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Optimal Hedging and Its Performance Based on A Lévy Driven Volatility Model

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***Abstract*—In this paper, we apply the martingale approach to address the mean-variance hedging problem in an incomplete market with coefficients driven by external random environmental (risk) factors of non-Gaissian Ornstein-Uhlenbeck (NGOU) processes. In constructing a hedging strategy and justifying it to be the global risk-minimizing one for our market, there are two steps involved. First, we derive a backward stochastic differential equation (BSDE) with jumps for the mean-value process of a given contingent claim under suitable terminal conditions including both European call and put options as special cases. Second, by combining the solution of the BSDE and the pure hedging coefficient obtained via the concept of opportunity-neutral measure, we get our optimal hedging strategy. Moreover, both analytical and numerical examples indicate that the optimal hedging errors obtained through the martingale approach are slightly smaller than (but quite consistent with) the optimal variances previously gained via the method of optimal control under some circumstances when the contingent claim reduces to a constant, which illustrate, to some extent, the reasonability of the widely concerned *ad hoc* decision manner used in the method of optimal control. In addition, these examples also indicate that, although perfect hedging is impossible in an incomplete market, the mean-variance hedging errors can be quite small in many cases as terminal time increases.**

I. Introduction

In this paper, we consider one of the major problems in financial engineering: the mean-variance hedging of contingent claims in an incomplete market defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. The market consists of $d+1$ primitive assets: one bond with fixed interest rate and d risky assets with price processes described by a generalized Black-Scholes model whose coefficients depend on the market mode (or regime caused by leverage effect, and etc.), or in other words, these coefficients are driven by external random risk factors of NGOU processes that are independent of the underlying d-dimensional Brownian motion. Our market model includes the Barndorff-Nielsen & Shephard (BNS) volatility model suggested by Barndorff-Nielsen and Shephard [2] and further studied in, such as, Benth *et al.* [3], Benth and Meyer-Brandis [4], and Lindberg [20] as a special case, and closely relates to the model considered in Delong and Klüppelberg [11]. The volatility level in these models are allowed to have sudden shifts in the upward direction, while decreasing exponentially between such shifts, and moreover, the empirical investigations on exchange rates for real market data in Barndorff-Nielsen and Shephard [2] demonstrate that such models fit the empirical auto-correlation and the leptokurtic behavior of log-return data remarkably well. However, these models induce incompleteness of the markets in the sense that it is typically impossible to perfect replication of contingent claims based on the bond and the d primitive risky assets. A criterion for determining a good hedging strategy is to minimize the mean squared hedging error over the set $\bar{\Theta}$ of all reasonable trading strategy processes, i.e., to solve the stochastic optimal control problem,

$$\inf_{u\in\bar{\Theta}} E\left[(v+(u\cdot D)(T)-H)^2\right], \tag{1.1}$$

where the random variable H is the discounted payoff of the claim, D is the discounted price process of d risky assets, v is the initial endowment and T is the time horizon. Mathematically speaking, one seeks to compute the orthogonal projection of $H-v$ on some space of stochastic integrals.

To solve the mean-variance hedging problem (1.1), we explicitly construct a trading strategy and thoroughly justify it to be the global risk-minimizing hedging strategy through the following procedure. First, we derive a BSDE with jumps and external random factors of NGOU type for the mean value process of the option H and show the unique existence of solution to the BSDE under suitable terminal conditions including both European call and put options as special cases. Second, by combining the solution to the BSDE and the pure hedging coefficient obtained through the concept of opportunity-neutral measure, we get the optimal hedging strategy for our market.

