)-(2.7). Let $T_k^{i,n}$ denote the partial sum of the exogenous interarrival time sequence at the station i for the n^{th} network, that is,

$$T_k^{i,n} = \sum_{l=1}^{k} \xi_i^n(l), \; i \in J \tag{2.21}$$

with $T_0^{i,n} \equiv 0$. Then we have the following lemma.

Lemma 4.2 *For each $k \geq 0$, $T_k^{i,n}$ is a $\mathcal{G}_t^n$-stopping time. Moreover, $0 = T_0^{i,n} < T_1^{i,n} < \cdots < T_k^{i,n} \to \infty$ as $k \to \infty$ a.s. for each n and $i \in J$.*

Proof. The first claim is an immediate conclusion of $\left\{T_k^{i,n} \leq t\right\} = \{E_i(t) \geq k\}$. The second claim follows from strong law of large number. For more detailed discussion, see page 57 and Theorem $T23$ in page 303 of Bremaud [4]. $\square$

Lemma 4.3 *Let $\mathcal{G}_{T_k^{(i,n)}-}$ denote the strict past at time $T_k^{i,n}$. Namely,*

$$\mathcal{G}_{T_k^{(i,n)}-} = \sigma\left(A_t \cap \{t < T_k^{i,n}\}, A_t \in \mathcal{G}_t^n, t \geq 0\right).$$

Then, (a) $T_k^{i,n}$ is $\mathcal{G}_{T_k^{(i,n)}-}$-measurable; (b) $\xi^{i,n}(k+1)$ is independent of $\mathcal{G}_{T_k^{(i,n)}-}$.

Proof. For (a), we know, by Lemma 4.2, that $T_k^{i,n}$ is a $\mathcal{G}_t^n$-stopping time. Then the claim directly follows from Theorem $T4$ in page 298 of Bremaud [4].

For (b), let $\tau_k^{i,n}$ denote the time at which the k^{th} external customer arrives at station i. Namely,

$$\tau_k^{i,n} = T_k^{i,n} + Y_{i+d}^n(\tau_k^{i,n}) \geq T_k^{i,n} \tag{2.22}$$

where $Y_{i+d}^n(\tau_k^{i,n})$ is the total blocking time experienced by the external arrival stream i at time $\tau_k^{i,n}$. Notice that $\xi^{i,n}(k+1)$ will be the actual working time for external generator i to generate the $(k+1)^{th}$ customer from the time $\tau_k^{i,n}$ on. Due to the independence assumptions, $\xi^{i,n}(k+1)$ is independent of the history of the network before the time $\tau_k^{i,n}$. Therefore, $\xi^{i,n}(k+1)$ is independent of the σ-field $\sigma\left(\mathcal{F}_t^{\tilde{Y}^n} \cap \{t < \tau_k^{i,n}\}\right)$ for each $t \geq 0$. Notice that

$$\begin{aligned}\{t < T_k^{i,n}\} &= \left(\{t < T_k^{i,n}\} \cap \{t \geq \tau_k^{i,n}\}\right) \cup \left(\{t < T_k^{i,n}\} \cap \{t < \tau_k^{i,n}\}\right) \\ &= \{t < T_k^{i,n}\} \cap \{t < \tau_k^{i,n}\}\end{aligned}$$

and

$$\begin{aligned}\sigma\left(\mathcal{F}_t^{\tilde{Y}^n} \cap \{t < T_k^{i,n}\}\right) &= \mathcal{F}_t^{\tilde{Y}^n} \cap \{t < T_k^{i,n}\} \\ &= \left\{A \cap \{t < \tau_k^{i,n}\} \cap \{t < T_k^{i,n}\}, A \in \mathcal{F}_t^{\tilde{Y}^n}\right\}.\end{aligned} \tag{2.23}$$

The first equality in (2.23) is due to Theorem 3 in page 8 in Chow and Teicher [8] since $\mathcal{F}_t^{\tilde{Y}^n} = \sigma(\tilde{Y}^n(s), s \leq t)$ is a σ-filed. Thus $\xi^{i,n}(k+1)$ is independent of $\sigma\left(\mathcal{F}_t^{\tilde{Y}^n} \cap \{t < T_k^{i,n}\}\right)$, and furthermore (b) is true. □

Theorem 4.3 *(Martingale Convergence Theorem). Under assumptions (2.1)-(2.5), we have that* $(\tilde{Q}^n, \tilde{X}^n, \tilde{Y}^n)$ *is weakly relatively compact and for any weak limit process* $(\tilde{Q}, \tilde{X}, \tilde{Y})$, $\tilde{X}(\cdot)$ *is a d-dimensional Brownian motion with the initial distribution* $\mathbf{P}\tilde{X}^{-1}(0) = \pi$ *and covariance matrix* Γ. *Moreover* $\tilde{X}(t) - \theta t$ *is a martingale with respect to the filtration* $\mathcal{F}_t = \sigma(\tilde{Q}(s), \tilde{Y}(s), s \leq t)$.

Proof. First, define

$$\tau_+^n(t) = \min\left\{T_k^{i,n} : T_k^{i,n} > t\right\}, \tag{2.24}$$

$$\tau_-^n(t) = \max\left\{T_k^{i,n} : T_k^{i,n} \leq t\right\}. \tag{2.25}$$

Then for each $i \in J$,

$$\begin{aligned}
&\lim_{n\to\infty} E[|\frac{1}{\sqrt{n}}(E_i^n(\tau_+^n(nt)) - \lambda_i^n \tau_+^n(nt)) - \tilde{E}_i^n(t)|] \\
&= \lim_{n\to\infty} E[|\frac{1}{\sqrt{n}}(1 - \lambda_i^n(\tau_+^n(nt) - nt)|] \\
&\le \frac{1}{\sqrt{n}} \lim_{n\to\infty} \lambda_i^n \; E[\tau_+^n(nt) - \tau_-^n(nt)] \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \lambda_i^n \; E[\xi_i^n(1)] = 0.
\end{aligned} \tag{2.26}$$

Similarly,

$$\lim_{n\to\infty} E[|\frac{1}{\sqrt{n}}(E_i^n(\tau_-^n(nt)) - \lambda_i^n \tau_-^n(nt)) - \tilde{E}_i^n(t)|] = 0. \tag{2.27}$$

Moreover,

$$\begin{aligned}
&E[\tilde{E}_i^n(T_{k+1}^{i,n}) - \tilde{E}_i^n(T_k^{i,n})|\mathcal{G}^n_{T_k^{i,n}}] \\
&= \frac{1}{\sqrt{n}}(1 - \lambda_i^n E[\xi_i^n(k+1)|\mathcal{G}^n_{T_k^{i,n}}]) = 0,
\end{aligned} \tag{2.28}$$

where the filtration $\{\mathcal{G}_t^n\}$ is defined in (2.20). Notice that for any $\{\mathcal{G}_t^n\}$-stopping time T and any random variable X such that $E[|X|] < \infty$,

$$E[E[X|\mathcal{G}_T^n]|\mathcal{G}_t^n]I_{\{T>t\}} = E[X|\mathcal{G}_t^n]I_{\{T>t\}} = E[XI_{\{T>t\}}|\mathcal{G}_t^n]. \tag{2.29}$$

Also, for each $i \in J$ and all $s, t \ge 0$,

$$\begin{aligned}
&E[\tilde{E}_i^n(t+s) - \tilde{E}_i^n(t)|\mathcal{G}_t^n] \\
= \; &E[\tilde{E}_i^n(t+s) - \frac{1}{\sqrt{n}}(E_i^n(\tau_-^n(n(t+s)) - \lambda_i^n \tau_-^n(n(t+s)))|\mathcal{G}_t^n] \\
&+E[\frac{1}{\sqrt{n}}(E_i^n(\tau_+^n(nt)) - \lambda_i^n \tau_+^n(nt)) - \tilde{E}_i^n(t)|\mathcal{G}_t^n] \\
&-\sum_k E[(\tilde{E}_i^n(T_{k+1}^{i,n}) - \tilde{E}_i^n(T_k^{i,n})) \; I_{\{nt<T_k^{i,n}\le n(t+s)\}}|\mathcal{G}_t^n] \\
= \; &E[\tilde{E}_i^n(t+s) - \frac{1}{\sqrt{n}}(E_i^n(\tau_-^n(n(t+s)) - \lambda_i^n \tau_-^n(n(t+s)))|\mathcal{G}_t^n] \\
&+E[\frac{1}{\sqrt{n}}(E_i^n(\tau_+^n(nt)) - \lambda_i^n \tau_+^n(nt)) - \tilde{E}_i^n(t)|\mathcal{G}_t^n] \\
&-\sum_k E[E[\tilde{E}_i^n(T_{k+1}^{i,n}) - \tilde{E}_i^n(T_m^{i,n})|\mathcal{G}^n_{T_k^{i,n}}] \; I_{\{nt<T_k^{i,n}\le n(t+s)\}}|\mathcal{G}_t^n]
\end{aligned} \tag{2.30}$$

Hence, from (2.26) to (2.29), we have

$$\lim_{n\to\infty} E[|\ E[\tilde{E}_i^n(t+s) - \tilde{E}_i^n(t)|\mathcal{G}_t^n]\ |] = 0. \tag{2.31}$$

Similarly,

$$\lim_{n\to\infty}\ E[|\ E[\tilde{S}_i^n(t+s) - \tilde{S}_i^n(t)|\mathcal{G}_t^n]\ |] = 0. \tag{2.32}$$

Next, by rescaling (1.12),

$$\tilde{Q}^n(t) = \tilde{Q}^n(0) + \tilde{X}^n(t) + R^{0,n}\tilde{Y}^{0,n}(t) + R^{b,n}\tilde{Y}^{b,n}(t), \tag{2.33}$$

and for each n, $(\tilde{Q}^n(t), \tilde{Y}^{0,n}(t), \tilde{Y}^{b,n}(t))$ has the properties (1.13) to (1.17) with the state space $\mathbf{S^n}$ given by

$$\mathbf{S^n} = \left\{ x \in R_+^d : x_i \le \tilde{b}_i^n = \frac{b_i^n}{\sqrt{n}} \right\}.$$

Furthermore from (1.7)-(1.9), we have

$$\begin{aligned}\tilde{X}_i^n(t) \quad &\equiv \quad \tilde{Q}^n(0) + \frac{1}{\sqrt{n}}\tilde{E}_i^n(n\bar{B}_i^{0,n}(t)) + \frac{1}{\sqrt{n}}\sum_{j<i,\sigma(j)=i}\tilde{S}_j^n(n\bar{B}_i^n(t)) \\ &\quad -\frac{1}{\sqrt{n}}\tilde{S}_i^n(n\bar{B}_i^n(t)) + \sqrt{n}\theta_i^n t.\end{aligned} \tag{2.34}$$

Then by (2.1)-(2.5), Fluid Limit Theorem, Theorem 4.4 of Billingsley [3] and Continuous Mapping Theorem, we have

$$\tilde{X}^n(t) \Rightarrow \tilde{X}(t) = \tilde{Q}(0) + \tilde{E}(t) - (I - P')\tilde{S}(t) + \theta t \tag{2.35}$$

where $\tilde{X}(t)$ is an d-dimensional Brownian motion with the initial random vector $\tilde{Q}(0)$, drift vector θ and $d \times d$ positive definite covariance matrix Γ given by

$$\Gamma = \Gamma^a + (I - P')\Gamma^s(I - P). \tag{2.36}$$

Notice that $R^n \to R$, and (N, R) satisfies conditions $(A.1)$ and $(A.2)$ introduced in Chapter 3. Then from Theorem 3.2, one can see that $(\tilde{Q}^n, \tilde{X}^n, \tilde{Y}^n)$ is weakly relatively compact. Therefore for any weak limit $(\tilde{Q},\tilde{X},\tilde{Y})$ of $(\tilde{Q}^n,\tilde{X}^n,\tilde{Y}^n)$, there exists a subsequence of $(\tilde{Q}^n,\tilde{X}^n,\tilde{Y}^n)$ which converges to $(\tilde{Q},\tilde{X},\tilde{Y})$. For notation convenience, we assume that the convergent subsequence is $(\tilde{Q}^n,\tilde{X}^n,\tilde{Y}^n)$ itself. Let

$$F^n(t,s)=(\tilde{E}^n(t+s)-\tilde{E}^n(t))-(I-P')(\tilde{S}^n(t+s)-\tilde{S}^n(t)).$$

Let $h(\cdot)$ be an arbitrary real-valued, bounded, and continuous function of its arguments and for arbitrary r; let $t_i \le t \le t+s, i \le r$. Define

$$\tilde{H}^n(t)=(\tilde{Q}^n(t),\tilde{Y}^n(t)),$$
$$\tilde{H}(t)=(\tilde{Q}(t),\tilde{Y}(t)).$$

Notice that

$$\tilde{E}_i^n(t)=\frac{1}{\sqrt{n}}\left(\sup\left\{k:\sum_{l=1}^{k}u_i(l)\le\lambda_i^n nt\right\}-\lambda_i^n nt\right),$$
$$\tilde{S}_i^n(t)=\frac{1}{\sqrt{n}}\left(\sup\left\{k:\sum_{l=1}^{k}v_i(l)\le\mu_i^n nt\right\}-\mu_i^n nt\right).$$

By (2.2), there exist some nonnegative constants C_1 and C_2 such that $C_1 \le \lambda_i^n, \mu_i^n \le C_2$. Thus for each fixed t, by (1.3), Theorem 7.3 and 7.4 in Chapter III in Gut [20], we have that $\{(\tilde{E}_i^n(t))^2\}$ and $\{(\tilde{S}_i^n(t))^2\}$ are uniformly integrable in terms of n. Then, by the weak convergence and (2.31)-(2.32), we have

$$\begin{aligned}
&\mid E[h(\tilde{H}(t_i),i\le r)(\tilde{X}(t+s)-\tilde{X}(t)-\theta s)\mid \qquad (2.37)\\
&=\mid \lim_{n\to\infty} E[h(\tilde{H}^n(t_i),i\le r)\ F^n(t,s)]\mid\\
&=\lim_{n\to\infty}\mid E[h(\tilde{H}^n(t_i),i\le r)E[F^n(t,s)|\mathcal{G}_t^n]\mid\\
&\le M\lim_{n\to\infty}E[\mid E[F^n(t,s)|\mathcal{G}_t^n]\mid]\\
&=0,
\end{aligned}$$

where M is some positive constant. The arbitrariness of $h(\cdot)$, t_i, k,t and $t+s$, implies that

$$E[\tilde{X}(t+s)-\tilde{X}(t)-\theta s|\mathcal{F}_u,u\le t]=0,$$

which means that $\tilde{X}(t) - \theta t$ is an $\{\mathcal{F}_t\}$-martingale. Thus we finish the proof. □

Proof of Theorem 4.1**.** It directly follows from Proposition 3.1, Theorem 3.3 and the above martingale convergence theorem. □

4.3 Extension to Tree-like Network

Consider a sequence of tree-like queueing networks described in Chapter 2. All of the processes and parameters associated with the n^{th} network will be indexed with an n in a convenient place. We suppose that the number d of stations is fixed and is independent of n. It is assumed that as $n \to \infty$,

$$\lambda_i^n \to \lambda_i \; , \; \mu_i^n \to u > 0 \tag{3.1}$$

$$\sqrt{n}(\lambda_i^n + \mu_j^n P_{ji} - \mu_i^n) \to \theta_i \; (j < i), \tag{3.2}$$

$$\frac{1}{\sqrt{n}} b_i^n \to b_i > 0 \text{ for } i = 1, ..., d. \tag{3.3}$$

Where we take $P_{ji} = 0$ if $i = 1$ and $\lambda_i^n = 0$ if $i > 1$. For the initial states $Q^n(0)$, assume that as $n \to \infty$,

$$\tilde{Q}^n(0) \equiv \frac{1}{\sqrt{n}} Q^n(0) \Rightarrow \tilde{Q}(0), \tag{3.4}$$

where $'' \Rightarrow ''$ denotes convergence in distribution. Moreover, due to Functional Central Limit Theorem, we have

$$\tilde{E}^n(\cdot) \equiv \frac{1}{\sqrt{n}} \hat{E}^n(n\cdot) = \sqrt{n}(\frac{E^n(n\cdot)}{n} - \lambda^n \cdot) \Rightarrow \tilde{E}(\cdot) \tag{3.5}$$

$$\tilde{S}^n(\cdot) \equiv \frac{1}{\sqrt{n}} \hat{S}^n(n\cdot) = \sqrt{n}(\frac{S^n(n\cdot)}{n} - \mu^n \cdot) \Rightarrow \tilde{S}(\cdot) \tag{3.6}$$

$$\tilde{\Phi}^{j,n}(\cdot) \equiv \frac{1}{\sqrt{n}} \hat{\Phi}^{j,n}([n\cdot]) = \sqrt{n}(\frac{\Phi^j([n\cdot])}{n} - P_j' \cdot) \Rightarrow \tilde{\Phi}^j(\cdot), \tag{3.7}$$

where [x] is the integer part of x, $\lambda^n = (\lambda_1^n, ..., \lambda_d^n)'$ and $\mu^n = (\mu_1^n, ..., \mu_d^n)'$. Φ^j does not change with n. $\tilde{E}(t)$, $\tilde{S}(t)$ and $\tilde{\Phi}^j(t)$ $(j = 1, ..., d)$ are independent d-dimensional zero-drift Brownian motion with covariance matrices Γ^a, Γ^s and Γ^j as follows,

$$\Gamma^a = \text{diag}(\lambda_1 c_{a,1}^2, ..., \lambda_d c_{a,d}^2) = \text{diag}(\lambda)\text{diag}(c_a^2),$$

$$\Gamma^s = \text{diag}(\mu_1 c_{s,1}^2, ..., \mu_d c_{s,d}^2) = \text{diag}(\mu)\text{diag}(c_s^2),$$

$$\Gamma_{lk}^j = \begin{cases} P_{jl}(1 - P_{jl}) & \text{if } l = k \\ -P_{jl}P_{jk} & \text{if } l \neq k \end{cases}$$

where P_j denotes the j^{th} row of P.

