Research Article

Mean-Variance Hedging Based on an Incomplete Market with External Risk Factors of Non-Gaussian OU Processes

Wanyang Dai

Department of Mathematics and State Key Laboratory of Novel Software Technology, Nanjing University, Nanjing 210093, China

Correspondence should be addressed to Wanyang Dai; nan5lu8@netra.nju.edu.cn

We prove the global risk optimality of the hedging strategy of contingent claim, which is explicitly (or called semiexplicitly) constructed for an incomplete financial market with external risk factors of non-Gaussian Ornstein-Uhlenbeck (NGOU) processes. Analytical and numerical examples are both presented to illustrate the effectiveness of our optimal strategy. Our study establishes the connection between our financial system and existing general semimartingale based discussions by justifying required conditions. More precisely, there are three steps involved. First, we firmly prove the no-arbitrage condition to be true for our financial market, which is used as an assumption in existing discussions. In doing so, we explicitly construct the square-integrable density process of the variance-optimal martingale measure (VOMM). Second, we derive a backward stochastic differential equation (BSDE) with jumps for the mean-value process of a given contingent claim. The unique existence of adapted strong solution to the BSDE is proved under suitable terminal conditions including both European call and put options as special cases. Third, by combining the solution of the BSDE and the VOMM, we reach the justification of the global risk optimality for our hedging strategy.

1. Introduction

In this paper, we justify the global risk optimality of the hedging strategy of contingent claim, which is explicitly constructed for an incomplete market defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. The financial market has $d+1$ primitive assets: one bond with constant interest rate and d risky assets. The price processes of the assets are described by a generalized Black-Scholes model with coefficients driven by the market regime caused by leverage effect, and so forth. The financial market model includes the Barndorff-Nielsen and Shephard (BNS) volatility model proposed by Barndorff-Nielsen and Shephard [1] and further studied in Benth et al. [2], Benth and Meyer-Brandis [3], Lindberg [4], and so forth as a particular case. Our model is closely related to the one considered in Delong and Klüppelberg [5]. As pointed out in Barndorff-Nielsen and Shephard [1], these models fit real market data quite well. Nevertheless, such models also induce incompleteness of the financial markets, which means that it is impossible to replicate perfectly contingent claims based on

the bond and the d primitive risky assets. A rule for designing a good hedging strategy is to minimize the mean squared hedging error over the set $\overline{\Theta}$ of all reasonable trading strategy processes:

$$\inf_{u \in \overline{\Theta}} E\left[(v + (u \cdot D)(T) - H)^2 \right], \tag{1}$$

where H is a random variable representing the discounted payoff of the claim, D is the discounted price process of d risky assets, v is the initial endowment, and T is the time horizon. Mathematically speaking, one seeks to compute the orthogonal projection of $H - v$ on the space $\overline{\Theta}$ of stochastic integrals.

To solve mean-variance hedging problem (1), we explicitly construct a trading strategy for the financial market and justify it to be the global risk-minimizing hedging strategy by using the following procedure.

First, we explicitly construct the square-integrable density process of a variance-optimal martingale measure (VOMM)

Q^*. As a result, the set of equivalent (local) martingale measures with square-integrable densities, that is,

$$
\begin{aligned}
&\mathcal{U}_2^e (D) \\
&\equiv \left\{ Q \sim P : \frac{dQ}{dP} \in L^2 (P) , \ D \text{ is a } Q\text{-local martingale} \right\},
\end{aligned} \tag{2}
$$

is nonempty. Hence, our market is arbitrage-free (e.g, [6]). Second, we derive a BSDE with jumps and external random factors of non-Gaussian Ornstein-Uhlenbeck (NGOU) type for the mean-value process of the option H (i.e., $E_{Q^*}[H \mid \mathcal{F}_t]$). The unique existence of adapted solution to the BSDE is proved under suitable terminal conditions including both European call and put options as special cases. Third, by combining the solution to the BSDE and the VOMM, we get the optimal hedging strategy for our market.

The BSDE and VOMM based procedure is a mixed method of two typical approaches in solving mean-variance hedging problem: martingale approach stemmed from Harrison and Kreps [7] and stochastic control approach that views the problem as a linear-quadratic control problem and employs BSDEs to describe the solution (see, e.g., Yong and Zhou). This procedure is structured for a general semimartingale in Cĕrný and Kallsen [8] and explicitly (or semiexplicitly) presented for the current market in Dai [9]. Some related and independent study can also be found in Jeanblanc et al. [10]. More precisely, we have the following literature review and technical comparisons.

A closely related (local) risk-minimizing problem was initially introduced by Föllmer and Sondermann [11] under complete information, who also suggested an approach for the computation of a minimizing strategy in an incomplete market by extending the martingale approach of Harrison and Kreps [7]. The basic idea of the approach was to introduce a measure of riskiness in terms of a conditional mean square error process where the discounted price process is a square-integrable martingale. Furthermore, the answer to the hedging problem is provided by the *Galtchouk-Kunita-Watanabe decomposition* of the claim. Then, this concept of local-risk minimization was further extended for the semimartingale case by Föllmer and Schweizer [12] and Schweizer [13, 14], where the minimal martingale measure and Föllmer-Schweizer (F-S) decomposition play a central role. Interested readers are referred to Fäollmer and Schweizer [15] and Schweizer [16] for more recent surveys about (local) risk minimization and mean-variance hedging.

Owing to the fact that one cares about the total hedging error and not the daily profit-loss ratios, the solution with respect to global risk minimization of the unconditional expected squared hedging error presented in (1) was considered (e.g., surveys in [16, 17]). Then, the study on global risk minimization was further developed by Cĕrný and Kallsen [8], who showed that hedging model (1) admits a solution in a very general class of arbitrage-free semimartingale markets where local-risk minimization may fail to be well defined. The key point of their approach is the introduction of the opportunity-neutral measure P^* that turns the dynamic asset allocation problem into a myopic one. Furthermore,

the minimal martingale measure relative to P^* coincides with the variance-optimal martingale measure relative to the original probability measure P. Recently, to overcome the difficulties appearing in Cĕrný and Kallsen [8] (i.e., a process N appearing in Definition 3.12 is very hard to find and the VOMM Q^* in Proposition 3.13 is notoriously difficult to determine), the authors in Jeanblanc et al. [10] developed a method via stochastic control and backward stochastic differential equations (BSDEs) to handle the mean-variance hedging problem for general semimartingales. Furthermore, the authors in Kallsen and Vierthauer [18] derived semiexplicit formulas for the optimal hedging strategy and the minimal hedging error by applying general structural results and Laplace transform techniques. In addition to these works, some related studies in both general theory and concrete results in specific setups for the mean-variance hedging problem can be found in works, such as, Arai [19], Chan et al. [20], Duffie and Richardson [21], Gourieroux et al. [22], Heath et al. [23], Laurent and Pham [24], and references therein.

Comparing with the above studies, our contribution of the current research is threefold. First, we firmly prove the no-arbitrage condition to be true for our financial market; that is, the set defined in (2) is nonempty. This condition is used as an assumption for the existence of the VOMM in existing discussions (e.g., [8, 10, 18–20]). In doing so, we explicitly (or called semiexplicitly) construct a measure through identifying its explicit density by the general structure presented in Cĕrný and Kallsen [8]. Then, we justify it to be the VOMM for our market model by proving the equivalent conditions given in Cĕrný and Kallsen [25]. Second, in applying our VOMM to obtain the optimal hedging strategy, we derive a BSDE with jumps for the mean-value process of the option H. Here, we lift the requirements that the contingent claims are bounded (e.g., [25, 26]) or satisfy Lipschitz condition (e.g., [20, 27]) to guarantee the corresponding integral-partial differential equation (IPDE) to have a classic or viscosity solution. Furthermore, the unique existence of an adapted solution to our derived BSDE is firmly proved under certain conditions while in the recent study of Jeanblanc et al. [10] such existence of an adapted solution to their constructed BSDE is only showed as an equivalent condition to guarantee the existence of an optimal strategy. More importantly, our BSDE can be solved by developing related numerical algorithms through the given terminal option H (see, e.g., [28]). Third, from the purpose of easy applications, our discussion is based on a multivariate financial market model, which is in contrast to existing studies (e.g., [8, 10, 18, 20]). Therefore, unlike the studies in Hubalek et al. [29] and Kallsen and Vierthauer [18], our option H is generally related to a multivariate terminal function and hence a BSDE involved approach is employed. Actually, whether one can extend the Laplace transform related method developed in Hubalek et al. [29] and Kallsen and Vierthauer [18] for single-variate terminal function to our general multivariate case is still an open problem.