The mean-variance hedging problem was initially introduced by Föllmer and Sondermann [13] under complete information, who also suggested an approach for the computation of hedging strategies in an incomplete market by extending the martingale approach of Harrison and Kreps [16]. The basic idea of the approach was to introduce a measure of riskiness in terms of a conditional mean square error process where the discounted price process is a square-integrable martingale and the answer to the hedging problem is provided by the *Galtchouk-Kunita-Watanabe decomposition* of the claim, and then this concept of local-risk minimization was further extended for

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the semimartingale case by Föllmer and Schweizer [14] and Schweizer [26], [27], where the minimal martingale measure and Föllmer-Schweizer (F-S) decomposition play a central role. As an alternative to local-risk minimization, the authors in Pham [22] and Schweizer [29] considered the solution in terms of global-risk minimization of the unconditional expected squared hedging error presented in (1.1), which is more attractive than local-risk minimization since after all one cares about the total hedging error and not the daily profit-loss ratios. Then the study on global-risk minimization was further developed by Cěrný and Kallsen [6], who showed that the hedging model (1.1) admits a solution in a very general class of arbitrage-free semimartingale markets where local-risk minimization may fail to be well defined. The key point of their approach is the introduction of the opportunity-neutral measure P^* that turns the dynamic asset allocation problem into a myopic one as stated previously.

Here, we point out that, when the interest rate is a random and unbounded process, the proper notion of admissibility for the hedging problem has not been worked out yet (see, e.g., Gourieroux *et al.* [15], Laurent and Pham [21]). So, in Dai [7], we turn to study the mean-variance portfolio selection problem via the method of optimal feedback control when the random variable H in (1.1) reduces to a constant, e.g., a prescribed daily expected return. Under the method, we can explicitly compute and justify optimal portfolios over an admissible set that is large enough to cover some important classes of strategies of practical interests such as the class of feedback controls of Markov type (see, e.g., Fleming and Rishel [12]). Moreover, the constructed strategy through optimal control is relatively easy to be implemented. Nevertheless, since the portfolio strategy is made in an *ad hoc* manner, a natural question is raised: how well the optimal strategy will be? So, in this paper, we also present both analytical and numerical examples to show that the differences between the optimal hedging errors obtained through the martingale approach and the optimal variance gained via the method of optimal control are quite small in some circumstances when the contingent claim reduces to a constant, which answer the question to some extent. Moreover, these examples also indicate that, although perfect hedging is impossible in an incomplete market, the mean-variance hedging errors can be quite small in many cases as terminal time increases.

The rest of the paper is organized as follows. In Section II, we formulate our financial market model. In Section III, we introduce the discounted price process and admissible strategies. In Section IV, we present our main theorem. In Section V, we present our analytical and numerical examples to conduct the comparisons between the martingale approach and the method of optimal feedback control. Finally, in Section VI, we conclude this paper with remarks.

II. The Financial Market Model

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a fixed complete probability space on which are defined a standard d-dimensional Brownian motion $W \equiv \{W(t), t \in [0,T]\}$ with $W(t) = (W_1(t), ..., W_d(t))'$ and $h-$dimensional subordinator $L \equiv \{L(t), t \in [0,T]\}$ with $L(t) \equiv (L_1(t), ..., L_h(t))'$ and càdlàg sample paths for some fixed $T \in [0,\infty)$ (see, e.g., Applebaum [1], Bertoin [5], and Sato [25] for more details about subordinators and Lévy processes), where the prime denotes the corresponding transpose of a matrix or a vector. Moreover, W, L and their components are assumed to be independent of each other. Related to the probability space, we suppose that there is a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t \equiv \sigma\{W(s), L(\lambda s) : 0 \leq s \leq t\}$ for each $t \in [0,T]$, $\lambda = (\lambda_1, ...\lambda_h)' > 0$, and $L(\lambda s) = (L_1(\lambda_1 s), ..., L_h(\lambda_h s))'$.