Theorem 4.4 *Under assumptions (3.1)-(3.7), we have*

$$\left(\frac{1}{\sqrt{n}}Q^n(n\cdot), \frac{1}{\sqrt{n}}Y^{0,n}(\cdot), \frac{1}{\sqrt{n}}Y^{b,n}(n\cdot)\right) \Rightarrow \left(\tilde{Q}(\cdot), \tilde{Y}^0(\cdot), \tilde{Y}^b(\cdot)\right) \text{ as } n \to \infty \tag{3.8}$$

where $\tilde{Q}(\cdot)$ together with $\tilde{Y}^0(\cdot)$ and $\tilde{Y}^b(\cdot)$ is an $(\mathbf{S}, \theta, \Gamma, R)$-semimartingale reflecting Brownian motion with the initial distribution $\mathbf{P}\tilde{Q}^{-1}(0) = \pi$ on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. $\{\mathcal{F}_t\}$ is the filtration generated by $\tilde{X}$, $\tilde{Y}$ and $\tilde{Y}^b$, augmented with all P-null sets. The process $(\tilde{Q}, \tilde{Y}^0, \tilde{Y}^b)$ is uniquely determined in distribution from the following equations.

$$\tilde{Q}_i(t) = \tilde{X}_i(t) + \sum_{j\in J} R_{ij}^0 \mu_j \tilde{Y}_j^0(t) + \sum_{j\in J} R_{ij}^b \tilde{Y}_j^b(t), \ P - a.s., \ t \geq 0,$$

$$0 \leq \tilde{Q}_i(t) \leq b_i, \ t \geq 0, \ i \in J,$$

$$\tilde{Q}(\cdot), \ \tilde{Y}^0(\cdot) \textit{ and } \tilde{Y}^b \textit{ are } \{\mathcal{F}_t\} - \textit{adapted},$$

$$\tilde{Y}_i^0(0) = 0, \ \tilde{Y}_i^0(\cdot) \textit{ is continuous and nondecreasing}, \ i \in J,$$

$$\tilde{Y}_i^b(0) = 0, \ \tilde{Y}_i^b(\cdot) \textit{ is continuous and nondecreasing}, \ i \in J,$$

$$\tilde{Y}^0(\cdot) \textit{ increases only at times } t \textit{ when } \tilde{Q}_i(t) = 0, \ i \in J,$$

$$\tilde{Y}_i^b(\cdot) \textit{ increases only at times } t \textit{ when } \tilde{Q}_i(t) = b_i, \ i \in J,$$

where

$$\begin{aligned} X_1(t) &= \tilde{Q}_1(0) + \tilde{E}_1(t) - \tilde{S}_1(t) + \theta_1 t, \\ \tilde{X}_i(t) &= \tilde{Q}_i(0) + \tilde{\Phi}_{ji}(S_j(B_j(t))) + P_{ji}\tilde{S}_j(B_j(t)) - \tilde{S}_i(B_i(t)) + \theta_i t,\ (i > 1), \end{aligned}$$

and R^0*,* R^b *are given by (2.8) and (2.9) in Chapter 2.* $\tilde{X}(t)$ *is an d-dimensional Brownian motion with drift* θ *and covariance matrix*

$$\Gamma = \Gamma^a + (I - P')\Gamma^s(I - P) + \sum_{i=1}^{d} \mu_i \Gamma^i. \tag{3.9}$$

4.4 Extension to Overflow Network with Feedback

Similar to previous discussion, consider a sequence of overflow queueing networks presented in Chapter 2. All of the processes and parameters associated with the n^{th} network will be indexed with an n in a convenient place. We suppose that the number d of stations is fixed and is independent of n. It is assumed that as $n \to \infty$,

$$\lambda_i^n \to \lambda_i\ ,\ \mu_i^n \to u > 0 \tag{4.1}$$

$$\sqrt{n}(\lambda_i^n + \sum_{j \neq i} \mu_j^n P_{ji} - \mu_i^n) \to \theta_i, \tag{4.2}$$

$$\frac{1}{\sqrt{n}} b_i^n \to b_i > 0 \text{ for } i = 1, ..., d \tag{4.3}$$

For the initial states $Q^n(0)$, we assume that as $n \to \infty$

$$\tilde{Q}^n(0) \equiv \frac{1}{\sqrt{n}} Q^n(0) \Rightarrow \tilde{Q}(0), \tag{4.4}$$

where $'' \Rightarrow ''$ denotes convergence in distribution. Moreover, due to the Functional Central Limit Theorem, we have

$$\tilde{E}^n(\cdot) \equiv \frac{1}{\sqrt{n}} \hat{E}^n(n\cdot) = \sqrt{n}(\frac{E^n(n\cdot)}{n} - \lambda^n\cdot) \Rightarrow \tilde{E}(\cdot) \tag{4.5}$$

$$\tilde{S}^n(\cdot) \equiv \frac{1}{\sqrt{n}} \hat{S}^n(n\cdot) = \sqrt{n}(\frac{S^n(n\cdot)}{n} - \mu^n\cdot) \Rightarrow \tilde{S}(\cdot) \tag{4.6}$$

$$\tilde{\Phi}^{j,n}(\cdot) \equiv \frac{1}{\sqrt{n}}\hat{\Phi}^{j,n}([n\cdot]) = \sqrt{n}(\frac{\Phi^j([n\cdot])}{n} - P_j'\cdot) \Rightarrow \tilde{\Phi}^j(\cdot), \tag{4.7}$$

$$\tilde{\bar{\Phi}}^{j,n}(\cdot) \equiv \frac{1}{\sqrt{n}}\hat{\bar{\Phi}}^{j,n}(n\cdot) = \sqrt{n}(\frac{1}{n}\bar{\Phi}^j([n\cdot]) - \bar{P}_j\cdot) \Rightarrow \tilde{\bar{\Phi}}^j(\cdot). \tag{4.8}$$

where [x] is the integer part of x, $\lambda^n = (\lambda_1^n, ..., \lambda_d^n)'$ and $\mu^n = (\mu_1^n, ..., \mu_d^n)'$. Φ^j does not change with n. $\tilde{E}(t)$, $\tilde{S}(t)$, $\tilde{\Phi}^j(t)$ and $\tilde{\bar{\Phi}}^j(t)$ $(j = 1, ..., d)$ are independent d-dimensional zero-drift Brownian motion with covariance matrices Γ^a, Γ^s, Γ^j and $\bar{\Gamma}^j$ as follows,

$$\Gamma^a = \mathrm{diag}(\lambda_1 c_{a,1}^2, ..., \lambda_d c_{a,d}^2) = \mathrm{diag}(\lambda)\mathrm{diag}(c_a^2),$$

$$\Gamma^s = \mathrm{diag}(\mu_1 c_{s,1}^2, ..., \mu_d c_{s,d}^2) = \mathrm{diag}(\mu)\mathrm{diag}(c_s^2),$$

$$\Gamma_{lk}^j = \begin{cases} P_{jl}(1 - P_{jl}), & \text{if } l = k, \\ -P_{jl}P_{jk}, & \text{if } l \neq k. \end{cases}$$

where P_j denotes the j^{th} row of P and $\bar{P}_j$ denotes the j^{th} row of $\bar{P}$.

Theorem 4.5 *Under assumptions (4.1)-(4.8), we have*

$$\left(\frac{1}{\sqrt{n}}Q^n(n\cdot), \frac{1}{\sqrt{n}}Y^{0,n}(\cdot), \frac{1}{\sqrt{n}}Y^{b,n}(n\cdot)\right) \Rightarrow \left(\tilde{Q}(\cdot), \tilde{Y}^0(\cdot), \tilde{Y}^b(\cdot)\right) \text{ as } n \to \infty \tag{4.9}$$

where $\tilde{Q}(\cdot)$ together with $\tilde{Y}^0(\cdot)$ and $\tilde{Y}^b(\cdot)$ is an $(\mathbf{S}, \theta, \Gamma, R)$-semimartingale reflecting Brownian motion with the initial distribution $\mathbf{P}\tilde{Q}^{-1}(0) = \pi$ on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. $\{\mathcal{F}_t\}$ is the filtration generated by $\tilde{X}$, $\tilde{Y}^0$ and $\tilde{Y}^b$, augmented with all P-null sets. The process $(\tilde{Q}, \tilde{Y}^0, \tilde{Y}^b)$ is uniquely determined in distribution from the following equations.

$$\tilde{Q}_i(t) = \tilde{X}_i(t) + \sum_{j\in J} R_{ij}^0 \mu_j \tilde{Y}_j^0(t) + \sum_{j\in J} R_{ij}^b \tilde{Y}_j^b(t), \; P\text{−}a.s., \; t \geq 0,$$

$$0 \leq \tilde{Q}_i(t) \leq b_i, \; t \geq 0, \; i \in J,$$

$$\tilde{Q}(\cdot), \; \tilde{Y}^0(\cdot) \text{ and } \tilde{Y}^b \text{ are } \{\mathcal{F}_t\}\text{−adapted},$$

$$\tilde{Y}_i^0(0) = 0, \; \tilde{Y}_i^0(\cdot) \text{ is continuous and nondecreasing}, \; i \in J,$$

$$\tilde{Y}_i^b(0) = 0,\ \tilde{Y}_i^b(\cdot) \text{ is continuous and nondecreasing},\ i \in J,$$

$$\tilde{Y}^0(\cdot) \text{ increases only at times } t \text{ when } \tilde{Q}_i(t) = 0,\ i \in J,$$

$$\tilde{Y}_i^b(\cdot) \text{ increases only at times } t \text{ when } \tilde{Q}_i(t) = b_i,\ i \in J,$$

where

$$\tilde{X}_i(t) = \tilde{Q}_i(0) + \tilde{E}_i(t) + \sum_{j \neq i} \tilde{\Phi}_{ji}(t) - \tilde{S}_i(t) + \sum_{j \neq i} P_{ji}\tilde{S}_j(t) + \theta_i t.$$

$R^0 = (I - P')$, $R^b = -(I - \bar{P}')$, *and* $\tilde{X}(t)$ *is an d-dimensional Brownian motion with drift* θ *and covariance matrix*

$$\Gamma = \Gamma^a + (I - P')\Gamma^s(I - P) + \sum_{i=1}^{d} \mu_i \Gamma^i.$$

□

The proof of this theorem is similar to the proof for intree-like network. Here we only give two remarks concerning the reflection matrix $R = (R^0, R^b)$.

Remark. The matrix NR satisfies condition $(A.1)$ and $(NR)'$ satisfies $(A.2)$, where N is defined in (1.20). In fact, the $2d \times 2d$ matrix NR can be written as

$$NR = \begin{pmatrix} I - P' & -(I - \bar{P}') \\ -(I - P') & I - \bar{P}' \end{pmatrix}. \tag{4.10}$$

Then, notice that there is a unique solution of $(R_+^d, I - P)$ -regulation problem for each $x \in C_{R^d}[0, \infty)$ with $x(0) \geq 0$ since the spectral radius is less than 1, see Harrison and Reiman [24]. Then by Theorem 1 in Bernard and EL Kharaibi [1], we know that $I - P$ is completely-$\mathcal{S}$. Moreover, by Lemma 3 in Reiman and Williams [37], $I - P'$ is also completely-$\mathcal{S}$. The same argument applies to matrix $I - \bar{P}'$. Then by the same procedure used in proving Lemma 2.1, we know that NR satisfies $(A.1)$.

Remark. Consider a two station network with $P_{21} = 1$,$P_{12} = 0$ and $\bar{P}_{12} = 1$,$\bar{P}_{21} = 0$, the reflection matrix around the vertex formed by faces F_2, F_3 is

$$R_{23} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Then by the same explanation as in intree-like case, the uniqueness of solution for the corresponding $(\mathbf{S}, R)$-regulation problem fails.

CHAPTER 5

Computing the Stationary Distribution of SRBM

5.1 A Basic Adjoint Relationship

Let θ be a d-dimensional vector, Γ be a $d \times d$ positive definite matrix and R be a $d \times 2d$ matrix. Let $\{P_x, x \in \mathbf{S}\}$ denote the unique family of probability measures on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ which makes $\tilde{W}$ an SRBM associated with the data $(\mathbf{S}, \theta, \Gamma, R)$ as before. For each $x \in \mathbf{S}$, let E_x be the expectation operator under P_x. For a probability measure π on $\mathbf{S}$, define

$$P_\pi(\cdot) \equiv \int_{\mathbf{S}} P_x(\cdot)\pi(dx) \tag{1.1}$$

and let E_π be the corresponding expectation operator. The integral in (1.1) is well defined due to the Feller continuity of $\{P_x, x \in \mathbf{S}\}$; see Theorem 1.3 in Dai and Williams [16].

Definition 5.1.1 *A stationary distribution for $\tilde{W}$ is a probability measure π on $(\mathbf{S}, \mathcal{B}_{\mathbf{S}})$ such that for every bounded Borel function f on $\mathbf{S}$ and every $t \geq 0$,*

$$\int_{\mathbf{S}} E_x\left[f(\tilde{W}(t))\right]\pi(dx) = \int_{\mathbf{S}} f(x)\pi(dx). \tag{1.2}$$

Two measures will be called equivalent if they are mutually absolutely continuous. The symbol $\approx$ will be used to denote the equivalence of measures.

Proposition 5.1 *There exists a unique stationary distribution π for the* $(\mathbf{S}, \theta, \Gamma, R)$*-SRBM $\tilde{W}$. Furthermore π is equivalent to Lebesgue measure on* $\mathbf{S}$.

Proof. Since the state space $\mathbf{S}$ is compact, the stationary distribution for $\tilde{W}$ exists, see Theorem 4.9.3 and the following remark in Ethier and Kurtz [18]. The rest of the proof is the same as in Harrison and Williams [25]. □

The following proposition establishes a basic adjoint relationship (BAR) for the stationary distribution of $\tilde{W}$. The basic adjoint relationship is the starting point for us to compute the stationary distribution numerically. First, we introduce more notation. Let $C_b^2(\mathbf{S})$ be the space of twice differentiable functions whose first and second order partials are continuous and bounded on $\mathbf{S}$. For each $f \in C_b^2(\mathbf{S})$, define the following differential operators

$$\mathcal{L}f \equiv \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\Gamma_{ij}\frac{\partial^2 f}{\partial x_i x_j} + \sum_{i=1}^{d}\theta_i\frac{\partial f}{\partial x_i} \ , \tag{1.3}$$

$$\mathcal{D}_i f(x) \equiv v_i \cdot \nabla f(x), \ for \ x \in F_i \ (i = 1, ...2d) \tag{1.4}$$

where v_i is the ith column of the reflection matrix R. Finally, let σ_i denote $(d-1)$-dimensional Lebesgue measure (surface measure) on face F_i and $\mathcal{B}_{F_i}$ be the Borel σ-field of F_i. Then, we have the following proposition.

Proposition 5.2 *Let π be the stationary distribution for the* $(\mathbf{S}, \theta, \Gamma, R)$*-SRBM $\tilde{W}$. Then for each $i = 1, ..., 2d$, there is a finite Borel measure $\beta_{\mathbf{i}}$ on face F_i such that $\beta_{\mathbf{i}} \approx \sigma_i$ and*

$$E_\pi\left\{\int_0^t I_A(\tilde{W}(s))d\tilde{Y}_i(s)\right\} = t\beta_{\mathbf{i}}(A), \ t \geq 0, \ A \in \mathcal{B}_{F_i}, \tag{1.5}$$

Furthermore, defining $d\pi/dx \equiv p_0$ and $d\beta_{\mathbf{i}}/d\sigma_i \equiv p_i$, $p \equiv (p_0, p_1, ..., p_{2d})$ satisfies the following basic adjoint relationship:

$$\int_{\mathbf{S}}(\mathcal{L}f \ p_0)dx + \sum_{i=1}^{2d}\int_{F_i}(\mathcal{D}_i f \ p_i)d\sigma_i = 0, \ for \ all \ f \in C_b^2(\mathbf{S}). \tag{1.6}$$

Conversely, if p_0 is a probability density function on $\mathbf{S}$ and p_i is an nonnegative integrable (with respect to σ_i) Borel function on F_i such that (1.6) holds, then p_0 is the stationary density of $\tilde{W}$ and $\beta_{\mathbf{i}}$ given by $d\beta_{\mathbf{i}} \equiv p_i d\sigma_i$ is the boundary measure defined in (1.5).