Note that our study in this paper establishes the connection between our financial system and existing general semimartingale based study in Cĕrný and Kallsen [8] since we can overcome the difficulties in Cĕrný and Kallsen [8] by

explicitly constructing the process N and the VOMM Q^* as mentioned earlier. Furthermore, our objective and discussion in this paper are different from the recent study of Jeanblanc et al. [10] since the authors in Jeanblanc et al. [10] did not aim to derive any concrete expression. Nevertheless, interested readers may make an attempt to extend the study in Jeanblanc et al. [10] and apply it to our financial market model to construct the corresponding explicit results.

Finally, when the random variable H in (1) is taken to be a constant (e.g., a prescribed daily expected return), the associated hedging problem reduces to a mean-variance portfolio selection problem as studied in Dai [30] by an alternative feedback control method. In this case, the optimal policies can be explicitly obtained by both the feedback control method in Dai [30] and the martingale method presented in the current paper. In the late method, the related BSDE is a degenerate one. From this constant option case, we can construct two insightful examples to provide the effective comparisons between the two methods. More precisely, our newly constructed hedging strategy can slightly outperform the feedback control based policy. However, the performance between the two methods is consistent in certain sense.

The remainder of the paper is organized as follows. We formulate our financial market model in Section 2 and present our main theorem in Section 3. Analytical and numerical examples are given in Section 4. Our main theorem is proven in Section 5. Finally, in Section 6, we conclude this paper with remarks.

2. The Financial Market

2.1. The Model. We use $(\Omega, \mathcal{F}, P)$ to denote a fixed complete probability space on which are defined a standard d-dimensional Brownian motion $W \equiv \{W(t), t \in [0, T]\}$ with $W(t) = (W_1(t), \ldots, W_d(t))'$ and h-dimensional subordinator $L \equiv \{L(t), t \in [0, T]\}$ with $L(t) \equiv (L_1(t), \ldots, L_h(t))'$ and Càdlàg sample paths for some fixed $T \in [0, \infty)$ (e.g., [31–33] for more details about subordinators and Lévy processes). The prime denotes the corresponding transpose of a matrix or a vector. Furthermore, W, L, and their components are assumed to be independent of each other. For each given $\lambda = (\lambda_1, \ldots, \lambda_h)' > 0$, we let $L(\lambda s) = (L_1(\lambda_1 s), \ldots, L_h(\lambda_h s))'$. Then, we suppose that there is a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ related to the probability space, where $\mathcal{F}_t \equiv \sigma\{W(s), L(\lambda s) : 0 \leq s \leq t\}$ for each $t \in [0, T]$.

The financial market under consideration is a multivariate Lévy-driven OU-type stochastic volatility model, which consists of $d + 1$ assets. One of the $d + 1$ assets is risk-free, whose price $S_0(t)$ is subject to the ordinary differential equation (ODE) with constant interest rate $r \geq 0$:

$$dS_0(t) = rS_0(t)\,dt, \qquad S_0(0) = s_0 > 0. \tag{3}$$

The other d assets are stocks whose vector price process $S(t) = (S_1(t), \ldots, S_d(t))'$ satisfies the following stochastic differential equation (SDE) for each $t \in [0, T]$:

$$dS(t) = \mathrm{diag}\left(S\left(t^-\right)\right)\left\{b\left(Y\left(t^-\right)\right)dt + \sigma\left(Y\left(t^-\right)\right)dW(t)\right\},$$
$$S(0) = s > 0. \tag{4}$$

Here and in the sequel, the $\mathrm{diag}(v)$ denotes the $d \times d$ diagonal matrix whose entries in the main diagonal are v_i with $i \in \{1, \ldots, d\}$ for a d-dimensional vector $v = (v_1, \ldots, v_d)'$ and all the other entries are zero. $Y(t)$ is a Lévy-driven OU-type process described by the following SDE:

$$dY(t) = -\Lambda Y\left(t^-\right)dt + dL(\lambda t),$$
$$Y(0) = y_0, \tag{5}$$

where $\Lambda = \mathrm{diag}(\lambda)$ and $y_0 = (y_{10}, \ldots, y_{h0})'$. Now, define

$$b(y) \equiv (b_1(y), \ldots, b_d(y))' : R_c^h \longrightarrow [0, \infty)^d,$$
$$\sigma(y) \equiv (\sigma_{mn}(y))_{d \times d} : R_c^h \longrightarrow (0, \infty)^{dd}, \tag{6}$$

where $R_c^h \equiv (c_1, \infty) \times \cdots \times (c_h, \infty)$ with $c_i = y_{i0}e^{-\lambda_i T}$. Thus, we can impose the following conditions related to the coefficients in (4)-(5).

(C1) The functions $b(y)$ and $\sigma(y)$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\|b(y)\| \leq A_b + B_b\|y\|, \tag{7}$$

$$\|\sigma(y)\sigma(y)'\| \leq A_\sigma + B_\sigma\|y\|, \tag{8}$$

$$\left\|\left(\sigma(y)\sigma(y)'\right)^{-1}\right\| \leq \frac{1}{b_\sigma\|y\|}, \tag{9}$$

where the norm $\|A\|$ takes the largest absolute value of all components of a vector A or all entries of a matrix A, and $A_b \geq 0$, $A_\sigma \geq 0$, $B_b \geq 0$, $B_\sigma \geq 0$, and $b_\sigma > 0$ are constants.

(C2) The derivatives $\partial b(y)/\partial y_i$ and $\partial(\sigma(y)\sigma(y)')^{-1}/\partial y_i$ for all $i \in \{1, \ldots, h\}$ are continuous in y and satisfy that, for each $y \in R_c^h$,

$$\left\|\frac{\partial b(y)}{\partial y_i}\right\| \leq \overline{A}_b + \overline{B}_b\|y\|,$$
$$\left\|\frac{\partial\left(\sigma(y)\sigma(y)'\right)^{-1}}{\partial y_i}\right\| \leq \overline{A}_\sigma + \overline{B}_\sigma\|y\|, \tag{10}$$

where $\overline{A}_b$, $\overline{A}_\sigma$, $\overline{B}_b$, and $\overline{B}_\sigma$ are some nonnegative constants.

We now introduce the conditions for each subordinator L_i with $i \in \{1, \ldots, h\}$, which can be represented by (e.g., Theorem 13.4 and Corollary 13.7 in [34])

$$L_i(t) = \int_{(0,t]} \int_{z_i > 0} z_i N_i(ds, dz_i), \quad t \geq 0. \tag{11}$$

Here and in the sequel, $N_i((0, t] \times A) \equiv \sum_{0 < s \leq t} I_A(L_i(s) - L_i(s^-))$ denotes a Poisson random measure with deterministic, time-homogeneous intensity measure $\nu_i(dz_i)ds$. $I_A(\cdot)$ is the index function over the set A. ν_i is the Lévy measure satisfying

$$\int_{z_i > 0} \left(e^{Cz_i} - 1\right)\nu_i(dz_i) < \infty \tag{12}$$

with C taken to be a sufficiently large positive constant to guarantee all of the related integrals in this paper meaningful. Note that the condition in (12) is on the integrability of the tails of the Lévy measures (readers are referred to ([9, 30, 35–37]) for the justification of its reasonability).