Our financial market is a multivariate Lévy-driven OU type stochastic volatility model, which consists of $d+1$ assets. One of the $d+1$ assets is supposed to be a risk-free account whose price $S_0(t)$ is subject to the ordinary differential equation (ODE) with constant interest rate $r \geq 0$,

$$\begin{cases} dS_0(t) = rS_0(t)dt, \\ S_0(0) = s_0 > 0, \end{cases} \tag{2.1}$$

and the other d assets are stocks whose vector price process $S(t) = (S_1(t), ..., S_d(t))'$ satisfies the following stochastic differential equation (SDE) for each $t \in [0,T]$,

$$\begin{cases} dS(t) = \text{diag}(S(t^-))\{b(Y(t^-))dt \\ \qquad\qquad +\sigma(Y(t^-))dW(t)\}, \\ S(0) = s > 0, \end{cases} \tag{2.2}$$

where the diag(v) denotes the $d \times d$ diagonal matrix whose entries in the main diagonal are v_i with $i \in \{1, ..., d\}$ for a d-dimensional vector $v = (v_1, ..., v_d)'$ and all the other entries are zero, and $Y(t)$ is a Lévy-driven OU type process described by the following SDE,

$$\begin{cases} dY(t) = -\Lambda Y(t^-)dt + dL(\lambda t), \\ Y(0) = y_0, \end{cases} \tag{2.3}$$

where $\Lambda = \text{diag}(\lambda)$ and $y_0 = (y_{10}, ..., y_{h0})'$. Now define

$$\begin{aligned} b(y) &\equiv (b_1(y), ..., b_d(y))' : R_c^h \to [0,\infty)^d, \\ \sigma(y) &\equiv (\sigma_{mn}(y))_{d\times d} : \quad R_c^h \to (0,\infty)^{dd}, \end{aligned}$$

where $R_c^h \equiv (c_1, \infty) \times ... \times (c_h, \infty)$ with $c_i = y_{i0}e^{-\lambda_i T}$. Then we can introduce the following conditions for the coefficients in (2.3)-(2.2):

C1. The functions $b(y)$ and $\sigma(y)$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\|b(y)\| \leq A_b + B_b\|y\|, \tag{2.4}$$

$$\|\sigma(y)\sigma(y)'\| \leq A_\sigma + B_\sigma\|y\|, \tag{2.5}$$

$$\left\|(\sigma(y)\sigma(y)')^{-1}\right\| \leq \frac{1}{b_\sigma\|y\|}, \tag{2.6}$$

where the norm $\|A\|$ takes the largest absolute value of all components of a vector A or all entries of a matrix A, and $A_b \geq 0, A_\sigma \geq 0, B_b \geq 0, B_\sigma \geq 0$, $b_\sigma > 0$ are constants.

C2. The derivatives $\frac{\partial b(y)}{\partial y_i}$ and $\frac{\partial(\sigma(y)\sigma(y)')^{-1}}{\partial y_i}$ for all $i \in \{1, ..., h\}$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\left\| \frac{\partial b(y)}{\partial y_i} \right\| \leq \bar{A}_b + \bar{B}_b \|y\|, \quad (2.7)$$

$$\left\| \frac{\partial(\sigma(y)\sigma(y)')^{-1}}{\partial y_i} \right\| \leq \bar{A}_\sigma + \bar{B}_\sigma \|y\|, \quad (2.8)$$

where $\bar{A}_b$, $\bar{A}_\sigma$, $\bar{B}_b$ and $\bar{B}_\sigma$ are some nonnegative constants.

We next impose conditions for each subordinator L_i with $i \in \{1, ..., h\}$, which can be represented by (see, e.g., Theorem 13.4 and Corollary 13.7 in Kallenberg [19])

$$L_i(t) = a_i t + \int_{(0,t]} \int_{z_i>0} z_i N_i(ds, dz_i), \ t \geq 0 \quad (2.9)$$

where $N_i((0,t] \times A) \equiv \sum_{0<s\leq t} I_A(L_i(s) - L_i(s^-))$ denotes a Poisson random measure with deterministic, time-homogeneous intensity measure $\nu_i(dz_i)ds$, $I_A(\cdot)$ is the index function over the set A, the constant a_i is taken to be zero, and ν_i is the Lévy measure satisfying

$$\int_{z_i>0} \left(e^{Cz_i} - 1\right) \nu_i(dz_i) < \infty \quad (2.10)$$

with C taken to be a large enough positive constant to guarantee all of the related integrals in this paper meaningful. Note that the condition in (2.10) is on the integrability of the tails of the Lévy measures. Due to the total wealth in the world is finite, the jump sizes corresponding to the subordinators $L_i(t)$ ($i \in \{1, ..., h\}$) associated with leverage effect and etc. should be bounded by some positive constant although it may be extremely large. Moreover, due to the limitations of computer hardware and softwares, all the random variables used in a computer simulation are truncated versions of the original ones if they are assumed to be heavy-tailed ones (see, e.g., Dai ([7], [8], [9])). So the assumption in (2.10) is practically meaningful in finance.