Proof. The necessary part is a direct generalization of results in Section 7 of Harrison and Williams [25]. The converse part is proved by Dai and Kurtz [14]. □

Finally, we rewrite (1.6) as a compact form which will be used in the next section. For $f \in C_b^2(\mathbf{S})$, let

$$\mathcal{A}f \equiv (\mathcal{L}f; \mathcal{D}_1 f, ..., \mathcal{D}_{2d} f)\,, \tag{1.7}$$

$$d\lambda \equiv (dx; d\sigma_1, ..., d\sigma_{2d})\,. \tag{1.8}$$

For a subset E of R^d, let $\mathcal{B}_E$ be the Borel σ-field of E and $\mathcal{B}(E)$ denote the set of functions which are $\mathcal{B}$-measurable. Let

$$\begin{aligned} L^j(\mathbf{S}, d\lambda) \;&\equiv\; \{g = (g_0; g_1, ..., g_{2d}) \in \mathcal{B}(\mathbf{S}) \times \mathcal{B}(F_1) \times \cdots \times \mathcal{B}(F_{2d}) : \qquad (1.9) \\ &\qquad \int_{\mathbf{S}} \mid g_0 \mid^j dx + \sum_{i=1}^{2d} \int_{F_i} \mid g_i \mid^j d\sigma_i < \infty\},\ j = 1, 2, ... \\ \int_{\mathbf{S}} g d\lambda \;&\equiv\; \int_{\mathbf{S}} g_0 dx + \sum_{i=1}^{2d} \int_{F_i} g_i d\sigma_i,\ \text{for } g \in L^1(\mathbf{S}, d\lambda). \qquad (1.10) \end{aligned}$$

For $g, h \in \mathcal{B}(\mathbf{S}) \times \mathcal{B}(F_1) \times \cdots \times \mathcal{B}(F_{2d})$, we put $g \cdot h \equiv (g_0 h_0; g_1 h_1, ..., g_{2d} h_{2d})$, and for $h > 0$, we put $g/h \equiv (g_0/h_0; g_1/h_1, ..., g_{2d}/h_{2d})$. With these notation, the basic adjoint relationship (1.6) can be rewritten as

$$\int_{\mathbf{S}} (\mathcal{A}f \cdot p) d\lambda = 0,\ \text{for } f \in C_b^2(\mathbf{S}). \tag{1.11}$$

5.2 A Least Squares Problem

In this section, we develop a least squares procedure to determine the stationary density p for an SRBM in a d-dimensional box. The procedure involves two major steps. We first convert the problem of solving (1.11) into a least squares problem, and then propose an algorithm to solve the least squares problem.

We begin with the compact form (1.11) of the basic adjoint relationship (1.6). Denote by $L^2 \equiv L^2(\mathbf{S}, d\lambda)$ all the square integrable functions on $\mathbf{S}$ with respect to $d\lambda$, taken with the usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Obviously, $\mathcal{A}f \in L^2$ for any $f \in C^2(\mathbf{S})$. Hence, we can define

$$H \equiv \text{the closure of } \left\{ \mathcal{A}f : f \in C_b^2(\mathbf{S}) \right\}, \tag{2.1}$$

where the closure is taken in L^2. If one assumes that the unknown density p is in L^2, then (1.6) simply means that $\mathcal{A}f$ is orthogonal to p for all $f \in C^2(\mathbf{S})$, or equivalently $p \in H^\perp$, where $H^\perp$ is the orthogonal space of H. Conversely, if $w \in H^\perp$, then w satisfies (1.11) and hence (1.6).

Let us suppose for the moment that the unknown density function p defined in Proposition 1.2 is in L^2. Namely, assume p_0 is square integrable in terms of Lebesgue measure in $\mathbf{S}$, and p_i is square integrable in terms of $(d-1)$-dimensional Lebesgue measure on F_i $(i = 1, 2, ..., 2d)$. For any $h^0 \notin H$, let $\bar{h}^0$ be the projection of h^0 onto H, that is,

$$\bar{h}^0 \equiv \text{argmin}_{h \in H} \|h^0 - h\|^2.$$

Such choice of h^0 exists because for $h^0 = (1, 1, ..., 1)$, we have

$$\int_{\mathbf{S}} (h^0 \cdot p) d\lambda \geq \int_{\mathbf{S}} p_0 dx = 1,$$

and therefore p is not orthogonal to h^0. It was conjectured in Dai and Harrison [13] that $h^0 - \bar{h}^0$ is almost surely nonnegative with respect to measure λ. It will be seen later that our numerical experiments support this conjecture. Then it follows Proposition 3 in Dai and Harrison [12] that

$$p = \kappa(h^0 - \bar{h}^0), \tag{2.2}$$

provided that $h^0 - \bar{h}^0 \geq 0$, and κ is some constant.

As we will see later, the assumption that p is in L^2 is *not* satisfied in all cases of practical interest. However, when that assumption is satisfied and if $h^0 - \bar{h}^0 \geq 0$, then the unknown stationary density is given by (2.2). We now define some quantities that are of interest in the queueing network applications of SRBM. Let

$$q_i = \int_{\mathbf{S}} (x_i\, p_0(x))dx, \quad (i = 1, 2, ..., d), \tag{2.3}$$

$$\delta_i = \int_{F_i} p_i(x) d\sigma_i, \qquad (i = 1, 2, ..., 2d). \tag{2.4}$$

where q_i denotes the long-run average value of Z_i, and δ_i represents the long-run average amount of pushing per unit of time needed on boundary F_i in order to keep Z inside the state space $\mathbf{S}$. That is, for each $x \in \mathbf{S}$ $(i = 1, 2, ..., 2d)$

$$\frac{E_x[Y_i(t)]}{t} \to \delta_i \quad \text{as } t \to \infty.$$

5.3 An Algorithm

Suppose that we can construct a sequence of finite dimensional subspaces H_n of H such that $H_n \uparrow H$ as $n \uparrow \infty$ ($H_n \uparrow H$ means that H_1, H_2 , ... are increasing and every $h \in H$ can be approximated by a sequence of h^n with $h^n \in H_n$ for each n). Let

$$h^n \equiv \text{argmin}_{h \in H_n} \|h^0 - h\|^2. \tag{3.1}$$

Again assume p is in L^2. It follows Proposition 4 in Dai and Harrison [12] that as $n \to \infty$,

$$\|\bar{h}^0 - h^n\|^2 \to 0, \tag{3.2}$$

$$w^n \equiv h^0 - h^n \to p \in L^2(S, d\lambda). \tag{3.3}$$

If $p \notin L^2$, as conjectured in Harrison and Dai [12], w^n converges to p weakly. Namely, for all $f \in C_b(S)$, as $n \to \infty$, we have

$$\int_{\mathbf{S}} f \cdot w^n d\lambda \to \int_{\mathbf{S}} f \cdot p d\lambda.$$

In the examples presented later, it will be seen that the algorithm works well even in this case.

There are many ways to choose the approximating subspaces H_n. Each of choice yields a different version of the algorithm. Here we choose H_n spanned by some finite element base functions, namely, some piecewise polynomials. To be specific and simple, we let $\mathbf{S} \equiv [0,1]^d = [0,1] \times [0,1] \times \cdots \times [0,1]$ and partition $\mathbf{S}$ into n^d equal d-dimensional boxes. Each box is called an element. Let $h = 1/n$, then we have $(n+1)^d$ grid points $(i_1 h, i_2 h, ..., i_d h)$ in $\mathbf{S}$ with $i_j = 0, 1, ..., n$ and $j = 1, 2, ..., d$.

We choose 2^d piecewise polynomials as base functions at each grid point $(i_1 h, i_2 h, ..., i_d h)$. Thus the total number of base functions is $N = 2^d (n+1)^d$ in $\mathbf{S}$. The basis is denoted by $\{f_i(x_1, x_2, ..., x_d)\}$ $(i = 1, ..., N)$. From (3.1) and (3.2), we know that w^n is the orthogonal complement of h^0 onto H_n and h^n is the projection of h^0 onto H_n. Thus there exist constants $a_1, a_2, ..., a_N$ such that

$$w^n = h^0 - \sum_{i=1}^{N} a_i \mathcal{A} f_i.$$

Notice that $\langle w^n, \mathcal{A} f_i \rangle = 0$ for $i = 1, ..., N$, and hence we obtain the following linear equation:

$$Aa = b, \tag{3.4}$$

where

$$A = (\langle \mathcal{A} f_i, \mathcal{A} f_j \rangle)_{1 \le i,j \le N},\ a = (a_1, ..., a_N)',\ b = (\langle h^0, \mathcal{A} f_1 \rangle, ..., \langle h^0, \mathcal{A} f_N \rangle)'.$$

The matrix A is positive definite, therefore, (3.4) has a unique solution a. Once we solve (3.4), we get an approximating value w^n of density p.

5.4 Finite Element Implementation

In this section, we describe a detailed procedure for the algorithm designed above. The whole procedure is to concretely compute the matrix A and the vector b in the linear equation (3.4).

5.4.1 The Hermite Base Functions

We choose the base functions $\{f_i(x_1, x_2, ..., x_d)\}_{i=1}^N$ as follows. First, let

$$\phi(x) = (|x|-1)^2(2|x|+1), \quad (-1 \le x \le 1), \tag{4.1}$$

$$\psi(x) = x(|x|-1)^2, \quad (-1 \le x \le 1). \tag{4.2}$$

It is easy to check that ϕ and ψ are C^1 functions on $[-1,1]$, with $\phi(-1) = \phi(1) = \phi'(-1) = \phi'(1) = 0$, $\phi(0) = 1$ and $\psi(-1) = \psi(1) = \psi'(-1) = \psi'(1) = 0$, $\psi'(0) = 1$. For $i = 0, 1, ..., n-1$, define

$$\phi_i(x) = \begin{cases} \phi(\frac{x-ih}{h}), & \text{if, } x \in [(i-1)h, (i+1)h] \cap [0,1] \\ 0, & \text{otherwise,} \end{cases} \tag{4.3}$$

and

$$\psi_i(x) = \begin{cases} h\psi(\frac{x-ih}{h}), & \text{if, } x \in [(i-1)h, (i+1)h] \cap [0,1] \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Functions ϕ_i and ψ_i are C^1 on $[0,1]$ with $\phi_i(ih) = 1$, $\phi_i'(ih) = 0$ and $\psi_i(ih) = 0$, $\psi_i'(ih) = 1$. Then for each grid point $(i_1h, i_2h, ..., i_dh)$ $(i_j = 0, 1, ..., n,\ j = 1, 2, ..., d)$, there are 2^d base functions of the form

$$f_{i_1,...,i_d}(x_1, ..., x_d) = \prod_{j=1}^{d} g_{i_j,r_j}(x_j),$$

where r_j is either 0 or 1, and

$$g_{i_j,r_j}(x_j) = \begin{cases} \phi_{i_j}(x_j), & \text{if } r_j = 0, \\ \psi_{i_j}(x_j), & \text{if } r_j = 1. \end{cases} \tag{4.5}$$

There are $(n+1)^d$ grid points in $\mathbf{S}$. Hence we have a total of $N = 2^d(n+1)^d$ base functions. Still we use the same notation $f_1(x_1, ..., x_d),\ ...\ , f_N(x_1, ..., x_d)$ as before to denote these particular base functions. Furthermore, these base functions $f_i(x_1, ..., x_d)$ can be ordered as

$$f_{N_{(n;i_1,...,i_d;r_1,...,r_d)}}(x_1, ..., x_d) = \prod_{j=1}^{d} g_{i_j,r_j}(x_j)$$

where

$$\begin{aligned} N_{(n;i_1,...,i_d;r_1,...,r_d)} &= 2^d \Big(i_d(n+1)^{d-1} + i_{d-1}(n+1)^{d-2} + ... + i_2(n+1) + i_1 \Big) \\ &\quad + 2^{d-1} r_d + 2^{d-2} r_{d-1} + ... + 2r_2 + r_1. \end{aligned} \qquad (4.6)$$

Let

$$H_n = \text{span}\{\mathcal{A} f_i(x_1, x_2, ..., x_d),\ i = 1, ..., N\}. \qquad (4.7)$$

Proposition 5.3

$$H_n \to H \quad \text{in } L^2. \qquad (4.8)$$

Proof. Without loss of generality, we only consider the case that $f \in C_b^2(\mathbf{S})$. Following Proposition 7.1 in the appendices of Ethier and Kurtz [18], for any given $\epsilon > 0$, there exists a polynomial g such that

$$\|\mathcal{A}f - \mathcal{A}g\| < \epsilon$$

where the norm $\|\cdot\|$ is taken in L^2 as before. For each n, let $\tilde{g}_n \in H_n$ be the finite element interpolation polynomial of g. Then by the interpolation error estimates given by Theorem 6.6 in page 269 of Oden and Reddy [33], we conclude that

$$\|\mathcal{A}g - \mathcal{A}\tilde{g}_n\| \le C\|g\|_{C^4(\mathbf{S})} h^2$$

where C is a constant independent of h, the norm $\|\cdot\|$ is taken in L^2, and

$$\|g\|_{C^4(\mathbf{S})} = \max_{x \in \mathbf{S}} \max_{0 \le \alpha \le 4} \left| \frac{\partial^\alpha g(x)}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_d}} \right|$$

with $\alpha_1+\alpha_2+\alpha_3+\alpha_4=\alpha$. Therefore, for large enough n, we have

$$\|\mathcal{A}f-\mathcal{A}\tilde{g}_n\|\leq 2\epsilon.$$

Thus we finish the proof of the proposition. □

Definition 5.4.1 *Two nodes* $(i_1h,...,i_dh)$ *and* $(j_1h,...,j_dh)$ *are neighbors if* $\max_{1\leq k\leq d}|i_k-j_k|\leq 1$. *Two indexes* i *and* j *for base functions* $f_i(x_1,...,x_d)$ *and* $f_j(x_1,...,x_d)$ *are neighbors if their corresponding nodes are neighboring each other.*

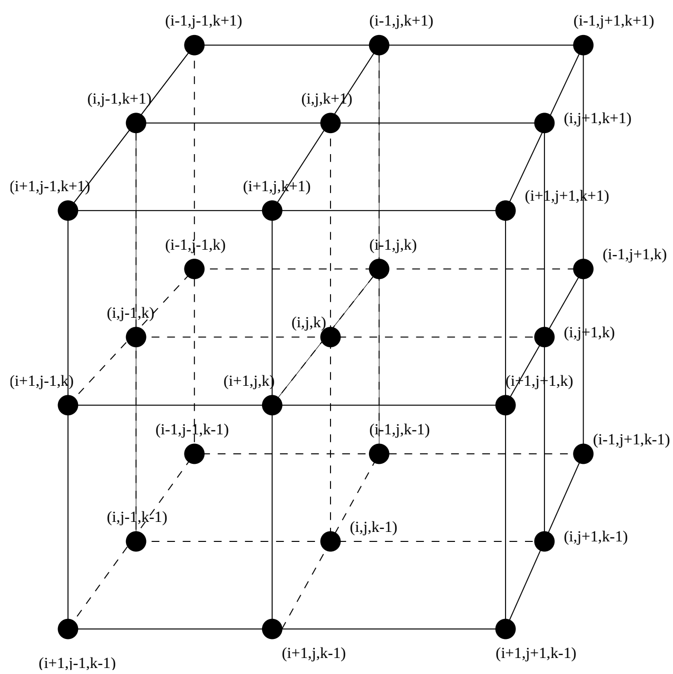

Figure 5.1: Nearest neighbors and next nearest neighbors of node (i,j,k)

It is obvious that $A_{ij}=0$ if i and j are not neighbors. Therefore, A is a sparse matrix. Now suppose that i and j for base functions $f_i(x_1,...,x_d)$ and $f_j(x_1,...,x_d)$ are neighbors, where $f_i(x_1,...,x_d)$ and $f_j(x_1,...,x_d)$ are given, as before, by

$$f_i(x_1,...,x_d)=f_{N_{(n;i_1,...,i_d;r_1,...,r_d)}}(x_1,...,x_d)=\prod_{u=1}^{d}g_{i_u,r_u}(x_u), \tag{4.9}$$

$$f_j(x_1,...,x_d)=f_{N_{(n;j_1,...,j_d;s_1,...,s_d)}}(x_1,...,x_d)=\prod_{u=1}^{d}g_{j_u,s_u}(x_u), \tag{4.10}$$

where $N_{(n;i_1,...,i_d;r_1,...,r_d)}$ and $N_{(n;j_1,...,j_d;s_1,...,s_d)}$ are defined as (4.6). From the definitions of operators $\mathcal{L}$ and $\mathcal{D}_k$ $(k = 1, ..., 2d)$, A_{ij} can be rewritten as

$$\begin{aligned}
A_{ij} &= \langle \mathcal{A}f_i, \mathcal{A}f_j \rangle \\
&= \frac{1}{4}\sum_{k,l=1}^{d}\sum_{p,q=1}^{d}\Gamma_{kl}\Gamma_{pq}\int_{\mathbf{S}}\frac{\partial^2 f_i(x)}{\partial x_k \partial x_l}\frac{\partial^2 f_j(x)}{\partial x_p \partial x_q}\,dx \\
&\quad +\frac{1}{2}\sum_{k,l=1}^{d}\sum_{p=1}^{d}\Gamma_{kl}\theta_p\int_{\mathbf{S}}\frac{\partial^2 f_i(x)}{\partial x_k \partial x_l}\frac{\partial f_j(x)}{\partial x_p}\,dx \\
&\quad +\frac{1}{2}\sum_{k=1}^{d}\sum_{p,q=1}^{d}\Gamma_{pq}\theta_k\int_{\mathbf{S}}\frac{\partial f_i(x)}{\partial x_k}\frac{\partial^2 f_j(x)}{\partial x_p \partial x_q}\,dx \\
&\quad +\sum_{k=1}^{d}\sum_{p=1}^{d}\theta_k\theta_p\int_{\mathbf{S}}\frac{\partial f_i(x)}{\partial x_k}\frac{\partial f_j(x)}{\partial x_p}\,dx \\
&\quad +\sum_{k=1}^{2d}\sum_{p,q=1}^{d}v_{kp}v_{kq}\int_{Fk}\frac{\partial f_i(x)}{\partial x_p}\frac{\partial f_j(x)}{\partial x_q}\,dx
\end{aligned} \tag{4.11}$$