2.2. Admissible Strategies. First, we use $D(t) = (D_1(t), \ldots, D_d(t))'$ to denote the associated d-dimensional discounted price process; that is, for each $m \in \{1, \ldots, d\}$,

$$D_m(t) = \frac{S_m(t)}{S_0(t)} = e^{-rt} S_m(t).\tag{13}$$

Furthermore, we define $L^2_{\mathscr{F}}([0, T], R^d, P)$ to be the set of all R^d-valued measurable stochastic processes $Z(t)$ adapted to $\{\mathscr{F}_t, t \in [0, T]\}$ such that $E[\int_0^T \|Z(t)\|^2 dt] < \infty$. Thus, it follows from Lemma 10 that $D(\cdot)$ is a continuous $\{\mathscr{F}_t\}$-semimartingale. In addition, $D(\cdot)$ is locally in $L^2_{\mathscr{F}}([0, T], R^d, P)$; that is, there is a localizing sequence of stopping times $\{\sigma_n\}$ with $n \in \mathscr{N} \equiv \{0, 1, 2, \ldots\}$ such that, for any $n \in \mathscr{N}$,

$$\sup \left\{ E\left[D^2(\tau)\right] : \text{all stopping } \tau \text{ time satisfying } \tau \le \sigma_n \right\}$$
$$< \infty.\tag{14}$$

Second, let $L(D)$ denote the set of D-integrable and predictable processes in the sense of Definition 6.17 in page 207 of Jacod and Shiryaev [38]. Furthermore, let $u_i(t)$ denote the number of shares invested in stock $i \in \{1, \ldots, d\}$ at time t and define $u(t) \equiv (u_1(t), \ldots, u_d(t))'$. Then, we have the following definitions concerning admissible strategies.

Definition 1. An R^d-valued trading strategy u is called simple if it is a linear combination of strategies $ZI_{(\tau_1, \tau_2]}$ where $\tau_1 \le \tau_2$ are stopping times dominated by σ_n for some $n \in \mathscr{N}$ and Z is a bounded $\mathscr{F}_{\tau_1}$-measurable random variable. Furthermore, the set of all such simple trading strategies is denoted by $\Theta(D)$.

Definition 2. A trading strategy $u \in L(D)$ is called admissible if there is a sequence $\{u^n, n \in \mathscr{N}\}$ of simple strategies such that $(u^n \cdot D)(t) \to (u \cdot D)(t)$ in probability as $n \to \infty$ for any $t \in [0, T]$ and $(u^n \cdot D)(T) \to (u \cdot D)(T)$ in $L^2(P)$ as $n \to \infty$. Furthermore, the set of all such admissible strategies is denoted by $\overline{\Theta}(D)$.

3. Main Theorem

First, for each $y \in R^h_c$, define

$$B(y) \equiv (b_1(y) - r, \ldots, b_d(y) - r)',\tag{15}$$

$$\rho(y) \equiv B(y)' \left[\sigma(y)\sigma(y)'\right]^{-1} B(y),\tag{16}$$

$$P(t, y) \equiv E_{t,y} \left[e^{-\int_t^T \rho(Y(s))ds} \right] > 0,\tag{17}$$

$$O(t) \equiv P(t, Y(t)),\tag{18}$$

$$a(t) \equiv \left(\text{diag}(D(t))\right)^{-1} \left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)^{-1}$$
$$\cdot B(t, Y(t^-)),\tag{19}$$

$$\widehat{Z}(t) \equiv \frac{O(t)\,\mathscr{E}(-a \cdot D)(t)}{O_0}, \quad O_0 = O(0).\tag{20}$$

Note that the process $a(\cdot)$ presented in (19) is corresponding to the adjustment process defined in Lemma 3.7 of Cěrný and Kallsen [8]. Furthermore, the process $\widehat{Z}(\cdot)$ presented in (20) is associated with the density process defined in Proposition 3.13 of Cěrný and Kallsen [8]. In addition, here and in the sequel, $\mathscr{E}(N) = \{\mathscr{E}(N)(t), t \in [0, T]\}$ denotes the stochastic exponential for a univariant continuous semimartingale $N = \{N(t), t \in [0, T]\}$ (e.g., pages 84-85 of [39]) with

$$\mathscr{E}(N)(t) = \exp\left\{N(t) - \frac{1}{2}[N, N](t)\right\},\tag{21}$$

where $[\cdot, \cdot]$ denotes the quadratic variation process of N.

Second, let $L^2_{\mathscr{F}, p}([0, T], R^d, P)$ denote the set of all R^d-valued predictable processes (see, e.g., Definition 5.2 in page 21 of [40]) and let $L^2_p([0, T], R^h, P)$ be the set of all R^h-valued predictable processes $\widetilde{Z}(t, z) = (\widetilde{Z}_1(t, z), \ldots, \widetilde{Z}_h(t, z))'$ satisfying

$$E\left[\sum_{i=1}^h \int_0^T \int_{z_i>0} \left|\widetilde{Z}_i(t, z)\right|^2 \nu_i(dz_i) dt\right] < \infty.\tag{22}$$

Furthermore, let

$$\overline{Z}(t) \equiv \frac{\widehat{Z}(t^-)}{\widehat{Z}(t)},\tag{23}$$

$$\overline{B}_i(Y(t^-)) \equiv \sum_{j=1}^d \left(\left(B(Y(t^-))' \left(\sigma(Y(t^-))\sigma(Y(t^-))'\right)^{-1}\right)\right)_j$$
$$\cdot \sigma_{ji}(Y(t^-)),\tag{24}$$

$$F(t, z_i) \equiv \frac{P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))}{P(t, Y(t^-))},\tag{25}$$

where e_i is the h-dimensional unit vector with the ith component one. Then, we define

$$g\left(t, V(t^-), \overline{V}(t), \widetilde{V}(t, \cdot), Y(t^-)\right)$$
$$\equiv -\sum_{i=1}^d \overline{V}_i(t) \overline{B}_i(Y(t^-))$$
$$+ \sum_{i=1}^h \int_{z_i>0} \left(\widetilde{V}_i(t, z_i) F(t, z_i) \overline{Z}(t)\right.$$
$$\left. + V(t^-)\left(F(t, z_i)\overline{Z}(t)\right)^2\right) \lambda_i \nu_i(dz_i).\tag{26}$$

Definition 3. For a given random variable H, a 3-tuple $(V, \overline{V}, \widetilde{V})$ is called a $\{\mathcal{F}_t\}$-adapted strong solution of the BSDE:

$$V(t) = H - \int_t^T g\left(s, V(s^-), \overline{V}(s), \widetilde{V}(s, \cdot), Y(s^-)\right) ds$$

$$- \int_t^T \sum_{i=1}^d \overline{V}_i(s) dW_i(s) \tag{27}$$

$$- \int_t^T \sum_{i=1}^h \int_{z_i > 0} \widetilde{V}_i(s, z_i) \widetilde{N}_i(\lambda_i ds, dz_i)$$

if $V \in L^2_{\mathcal{F}}([0,T], R, P)$ is a Càdlàg process, $\overline{V} = (\overline{V}_1, \ldots, \overline{V}_d) \in L^2_{\mathcal{F}, p}([0,T], R^d, P)$, $\widetilde{V} = (\widetilde{V}_1, \ldots, \widetilde{V}_h) \in L^2_p([0,T], R^h, P)$, and (27) holds a.s., where

$$\widetilde{N}_i(\lambda_i dt, dz_i) \equiv N_i(\lambda_i dz_i, dt) - \lambda_i \nu_i(dz_i) dt \tag{28}$$
$$\text{for each } i \in \{1, \ldots, h\}.$$

To impose suitable condition on the option H, we use $L^\gamma_{\mathcal{F}_T}(\Omega, R^d, P)$ for a positive integer γ to denote the set of all R^d-valued, $\mathcal{F}_T$-measurable random variables $\xi \in R^d$ satisfying $E[\|\xi\|^\gamma] < \infty$.