III. Discounted Price Process and Admissible Strategies

Let $D(t) = (D_1(t), ..., D_d(t))'$ be the corresponding d-dimensional discounted price process, i.e., for $m \in \{1, ..., d\}$,

$$D_m(t) = \frac{S_m(t)}{S_0(t)} = e^{-rt} S_m(t). \quad (3.1)$$

Moreover, let $L^2_{\mathcal{F}}\left([0,T], R^d, P\right)$ denote the set of all R^d-valued measurable stochastic processes $Z(t)$ adapted to $\{\mathcal{F}_t, t \in [0,T]\}$ such that $E\left[\int_0^T \|Z(t)\|^2 dt\right] < \infty$. Then we have the following lemma.

Lemma 3.1: Under conditions **C1**, **C2** and (2.10), $D(\cdot)$ is a continuous $\{\mathcal{F}_t\}$-semimartingale, i.e.,

$$D(\cdot) = D_0 + M^D(\cdot) + B^D(\cdot), \quad (3.2)$$

where $M^D(\cdot)$ and $B^D(\cdot)$ are a $\{\mathcal{F}_t\}$-martingale and a predictable process of finite variation respectively. Moreover, $D(\cdot)$ is locally in $L^2_{\mathcal{F}}([0,T],\ R^d, P)$ in the sense that there is a localizing sequence of stopping times $\{\sigma_n\}$ with $n \in \mathcal{N} \equiv \{0, 1, 2, ...\}$ such that, for any $n \in \mathcal{N}$,

$$\sup\{E\left[D^2(\tau)\right] : \text{all stopping } \tau \text{ time satisfying } \tau \leq \sigma_n\} < \infty.$$

The above lemma will be proved elsewhere due to its length. Instead, we let $L(D)$ denote the set of D-integrable and predictable processes in the sense of Definition 6.17 in page 207 of Jacod and Shiryaev [18]. Further, let $u_i(t)$ denote the number of shares invested in stock $i \in \{1, ..., d\}$ at time t and define $u(t) \equiv (u_1(t), ..., u_d(t))'$, then we have the following definitions concerning admissible strategies.

Definition 3.1: A R^d-valued trading strategy u is called simple if it is a linear combination of strategies $ZI_{(\tau_1,\tau_2]}$ where $\tau_1 \leq \tau_2$ are stopping times dominated by σ_n for some $n \in \mathcal{N}$ and Z is a bounded $\mathcal{F}_{\tau_1}$-measurable random variable. Moreover, the set of all such simple trading strategies is denoted by $\Theta(D)$.

Definition 3.2: A trading strategy $u \in L(D)$ is called admissible if there is a sequence $\{u^n, n \in \mathcal{N}\}$ of simple strategies such that: $(u^n \cdot D)(t) \to (u \cdot D)(t)$ in probability as $n \to \infty$ for any $t \in [0,T]$, and $(u^n \cdot D)(T) \to (u \cdot D)(T)$ in $L^2(P)$ as $n \to \infty$. Moreover, the set of all such admissible strategies is denoted by $\bar{\Theta}(D)$.

IV. Optimal Hedging and Main Theorem

Before stating our main theorem, we introduce more notations for convenience.