Each term in A_{ij} can be calculated explicitly. For notational convenience, define

$$\begin{aligned}
I_1 &= \int_{\mathbf{S}}\frac{\partial^2 f_i(x)}{\partial x_k \partial x_l}\frac{\partial^2 f_j(x)}{\partial x_p \partial x_q}\,dx. \\
I_2 &= \int_{\mathbf{S}}\frac{\partial^2 f_i(x)}{\partial x_k \partial x_l}\frac{\partial f_j(x)}{\partial x_p}\,dx \\
I_3 &= \int_{\mathbf{S}}\frac{\partial f_i(x)}{\partial x_k}\frac{\partial^2 f_j(x)}{\partial x_p \partial x_q}\,dx \\
I_4 &= \int_{\mathbf{S}}\frac{\partial f_i(x)}{\partial x_k}\frac{\partial f_j(x)}{\partial x_p}\,dx \\
Ib_k &= \int_{F_k}\frac{\partial f_i(x)}{\partial x_p}\frac{\partial f_j(x)}{\partial x_q}\,dx
\end{aligned}$$

Finally, we can calculate b using the same idea. The value of b depends on the choice of h^0. In our implementation, we use $h^0 = (1, 1, ..., 1)$. Then, we have

$$\begin{aligned}
b_i &= \langle h^0, \mathcal{A}f_i \rangle \\
&= \frac{1}{2}\sum_{k,l=1}^{d}\Gamma_{kl}\int_{\mathbf{S}}\frac{\partial^2 f_i(x)}{\partial x_k \partial x_l}dx + \sum_{k=1}^{d}\theta_k\int_{\mathbf{S}}\frac{\partial f_i}{\partial x_k}dx + \sum_{k=1}^{2d}\sum_{l=1}^{d}v_{kl}\int_{\mathbf{S}}\frac{\partial f_i(x)}{\partial x_l}dx.
\end{aligned} \tag{4.12}$$

5.4.2 Calculation of Basic Integrals

In order to explicitly calculate the integrals I_1, I_2, I_3, I_4 and Ib_k, we first calculate some basic one variable integrals. Noticing (4.9) and (4.10), we have

$$\frac{\partial^2 f_i(x)}{\partial x_k \partial x_l} = \begin{cases} g''_{i_k,r_k}(x_k) \prod_{u\neq k}^d g_{i_u,r_u}(x_u), & \text{if } k = l, \\ g'_{i_k,r_k}(x_k) g'_{i_l,r_l}(x_l) \prod_{u\neq k, u\neq l}^d g_{i_u,r_u}(x_u), & \text{if } k \neq l, \end{cases} \tag{4.13}$$

$$\frac{\partial f_i(x)}{\partial x_k} = g'_{i_k,r_k}(x_k) \prod_{u\neq k}^d g_{i_u,r_u}(x_u). \tag{4.14}$$

Similarly, we have

$$\frac{\partial^2 f_j(x)}{\partial x_p \partial x_q} = \begin{cases} g''_{j_p,s_p}(x_p) \prod_{u\neq p}^d g_{j_u,r_u}(x_u), & \text{if } p = q, \\ g'_{j_p,s_p}(x_p) g'_{j_q,s_q}(x_q) \prod_{u\neq p, u\neq q}^d g_{j_u,s_u}(x_u), & \text{if } p \neq q, \end{cases} \tag{4.15}$$

$$\frac{\partial f_j(x)}{\partial x_p} = g'_{j_p,s_p}(x_p) \prod_{u\neq p}^d g_{j_u,s_u}(x_u). \tag{4.16}$$

Then the following basic integrals will be used to determine I_1, I_2, I_3, I_4 and Ib_k.

$$\int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u$$
$$= \begin{cases} 0, & \text{if } |i_u - j_u| > 1, \\ \int_0^1 g_{i_u,r_u}(x_u) g_{i_u+1,s_u}(x_u)\, dx_u, & \text{if } j_u = i_u + 1, \\ \int_0^1 g_{i_u,r_u}(x_u) g_{i_u,s_u}(x_u)\, dx_u, & \text{if } j_u = i_u, \\ \int_0^1 g_{i_u,r_u}(x_u) g_{i_u-1,s_u}(x_u)\, dx_u, & \text{if } j_u = i_u - 1. \end{cases} \tag{4.17}$$

Then, by (4.3)-(4.5), we have

$$\int_0^1 g_{i_u,r_u}(x_u) g_{i_u+1,s_u}(x_u)\, dx_u$$
$$= \int_{i_u h}^{(i_u+1)h} g_{i_u,r_u}(x_u) g_{i_u+1,s_u}(x_u)\, dx_u$$
$$= \begin{cases} h \int_0^1 \phi(y)\phi(y-1)\, dy = \frac{9}{70} h, & \text{if } r_u = 0, s_u = 0, \\ h^2 \int_0^1 \phi(y)\psi(y-1)\, dy = -\frac{13}{420} h^2, & \text{if } r_u = 0, s_u = 1, \\ h^2 \int_0^1 \psi(y)\phi(y-1)\, dy = \frac{13}{420} h^2, & \text{if } r_u = 1, s_u = 0, \\ h^3 \int_0^1 \psi(y)\psi(y-1)\, dy = -\frac{1}{140} h^3, & \text{if } r_u = 1, s_u = 1. \end{cases}$$

$$\begin{aligned}
&\int_0^1 g_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\ dx_u \\
&= \int_{((i_u-1)h)\wedge 0}^{i_u h} g_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\ dx_u + \int_{i_u h}^{((i_u+1)h)\vee 1} g_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\ dx_u \\
&= \begin{cases}
2h\int_0^1 \phi(y)\phi(y)\ dy = \frac{26}{35}h, & \text{if } r_u=0, s_u=0, i_u\neq 0, n, \\
h\int_0^1 \phi(y)\phi(y)\ dy = \frac{13}{35}h, & \text{if } r_u=0, s_u=0, i_u=0, n, \\
h^2\int_{-1}^1 \phi(y)\psi(y)\ dy = 0, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u\neq 0, n \\
h^2\int_0^1 \phi(y)\psi(y)\ dy = \frac{11}{210}h^2, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u=0 \\
h^2\int_{-1}^0 \phi(y)\psi(y)\ dy = -\frac{11}{210}h^2, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u=n \\
2h^3\int_0^1 \psi(y)\psi(y)\ dy = \frac{2}{105}h^3, & \text{if } r_u=1, s_u=1, x_u\neq 0, n, \\
h^3\int_0^1 \psi(y)\psi(y)\ dy = \frac{1}{105}h^3, & \text{if } r_u=1, s_u=1, x_u=0, n.
\end{cases}
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 g_{i_u,r_u}(x_u)g_{i_u-1,s_u}(x_u)\ dx_u \\
&= \int_{(i_u-1)h}^{i_u h} g_{i_u,r_u}(x_u)g_{i_u-1,s_u}(x_u)\ dx_u \\
&= \begin{cases}
h\int_0^1 \phi(y-1)\phi(y)\ dy = \frac{9}{70}h, & \text{if } r_u=0, s_u=0, \\
h^2\int_0^1 \phi(y-1)\psi(y)\ dy = \frac{13}{420}h^2, & \text{if } r_u=0, s_u=1, \\
h^2\int_0^1 \psi(y-1)\phi(y)\ dy = -\frac{13}{420}h^2, & \text{if } r_u=1, s_u=0, \\
h^3\int_0^1 \psi(y-1)\psi(y)\ dy = -\frac{1}{140}h^3, & \text{if } r_u=1, s_u=1.
\end{cases}
\end{aligned}$$

Similarly, one can explicitly calculate the following integrals.

$$\begin{aligned}
&\int_0^1 g_{i_u,r_u}(x_u)g'_{j_u,s_u}(x_u)\ dx_u \\
&= \begin{cases}
0, & \text{if } |i_u-j_u|>1, \\
\int_0^1 g_{i_u,r_u}(x_u)g'_{i_u+1,s_u}(x_u)\ dx_u, & \text{if } j_u=i_u+1, \\
\int_0^1 g_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\ dx_u, & \text{if } j_u=i_u, \\
\int_0^1 g_{i_u,r_u}(x_u)g'_{i_u-1,s_u}(x_u)\ dx_u, & \text{if } j_u=i_u-1.
\end{cases}
\end{aligned} \tag{4.18}$$

By (4.3)-(4.5), we have

$$\int_0^1 g_{i_u,r_u}(x_u)g'_{i_u+1,s_u}(x_u)\ dx_u$$

$$= \int_{i_u h}^{(i_u+1)h} g_{i_u,r_u}(x_u) g'_{i_u+1,s_u}(x_u)\, dx_u$$

$$= \begin{cases} \int_0^1 \phi(y)\phi'(y-1)\, dy = \frac{1}{2}, & \text{if } r_u = 0, s_u = 0, \\ h\int_0^1 \phi(y)\psi'(y-1)\, dy = -\frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, \\ h\int_0^1 \psi(y)\phi'(y-1)\, dy = \frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, \\ h^2\int_0^1 \psi(y)\psi'(y-1)\, dy = -\frac{1}{60}h^2, & \text{if } r_u = 1, s_u = 1. \end{cases}$$

$$\int_0^1 g_{i_u,r_u}(x_u) g'_{i_u,s_u}(x_u)\, dx_u$$

$$= \int_{((i_u-1)h)\wedge 0}^{i_u h} g_{i_u,r_u}(x_u) g'_{i_u,s_u}(x_u)\, dx_u + \int_{i_u h}^{((i_u+1)h)\vee 1} g_{i_u,r_u}(x_u) g'_{i_u,s_u}(x_u)\, dx_u$$

$$= \begin{cases} \int_{-1}^1 \phi(y)\phi'(y)\, dy = 0, & \text{if } r_u = 0, s_u = 0, i_u \neq 0, n, \\ \int_0^1 \phi(y)\phi'(y)\, dy = -\frac{1}{2}, & \text{if } r_u = 0, s_u = 0, i_u = 0, \\ \int_{-1}^0 \phi(y)\phi'(y)\, dy = \frac{1}{2}, & \text{if } r_u = 0, s_u = 0, i_u = n, \\ h\int_{-1}^1 \phi(y)\psi'(y)\, dy = \frac{1}{5}h, & \text{if } r_u = 0, s_u = 1, i_u \neq 0, n \\ h\int_0^1 \phi(y)\psi'(y)\, dy = \frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, i_u = 0 \\ h\int_{-1}^0 \phi(y)\psi'(y)\, dy = \frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, i_u = n \\ h\int_{-1}^1 \psi(y)\phi'(y)\, dy = -\frac{1}{5}h, & \text{if } r_u = 1, s_u = 0, i_u \neq 0, n \\ h\int_0^1 \psi(y)\phi'(y)\, dy = -\frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, i_u = 0 \\ h\int_{-1}^0 \psi(y)\phi'(y)\, dy = -\frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, i_u = n \\ h^2\int_{-1}^1 \psi(y)\psi'(y)\, dy = 0, & \text{if } r_u = 1, s_u = 1, x_u \neq 0, n, \\ h^2\int_0^1 \psi(y)\psi'(y)\, dy = 0, & \text{if } r_u = 1, s_u = 1, x_u = 0, \\ h^2\int_0^1 \psi(y)\psi'(y)\, dy = 0, & \text{if } r_u = 1, s_u = 1, x_u = n. \end{cases}$$

$$\int_0^1 g_{i_u,r_u}(x_u) g'_{i_u-1,s_u}(x_u)\, dx_u$$

$$= \int_{(i_u-1)h}^{i_u h} g_{i_u,r_u}(x_u) g'_{i_u-1,s_u}(x_u)\, dx_u$$

$$= \begin{cases} \int_0^1 \phi(y-1)\phi'(y)\ dy = -\frac{1}{2}, & \text{if } r_u = 0, s_u = 0, \\ h\int_0^1 \phi(y-1)\psi'(y)\ dy = -\frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, \\ h\int_0^1 \psi(y-1)\phi'(y)\ dy = \frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, \\ h^2\int_0^1 \psi(y-1)\psi'(y)\ dy = \frac{1}{60}h^2, & \text{if } r_u = 1, s_u = 1. \end{cases}$$

Now we calculate the following integral.

$$\begin{aligned} &\int_0^1 g_{i_u,r_u}(x_u)g''_{j_u,s_u}(x_u)\ dx_u \\ &= \begin{cases} 0, & \text{if } |i_u - j_u| > 1, \\ \int_0^1 g_{i_u,r_u}(x_u)g''_{i_u+1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u + 1, \\ \int_0^1 g_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u, \\ \int_0^1 g_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u - 1. \end{cases} \end{aligned} \tag{4.19}$$

By (4.3)-(4.5), we have

$$\begin{aligned} &\int_0^1 g_{i_u,r_u}(x_u)g''_{i_u+1,s_u}(x_u)\ dx_u \\ &= \int_{i_u h}^{(i_u+1)h} g_{i_u,r_u}(x_u)g''_{i_u+1,s_u}(x_u)\ dx_u \\ &= \begin{cases} \frac{1}{h}\int_0^1 \phi(y)\phi''(y-1)\ dy = \frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, \\ \int_0^1 \phi(y)\psi''(y-1)\ dy = -\frac{1}{10}, & \text{if } r_u = 0, s_u = 1, \\ \int_0^1 \psi(y)\phi''(y-1)\ dy = \frac{1}{10}, & \text{if } r_u = 1, s_u = 0, \\ h\int_0^1 \psi(y)\psi''(y-1)\ dy = \frac{1}{30}h, & \text{if } r_u = 1, s_u = 1. \end{cases} \end{aligned}$$

$$\begin{aligned} &\int_0^1 g_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u \\ &= \int_{((i_u-1)h)\wedge 0}^{i_u h} g_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u + \int_{i_u h}^{((i_u+1)h)\vee 1} g_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u \end{aligned}$$

$$
= \begin{cases}
\frac{1}{h}\int_{-1}^{1}\phi(y)\phi''(y)\,dy = -\frac{12}{5h}, & \text{if } r_u = 0, s_u = 0, i_u \neq 0, n,\\
\frac{1}{h}\int_{0}^{1}\phi(y)\phi''(y)\,dy = -\frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, i_u = 0,\\
\frac{1}{h}\int_{-1}^{0}\phi(y)\phi'(y)\,dy = -\frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, i_u = n,\\
\int_{-1}^{1}\phi(y)\psi''(y)\,dy = 0, & \text{if } r_u = 0, s_u = 1, i_u \neq 0, n\\
\int_{0}^{1}\phi(y)\psi''(y)\,dy = -\frac{11}{10}, & \text{if } r_u = 0, s_u = 1, i_u = 0\\
\int_{-1}^{0}\phi(y)\psi''(y)\,dy = \frac{11}{10}, & \text{if } r_u = 0, s_u = 1, i_u = n\\
\int_{-1}^{1}\psi(y)\phi''(y)\,dy = 0, & \text{if } r_u = 1, s_u = 0, i_u \neq 0, n\\
\int_{0}^{1}\psi(y)\phi''(y)\,dy = -\frac{1}{10}, & \text{if } r_u = 1, s_u = 0, i_u = 0\\
\int_{-1}^{0}\psi(y)\phi''(y)\,dy = \frac{1}{10}, & \text{if } r_u = 1, s_u = 0, i_u = n\\
h\int_{-1}^{1}\psi(y)\psi''(y)\,dy = -\frac{4}{15}h, & \text{if } r_u = 1, s_u = 1, x_u \neq 0, n,\\
h\int_{0}^{1}\psi(y)\psi''(y)\,dy = -\frac{2}{15}h, & \text{if } r_u = 1, s_u = 1, x_u = 0,\\
h\int_{-1}^{0}\psi(y)\psi''(y)\,dy = -\frac{2}{15}h, & \text{if } r_u = 1, s_u = 1, x_u = n.
\end{cases}
$$

$$
\begin{aligned}
&\int_0^1 g_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\,dx_u\\
&= \int_{(i_u-1)h}^{i_u h} g_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\,dx_u\\
&= \begin{cases}
\frac{1}{h}\int_0^1 \phi(y-1)\phi''(y)\,dy = \frac{6}{5h}, & \text{if } r_u = 0, s_u = 0,\\
\int_0^1 \phi(y-1)\psi''(y)\,dy = \frac{1}{10}, & \text{if } r_u = 0, s_u = 1,\\
\int_0^1 \psi(y-1)\phi''(y)\,dy = -\frac{1}{10}, & \text{if } r_u = 1, s_u = 0,\\
h\int_0^1 \psi(y-1)\psi''(y)\,dy = \frac{1}{30}h, & \text{if } r_u = 1, s_u = 1.
\end{cases}
\end{aligned}
$$

Now we compute the following integral.

$$
\begin{aligned}
&\int_0^1 g'_{i_u,r_u}(x_u)g_{j_u,s_u}(x_u)\,dx_u\\
&= \begin{cases}
0, & \text{if } |i_u - j_u| > 1,\\
\int_0^1 g'_{i_u,r_u}(x_u)g_{i_u+1,s_u}(x_u)\,dx_u, & \text{if } j_u = i_u + 1,\\
\int_0^1 g'_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\,dx_u, & \text{if } j_u = i_u,\\
\int_0^1 g'_{i_u,r_u}(x_u)g_{i_u-1,s_u}(x_u)\,dx_u, & \text{if } j_u = i_u - 1.
\end{cases}
\end{aligned}
\tag{4.20}
$$