Assumption 4. Consider $H \in L^4_{\mathcal{F}_T}(\Omega, R, P)$ and there exists a sequence of random variables $H_{\tau_n} \in L^2_{\mathcal{F}_{T \wedge \tau_n}}(\Omega, R, P)$ satisfying $H_{\tau_n} \to H$ in L^2 as $n \to \infty$ and $H_{\tau_n}(\omega) = H(\omega)$ for all $\omega \in \{\omega, \tau_n(\omega) \geq T\}$, where $\{\tau_n\}$ is a sequence of nondecreasing $\{\mathcal{F}_t\}$-stopping times satisfying $\tau_n \to \infty$ a.s. as $n \to \infty$.

As pointed out in Dai [9], under conditions (C1), (C2), and (12), the discounted European call and put options satisfy Assumption 4. Now, we can state our main theorem of the paper as follows.

Theorem 5. *Under conditions (C1), (C2), and (12) and Assumption 4, let $(V, \overline{V}, \widetilde{V})$ be the unique $\{\mathcal{F}_t\}$-adapted strong solution of the BSDE in (27). Then, the optimal hedging strategy $\phi \in \overline{\Theta}(D)$ for (1) is given by*

$$\phi(t) = \xi(t) - \left(v + \Psi(t^-) - V(t^-)\right) a(t), \tag{29}$$

where the pure hedge coefficient ξ is given by

$$\xi(t) = \left(\bar{c}^{D^*}(t)\right)^{-1} \left(\bar{c}^{DV^*}(t)\right), \tag{30}$$

$$\bar{c}^{D^*}(t) = \text{diag}(D(t)) \left(\sigma(Y(t^-)) \sigma(Y(t^-))'\right) \text{diag}(D(t)), \tag{31}$$

$$\bar{c}^{DV^*}(t) = \left(\sum_{i=1}^d D_1(t) \sigma_{1i}(Y(t^-)) \overline{V}_i(t), \ldots, \right.$$
$$\left. \sum_{i=1}^d D_d(t) \sigma_{di}(Y(t^-)) \overline{V}_i(t)\right)'. \tag{32}$$

In addition, Ψ is the unique solution of the SDE

$$\Psi(t) = \left(\left(\xi - (v - V_-) a\right) \cdot D\right)(t) - \left(\Psi_- \cdot (a \cdot D)\right)(t). \tag{33}$$

Remark 6. The process $V(\cdot)$ appearing in Theorem 5 is actually the conditional mean-value process:

$$V(t) = E_{Q^*}[H \mid \mathcal{F}_t] \quad \text{with } dQ^* \equiv \widehat{Z}(T) dP. \tag{34}$$

Since it is not easy to be computed directly as the Markovian based conditional process $O(t, Y(t))$, we turn to use the BSDE in (27) to evaluate the process $V(t)$, which is convenient for us to design the optimal hedging policy as explained in Section 1 of the paper.

The proof of Theorem 5 will be provided in Section 5.

4. Performance Comparisons

The material in this section is partially reported in the short conference version of the current paper (see, [9]). To be convenient and clear for readers, we refine it here. Note that the interest rate r in (3) here is taken to be zero. Furthermore, the financial market is assumed to be self-financing, which implies that $X(t) = v + (u \cdot D)(t)$. In addition, the terminal option H is taken to be a constant p; that is, $H = p$. In this case, the optimal policies can be explicitly obtained by the feedback control method studied in Dai [30] and the martingale method presented in the current paper. In the late method, the related BSDE is a degenerate one, which can be easily observed from (34) in Remark 6. However, from this constant option $H = p$, we can construct two insightful examples to provide the effective comparisons between the two methods.

More precisely, by (18) in Theorem 3.1 of Dai [30], we know that the terminal variance under the optimal policy stated in (15) of Theorem 3.1 of Dai [30] is given by

$$\text{Var}(X^*(T)) = \frac{P(0, y_0)}{1 - P(0, y_0)} (p - v)^2. \tag{35}$$

In addition, by using Theorem 5 in the current paper and Theorem 4.12 in Černý and Kallsen [8], we know that the hedging error under the optimal policy in (29) is given by

$$\text{Herr} = P(0, y_0)(p - v)^2. \tag{36}$$

For the purpose of performance comparisons, we calculate the differences between the optimal terminal variances in (35) and the optimal hedging errors in (36); that is,

$$\text{Error} = \text{Var}(X^*(T)) - \text{Herr}$$
$$= \frac{(P(0, y_0))^2}{1 - P(0, y_0)} (p - v)^2 > 0. \tag{37}$$

The result shown in the last inequality of (37) is intuitively right since the optimal strategy in (29) is taken over a general decision set given in Definition 2 and the one in (15) of Theorem 3.1 of Dai [30] is taken in an *ad hoc* approach. Nevertheless, the errors are very small as displayed in the following numerical examples.
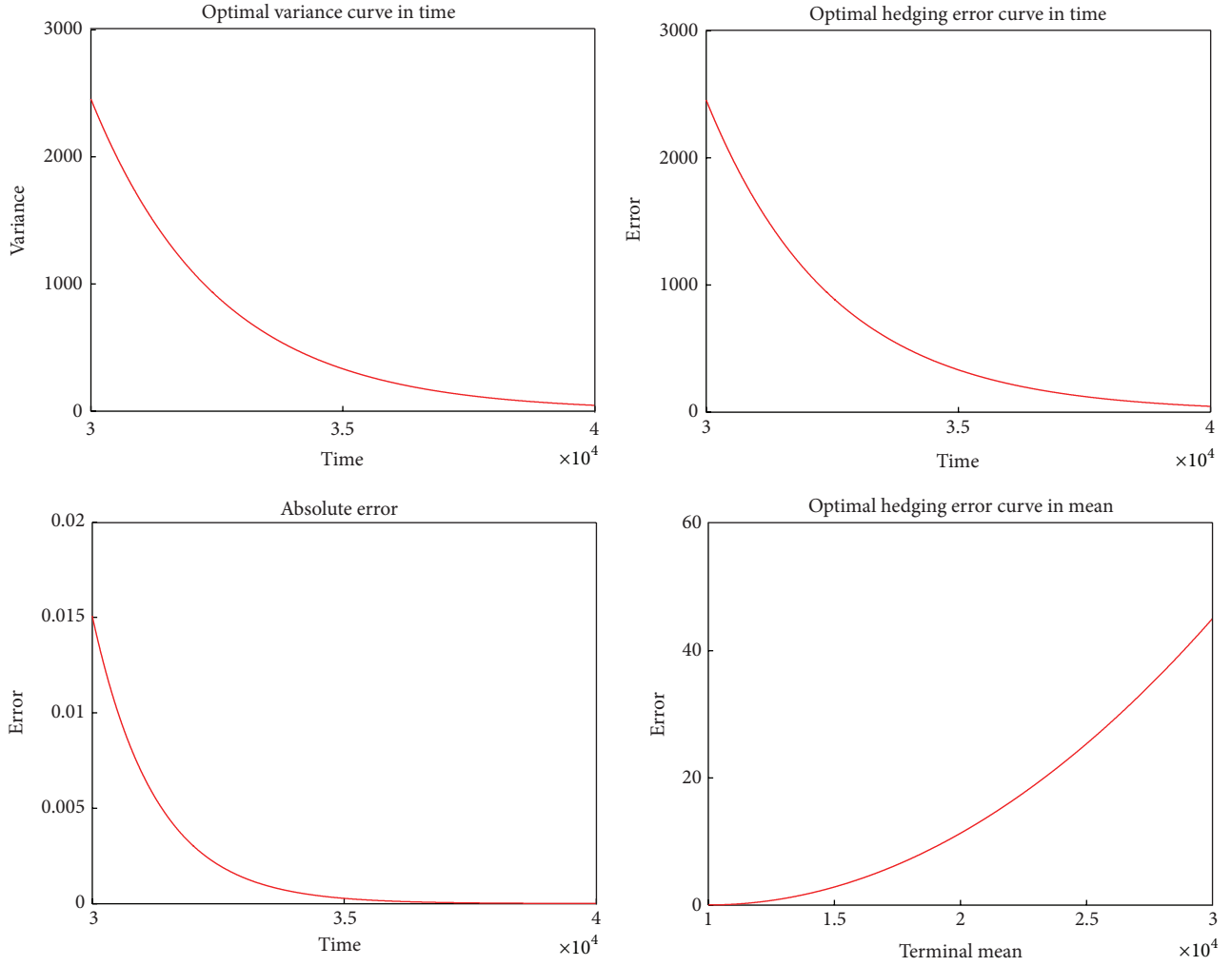
FIGURE 1: Errors using Black-Scholes model with $r = 0$, $v = 10000$, $p = 30000$, $T = 40000$, $\alpha = 2$, and $\beta = 100$.