First, let $\mathcal{E}(N) = \{\mathcal{E}(N)(t), t \in [0,T]\}$ be the stochastic exponential for a univariant continuous semimartingale $N = \{N(t), t \in [0,T]\}$ (see, e.g., pages 84-85 of Protter [24]) with

$$\mathcal{E}(N)(t) = \exp\left\{N(t) - \frac{1}{2}[N,N](t)\right\} \quad (4.1)$$

where $[\cdot,\cdot]$ denote the quadratic variation process of N

Second, for each $y \in R_c^h$, define

$$B(y) \equiv (b_1(y) - r, ..., b_d(y) - r)', \quad (4.2)$$

$$\rho(y) \equiv B(y)' \left[\sigma(y)\sigma(y)'\right]^{-1} B(y), \quad (4.3)$$

$$P(t,y) \equiv E_{t,y}\left[e^{-\int_t^T \rho(Y(s))ds}\right] > 0, \quad (4.4)$$

$$O(t) \equiv P(t, Y(t)), \quad (4.5)$$

$$a(t) \equiv (\text{diag}(D(t)))^{-1} \left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)^{-1} B(t, Y(t^-)), \quad (4.6)$$

$$\hat{Z}(t) \equiv \frac{O(t)\mathcal{E}(-a \cdot D)(t)}{O_0}, \ O_0 = O(0), \quad (4.7)$$

where $\text{diag}(D(t))$ denotes the diagonal matrix with diagonal entries given by $D_i(t)$ for $i \in \{1, ..., d\}$.

Third, let $L^2_{\mathcal{F},p}\left([0,T], R^d, P\right)$ denote the set of all R^d-valued predictable processes (see, e.g., Definition 5.2 in page 21 of Ikeda and Watanabe [17]) and let $L^2_p([0,T],\ R^h, P)$

be the set of all R^h-valued predictable processes $\tilde{Z}(t,z) = (\tilde{Z}_1(t,z), ..., \tilde{Z}_h(t,z))'$ satisfying

$$E\left[\sum_{i=1}^{h}\int_0^T\int_{z_i>0}\left|\tilde{Z}_i(t,z)\right|^2\nu_i(dz_i)dt\right] < \infty.$$

and define

$$\bar{Z}(t) \equiv \frac{\hat{Z}(t^-)}{\hat{Z}(t)}, \tag{4.8}$$

$$\bar{B}_i(Y(t^-)) \equiv \sum_{j=1}^{d}\Big(\big(B(Y(t^-))'\big(\sigma(Y(t^-)) \tag{4.9}$$

$$\sigma(Y(t^-))'\big)^{-1}\Big)\Big)_j\sigma_{ji}(Y(t^-)). \tag{4.10}$$

Moreover, for a càdlàg process $V \in L^2_{\mathcal{F}}([0,T],R,P)$, predictable processes $\bar{V} = (\bar{V}_1, ..., \bar{V}_d) \in L^2_{\mathcal{F},p}([0,T],R^d,P)$ and $\tilde{V} = (\tilde{V}_1, ..., \tilde{V}_h) \in L^2_p([0,T],R^h,P)$, define

$$g\left(t, V(t^-), \bar{V}(t), \tilde{V}(t,\cdot), Y(t^-)\right) \tag{4.11}$$

$$\equiv -\sum_{i=1}^{d}\bar{V}_i(t)\bar{B}_i(Y(t^-)) + \sum_{i=1}^{h}\int_{z_i>0}\Big(\tilde{V}_i(t,z_i)F(t,z_i)\bar{Z}(t) + V(t^-)\left(F(t,z_i)\bar{Z}(t)\right)^2\Big)\lambda_i\nu_i(dz_i).$$

Fourth, corresponding to Definition 3.16 in Cĕrný and Kallsen [6], we introduce the following definition.

Definition 4.1: A probability measure $P^* \sim P$ is called opportunity-neutral probability measure if it is defined in terms of the density process

$$Z^{P^*}(t) \equiv \frac{O(t)}{E[O_0]\,\mathcal{E}(B^K)(t)} \tag{4.12}$$

for each $t \in [0,T]$, where $B^K(t) = \int_0^t \rho(Y(s^-))ds$.