By (4.3)-(4.5), we have

$$
\begin{aligned}
&\int_0^1 g'_{i_u,r_u}(x_u) g_{i_u+1,s_u}(x_u)\, dx_u \\
&= \int_{i_u h}^{(i_u+1)h} g'_{i_u,r_u}(x_u) g_{i_u+1,s_u}(x_u)\, dx_u \\
&= \begin{cases}
\int_0^1 \phi'(y)\phi(y-1)\, dy = -\frac{1}{2}, & \text{if } r_u = 0, s_u = 0, \\
h\int_0^1 \phi'(y)\psi(y-1)\, dy = \frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, \\
h\int_0^1 \psi'(y)\phi(y-1)\, dy = -\frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, \\
h^2\int_0^1 \psi'(y)\psi(y-1)\, dy = \frac{1}{60}h^2, & \text{if } r_u = 1, s_u = 1.
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 g'_{i_u,r_u}(x_u) g_{i_u,s_u}(x_u)\, dx_u \\
&= \int_{((i_u-1)h)\wedge 0}^{i_u h} g'_{i_u,r_u}(x_u) g_{i_u,s_u}(x_u)\, dx_u + \int_{i_u h}^{((i_u+1)h)\vee 1} g'_{i_u,r_u}(x_u) g_{i_u,s_u}(x_u)\, dx_u \\
&= \begin{cases}
\int_{-1}^1 \phi'(y)\phi(y)\, dy = 0, & \text{if } r_u = 0, s_u = 0, i_u \neq 0, n, \\
\int_0^1 \phi'(y)\phi(y)\, dy = -\frac{1}{2}, & \text{if } r_u = 0, s_u = 0, i_u = 0, \\
\int_{-1}^0 \phi'(y)\phi(y)\, dy = \frac{1}{2}, & \text{if } r_u = 0, s_u = 0, i_u = n, \\
h\int_{-1}^1 \phi'(y)\psi(y)\, dy = -\frac{1}{5}h, & \text{if } r_u = 0, s_u = 1, i_u \neq 0, n \\
h\int_0^1 \phi'(y)\psi(y)\, dy = -\frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, i_u = 0 \\
h\int_{-1}^0 \phi'(y)\psi(y)\, dy = -\frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, i_u = n \\
h\int_{-1}^1 \psi'(y)\phi(y)\, dy = \frac{1}{5}h, & \text{if } r_u = 1, s_u = 0, i_u \neq 0, n \\
h\int_0^1 \psi'(y)\phi(y)\, dy = \frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, i_u = 0 \\
h\int_{-1}^0 \psi'(y)\phi(y)\, dy = \frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, i_u = n \\
h^2\int_{-1}^1 \psi'(y)\psi(y)\, dy = 0, & \text{if } r_u = 1, s_u = 1, x_u \neq 0, n, \\
h^2\int_0^1 \psi'(y)\psi(y)\, dy = 0, & \text{if } r_u = 1, s_u = 1, x_u = 0, \\
h^2\int_0^1 \psi'(y)\psi(y)\, dy = 0, & \text{if } r_u = 1, s_u = 1, x_u = n.
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 g'_{i_u,r_u}(x_u) g_{i_u-1,s_u}(x_u)\, dx_u \\
&= \int_{(i_u-1)h}^{i_u h} g'_{i_u,r_u}(x_u) g_{i_u-1,s_u}(x_u)\, dx_u
\end{aligned}
$$

$$= \begin{cases} \int_0^1 \phi'(y-1)\phi(y)\, dy = \frac{1}{2}, & \text{if } r_u = 0, s_u = 0, \\ h\int_0^1 \phi'(y-1)\psi(y)\, dy = \frac{1}{10}h, & \text{if } r_u = 0, s_u = 1, \\ h\int_0^1 \psi'(y-1)\phi(y)\, dy = -\frac{1}{10}h, & \text{if } r_u = 1, s_u = 0, \\ h^2\int_0^1 \psi'(y-1)\psi(y)\, dy = -\frac{1}{60}h^2, & \text{if } r_u = 1, s_u = 1. \end{cases}$$

Now we calculate the following integral.

$$\begin{aligned} &\int_0^1 g''_{i_u,r_u}(x_u)g_{j_u,s_u}(x_u)\, dx_u \\ &= \begin{cases} 0, & \text{if } |i_u - j_u| > 1, \\ \int_0^1 g''_{i_u,r_u}(x_u)g_{i_u+1,s_u}(x_u)\, dx_u, & \text{if } j_u = i_u + 1, \\ \int_0^1 g''_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\, dx_u, & \text{if } j_u = i_u, \\ \int_0^1 g''_{i_u,r_u}(x_u)g_{i_u-1,s_u}(x_u)\, dx_u, & \text{if } j_u = i_u - 1. \end{cases} \end{aligned} \tag{4.21}$$

By (4.3)-(4.5), we have

$$\begin{aligned} &\int_0^1 g''_{i_u,r_u}(x_u)g_{i_u+1,s_u}(x_u)\, dx_u \\ &= \int_{i_u h}^{(i_u+1)h} g''_{i_u,r_u}(x_u)g_{i_u+1,s_u}(x_u)\, dx_u \\ &= \begin{cases} \frac{1}{h}\int_0^1 \phi''(y)\phi(y-1)\, dy = \frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, \\ \int_0^1 \phi''(y)\psi(y-1)\, dy = -\frac{1}{10}, & \text{if } r_u = 0, s_u = 1, \\ \int_0^1 \psi''(y)\phi(y-1)\, dy = \frac{1}{10}, & \text{if } r_u = 1, s_u = 0, \\ h\int_0^1 \psi''(y)\psi(y-1)\, dy = \frac{1}{30}h, & \text{if } r_u = 1, s_u = 1. \end{cases} \end{aligned}$$

$$\begin{aligned} &\int_0^1 g''_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\, dx_u \\ &= \int_{((i_u-1)h)\wedge 0}^{i_u h} g''_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\, dx_u + \int_{i_u h}^{((i_u+1)h)\vee 1} g''_{i_u,r_u}(x_u)g_{i_u,s_u}(x_u)\, dx_u \end{aligned}$$

$$= \begin{cases} \frac{1}{h}\int_{-1}^{1} \phi''(y)\phi(y)\ dy = -\frac{12}{5h}, & \text{if } r_u = 0, s_u = 0, i_u \neq 0, n, \\ \frac{1}{h}\int_{0}^{1} \phi''(y)\phi(y)\ dy = -\frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, i_u = 0, \\ \frac{1}{h}\int_{-1}^{0} \phi''(y)\phi(y)\ dy = -\frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, i_u = n, \\ \int_{-1}^{1} \phi''(y)\psi(y)\ dy = 0, & \text{if } r_u = 0, s_u = 1, i_u \neq 0, n \\ \int_{0}^{1} \phi''(y)\psi(y)\ dy = -\frac{1}{10}, & \text{if } r_u = 0, s_u = 1, i_u = 0 \\ \int_{-1}^{0} \phi''(y)\psi(y)\ dy = \frac{1}{10}, & \text{if } r_u = 0, s_u = 1, i_u = n \\ \int_{-1}^{1} \psi''(y)\phi(y)\ dy = 0, & \text{if } r_u = 1, s_u = 0, i_u \neq 0, n \\ \int_{0}^{1} \psi''(y)\phi(y)\ dy = -\frac{11}{10}, & \text{if } r_u = 1, s_u = 0, i_u = 0 \\ \int_{-1}^{0} \psi''(y)\phi(y)\ dy = \frac{11}{10}, & \text{if } r_u = 1, s_u = 0, i_u = n \\ h\int_{-1}^{1} \psi''(y)\psi(y)\ dy = -\frac{4}{15}h, & \text{if } r_u = 1, s_u = 1, x_u \neq 0, n, \\ h\int_{0}^{1} \psi''(y)\psi(y)\ dy = -\frac{2}{15}h, & \text{if } r_u = 1, s_u = 1, x_u = 0, \\ h\int_{-1}^{0} \psi''(y)\psi(y)\ dy = -\frac{2}{15}h, & \text{if } r_u = 1, s_u = 1, x_u = n. \end{cases}$$

$$\begin{aligned} &\int_0^1 g''_{i_u,r_u}(x_u) g_{i_u-1,s_u}(x_u)\ dx_u \\ &= \int_{(i_u-1)h}^{i_u h} g''_{i_u,r_u}(x_u) g_{i_u-1,s_u}(x_u)\ dx_u \\ &= \begin{cases} \frac{1}{h}\int_0^1 \phi''(y-1)\phi(y)\ dy = \frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, \\ \int_0^1 \phi''(y-1)\psi(y)\ dy = \frac{1}{10}, & \text{if } r_u = 0, s_u = 1, \\ \int_0^1 \psi''(y-1)\phi(y)\ dy = -\frac{1}{10}, & \text{if } r_u = 1, s_u = 0, \\ h\int_0^1 \psi''(y-1)\psi(y)\ dy = \frac{1}{30}h, & \text{if } r_u = 1, s_u = 1. \end{cases} \end{aligned}$$

Now we begin to calculate the following integral.

$$\begin{aligned} &\int_0^1 g'_{i_u,r_u}(x_u) g'_{j_u,s_u}(x_u)\ dx_u \\ &= \begin{cases} 0, & \text{if } |i_u - j_u| > 1, \\ \int_0^1 g'_{i_u,r_u}(x_u) g'_{i_u+1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u + 1, \\ \int_0^1 g'_{i_u,r_u}(x_u) g'_{i_u,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u, \\ \int_0^1 g'_{i_u,r_u}(x_u) g'_{i_u-1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u - 1. \end{cases} \end{aligned} \tag{4.22}$$

By (4.3)-(4.5), we have

$$
\begin{aligned}
&\int_0^1 g'_{i_u,r_u}(x_u)g'_{i_u+1,s_u}(x_u)\ dx_u\\
&=\int_{i_uh}^{(i_u+1)h} g'_{i_u,r_u}(x_u)g'_{i_u+1,s_u}(x_u)\ dx_u\\
&=\begin{cases}
\frac{1}{h}\int_0^1 \phi'(y)\phi'(y-1)\ dy=-\frac{6}{5h}, & \text{if } r_u=0, s_u=0,\\
\int_0^1 \phi'(y)\psi'(y-1)\ dy=\frac{1}{10}, & \text{if } r_u=0, s_u=1,\\
\int_0^1 \psi'(y)\phi'(y-1)\ dy=-\frac{1}{10}, & \text{if } r_u=1, s_u=0,\\
h\int_0^1 \psi'(y)\psi'(y-1)\ dy=-\frac{1}{30}h, & \text{if } r_u=1, s_u=1.
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 g'_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\ dx_u\\
&=\int_{((i_u-1)h)\wedge 0}^{i_uh} g'_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\ dx_u+\int_{i_uh}^{((i_u+1)h)\vee 1} g'_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\ dx_u\\
&=\begin{cases}
\frac{1}{h}\int_{-1}^1 \phi'(y)\phi'(y)\ dy=\frac{12}{5h}, & \text{if } r_u=0, s_u=0, i_u\neq 0,n,\\
\frac{1}{h}\int_0^1 \phi'(y)\phi'(y)\ dy=\frac{6}{5h}, & \text{if } r_u=0, s_u=0, i_u=0,\\
\frac{1}{h}\int_{-1}^0 \phi'(y)\phi'(y)\ dy=\frac{6}{5h}, & \text{if } r_u=0, s_u=0, i_u=n,\\
\int_{-1}^1 \phi'(y)\psi'(y)\ dy=0, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u\neq 0,n\\
\int_0^1 \phi'(y)\psi'(y)\ dy=\frac{1}{10}, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u=0\\
\int_{-1}^0 \phi'(y)\psi'(y)\ dy=-\frac{1}{10}, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u=n\\
h\int_{-1}^1 \psi'(y)\psi'(y)\ dy=\frac{4}{15}h, & \text{if } r_u=1, s_u=1, x_u\neq 0,n,\\
h\int_0^1 \psi'(y)\psi'(y)\ dy=\frac{2}{15}h, & \text{if } r_u=1, s_u=1, x_u=0,\\
h\int_0^1 \psi'(y)\psi'(y)\ dy=\frac{2}{15}h, & \text{if } r_u=1, s_u=1, x_u=n.
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 g'_{i_u,r_u}(x_u)g'_{i_u-1,s_u}(x_u)\ dx_u\\
&=\int_{(i_u-1)h}^{i_uh} g'_{i_u,r_u}(x_u)g'_{i_u-1,s_u}(x_u)\ dx_u
\end{aligned}
$$

$$= \begin{cases} \frac{1}{h}\int_0^1 \phi'(y-1)\phi'(y)\ dy = -\frac{6}{5h}, & \text{if } r_u = 0, s_u = 0, \\ \int_0^1 \phi'(y-1)\psi'(y)\ dy = -\frac{1}{10}, & \text{if } r_u = 0, s_u = 1, \\ \int_0^1 \psi'(y-1)\phi'(y)\ dy = \frac{1}{10}, & \text{if } r_u = 1, s_u = 0, \\ h\int_0^1 \psi'(y-1)\psi'(y)\ dy = -\frac{1}{30}h, & \text{if } r_u = 1, s_u = 1. \end{cases}$$

Now we start to calculate the following integral

$$\begin{aligned} &\int_0^1 g'_{i_u,r_u}(x_u) g''_{j_u,s_u}(x_u)\ dx_u \\ &= \begin{cases} 0, & \text{if } |i_u - j_u| > 1, \\ \int_0^1 g'_{i_u,r_u}(x_u) g''_{i_u+1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u + 1, \\ \int_0^1 g'_{i_u,r_u}(x_u) g''_{i_u,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u, \\ \int_0^1 g'_{i_u,r_u}(x_u) g''_{i_u-1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u - 1. \end{cases} \end{aligned} \tag{4.23}$$

By (4.3)-(4.5), we have

$$\begin{aligned} &\int_0^1 g'_{i_u,r_u}(x_u) g''_{i_u+1,s_u}(x_u)\ dx_u \\ &= \int_{i_u h}^{(i_u+1)h} g'_{i_u,r_u}(x_u) g''_{i_u+1,s_u}(x_u)\ dx_u \\ &= \begin{cases} \frac{1}{h^2}\int_0^1 \phi'(y)\phi''(y-1)\ dy = 0, & \text{if } r_u = 0, s_u = 0, \\ \frac{1}{h}\int_0^1 \phi'(y)\psi''(y-1)\ dy = -\frac{1}{h}, & \text{if } r_u = 0, s_u = 1, \\ \frac{1}{h}\int_0^1 \psi'(y)\phi''(y-1)\ dy = \frac{1}{h}, & \text{if } r_u = 1, s_u = 0, \\ \int_0^1 \psi'(y)\psi''(y-1)\ dy = -\frac{1}{2}, & \text{if } r_u = 1, s_u = 1. \end{cases} \end{aligned}$$

$$\begin{aligned} &\int_0^1 g'_{i_u,r_u}(x_u) g''_{i_u,s_u}(x_u)\ dx_u \\ &= \int_{((i_u-1)h)\wedge 0}^{i_u h} g'_{i_u,r_u}(x_u) g''_{i_u,s_u}(x_u)\ dx_u + \int_{i_u h}^{((i_u+1)h)\vee 1} g'_{i_u,r_u}(x_u) g''_{i_u,s_u}(x_u)\ dx_u \end{aligned}$$

$$= \begin{cases} \frac{1}{h^2}\int_{-1}^{1}\phi'(y)\phi''(y)\,dy = 0, & \text{if } r_u = 0, s_u = 0, i_u \neq 0, n, \\ \frac{1}{h^2}\int_{0}^{1}\phi'(y)\phi''(y)\,dy = 0, & \text{if } r_u = 0, s_u = 0, i_u = 0, \\ \frac{1}{h^2}\int_{-1}^{0}\phi'(y)\phi''(y)\,dy = 0, & \text{if } r_u = 0, s_u = 0, i_u = n, \\ \frac{1}{h}\int_{-1}^{1}\phi'(y)\psi''(y)\,dy = \frac{2}{h}, & \text{if } r_u = 0, s_u = 1, i_u \neq 0, n \\ \frac{1}{h}\int_{0}^{1}\phi'(y)\psi''(y)\,dy = \frac{1}{h}, & \text{if } r_u = 0, s_u = 1, i_u = 0 \\ \frac{1}{h}\int_{-1}^{0}\phi'(y)\psi''(y)\,dy = \frac{1}{h}, & \text{if } r_u = 0, s_u = 1, i_u = n \\ \frac{1}{h}\int_{-1}^{1}\psi'(y)\phi''(y)\,dy = -\frac{2}{h}, & \text{if } r_u = 1, s_u = 0, i_u \neq 0, n \\ \frac{1}{h}\int_{0}^{1}\psi'(y)\phi''(y)\,dy = -\frac{1}{h}, & \text{if } r_u = 1, s_u = 0, i_u = 0 \\ \frac{1}{h}\int_{-1}^{0}\psi'(y)\phi''(y)\,dy = -\frac{1}{h}, & \text{if } r_u = 1, s_u = 0, i_u = n \\ \int_{-1}^{1}\psi'(y)\psi''(y)\,dy = 0, & \text{if } r_u = 1, s_u = 1, x_u \neq 0, n, \\ \int_{0}^{1}\psi'(y)\psi''(y)\,dy = -\frac{1}{2}, & \text{if } r_u = 1, s_u = 1, x_u = 0, \\ \int_{0}^{1}\psi'(y)\psi''(y)\,dy = \frac{1}{2}, & \text{if } r_u = 1, s_u = 1, x_u = n. \end{cases}$$

$$\begin{aligned} &\int_0^1 g'_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\,dx_u \\ &= \int_{(i_u-1)h}^{i_u h} g'_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\,dx_u \\ &= \begin{cases} \frac{1}{h^2}\int_0^1 \phi'(y-1)\phi''(y)\,dy = 0, & \text{if } r_u = 0, s_u = 0, \\ \frac{1}{h}\int_0^1 \phi'(y-1)\psi''(y)\,dy = -\frac{1}{h}, & \text{if } r_u = 0, s_u = 1, \\ \frac{1}{h}\int_0^1 \psi'(y-1)\phi''(y)\,dy = \frac{1}{h}, & \text{if } r_u = 1, s_u = 0, \\ \int_0^1 \psi'(y-1)\psi''(y)\,dy = \frac{1}{2}, & \text{if } r_u = 1, s_u = 1. \end{cases} \end{aligned}$$