Example 7. Here, we suppose that the financial market is given by the Black-Scholes model:

$$dD(t) = D(t)\left(\alpha dt + \beta dB(t)\right), \tag{38}$$

where α and β are given constants. Owing to Definition 2.1.4(b) in pages 273-274 of Øksendal [41], the option $H = p$ (a positive constant) is not attainable and hence the associated hedging error can not be zero if the initial endowment $v \neq p$. However, by the simulated results displayed in Figures 1 and 2, we see that the absolute error between the optimal variance based on the policy in (15) of Theorem 3.1 of Dai [30] and the optimal hedging error based on the strategy in (29) approaches zero as the terminal time increases. The rate of convergence is heavily dependent on the volatility β. If β is relatively large, the difference requires more time to reach zero. Nevertheless, if the millisecond is employed to represent the time unit in a supercomputer based trading system, the required time for the convergence makes sense in practice.

Example 8. Here, we assume that the financial market is presented by the BNS model:

$$dD(t) = D(t)\left(\left(\alpha + \beta Y\left(t^-\right)\right)dt + \sqrt{Y\left(t^-\right)}dB(t)\right), \tag{39}$$

where α and β are given constants. Furthermore, owing to the remarks to the condition in (12) and owing to the discussions in Dai [35], we suppose that the driving subordinator $L(\lambda\cdot)$ with $\lambda = 1$ to the SDE in (5) is a compound Poisson process. The interarrival times of the process are exponentially distributed with mean $1/\mu$ and the jump sizes of the process are also exponentially distributed with mean $1/\mu_1$. By the simulated results displayed in Figure 3, we see that similar illustration displayed in Example 7 also makes sense for the current example, where δ appearing in Figure 3 is the length of equally divided subintervals of $[0, T]$. In addition, by the simulated results, we also see that, although perfect hedging is impossible in an incomplete market, the mean-variance hedging errors can be very small in many cases when terminal time increases.
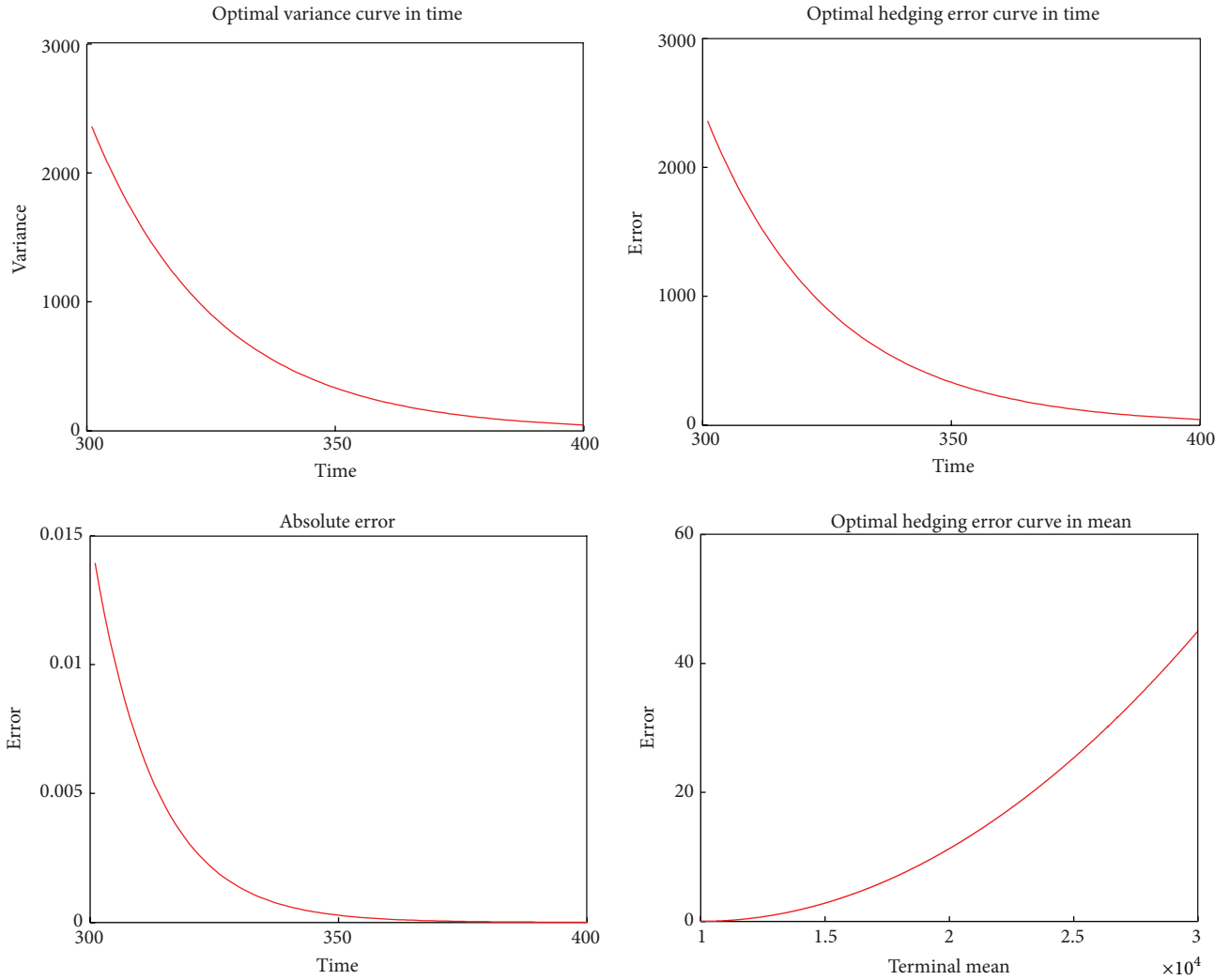
FIGURE 2: Errors using Black-Scholes model with $y_0 = 10$, $r = 0$, $v = 10000$, $p = 30000$, $T = 400$, $\alpha = 2$, and $\beta = 10$.

5. Proof of Theorem 5

The proof consists of four parts presented in the subsequent four subsections: the justification of a proposition related to the discounted price process, the demonstration of a proposition related to the VOMM, the illustration of unique existence of solution to a type of BSDEs with jumps, and the remaining proof of Theorem 5.

5.1. The Proposition Related to the Discounted Price Process

Proposition 9. *Under conditions (C1), (C2), and (12), one has that $D(\cdot)$ is a continuous $\{\mathcal{F}_t\}$-semimartingale; that is,*

$$D(\cdot) = D_0 + M^D(\cdot) + B^D(\cdot), \tag{40}$$

where $M^D(\cdot)$ and $B^D(\cdot)$ are an $\{\mathcal{F}_t\}$-martingale and a predictable process of finite variation, respectively. Furthermore, $D(\cdot)$ is locally in $L^2_{\mathcal{F}}([0,T], R^d, P)$ in the sense as stated in (14).

We divide the proof of the proposition into two parts. First, we have the following lemma.