Fifth, let $L^{\gamma}_{\mathcal{F}_T}(\Omega, R^d, P)$ for a positive integer γ denote the set of all R^d-valued, $\mathcal{F}_T$-measurable random variables $\xi \in R^d$ satisfying $E[\|\xi\|^{\gamma}] < \infty$. Then we can impose the following condition on the option H.

Assumption 4.1: $H \in L^4_{\mathcal{F}_T}(\Omega, R, P)$ and there exists a sequence of random variables $H_{\tau_n} \in L^2_{\mathcal{F}_{T\wedge\tau_n}}(\Omega, R, P)$ satisfying $H_{\tau_n} \to H$ in L^2 as $n \to \infty$ and $H_{\tau_n}(\omega) = H(\omega)$ for all $\omega \in \{\omega, \tau_n(\omega) \geq T\}$, where $\{\tau_n\}$ is a sequence of nondecreasing $\{\mathcal{F}_t\}$-stopping times satisfying $\tau_n \to \infty$ a.s. as $n \to \infty$.

Theorem 4.1: Under conditions **C1**, **C2**, (2.10), and Assumption 4.1, the optimal hedging strategy $\phi \in \bar{\Theta}(D)$ for (1.1) is given by

$$\phi(t) = \xi(t) - (v + \Psi(t^-) - V(t^-))a(t) \tag{4.13}$$

where $V(t)$ is given by the unique $\{\mathcal{F}_t\}$-adapted solution $(V, \bar{V}, \tilde{V})$ of the BSDE

$$V(t) = H - \int_t^T g(s, V(s^-), \bar{V}(s), \tilde{V}(s,\cdot), Y(s^-))ds - \int_t^T\sum_{i=1}^{d}\bar{V}_i(s)dW_i(s) - \int_t^T\sum_{i=1}^{h}\int_{z_i>0}\tilde{V}_i(s,z_i)\tilde{N}_i(\lambda_i ds, dz_i) \tag{4.14}$$

with $\tilde{N}_i(\lambda_i dt, dz_i) \equiv N_i(\lambda_i dz_i, dt) - \lambda_i\nu_i(dz_i)dt$ and g given by (4.11). Furthermore, the pure hedge coefficient ξ is given by

$$\xi(t) = \left(\tilde{c}^{D^*}(t)\right)^{-1}\left(\tilde{c}^{DV^*}(t)\right) \tag{4.15}$$

where $*$ corresponds to P^* and

$$\tilde{c}^{D^*}(t) = \text{diag}(D(t))\left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)\text{diag}(D(t)), \tag{4.16}$$

$$\tilde{c}^{DV^*}(t) = \left(\sum_{i=1}^{d}D_1(t)\sigma_{1i}(Y(t^-))\bar{V}_i(t), ..., \sum_{i=1}^{d}D_d(t)\sigma_{di}(Y(t^-))\bar{V}_i(t)\right)'. \tag{4.17}$$

Finally, Ψ is the unique solution of the following SDE

$$\Psi(t) = ((\xi - (v - V_-)a)\cdot D)(t) - (\Psi_-\cdot(a\cdot D))(t). \tag{4.18}$$

The proof of Theorem 4.1 will be provided elsewhere due to its length.

Example 4.1: Under conditions **C1**, **C2** and (2.10), the following options satisfy Assumption 4.1: The discounted European call option, i.e., for some $m \in \{1, ..., d\}$, $H = e^{-rT}(S_m(T) - C_e)^+$; The discounted European put option, i.e., for some $m \in \{1, ..., d\}$, $H = e^{-rT}(C_e - S_m(T))^+$ where $C_e > 0$ is the exercise price.