Now we start to calculate the following integral

$$\begin{aligned} &\int_0^1 g''_{i_u,r_u}(x_u)g'_{j_u,s_u}(x_u)\,dx_u \\ &= \begin{cases} 0, & \text{if } |i_u - j_u| > 1, \\ \int_0^1 g''_{i_u,r_u}(x_u)g'_{i_u+1,s_u}(x_u)\,dx_u, & \text{if } j_u = i_u + 1, \\ \int_0^1 g''_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\,dx_u, & \text{if } j_u = i_u, \\ \int_0^1 g''_{i_u,r_u}(x_u)g'_{i_u-1,s_u}(x_u)\,dx_u, & \text{if } j_u = i_u - 1. \end{cases} \end{aligned} \tag{4.24}$$

By (4.3)-(4.5), we have

$$\begin{aligned}
&\int_0^1 g''_{i_u,r_u}(x_u)g'_{i_u+1,s_u}(x_u)\,dx_u \\
&= \int_{i_uh}^{(i_u+1)h} g''_{i_u,r_u}(x_u)g'_{i_u+1,s_u}(x_u)\,dx_u \\
&= \begin{cases}
\frac{1}{h^2}\int_0^1 \phi''(y)\phi'(y-1)\,dy = 0, & \text{if } r_u=0, s_u=0, \\
\frac{1}{h}\int_0^1 \phi''(y)\psi'(y-1)\,dy = \frac{1}{h}, & \text{if } r_u=0, s_u=1, \\
\frac{1}{h}\int_0^1 \psi''(y)\phi'(y-1)\,dy = -\frac{1}{h}, & \text{if } r_u=1, s_u=0, \\
\int_0^1 \psi''(y)\psi'(y-1)\,dy = \frac{1}{2}, & \text{if } r_u=1, s_u=1.
\end{cases}
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 g''_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\,dx_u \\
&= \int_{((i_u-1)h)\wedge 0}^{i_uh} g''_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\,dx_u + \int_{i_uh}^{((i_u+1)h)\vee 1} g''_{i_u,r_u}(x_u)g'_{i_u,s_u}(x_u)\,dx_u \\
&= \begin{cases}
\frac{1}{h^2}\int_{-1}^1 \phi''(y)\phi'(y)\,dy = 0, & \text{if } r_u=0, s_u=0, i_u\neq 0,n, \\
\frac{1}{h^2}\int_0^1 \phi''(y)\phi'(y)\,dy = 0, & \text{if } r_u=0, s_u=0, i_u=0, \\
\frac{1}{h^2}\int_{-1}^0 \phi''(y)\phi'(y)\,dy = 0, & \text{if } r_u=0, s_u=0, i_u=n, \\
\frac{1}{h}\int_{-1}^1 \phi''(y)\psi'(y)\,dy = -\frac{2}{h}, & \text{if } r_u=0, s_u=1, i_u\neq 0,n \\
\frac{1}{h}\int_0^1 \phi''(y)\psi'(y)\,dy = -\frac{1}{h}, & \text{if } r_u=0, s_u=1, i_u=0 \\
\frac{1}{h}\int_{-1}^0 \phi''(y)\psi'(y)\,dy = -\frac{1}{h}, & \text{if } r_u=0, s_u=1, i_u=n \\
\frac{1}{h}\int_{-1}^1 \psi''(y)\phi'(y)\,dy = \frac{2}{h}, & \text{if } r_u=1, s_u=0, i_u\neq 0,n \\
\frac{1}{h}\int_0^1 \psi''(y)\phi'(y)\,dy = \frac{1}{h}, & \text{if } r_u=1, s_u=0, i_u=0 \\
\frac{1}{h}\int_{-1}^0 \psi''(y)\phi'(y)\,dy = \frac{1}{h}, & \text{if } r_u=1, s_u=0, i_u=n \\
\int_{-1}^1 \psi''(y)\psi'(y)\,dy = 0, & \text{if } r_u=1, s_u=1, x_u\neq 0,n, \\
\int_0^1 \psi''(y)\psi'(y)\,dy = -\frac{1}{2}, & \text{if } r_u=1, s_u=1, x_u=0, \\
\int_0^1 \psi''(y)\psi'(y)\,dy = \frac{1}{2}, & \text{if } r_u=1, s_u=1, x_u=n.
\end{cases}
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 g''_{i_u,r_u}(x_u)g'_{i_u-1,s_u}(x_u)\,dx_u \\
&= \int_{(i_u-1)h}^{i_uh} g''_{i_u,r_u}(x_u)g'_{i_u-1,s_u}(x_u)\,dx_u
\end{aligned}$$

$$= \begin{cases} \frac{1}{h^2}\int_0^1 \phi''(y-1)\phi'(y)\ dy = 0, & \text{if } r_u = 0, s_u = 0, \\ \frac{1}{h}\int_0^1 \phi''(y-1)\psi'(y)\ dy = \frac{1}{h}, & \text{if } r_u = 0, s_u = 1, \\ \frac{1}{h}\int_0^1 \psi''(y-1)\phi'(y)\ dy = -\frac{1}{h}, & \text{if } r_u = 1, s_u = 0, \\ \int_0^1 \psi''(y-1)\psi'(y)\ dy = -\frac{1}{2}, & \text{if } r_u = 1, s_u = 1. \end{cases}$$

Finally, we compute the following integral

$$\int_0^1 g''_{i_u,r_u}(x_u)g''_{j_u,s_u}(x_u)\ dx_u$$
$$= \begin{cases} 0, & \text{if } |i_u - j_u| > 1, \\ \int_0^1 g''_{i_u,r_u}(x_u)g''_{i_u+1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u + 1, \\ \int_0^1 g''_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u, \\ \int_0^1 g''_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\ dx_u, & \text{if } j_u = i_u - 1. \end{cases} \tag{4.25}$$

By (4.3)-(4.5), we have

$$\int_0^1 g''_{i_u,r_u}(x_u)g''_{i_u+1,s_u}(x_u)\ dx_u$$
$$= \int_{i_u h}^{(i_u+1)h} g''_{i_u,r_u}(x_u)g''_{i_u+1,s_u}(x_u)\ dx_u$$
$$= \begin{cases} \frac{1}{h^3}\int_0^1 \phi''(y)\phi''(y-1)\ dy = -\frac{12}{h^3}, & \text{if } r_u = 0, s_u = 0, \\ \frac{1}{h^2}\int_0^1 \phi''(y)\psi''(y-1)\ dy = \frac{6}{h^2}, & \text{if } r_u = 0, s_u = 1, \\ \frac{1}{h^2}\int_0^1 \psi''(y)\phi''(y-1)\ dy = -\frac{6}{h^2}, & \text{if } r_u = 1, s_u = 0, \\ \frac{1}{h}\int_0^1 \psi''(y)\psi''(y-1)\ dy = \frac{2}{h}, & \text{if } r_u = 1, s_u = 1. \end{cases}$$

$$\int_0^1 g''_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u$$
$$= \int_{((i_u-1)h)\wedge 0}^{i_u h} g'_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u + \int_{i_u h}^{((i_u+1)h)\vee 1} g'_{i_u,r_u}(x_u)g''_{i_u,s_u}(x_u)\ dx_u$$

$$= \begin{cases} \frac{1}{h^3}\int_{-1}^{1}\phi''(y)\phi''(y)\,dy = \frac{24}{h^3}, & \text{if } r_u=0, s_u=0, i_u\neq 0,n,\\ \frac{1}{h^3}\int_{0}^{1}\phi''(y)\phi''(y)\,dy = \frac{12}{h^3}, & \text{if } r_u=0, s_u=0, i_u=0,\\ \frac{1}{h^3}\int_{-1}^{0}\phi''(y)\phi''(y)\,dy = \frac{12}{h^3}, & \text{if } r_u=0, s_u=0, i_u=n,\\ \frac{1}{h^2}\int_{-1}^{1}\phi''(y)\psi''(y)\,dy = 0, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u\neq 0,n\\ \frac{1}{h^2}\int_{0}^{1}\phi''(y)\psi''(y)\,dy = \frac{6}{h^2}, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u=0\\ \frac{1}{h^2}\int_{-1}^{0}\phi''(y)\psi''(y)\,dy = -\frac{6}{h^2}, & \text{if } r_u=0, s_u=1, \text{or } r_u=1, s_u=0, i_u=n\\ \frac{1}{h}\int_{-1}^{1}\psi''(y)\psi''(y)\,dy = \frac{8}{h}, & \text{if } r_u=1, s_u=1, x_u\neq 0,n,\\ \frac{1}{h}\int_{0}^{1}\psi''(y)\psi''(y)\,dy = \frac{4}{h}, & \text{if } r_u=1, s_u=1, x_u=0,\\ \frac{1}{h}\int_{0}^{1}\psi''(y)\psi''(y)\,dy = \frac{4}{h}, & \text{if } r_u=1, s_u=1, x_u=n. \end{cases}$$

$$\begin{aligned} &\int_0^1 g''_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\,dx_u\\ &= \int_{(i_u-1)h}^{i_uh} g''_{i_u,r_u}(x_u)g''_{i_u-1,s_u}(x_u)\,dx_u\\ &= \begin{cases} \frac{1}{h^3}\int_0^1 \phi''(y-1)\phi''(y)\,dy = -\frac{12}{h^3}, & \text{if } r_u=0, s_u=0,\\ \frac{1}{h^2}\int_0^1 \phi''(y-1)\psi''(y)\,dy = -\frac{6}{h^2}, & \text{if } r_u=0, s_u=1,\\ \frac{1}{h^2}\int_0^1 \psi''(y-1)\phi''(y)\,dy = \frac{6}{h^2}, & \text{if } r_u=1, s_u=0,\\ \frac{1}{h}\int_0^1 \psi'(y-1)\psi''(y)\,dy = \frac{2}{h}, & \text{if } r_u=1, s_u=1. \end{cases} \end{aligned}$$

5.4.3 Calculating Integral I_1

Recall the definition of integral I_1

$$I_1 = \int_{\mathbf{S}} \frac{\partial^2 f_i(x)}{\partial x_k \partial x_l}\frac{\partial^2 f_j(x)}{\partial x_p \partial x_q}\,dx.$$

Then we can divide the calculation into the following several cases.

Case 1. $k=l$ and $p=q$. Then if $k=p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u\neq k}^{d}\int_0^1 g_{i_u,r_u}(x_u)g_{j_u,s_u}(x_u)\,dx_u\right)\\ &\quad\times\left(\int_0^1 g''_{i_k,r_k}(x_k)g''_{j_k,s_k}(x_k)\,dx_k\right), \end{aligned}$$

and if $k \neq p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g''_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_p,r_p}(x_p) g''_{j_p,s_p}(x_p)\, dx_p \right), \end{aligned}$$

Case 2. $k = l$, $p \neq q$. Then if $k = p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g''_{i_k,r_k}(x_k) g'_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_q,r_q}(x_q) g'_{j_q,s_q}(x_q)\, dx_q \right), \end{aligned}$$

and if $k = q$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g''_{i_k,r_k}(x_k) g'_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p \right), \end{aligned}$$

Case 3. $k \neq l$, $p = q$. Then if $k = p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g''_{j_k,s_k}(x_k)\, dx_k \int_0^1 g'_{i_l,r_l}(x_l) g_{j_l,s_l}(x_l)\, dx_l \right), \end{aligned}$$

and if $l = p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_l,r_l}(x_l) g''_{j_l,s_l}(x_l)\, dx_l \int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \right), \end{aligned}$$

Case 4. $k \neq l$, $p \neq q$. Then if $k = p$ and $l = q$, or $k = q$ and $l = p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g'_{j_k,s_k}(x_k)\, dx_k \int_0^1 g'_{i_l,r_l}(x_l) g'_{j_l,s_l}(x_l)\, dx_l \right), \end{aligned}$$

and if $k = p$ and $l \neq q$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u\right) \left(\int_0^1 g'_{i_l,r_l}(x_l) g'_{j_l,s_l}(x_l)\, dx_l\right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p\right), \end{aligned}$$

and if $k \neq p$ and $l = q$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u\right) \left(\int_0^1 g'_{i_l,r_l}(x_l) g'_{j_l,s_l}(x_l)\, dx_l\right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p\right), \end{aligned}$$

and if $k = q$ and $l \neq p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u\right) \left(\int_0^1 g'_{i_k,r_k}(x_k) g'_{j_k,s_k}(x_k)\, dx_k\right) \\ &\quad \times \left(\int_0^1 g'_{i_l,r_l}(x_l) g_{j_l,s_l}(x_l)\, dx_l \int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p\right), \end{aligned}$$

and if $k \neq q$ and $l = p$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u\right) \left(\int_0^1 g'_{i_l,r_l}(x_l) g'_{j_l,s_l}(x_l)\, dx_l\right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_q,r_q}(x_q) g'_{j_q,s_q}(x_q)\, dx_q\right), \end{aligned}$$

finally if $k \neq p, q$ and $l \neq p, q$, we have

$$\begin{aligned} I_1 &= \left(\prod_{u \neq k,l,p,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u\right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g'_{i_l,r_l}(x_l) g_{j_l,s_l}(x_l)\, dx_l\right) \\ &\quad \times \left(\int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p \int_0^1 g_{i_q,r_q}(x_q) g'_{j_q,s_q}(x_q)\, dx_q\right). \end{aligned}$$

5.4.4 Calculating Integral I_2

Recall the definition of integral I_2

$$I_2 = \int_{\mathbf{S}} \frac{\partial^2 f_i(x)}{\partial x_k \partial x_l} \frac{\partial f_j(x)}{\partial x_p}\, dx.$$

Then the calculation can be divided into the following several cases.

Case 1. $k = l = p$. Then we have

$$\begin{aligned} I_2 &= \left(\prod_{u \neq k}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g''_{i_k,r_k}(x_k) g'_{j_k,s_k}(x_k)\, dx_k \right). \end{aligned}$$

Case 2. $k = l \neq p$. Then we have

$$\begin{aligned} I_2 &= \left(\prod_{u \neq k,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g''_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p \right). \end{aligned}$$

Case 3. $k \neq l$ and $k = p$. Then we have

$$\begin{aligned} I_2 &= \left(\prod_{u \neq k,l}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g'_{j_k,s_k}(x_k)\, dx_k \int_0^1 g'_{i_l,r_l}(x_l) g_{j_l,s_l}(x_l)\, dx_l \right). \end{aligned}$$

Case 4. $k \neq l$ and $l = p$. Then we have

$$\begin{aligned} I_2 &= \left(\prod_{u \neq k,l}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g'_{i_l,r_l}(x_l) g'_{j_l,s_l}(x_l)\, dx_l \right). \end{aligned}$$

Case 5. $k \neq l \neq p$. Then we have

$$\begin{aligned} I_2 &= \left(\prod_{u \neq k,l,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \left(\int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \right) \\ &\quad \times \left(\int_0^1 g'_{i_l,r_l}(x_l) g_{j_l,s_l}(x_l)\, dx_l \int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p \right). \end{aligned}$$

5.4.5 Calculating Integral I_3

Recall the definition of integral I_3

$$I_3 = \int_{\mathbf{S}} \frac{\partial f_i(x)}{\partial x_k} \frac{\partial f_j^2(x)}{\partial x_p \partial x_q}\, dx.$$

Then the calculation can be divided into the following several cases.

Case 1. $k = p = q$. Then we have

$$\begin{aligned} I_3 &= \left(\prod_{u \neq k}^{d} \int_0^1 g_{i_u, r_u}(x_u) g_{j_u, s_u}(x_u) \, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k, r_k}(x_k) g''_{j_k, s_k}(x_k) \, dx_k \right). \end{aligned}$$

Case 2. $k \neq p = q$. Then we have

$$\begin{aligned} I_3 &= \left(\prod_{u \neq k,p}^{d} \int_0^1 g_{i_u, r_u}(x_u) g_{j_u, s_u}(x_u) \, dx_u \right) \\ &\quad \times \left(\int_0^1 g''_{i_k, r_k}(x_k) g_{j_k, s_k}(x_k) \, dx_k \int_0^1 g_{i_p, r_p}(x_p) g'_{j_p, s_p}(x_p) \, dx_p \right). \end{aligned}$$

Case 3. $k = p$ and $p \neq q$. Then we have

$$\begin{aligned} I_3 &= \left(\prod_{u \neq k,q}^{d} \int_0^1 g_{i_u, r_u}(x_u) g_{j_u, s_u}(x_u) \, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k, r_k}(x_k) g'_{j_k, s_k}(x_k) \, dx_k \int_0^1 g_{i_q, r_q}(x_q) g'_{j_q, s_q}(x_q) \, dx_q \right). \end{aligned}$$

Case 4. $k = q$ and $p \neq q$. Then we have

$$\begin{aligned} I_3 &= \left(\prod_{u \neq k,p}^{d} \int_0^1 g_{i_u, r_u}(x_u) g_{j_u, s_u}(x_u) \, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k, r_k}(x_k) g'_{j_k, s_k}(x_k) \, dx_k \int_0^1 g_{i_p, r_p}(x_p) g'_{j_p, s_p}(x_p) \, dx_p \right). \end{aligned}$$

Case 5. $k \neq p \neq q$. Then we have

$$\begin{aligned} I_3 &= \left(\prod_{u \neq k,p,q}^{d} \int_0^1 g_{i_u, r_u}(x_u) g_{j_u, s_u}(x_u) \, dx_u \right) \left(\int_0^1 g'_{i_k, r_k}(x_k) g_{j_k, s_k}(x_k) \, dx_k \right) \\ &\quad \times \left(\int_0^1 g_{i_p, r_p}(x_p) g'_{j_p, s_p}(x_p) \, dx_p \int_0^1 g_{i_q, r_q}(x_q) g'_{j_q, s_q}(x_q) \, dx_q \right). \end{aligned}$$

5.4.6 Calculating Integral I_4

Recall the definition of integral I_4

$$I_4 = \int_{\mathbf{S}} \frac{\partial f_i(x)}{\partial x_k} \frac{\partial f_j(x)}{\partial x_p} \, dx.$$

Then the calculation can be divided into the following several cases.