Lemma 10. *Under (12), the unique adapted solution to the SDE in (5) for each $\widehat{t} > t$, $i \in \{1, \dots, h\}$, and $y \in (0, \infty)^h$ is given by*

$$Y_i(\widehat{t}) = y_i e^{-\lambda_i(\widehat{t}-t)} + \int_t^{\widehat{t}} e^{-\lambda_i(s-t)} dL_i(\lambda_i s) \geq y_i e^{-\lambda_i \widehat{t}}, \tag{41}$$

$$Y_i(t) = y_i.$$

Furthermore, under conditions (C1), (C2), and (12), there is a unique solution $(S_0(t), S(t)')$ for (4)-(5), which is an $\{\mathcal{F}_t\}$-adapted and continuous semimartingale with

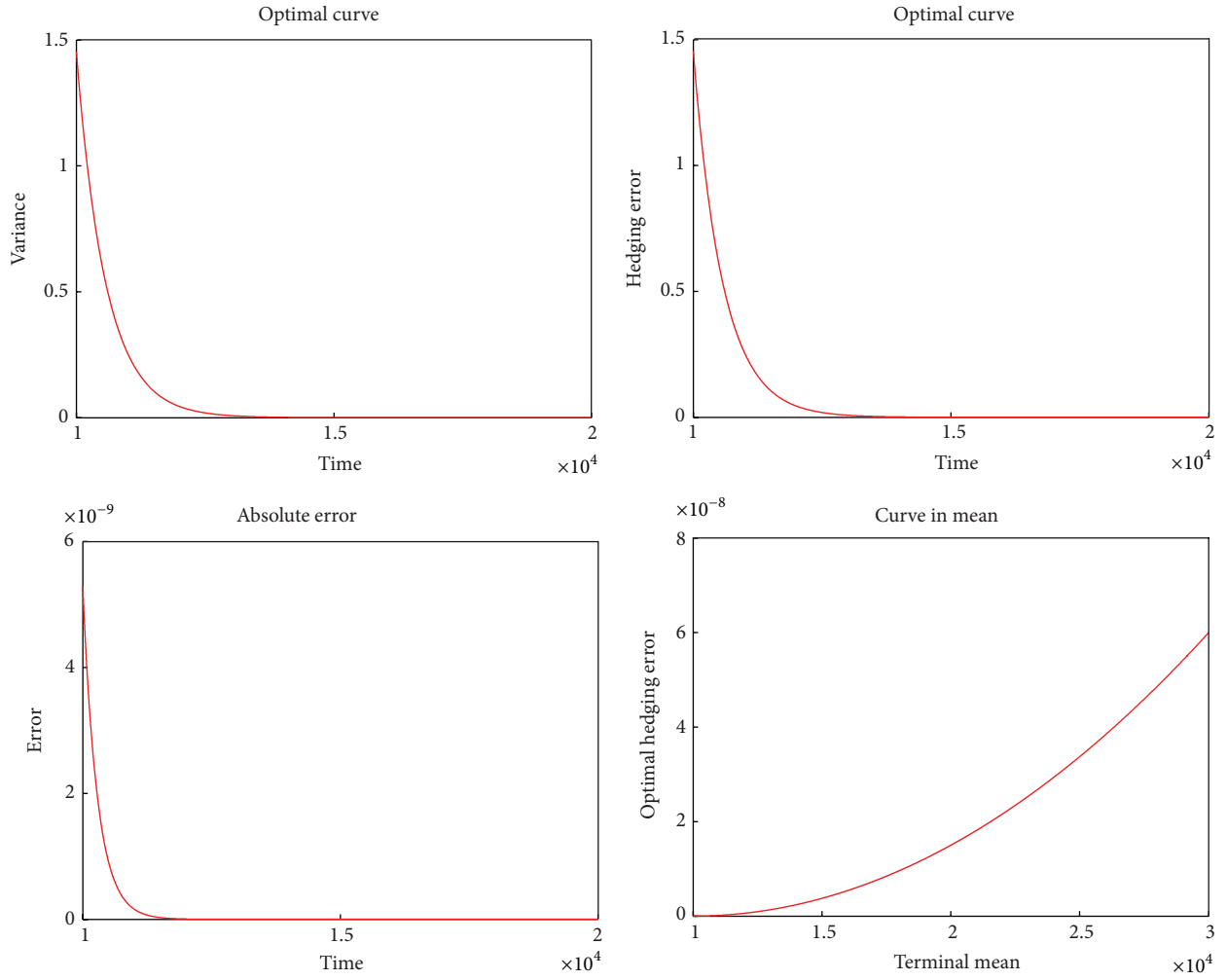$$S(\cdot) \in L^2_{\mathcal{F}}([0,T], R^d, P). \tag{42}$$

FIGURE 3: Errors using BNS model with $y_0 = 10$, $r = 0$, $v = 10000$, $p = 30000$, $T = 200$, $\delta = 0.01$, $\alpha = 0.5$, $\beta = 0.02$, $\mu = 10$, and $\mu_1 = 8$.

In addition, for each $m \in \{1, \ldots, d\}$,

$$
S_m(t) = S_m(0) \exp \left\{ \int_0^t \left[b_m(Y(s^-)) - \frac{1}{2} \sum_{n=1}^d \sigma_{mn}^2(Y(s^-)) \right] ds \right.
$$
$$
\left. + \int_0^t \sum_{n=1}^d \sigma_{mn}(Y(s^-)) \, dW_n(s) \right\}.
$$
(43)

Proof. The claim concerning (41) directly follows from pages 316-317 in Applebaum [31]. Furthermore, owing to conditions (C1) and (C2), we know that our market given by (4)-(5) satisfies the conditions as required by Lemma 4.1 in Dai [30]. Thus, our market has a unique solution, which is $\{\mathcal{F}_t\}$-adapted, continuous, and mean square-integrable as stated in Lemma 10. In order to prove (43), let

$$
X_m(t) = \int_0^t \alpha_m(Y(s^-)) \, ds + \int_0^t \beta_m(Y(s^-))' \, dW(s),
$$
(44)

where, for any $s \in [0, T]$,

$$
\alpha_m(Y(s^-)) = b_m(Y(s^-)) - \frac{1}{2} \sum_{n=1}^d \sigma_{mn}^2(Y(s^-)),
$$
$$
\beta_m(Y(s^-)) = (\sigma_{m1}(Y(s^-)), \ldots, \sigma_{md}(Y(s^-)))'.
$$
(45)

Then, by condition (C1), there exists some nonnegative constant D_1 such that

$$
E\left[\int_0^T |\alpha_m(Y(s^-))| \, ds \right]
$$
$$
\leq D_1 T + \left(B_b + \frac{1}{2} B_\sigma \right) T e^{\sum_{i=1}^h y_{i0}} \prod_{i=1}^h E\left[e^{L_i(\lambda_i T)} \right]
$$
(46)
$$
< \infty,
$$

where we have used the facts that $L(\lambda t)$ is nonnegative and nondecreasing in t, the independence assumption among $L_i(\lambda_i \cdot)$ for $i \in \{1, \ldots, h\}$, and

$$a + b \| L(\lambda t) \|$$

$$\leq \left(\frac{1}{\epsilon} \vee a \right) e^{b\epsilon \| L(\lambda t) \|} \quad \text{for any } a \geq 0, \ b \geq 0, \ \epsilon > 0, \tag{47}$$

$$Y_i^{t, y_i}(\hat{t}) \leq y_i + L_i(\lambda_i \hat{t}) - L_i(\lambda_i t) \quad \text{for any } \hat{t} \geq t, \tag{48}$$

$$E \left[e^{CL_i(\lambda_i t)} \right] = \exp \left(\lambda_i t \int_{z_i > 0} \left(e^{Cz_i} - 1 \right) \nu_i(dz_i) \right) < \infty. \tag{49}$$

Similarly, we can show that

$$E \left[\int_0^T \beta_m^2 (Y(s^-)) \, ds \right] < \infty. \tag{50}$$

Note that $W(\cdot)$ and $L_i(\lambda_i \cdot)$ for $i \in \{1, \ldots, h\}$ are independent; W is $\{\mathscr{F}_t, t \in [0, T]\}$-martingale; $\alpha_m(Y(t^-))$ and $\beta_m(Y(t^-))$ are $\mathscr{F}_t$-adapted. Then, it follows from Definition 4.1.1 in Øksendal [41] and the associated Itô's formula (e.g., Theorem 4.1.2 in [41]) that $S_m(t)$ given in (43) for each m is the unique solution of (4).