V. Numerical Examples and Comparisons

In this section, the interest rate r in (2.1) is taken to be zero and the financial market is supposed to be self-financing so that $X(t) = v + (u\cdot D)(t)$. Then it follows from (18) in Theorem 3.1 of Dai [7] that the terminal variance under the optimal policy stated in (15) of Theorem 3.1 of Dai [7] is given by

$$Var(X^*(T)) = \frac{P(0,y_0)}{1 - P(0,y_0)}(p - v)^2. \tag{5.1}$$

Moreover, it follows from Theorem 4.1 in the current paper and Theorem 4.12 in Cĕrný and Kallsen [6] that the hedging error under the optimal policy in (4.13) is given by

$$Herr = P(0,y_0)(p - v)^2. \tag{5.2}$$

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In the following numerical examples, we compute the differences between the optimal terminal variances in (5.1) and the optimal hedging errors in (5.2), i.e.,

$$\begin{aligned} Error &= Var(X^*(T)) - Herr \qquad (5.3) \\ &= \frac{(P(0,y_0))^2}{1-P(0,y_0)}(p-v)^2 \\ &> 0. \end{aligned}$$

The observation in the inequality of (5.3) is intuitively correct since the optimal strategy in (4.13) is taken over a general strategy set given in Definition 3.2 and the optimal strategy in (15) of Theorem 3.1 of Dai [7] is taken in an ad-hoc manner. However, the Errors are quite small as shown in the following numerical examples.

Example 5.1: In this example, we assume that the financial market is presented by the well-known Black-Scholes model

$$dD(t) = D(t)(\alpha dt + \beta dB(t))$$

where α and β are some constants. From the simulated numerical results shown in Figures 1 and 2, we can see that the absolute error between the optimal variance based on the policy in (15) of Theorem 3.1 of Dai [7] and the optimal hedging error based on the strategy in (4.13) tends to zero when the terminal time increases. The convergence rate heavily depends on the volatility β. When β is relatively large, the difference needs more time to reach zero. However, if the millisecond is used to denote the time unit in a supercomputer based trading system, the required time for the convergence is still practical meaningful.

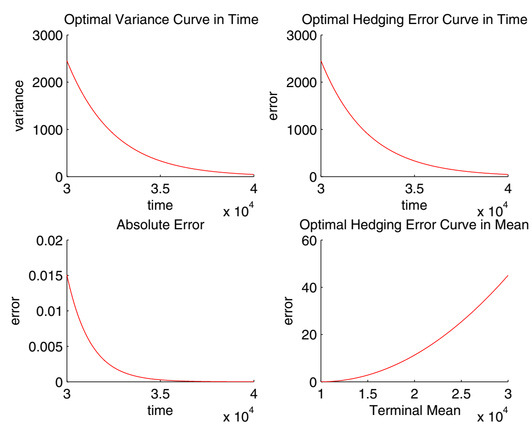

Fig. 1. The errors based on Black-Scholes model with $r = 0$, $v = 10000$, $p = 30000$, $T = 40000$, $\alpha = 2$, $\beta = 100$.

Example 5.2: In this example, we suppose that the financial market is described by the well-known BNS model

$$dD(t) = D(t)((\alpha + \beta Y(t^-))dt + \sqrt{Y(t^-)}dB(t))$$

where α and β are some constants. Moreover, due to the remarks to the condition in (2.10) and due to the discussions in Dai [8], we assume that the driving subordinator $L(\lambda\cdot)$ with $\lambda = 1$ to the SDE in (2.3) is a compound Poisson

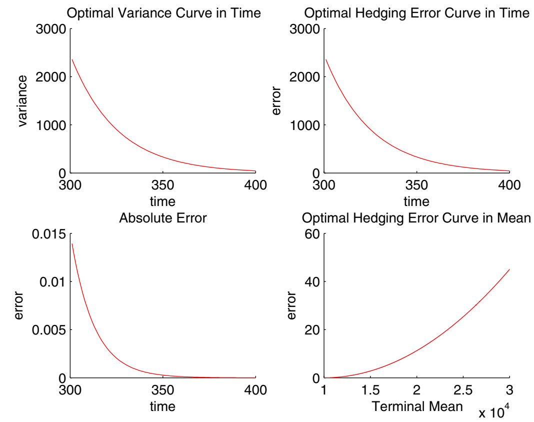

Fig. 2. The errors based on Black-Scholes model with $y_0 = 10$, $r = 0$, $v = 10000$, $p = 30000$, $T = 400$, $\alpha = 2$, $\beta = 10$.