Case 1. $k = p$. Then we have

$$\begin{aligned} I_4 &= \left(\prod_{u \neq k}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g'_{j_k,s_k}(x_k)\, dx_k \right). \end{aligned}$$

Case 2. $k \neq p$. Then we have

$$\begin{aligned} I_4 &= \left(\prod_{u \neq k,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_k,r_k}(x_k) g_{j_k,s_k}(x_k)\, dx_k \int_0^1 g_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p \right). \end{aligned}$$

5.4.7 Calculating Integral Ib_k

Recall the definition of integral Ib_k

$$Ib_k = \int_{F_k} \frac{\partial f_i(x)}{\partial x_p} \frac{\partial f_j(x)}{\partial x_q}\, dx$$

For boundaries $k = 1$ to $k = d$, the calculation can be divided into the following several cases.

Case 1. $p = q = k$. Then we have

$$Ib_k = \left(g'_{i_k,r_k}(0) g'_{j_k,s_k}(0) \right) \left(\prod_{u \neq k}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right)$$

Case 2. $p = q \neq k$. Then we have

$$\begin{aligned} Ib_k &= \left(g_{i_k,r_k}(0) g_{j_k,s_k}(0) \right) \left(\prod_{u \neq k,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p \right). \end{aligned}$$

Case 3. $p = k$, $q \neq k$. Then we have

$$\begin{aligned} Ib_k &= \left(g'_{i_k,r_k}(0) g_{j_k,s_k}(0) \right) \left(\prod_{u \neq k,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g_{i_q,r_q}(x_q) g'_{j_q,s_q}(x_q)\, dx_q \right). \end{aligned}$$

Case 4. $p \neq k$, $q = k$. Then we have

$$\begin{aligned} Ib_k &= \left(g_{i_k,r_k}(0) g'_{j_k,s_k}(0)\right) \left(\prod_{u \neq k,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_p,r_p}(x_p) g_{j_p,s_p}(x_p)\, dx_p \right). \end{aligned}$$

Case 5. $p \neq q$, $p \neq k$ and $q \neq k$. Then we have

$$\begin{aligned} Ib_k &= \left(g_{i_k,r_k}(0) g_{j_k,s_k}(0)\right) \left(\prod_{u \neq k,p,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_p,r_p}(x_p) g_{j_p,s_p}(x_p)\, dx_p \int_0^1 g_{i_q,r_q}(x_q) g'_{j_q,s_q}(x_q)\, dx_q \right). \end{aligned}$$

For boundaries $k = d$ to $k = 2d$, the calculation can be divided into the following several cases.

Case 1. $p = q = k$. Then we have

$$Ib_k = \left(g'_{i_k,r_k}(1) g'_{j_k,s_k}(1)\right) \left(\prod_{u \neq k}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right)$$

Case 2. $p = q \neq k$. Then we have

$$\begin{aligned} Ib_k &= \left(g'_{i_k,r_k}(1) g'_{j_k,s_k}(1)\right) \left(\prod_{u \neq k,p}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right), \\ &\quad \times \left(\int_0^1 g'_{i_p,r_p}(x_p) g'_{j_p,s_p}(x_p)\, dx_p \right) \end{aligned}$$

Case 3. $p = k$, $q \neq k$. Then we have

$$\begin{aligned} Ib_k &= \left(g'_{i_k,r_k}(1) g_{j_k,s_k}(1)\right) \left(\prod_{u \neq k,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g_{i_q,r_q}(x_q) g'_{j_q,s_q}(x_q)\, dx_q \right). \end{aligned}$$

Case 4. $p \neq k$, $q = k$. Then we have

$$\begin{aligned} Ib_k &= \left(g_{i_k,r_k}(1) g'_{j_k,s_k}(1)\right) \left(\prod_{u \neq k,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_p,r_p}(x_p) g_{j_p,s_p}(x_p)\, dx_p \right). \end{aligned}$$

Case 5. $p \neq q$, $p \neq k$ and $q \neq k$. Then we have

$$\begin{aligned} Ib_k &= (g_{i_k,r_k}(1)g_{j_k,s_k}(1)) \left(\prod_{u \neq k,p,q}^{d} \int_0^1 g_{i_u,r_u}(x_u) g_{j_u,s_u}(x_u)\, dx_u \right) \\ &\quad \times \left(\int_0^1 g'_{i_p,r_p}(x_p) g_{j_p,s_p}(x_p)\, dx_p \int_0^1 g_{i_q,r_q}(x_q) g'_{j_q,s_q}(x_q)\, dx_q \right). \end{aligned}$$

5.5 Numerical Comparisons

5.5.1 Comparison with SC Solutions

In this subsection we compare results obtained with our algorithm against a known analytic solution for a special case of SRBM. In this special case, we take $\theta = 0$ and $\Gamma = 2I$ (I is the 2×2 identity matrix. The corresponding reflection matrix is

$$R = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \end{pmatrix}$$

As discussed in Section 2.5 of Dai and Harrison [11], the density $p \notin L^2$. Readers will see that our algorithm gives accurate approximations even in this case. This is consistent with the conjecture in Dai and Harrison [12].

Table 5.1 compares two different estimates of q_1, q_2, δ_1, δ_2, δ_3 and δ_4. The FEM estimate is obtained with our algorithm, using $n = 14$. The SC estimate was obtained by Trefethen and Williams [39] using a software package called SCPACK. The row DIFF gives the differences between the SC estimates and our finite element estimates.

5.5.2 Comparisons with $2D$ Exponential Solutions.

We first give a criterion for the stationary density p to be of exponential form in a two-dimensional RBM. Under the criterion, the stationary density is of exponential form and all of the performance measures have explicit formulas. Thus, we can compare our

$\Gamma = 2I, \theta = 0.0, n = 14$						
	q_1	q_2	δ_1	δ_2	δ_3	δ_4
FEM	0.551442	0.448558	0.805350	1.610701	1.610701	0.805350
SC	0.551506	0.448494	0.805295	1.610589	1.610589	0.805295
DIFF	-0.000054	0.000064	0.000065	0.000112	0.000112	0.000065

Table 5.1: Comparisons with SCPACK.

finite element estimates with these densities and corresponding performance measures. Let the reflection matrix R be

$$R = \begin{pmatrix} 1 & t_2 & -1 & t_4 \\ t_1 & 1 & t_3 & -1 \end{pmatrix} \tag{5.1}$$

Then we have the following Proposition proved in Chapter 2 of Dai [11].

Proposition 5.4 *The stationary density p_0 is of exponential form if and only if*

$$\begin{cases} t_1\Gamma_{11} + t_2\Gamma_{22} = 2\Gamma_{21}, \\ t_3 = -t_1,\ t_4 = -t_2. \end{cases} \tag{5.2}$$

In this case, the stationary density is an exponential function

$$x \to c \cdot exp(\lambda \cdot x), \tag{5.3}$$

where

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ with } \lambda_1 = \frac{2(\theta_1 - t_2\theta_2)}{(1-t_1t_2)\Gamma_{11}} \text{ and } \lambda_2 = \frac{2(\theta_2 - t_1\theta_1)}{(1-t_1t_2)\Gamma_{22}} \tag{5.4}$$

and c is a normalizing constant such that $\int_{\mathbf{S}} p_0(x)dx = 1$.

Remark. The denominators in the expressions for λ_1 and λ_2 are not zero because $1 - t_1t_2 = (t_1^2\Gamma_{11} - 2t_1\Gamma_{12} + \Gamma_{22})/\Gamma_{22} > 0$ by the positive definiteness of Γ.

Let k_1 and k_2 satisfy

$$c_1 \int_0^1 e^{\lambda_1 x_1} dx_1 = 1,\ c_2 \int_0^1 e^{\lambda_2 x_2} dx_2 = 1.$$

Where c_1c_2 is the normalizing constant for the density p_0 and the mean vector $(q_1, q_2)'$ is given by

$$q_1 = c_1 \int_0^1 x_1 e^{\lambda_1 x_1} dx_1, \quad q_2 = c_2 \int_0^1 x_2 e^{\lambda_2 x_2} dx_2. \tag{5.5}$$

In the case that $\Gamma = I$, (5.2) shows that one must choose $t_1 = t_4$, $t_2 = -t_4$ and $t_3 = -t_4$ to assure an exponential stationary distribution. Tables 5.2 through 5.3 give computational results for $t_4 = 0.0$. Table 5.2 presents the estimates of q_1 and q_2 with our algorithm for various test problems having exponential stationary distributions. The columns of θ_1 and θ_2 correspond to different choices of the drift vector $\theta = (\theta_1, \theta_2)'$, and the columns labeled q_1-error and q_2-error give differences between estimates computed with our algorithm and the exact values derived from (5.5). Table 5.3 gives the density estimates with our algorithm for $\theta_1 = -1.0$ and $\theta_2 = 1.0$, and table 5.4 is the corresponding error estimates with the exact values derived from (5.3). Table 5.5 through 5.7 repeat the above procedure for $t_4 = 1.0$.

In the case that $\Gamma \neq I$, we select a positive definite matrix Γ with $\Gamma_{11} = 4.0$, $\Gamma_{22} = 1.0$ and $\Gamma_{12} = 0.5$, and $t_4 = 1.0$. Considering condition (5.2), we have $t_1 = 0.5$, $t_2 = -1$, and $t_3 = -0.5$. Under these parameters, tables 5.8 through 5.10 repeat the procedure in previous paragraph.

5.5.3 Comparisons with $3D$ Exponential Solutions

In this subsection, we use some special 3-dimensional SRBM whose stationary density has explicit formula to compare with our finite element estimates. Let the reflection matrix be

$$R = \begin{pmatrix} 0 & 1 & t_3 & 0 & -1 & t_6 \\ 0 & t_2 & 1 & 0 & t_5 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{5.6}$$

Suppose $\Gamma_{i3} = 0$ $(i = 1, 2)$ and

$$\begin{cases} t_2\Gamma_{11} + t_3\Gamma_{22} = 2\Gamma_{21}, \\ t_5 = -t_2,\ t_6 = -t_3. \end{cases} \tag{5.7}$$

Then we have the stationary density p_0 for the SRBM is an exponential function

$$x \to c \cdot exp(\lambda \cdot x), \tag{5.8}$$

where

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \text{ with } \lambda_1 = \frac{2(\theta_1 - t_2\theta_2)}{(1 - t_3t_2)\Gamma_{11}},\ \lambda_2 = \frac{2(\theta_2 - t_3\theta_1)}{(1 - t_3t_2)\Gamma_{22}},\ \lambda_3 = \frac{2\theta_3}{\Gamma_{33}}, \tag{5.9}$$

and k is a normalizing constant such that $\int_{\mathbf{S}} p_0(x)dx = 1$. Let c_1, c_2 and c_3 satisfy

$$c_1 \int_0^1 e^{\lambda_1 x_1} dx_1 = 1,\ c_2 \int_0^1 e^{\lambda_2 x_2} dx_2 = 1,\ c_3 \int_0^1 e^{\lambda_3 x_3} dx_3 = 1.$$

Where $c_1c_2c_3$ is the normalizing constant for the density p_0 and the mean vector $(q_1, q_2, q_3)'$ is given by

$$q_1 = c_1 \int_0^a x_1 e^{\lambda_1 x_1} dx_1,\ q_2 = c_2 \int_0^b x_2 e^{\lambda_2 x_2} dx_2,\ q_3 = c_3 \int_0^c x_3 e^{\lambda_3 x_3} dx_3. \tag{5.10}$$

For $\Gamma = I$, $t_2 = 1.0$, Table 5.11 gives the estimates of q_1, q_2 and q_3 with our algorithm for various test problems having exponential stationary distributions. The columns θ_1, θ_2 and θ_3 correspond to different choices of the drift vector θ. In Table 5.12, the columns q_1-error, q_2-error and q_3-error present differences between estimates computed with our algorithm and the exact values derived from (5.10). Tables 5.13 and 5.14 present the estimated density function with our algorithm for $\theta_1 = 1.0$, $\theta_2 = 1.0$ and $\theta_3 = -0.5$. Tables 5.15 and 5.16 give the corresponding error estimates with the exact values derived from (5.8).

Finally, it should be mentioned that currently, we use a solver for the linear equation (3.4). It requires large run-time memory and limits our implementation for

$d = 3$ and $n > 6$. As mentioned before, our coefficient matrix A is a large sparse matrix. The solver did not take advantage of the sparsity of matrix A. If we employ a sparse matrix solver, we expect to solve large problems.

$\Gamma = I, t_4 = 0.0, n = 14$					
θ_1	θ_2	q_1	q_2	q_1-error	q_2-error
0.0	0.0	0.500000	0.500000	$-9.436896e-16$	$-7.771561e-16$
0.5	0.5	0.581977	0.581977	$5.727013e-08$	$5.727194e-08$
-0.5	-0.5	0.418023	0.418023	$-5.726244e-08$	$-5.725044e-08$
-0.5	0.0	0.418023	0.500000	$-1.806236e-09$	$-5.512535e-11$
1.0	1.0	0.656518	0.656518	$8.955172e-07$	$8.954953e-07$
-1.0	1.0	0.343482	0.656518	$-8.954984e-07$	$8.954621e-07$
2.0	2.0	0.768663	0.768663	$4.513523e-05$	$4.513522e-05$
2.0	-2.0	0.768663	0.231337	$4.513523e-05$	$-4.513551e-05$
0.0	-2.0	0.500000	0.231342	$-1.841194e-10$	$-4.056480e-05$
3.0	-3.0	0.835867	0.164133	$4.259646e-04$	$-4.259631e-04$
4.0	-4.0	0.875610	0.124390	$1.744794e-03$	$-1.744796e-03$

Table 5.2: Mean comparisons with exponential solutions in unit square.

$\Gamma = I, t_4 = 0.0, \theta_1 = -1.0, \theta_2 = 1.0, n = 14$						
$p_0(i,j)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	0.71469	0.47628	0.31792	0.21004	0.13639	0.10476
1/5	1.08675	0.72428	0.48517	0.32605	0.21767	0.13639
2/5	1.61456	1.08014	0.72458	0.48687	0.32605	0.21004
3/5	2.40837	1.61284	1.08135	0.72458	0.48517	0.31792
4/5	3.58309	2.40321	1.61284	1.08014	0.72428	0.47628
1.0	5.37019	3.58309	2.40837	1.61456	1.08675	0.71469

Table 5.3: Estimated density function.

$\Gamma = I, t_4 = 0.0, \theta_1 = -1.0, \theta_2 = 1.0, n = 14$						
$p_0(i,j)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	$-9.37e{-03}$	$-9.07e{-03}$	$-7.41e{-03}$	$-8.04e{-03}$	$-9.79e{-03}$	$6.76e{-03}$
1/5	$6.58e{-03}$	$2.20e{-04}$	$-1.83e{-04}$	$7.03e{-04}$	$-4.13e{-04}$	$-9.79e{-03}$
2/5	$3.12e{-03}$	$-3.57e{-05}$	$5.20e{-04}$	$1.51e{-03}$	$7.03e{-04}$	$-8.04e{-03}$
3/5	$4.39e{-03}$	$1.41e{-03}$	$1.17e{-03}$	$5.20e{-04}$	$-1.83e{-04}$	$-7.41e{-03}$
4/5	$-3.21e{-03}$	$-7.57e{-04}$	$1.41e{-03}$	$-3.57e{-05}$	$2.20e{-04}$	$-9.07e{-03}$
1.0	$2.00e{-02}$	$-3.21e{-03}$	$4.39e{-03}$	$3.12e{-03}$	$6.58e{-03}$	$-9.37e{-03}$

Table 5.4: Error estimates with exponential solutions.