Now, we show that $S_m(\cdot)$ for each $m \in \{1, \ldots, d\}$ is a square-integrable $\{\mathscr{F}_t\}$-semimartingale. To do so, we rewrite (4) in its integral form

$$S_m(t) = S_m(0) + \int_0^t S_m(s) b_m(Y(s^-)) \, ds$$

$$+ \int_0^t S_m(s) \sum_{n=1}^d \sigma_{mn}(Y(s^-)) \, dW_n(s). \tag{51}$$

Then, the third term on the right-hand side of (51) is a square-integrable $\{\mathscr{F}_t\}$-martingale. In fact, it follows from (41) that, for each $i \in \{1, \ldots, h\}$ and $\hat{t} > t$,

$$\lambda_i \int_t^{\hat{t}} Y_i^{(t, y_i)}(s) \, ds = y_i + L_i(\lambda_i \hat{t}) - L_i(\lambda_i t) - Y_i^{(t, y_i)}(\hat{t})$$

$$\leq y_i + L_i(\lambda_i \hat{t}) - L_i(\lambda_i t)$$

$$= y_i + L_i(\lambda_i(\hat{t} - t)), \tag{52}$$

where the last equality in (52) holds in distribution. Thus, it follows from condition (C1) and (43) in Lemma 10 that

$$E \left[\int_0^T \left(S_m(s) \sum_{n=1}^d \sigma_{mn}(Y(s^-)) \right)^2 ds \right]$$

$$\leq ds_m^2 CT^{1/2} \left(E \left[e^{C\|L(\lambda T)\|} \right] \right)^{1/2} < \infty, \tag{53}$$

where C is some positive constant and we have used Theorem 39 in page 138 of Protter [39] and condition (12). Therefore, by Theorem 4.40(b) in page 48 of Jacod and Shiryaev [38], we know that the third term in (51) is a square-integrable $\{\mathscr{F}_t\}$-martingale.

Furthermore, by the same method, we can show that the second term on the right-hand side of (51) is of finite variation a.s. and is square-integrable over $[0, T]$. Therefore, we conclude that $S_m(\cdot)$ for each $m \in \{1, \ldots, d\}$ is a square-integrable $\{\mathscr{F}_t\}$-semimartingale. Hence, we complete the proof of Lemma 10. □

Proof of Proposition 9. It follows from Lemma 10 and Itô's formula that, for each $m \in \{1, \ldots, d\}$,

$$B_m^D(t) = \int_0^t D_m(s) \left(b_m(Y(s^-)) - r \right) ds, \tag{54}$$

$$M_m^D(t) = \int_0^t D_m(s) \sum_{n=1}^d \sigma_{mn}(Y(s^-)) \, dW_n(s). \tag{55}$$

Note that, by similar calculation as in (53), we have

$$E \left[\int_0^t \left(D_m(s) \sum_{n=1}^d \sigma_{mn}(Y(s^-)) \right)^2 ds \right] < \infty \tag{56}$$

for all $t \in [0, T]$. Thus, it follows from Theorem 4.40(b) in page 48 of Jacod and Shiryaev [38] that M^D is an $\{\mathscr{F}_t\}$-martingale. Furthermore, it follows from a similar explanation with the end of the proof for Lemma 10 that B^D is a predictable process of finite variation and is square-integrable. Thus, we know that D is a continuous $\{\mathscr{F}_t\}$-semimartingale. Moreover, it is locally in $L^2(P)$ since we may take $\sigma_n \equiv \inf \{ \tau : D^2(\tau) \geq n \}$ as the sequence of localizing times. Hence, we complete the proof of Proposition 9. □

5.2. A Proposition Related to the VOMM. First of all, we use $\mathscr{P}_D(\overline{\Theta})(D)$ to denote the set of all signed $\overline{\Theta}$-martingale measures in the sense that $Q(\Omega) = 1$ and $Q \ll P$ with

$$\frac{dQ}{dP} \in L^2(P), \quad E \left[\frac{dQ}{dP}(u \cdot D)(T) \right] = 0 \tag{57}$$

for a signed measure Q on $(\Omega, \mathscr{F})$ and all $u \in \overline{\Theta}(D)$. Then, we have the following proposition.

Proposition 11. *Under conditions (C1), (C2), and (12), the following claims are true:*

(1) *$\widehat{Z}$ is a $\{\mathscr{F}_t\}$-martingale, where $\widehat{Z}(\cdot)$ is given in (20);*

(2) *the measure Q^* defined in (34) is an equivalent martingale measure (EMM) and $Q^* \in \mathscr{U}_2^e(D)$ that is defined in (2);*

(3) *the measure Q^* is the VOMM in the sense that*

$$\text{Var} \left(\frac{dQ^*}{dP} \right) = \min_{Q \in \mathscr{P}_D(\overline{\Theta})} \text{Var} \left(\frac{dQ}{dP} \right). \tag{58}$$

We divide the proof of the proposition into demonstrating six lemmas as follows.

Lemma 12. *Under conditions (C1), (C2), and (12), $P(t, y)$ defined in (17) is a solution of the following IPDE:*

$$\frac{\partial}{\partial t} P(t, y)$$

$$= \rho(y) P(t, y) + \sum_{i=1}^{h} \lambda_i y_i \frac{\partial}{\partial y_i} P(t, y)$$

$$- \sum_{i=1}^{h} \lambda_i \int_{z_i > 0} \left(P(t, y + z_i e_i) - P(t, y) \right) \nu_i(dz_i),$$

$$P(T, y) = 1$$

$$(59)$$

for $y \in R_c^h$. Furthermore, one has

$$P(t, y) \in C^{1,1}\left([0, T] \times R_c^h, R^1\right), \quad (60)$$

$$E\left[\int_0^T |P(t, Y(t^-))|^2 dt\right] < \infty, \quad (61)$$

$$\sum_{i=1}^{h} E\left[\int_0^T \int_{z_i > 0} |P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))|^2 \right.$$

$$(62)$$

$$\left. \cdot \nu(dz_i) dt \right] < \infty.$$

Proof. It follows from conditions (C1), (C2), and (41) that, for each $i \in \{1, \ldots, h\}$,

$$\|\rho(Y(t))\| \le A_\rho + B_\rho \|Y(t)\|, \quad (63)$$

$$\left\|\frac{\partial \rho(Y(t))}{\partial y_i}\right\| \le \overline{A}_1 + \overline{A}_2 \|Y(t)\|$$

$$+ \overline{A}_3 \|Y(t)\|^2 + \overline{A}_4 \|Y(t)\|^3,$$

$$(64)$$

where $\overline{A}_i$ for $i \in \{1, 2, 3, 4\}$ are some nonnegative constants and A_ρ and B_ρ are given by

$$A_\rho = \frac{2(A_b + r) B_b}{b_\sigma} + \frac{(A_b + r)^2}{b_\sigma K}, \qquad B_\rho = \frac{B_\sigma^2}{b_\sigma}, \quad (65)$$

with $K = \min\{y_{i0} e^{-\lambda_i T}, i = 1, \ldots, h\}$. Then, based on an idea as used in Benth at al. [2], we can prove Lemma 12 by the following four steps.