process whose interarrival times are exponentially distributed with mean $1/\mu$ and whose jump sizes are also exponentially distributed with mean $1/\mu_1$. From the simulated results in Figure 3, we can see that the similar explanation presented in Example 5.1 is also true for the current example, where δ appeared in Figure 3 is the length of equally divided subintervals of $[0, T]$. Moreover, from the simulated results, we also can see that, although perfect hedging is impossible in an incomplete market, the mean-variance hedging errors can be quite small in many cases as terminal time increases. In addition, based on the solution to (2.3), we present a sample path of $Y(t^-)$ in Figure 4 to provide some insight about the stationarity and evolving of the volatility process.

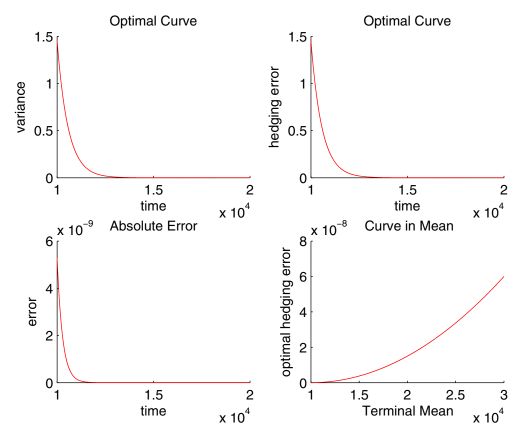

Fig. 3. The errors based on BNS model with $y_0 = 10$, $r = 0$, $v = 10000$, $p = 30000$, $T = 200$, $\delta = 0.01$, $\alpha = 0.5$, $\beta = 0.02$, $\mu = 10$, $\mu_1 = 8$.

VI. Conclusion

In this paper, we apply the martingale approach to address the mean-variance hedging problem in an incomplete market with coefficients driven by external random environmental (risk)

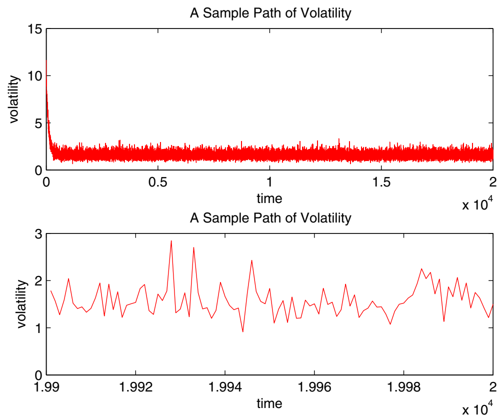

Fig. 4. The volatility process for BNS model with $y_0 = 10$, $r = 0$, $v = 1000$, $T = 200$, $\delta = 0.01$, $\mu = 0.5$, and $\mu_1 = 8$.

factors of NGOU processes. Both analytical and numerical examples indicate that the optimal hedging errors obtained through the martingale approach are slightly smaller than (but quite consistent with) the optimal variances previously gained via the method of optimal control under some circumstances when the contingent claim reduces to a constant, which illustrate, to some extent, the reasonability of the widely concerned ad hoc decision manner used in the method of optimal control. Moreover, these examples also indicate that, although perfect hedging is impossible in an incomplete market, the mean-variance hedging errors can be quite small in many cases as terminal time increases. In constructing a mean-variance hedging strategy and justifying it to be the global risk-minimizing one for our market, there are two steps involved. First, we derive a BSDE with jumps for the mean-value process of a given contingent claim under suitable terminal conditions including both European call and put options as special cases. Second, by combining the solution of the BSDE and the pure hedging coefficient obtained via the concept of opportunity-neutral measure, we get our optimal hedging strategy. Here we remark that, due to the discussions in Pigorsch and Stelzer [23] and references therein, our discussion in this paper can be extended to the cases that the external risk factors in (2.3) are correlated in certain manners. For the simplicity of notation, we keep the presentation of the paper in the current way.

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