$\Gamma = I, t_4 = 1.0, n = 14$					
θ_1	θ_2	q_1	q_2	q_1-error	q_2-error
0.0	0.0	0.500000	0.500000	$-9.436896e{-16}$	$2.220446e{-16}$
0.5	0.5	0.581972	0.500006	$-4.894779e{-06}$	$6.089591e{-06}$
−0.5	−0.5	0.418028	0.499994	$4.894780e{-06}$	$-6.089590e{-06}$
−0.5	0.0	0.458506	0.541489	$7.895435e{-08}$	$-5.417237e{-06}$
1.0	1.0	0.656511	0.500016	$-5.823915e{-06}$	$1.617267e{-05}$
−1.0	1.0	0.499984	0.656511	$-1.617269e{-05}$	$-5.823903e{-06}$
2.0	2.0	0.768675	0.500073	$5.786730e{-05}$	$7.318189e{-05}$
2.0	−2.0	0.500073	0.231325	$7.318187e{-05}$	$-5.786730e{-05}$
0.0	−2.0	0.343498	0.343480	$1.501171e{-05}$	$-3.024782e{-06}$
3.0	−3.0	0.500265	0.164066	$2.654667e{-04}$	$-4.934022e{-04}$
4.0	−4.0	0.500808	0.124316	$8.077718e{-04}$	$-1.819233e{-03}$

Table 5.5: Mean comparisons with exponential solutions.

$\Gamma = I, t_4 = 0.0, \theta_1 = -1.0, \theta_2 = 1.0, n = 14$						
$p_0(i,j)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	0.32207	0.46686	0.69633	1.03875	1.54635	2.34062
1/5	0.31354	0.46704	0.69670	1.03951	1.55054	2.31367
2/5	0.31372	0.46772	0.69753	1.04007	1.55053	2.31165
3/5	0.31375	0.46847	0.69856	1.04134	1.55129	2.31184
4/5	0.31203	0.46680	0.69666	1.03983	1.54957	2.31728
1.0	0.30405	0.46033	0.68705	1.02518	1.51806	2.20096

Table 5.6: Estimated density function.

$\Gamma = I, t_4 = 1.0, \theta_1 = -1.0, \theta_2 = 1.0, n = 14$						
$p_0(i,j)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	$9.03e{-03}$	$-1.36e{-04}$	$-3.40e{-04}$	$-5.67e{-04}$	$-4.12e{-03}$	$2.75e{-02}$
1/5	$5.00e{-04}$	$4.90e{-05}$	$2.88e{-05}$	$1.99e{-04}$	$6.95e{-05}$	$6.35e{-04}$
2/5	$6.84e{-04}$	$7.28e{-04}$	$8.52e{-04}$	$7.55e{-04}$	$5.79e{-05}$	$-1.38e{-03}$
3/5	$7.13e{-04}$	$1.47e{-03}$	$1.89e{-03}$	$2.02e{-03}$	$8.12e{-04}$	$-1.20e{-03}$
4/5	$-1.01e{-03}$	$-1.97e{-04}$	$-1.67e{-05}$	$5.20e{-04}$	$-9.03e{-04}$	$4.24e{-03}$
1.0	$-8.98e{-03}$	$-6.66e{-03}$	$-9.62e{-03}$	$-1.41e{-02}$	$-3.24e{-02}$	$-1.12e{-01}$

Table 5.7: Error estimates with exponential solution.

$\Gamma_{11} = 4.0, \Gamma_{22} = 1.0, \Gamma_{12} = 0.5, t_4 = 1.0, n = 14$					
θ_1	θ_2	q_1	q_2	q_1-error	q_2-error
0.0	0.0	0.500000	0.500000	$2.258027e{-}12$	$1.094236e{-}12$
0.5	1.0	0.541468	0.581922	$-2.635217e{-}05$	$-5.460376e{-}05$
-1.0	1.0	0.499949	0.656425	$-5.084222e{-}05$	$-9.202287e{-}05$
2.0	-1.0	0.527794	0.300438	$6.736003e{-}05$	$1.064767e{-}04$

Table 5.8: Mean estimations for $\Gamma \neq I$.

$\Gamma_{11} = 4.0, \Gamma_{22} = 1.0, \Gamma_{12} = 0.5, t_4 = 1.0, n = 14$						
$p_0(i, j)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	0.23825	0.46667	0.69202	1.03394	1.54391	2.49797
1/5	0.21623	0.46844	0.69551	1.03969	1.55041	2.39347
2/5	0.30060	0.46708	0.69772	1.04296	1.55377	2.34769
3/5	0.31754	0.46714	0.69826	1.04517	1.55947	2.36926
4/5	0.33629	0.46708	0.69751	1.04326	1.54681	2.68646
1.0	0.36964	0.46996	0.69665	1.03855	1.53722	2.28198

Table 5.9: Estimated density function for $\Gamma \neq I$.

$\Gamma_{11} = 4.0, \Gamma_{22} = 1.0, \Gamma_{12} = 0.5, t_4 = 1.0, n = 14$						
$p_0(i,j)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	$-7.47e{-02}$	$-3.22e{-04}$	$-4.65e{-03}$	$-5.37e{-03}$	$-6.56e{-03}$	$1.84e{-01}$
1/5	$-9.68e{-02}$	$1.44e{-03}$	$-1.16e{-03}$	$3.74e{-04}$	$-5.98e{-05}$	$8.04e{-02}$
2/5	$-1.24e{-02}$	$8.89e{-05}$	$1.04e{-03}$	$3.65e{-03}$	$3.29e{-03}$	$3.46e{-02}$
3/5	$4.50e{-03}$	$1.51e{-04}$	$1.58e{-03}$	$5.85e{-03}$	$9.00e{-03}$	$5.62e{-02}$
4/5	$2.32e{-02}$	$9.01e{-05}$	$8.36e{-04}$	$3.94e{-03}$	$-3.66e{-03}$	$3.73e{-01}$
1.0	$5.66e{-02}$	$2.97e{-03}$	$-2.39e{-05}$	$-7.62e{-04}$	$-1.32e{-02}$	$-3.10e{-02}$

Table 5.10: Error estimates for $\Gamma \neq I$.

$\Gamma = I, t_2 = 1.0, n = 6$					
θ_1	θ_2	θ_3	q_1	q_2	q_3
0.0	0.0	0.0	0.500014	0.500002	0.499972
1.0	0.5	0.0	0.620499	0.458605	0.499905
1.0	0.5	-0.5	0.620459	0.458601	0.417985
-1.0	0.5	0.5	0.458568	0.620454	0.581826
-1.0	0.5	1.0	0.458660	0.620471	0.656142
1.0	-1.0	-0.5	0.500177	0.343629	0.417883

Table 5.11: Estimated means in unit cube.

$\Gamma = I, t_2 = 1.0, n = 6$					
θ_1	θ_2	θ_3	q_1-error	q_2-error	q_3-error
0.0	0.0	0.0	$1.390652e{-05}$	$1.549133e{-06}$	$-2.817109e{-05}$
1.0	0.5	0.0	$-5.131732e{-05}$	$9.894971e{-05}$	$-9.500724e{-05}$
1.0	0.5	-0.5	$-9.070568e{-05}$	$9.469117e{-05}$	$-3.851720e{-05}$
-1.0	0.5	0.5	$6.201958e{-05}$	$-9.627116e{-05}$	$-1.506037e{-04}$
-1.0	0.5	1.0	$1.539639e{-04}$	$-7.880091e{-05}$	$-3.748214e{-04}$
1.0	-1.0	-0.5	$1.773108e{-04}$	$1.463151e{-04}$	$-1.403901e{-04}$

Table 5.12: Mean comparisons with exponential solutions.

$\Gamma = I, t_2 = 1.0, \theta_1 = 1.0, \theta_2 = -1.0, \theta_3 = -0.5, n = 6$						
$k = 0.0$						
$p_0(i,j,k)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	3.26886	2.56206	1.66711	1.10260	0.68539	0.12973
1/5	3.61918	2.38741	1.63753	1.10459	0.77480	0.44912
2/5	3.65398	2.42186	1.65617	1.11847	0.77158	0.51030
3/5	3.65419	2.42323	1.65606	1.11866	0.77248	0.50295
4/5	3.62649	2.45744	1.68804	1.14275	0.79836	0.50589
1.0	3.78717	2.46150	1.66640	1.10520	0.69802	0.72862
$k = 1/5$						
0.0	2.52478	1.88806	1.30120	0.87792	0.44751	0.61430
1/5	3.02124	2.00631	1.35274	0.90251	0.60701	0.42035
2/5	3.00395	2.01298	1.35780	0.90818	0.60974	0.41414
3/5	3.00219	2.00934	1.35102	0.90252	0.60801	0.41585
4/5	3.00301	2.00943	1.34848	0.89828	0.60680	0.41479
1.0	2.89777	1.96249	1.31048	0.87536	0.59243	0.65070
$k = 2/5$						
0.0	2.03619	1.53155	1.05199	0.71012	0.48055	0.40018
1/5	2.46541	1.64030	1.10584	0.73890	0.49254	0.33165
2/5	2.45118	1.64817	1.11220	0.74579	0.49846	0.33035
3/5	2.44974	1.64802	1.10914	0.74315	0.49861	0.33092
4/5	2.44493	1.64732	1.10770	0.74087	0.49865	0.32857
1.0	2.33145	1.65065	1.10220	0.73714	0.49680	0.42950

Table 5.13: Estimated density function in unit cube

$\Gamma = I, t_2 = 1.0, \theta_1 = 1.0, \theta_2 = -1.0, \theta_3 = -0.5, n = 6$						
$k = 3/5$						
$p_0(i,j,k)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	1.64555	1.24513	0.85511	0.57658	0.38585	0.27220
1/5	2.02015	1.34121	0.90575	0.60753	0.40562	0.26480
2/5	2.00725	1.34917	0.91309	0.61564	0.41223	0.26564
3/5	2.00543	1.34932	0.91059	0.61351	0.41163	0.26326
4/5	1.99618	1.34559	0.90702	0.61024	0.41012	0.25973
1.0	1.84864	1.35894	0.90590	0.60726	0.40752	0.24593
$k = 4/5$						
0.0	1.32575	1.01985	0.70195	0.46938	0.30918	0.17769
1/5	1.68021	1.10049	0.74238	0.49642	0.32959	0.20545
2/5	1.66644	1.10747	0.74947	0.50469	0.33614	0.20794
3/5	1.66421	1.10660	0.74564	0.50119	0.33346	0.20351
4/5	1.65192	1.10192	0.74111	0.49708	0.33035	0.19941
1.0	1.44717	1.11511	0.73710	0.49216	0.32635	0.12909
$k = 1.0$						
0.0	1.07242	0.97887	0.62491	0.41554	0.27873	0.15892
1/5	1.32177	0.83866	0.57767	0.38268	0.24795	0.19181
2/5	1.34791	0.86288	0.59208	0.39503	0.25801	0.18995
3/5	1.35774	0.86234	0.58843	0.39137	0.25489	0.18722
4/5	1.39311	0.85596	0.58567	0.38931	0.25151	0.18662
1.0	1.19958	0.88265	0.60937	0.41392	0.28677	0.14216

Table 5.14: Estimated density function in unit cube (continued)

$\Gamma = I, t_2 = 1.0, \theta_1 = 1.0, \theta_2 = -1.0, \theta_3 = -0.5, n = 6$						
$k = 0.0$						
$p_0(i,j,k)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	$-3.90e{-01}$	$1.09e{-01}$	$2.29e{-02}$	$4.80e{-04}$	$-5.34e{-02}$	$-3.65e{-01}$
1/5	$-4.00e{-02}$	$-6.54e{-02}$	$-6.64e{-03}$	$2.47e{-03}$	$3.60e{-02}$	$-4.61e{-02}$
2/5	$-5.19e{-03}$	$-3.10e{-02}$	$1.20e{-02}$	$1.64e{-02}$	$3.28e{-02}$	$1.51e{-02}$
3/5	$-4.98e{-03}$	$-2.96e{-02}$	$1.19e{-02}$	$1.65e{-02}$	$3.37e{-02}$	$7.74e{-03}$
4/5	$-3.27e{-02}$	$4.63e{-03}$	$4.39e{-02}$	$4.06e{-02}$	$5.96e{-02}$	$1.07e{-02}$
1.0	$1.28e{-01}$	$8.69e{-03}$	$2.22e{-02}$	$3.08e{-03}$	$-4.07e{-02}$	$2.33e{-01}$
$k = 1/5$						
0.0	$-4.71e{-01}$	$-1.20e{-01}$	$-4.49e{-02}$	$-2.44e{-02}$	$-1.57e{-01}$	$2.09e{-01}$
1/5	$2.54e{-02}$	$-1.88e{-03}$	$6.61e{-03}$	$1.71e{-04}$	$2.15e{-03}$	$1.49e{-02}$
2/5	$8.08e{-03}$	$4.79e{-03}$	$1.17e{-02}$	$5.84e{-03}$	$4.88e{-03}$	$8.70e{-03}$
3/5	$6.32e{-03}$	$1.15e{-03}$	$4.89e{-03}$	$1.76e{-04}$	$3.15e{-03}$	$1.04e{-02}$
4/5	$7.13e{-03}$	$1.24e{-03}$	$2.35e{-03}$	$-4.06e{-03}$	$1.95e{-03}$	$9.35e{-03}$
1.0	$-9.81e{-02}$	$-4.57e{-02}$	$-3.57e{-02}$	$-2.70e{-02}$	$-1.24e{-02}$	$2.45e{-01}$
$k = 2/5$						
0.0	$-4.17e{-01}$	$-1.13e{-01}$	$-5.01e{-02}$	$-2.86e{-02}$	$-1.47e{-02}$	$6.82e{-02}$
1/5	$1.26e{-02}$	$-3.87e{-03}$	$3.72e{-03}$	$1.24e{-04}$	$-2.68e{-03}$	$-3.01e{-04}$
2/5	$-1.63e{-03}$	$4.00e{-03}$	$1.01e{-02}$	$7.02e{-03}$	$3.24e{-03}$	$-1.60e{-03}$
3/5	$-3.08e{-03}$	$3.85e{-03}$	$7.02e{-03}$	$4.38e{-03}$	$3.39e{-03}$	$-1.04e{-03}$
4/5	$-7.88e{-03}$	$3.15e{-03}$	$5.58e{-03}$	$2.10e{-03}$	$3.44e{-03}$	$-3.39e{-03}$
1.0	$-1.21e{-01}$	$6.48e{-03}$	$8.46e{-05}$	$-1.63e{-03}$	$1.59e{-03}$	$9.75e{-02}$

Table 5.15: Error estimations with exponential solutions

$\Gamma = I, t_2 = 1.0, \theta_1 = 1.0, \theta_2 = -1.0, \theta_3 = -0.5, n = 6$						
$k = 3/5$						
$p_0(i,j,k)$	0.0	1/5	2/5	3/5	4/5	1.0
0.0	$-3.63e{-01}$	$-1.01e{-01}$	$-4.72e{-02}$	$-2.83e{-02}$	$-1.96e{-02}$	$4.24e{-04}$
1/5	$1.20e{-02}$	$-4.92e{-03}$	$3.41e{-03}$	$2.67e{-03}$	$1.75e{-04}$	$-6.98e{-03}$
2/5	$-9.42e{-04}$	$3.04e{-03}$	$1.07e{-02}$	$1.08e{-02}$	$6.78e{-03}$	$-6.14e{-03}$
3/5	$-2.76e{-03}$	$3.19e{-03}$	$8.25e{-03}$	$8.65e{-03}$	$6.18e{-03}$	$-8.52e{-03}$
4/5	$-1.20e{-02}$	$-5.41e{-04}$	$4.68e{-03}$	$5.39e{-03}$	$4.67e{-03}$	$-1.21e{-02}$
1.0	$-1.60e{-01}$	$1.28e{-02}$	$3.56e{-03}$	$2.40e{-03}$	$2.07e{-03}$	$-2.59e{-02}$
$k = 4/5$						
0.0	$-3.18e{-01}$	$-8.23e{-02}$	$-3.68e{-02}$	$-2.58e{-02}$	$-2.28e{-02}$	$-4.48e{-02}$
1/5	$3.60e{-02}$	$-1.63e{-03}$	$3.61e{-03}$	$1.20e{-03}$	$-2.36e{-03}$	$-1.71e{-02}$
2/5	$2.23e{-02}$	$5.35e{-03}$	$1.07e{-02}$	$9.47e{-03}$	$4.19e{-03}$	$-1.46e{-02}$
3/5	$2.00e{-02}$	$4.48e{-03}$	$6.87e{-03}$	$5.97e{-03}$	$1.51e{-03}$	$-1.90e{-02}$
4/5	$7.75e{-03}$	$-1.98e{-04}$	$2.33e{-03}$	$1.87e{-03}$	$-1.60e{-03}$	$-2.31e{-02}$
1.0	$-1.97e{-01}$	$1.30e{-02}$	$-1.67e{-03}$	$-3.06e{-03}$	$-5.61e{-03}$	$-9.34e{-02}$
$k = 1.0$						
0.0	$-2.74e{-01}$	$7.65e{-02}$	$2.01e{-02}$	$1.01e{-02}$	$6.95e{-03}$	$-2.33e{-02}$
1/5	$-2.44e{-02}$	$-6.37e{-02}$	$-2.72e{-02}$	$-2.28e{-02}$	$-2.38e{-02}$	$9.63e{-03}$
2/5	$1.78e{-03}$	$-3.95e{-02}$	$-1.28e{-02}$	$-1.04e{-02}$	$-1.38e{-02}$	$7.77e{-03}$
3/5	$1.16e{-02}$	$-4.00e{-02}$	$-1.64e{-02}$	$-1.41e{-02}$	$-1.69e{-02}$	$5.04e{-03}$
4/5	$4.70e{-02}$	$-4.64e{-02}$	$-1.92e{-02}$	$-1.61e{-02}$	$-2.03e{-02}$	$4.44e{-03}$
1.0	$-1.47e{-01}$	$-1.97e{-02}$	$4.52e{-03}$	$8.47e{-03}$	$1.50e{-02}$	$-4.00e{-02}$

Table 5.16: Error estimations with exponential solutions (continued).

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