First, by direct calculation, we know that $P(t, y)$ is finite for any $(t, y) \in [0, T] \times R_c^h$; that is,

$$P(t, y) \le \exp\left(K_1 (T - t) + B_\rho \sum_{i=1}^{h} \frac{y_i}{\lambda_i}\right) < \infty, \quad (66)$$

where the nonnegative constant K_1 is given by

$$K_1 = A_\rho + \sum_{i=1}^{h} \lambda_i \int_{z_i > 0} \left(e^{B_\rho z_i / \lambda_i} - 1\right) \nu_i(dz_i). \quad (67)$$

Second, we prove that $P \in C^{0,1}([0, T] \times R_c^h, R^1)$ and the mapping $(t, y) \to (\partial P / \partial y_i)(t, y)$ for each $i \in \{1, \ldots, h\}$ is continuous. The continuity of $P(\cdot, y)$ for each $y \in R_c^h$ can be shown as follows. Owing to condition (7) and fact (52), we know that

$$\exp\left(\int_t^T \rho\left(Y^{t,y}(s)\right) ds\right)$$

$$\le \exp\left(A_\rho T + \sum_{i=1}^{h} \frac{B_\rho}{\lambda_i} \left(y_i + L_i(\lambda_i T)\right)\right).$$

$$(68)$$

By (12) and (49), we know that the function on the right-hand side of (68) is integrable for each fixed $y \in R_c^h$. Then, it follows from Lebesgue's dominated convergence theorem that $P(t, y)$ for each y is continuous in terms of $t \in [0, T]$.

Next, we show that $(\partial P / \partial y_i)(t, \cdot)$ with $i \in \{1, \ldots, h\}$ for all $t \in [0, T]$ exist and are continuous. In fact, consider an arbitrary but fixed point y and take a compact set $U \subset R_c^h$ such that y is in the interior of U. Note that all points in U can be assumed to be bounded by some positive constant M. Thus, by (64), (41), (48), and (47), we have, for all $s \ge t$,

$$\left|\frac{\partial}{\partial y_i} \rho\left(Y^{t,y}(s)\right)\right| \le \left(\sum_{i=1}^{4} \overline{A}_i\right) e^{3hM + 3 \sum_{i=1}^{h} L_i(\lambda_i T)}, \quad (69)$$

where $Y^{t,y}(s)$ denotes the process with the initial value y at time t. Owing to (12) and (49), the function on the right-hand side of (69) is integrable. Thus, it follows from Theorem 2.27(b) in Folland [42] that the partial derivative of $\int_t^T \rho(Y^{t,y}(s)) ds$ in terms of y_i for each $i \in \{1, \ldots, h\}$ exists. Hence, we have

$$\left|\frac{\partial}{\partial y_i} \left(e^{\int_t^T \rho(Y^{t,y}(s)) ds}\right)\right|$$

$$\le T\left(\left(\sum_{i=1}^{4} \overline{A}_i\right) e^{(A_\rho T + 3hM + B_\rho \sum_{i=1}^{h}(1/\lambda_i)) + \sum_{i=1}^{h}(3 + B_\rho/\lambda_i) L_i(\lambda_i T)}\right).$$

$$(70)$$

Again, by (12) and (49), we know that the function on the right-hand side of (70) is integrable. Therefore, by Theorem 2.27(b) in Folland [42], we can conclude that $P(t, y)$ is differentiable with respect to $y \in R_c^h$. Furthermore, by (41), (70), and Lebesgue's dominated convergence theorem, we obtain that the mapping $(t, y) \to (\partial P / \partial y_i)(t, y)$ for each $i \in \{1, \ldots, h\}$ is continuous. Hence, $P(t, y) \in C^{0,1}([0, T] \times R_c^h, R^1)$.

Third, we prove the square-integrable property (62) to be true. In fact, it follows from condition (12) that $\nu_i(\cdot)$ ($i \in \{1, \ldots, h\}$) is a σ-finite measure since $\nu_i([\epsilon, \infty)) < \infty$ for any $\epsilon > 0$. In addition, it is easy to see that the nonnegative function $|P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))|^2$ is a measurable one on the product space $[0, T] \times R_c^h \times \Omega$. Hence, by

the mean-value theorem, (69), (70), Jensen's inequality, and the differentiability of $P(t, y)$ in y, we have

$$
E\left[\int_0^T \int_{z_i>0} |P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-))|^2 \nu_i(dz_i) dt\right]
$$

$$
\leq K_3 K_4 \left(e^{(6+2B_\rho/\lambda_i)} \int_{0<z_i<1} z_i^2 \nu_i(dz_i)\right.
$$

$$
\left. + \int_{z_i\geq 1} \left(e^{(8+2B_\rho/\lambda_i)z_i} - 1\right)\nu_i(dz_i) + \int_{z_i\geq 1} \nu_i(dz_i)\right)
$$

$$
< \infty,
$$

(71)

where K_3 and K_4 are some positive constants. Furthermore, it follows from (66), (48), and (12) that (61) is true.

Fourth, we prove that $P(t, y)$ satisfies the IPDE (59). In fact, for each $t \in [0, T)$, it follows from the time-homogeneity of Y that

$$
g(T - t, y) \equiv E_{0,y}\left[e^{-\int_0^{T-t} \rho(Y(s))ds}\right]
$$

$$
= E_{t,y}\left[e^{-\int_t^T \rho(Y(s))ds}\right] = P(t, y).
$$

(72)

Since $P(t, y) \in C^{0,1}([0, T] \times R_c^h)$, it follows from Itô's formula (see, e.g., Theorem 1.14 and Theorem 1.16 in pages 6–9 of [43]) that, for each fixed t,

$$
g(T - t, Y^{0,y}(l))
$$

$$
= g(T - t, y)
$$

$$
- \sum_{i=1}^h \lambda_i \int_0^l Y_i^{0,y_i}(s^-) \frac{\partial g}{\partial y_i}(T - t, Y^{0,y}(s^-)) ds
$$

$$
+ \sum_{i=1}^h \int_0^l \int_{z_i>0} \left(g(T - t, Y^{0,y}(s^-) + z_i e_i)\right.
$$

$$
\left. - g(T - t, Y^{0,y}(s^-))\right) N_i(\lambda_i ds, dz_i).
$$

(73)

Furthermore, let $\hat{g}(t, z_i, \omega) \equiv g(T - t, Y^{0,y}(s^-, \omega) + z_i e_i) - g(T - t, Y^{0,y}(s^-), \omega)$ for each $z_i \in (0, \infty)$, $i \in \{1, \ldots, h\}$ and $\omega \in \Omega$. Then, $\hat{g}$ is $\{\mathscr{F}_t\}$-predictable. Thus, owing to (62) (here we need to use an arbitrary but fixed y to replace y_0), it follows from Theorem 4.2.3 in Applebaum [31] (or the explanation in pages 61-62 of [40]) that the last term in (73) is a semimartingale. Thus, taking expectations on both sides of (73), we get

$$
\frac{E\left[g(T - t, Y^{0,y}(l))\right] - g(T - t, y)}{l}
$$

$$
= \sum_{i=1}^h \frac{\lambda_i}{l} \int_0^l E\left[Y_i^{0,y_i}(s^-) \frac{\partial g}{\partial y_i}(T - t, Y^{0,y}(s^-))\right] ds
$$

$$
- \sum_{i=1}^h \frac{\lambda_i}{l} \int_0^l \int_{z_i>0} E\left[g(T - t, Y^{0,y}(s^-) + z_i e_i)\right.
$$

$$
\left. - g(T - t, Y^{0,y}(s^-))\right] \nu_i(dz_i) ds.
$$

(74)

Then, by letting $l \downarrow 0$, we know that $P(t, \cdot)$ is in the domain of the infinitesimal generator of Y, which is denoted by $\mathscr{A}$; that is,

$$
\mathscr{A} g(T - t, y)
$$

$$
= \sum_{i=1}^h \lambda_i y_i \frac{\partial}{\partial y_i} g(T - t, y)
$$

$$
- \sum_{i=1}^h \lambda_i \int_{z_i>0} (g(T - t, y + z_i e_i) - g(T - t, y)) \nu_i(dz_i).
$$

(75)

Now, by (66), we see that $g(T-t, y) = P(t, Y^{0,y}(l)) \in L^2(\Omega, P)$ for each $t \in [0, T)$ and all l in a neighborhood of zero such that $t - l \leq T$. Thus, we have

$$
E_{0,y}[g(T - t, Y(l))] = E_{0,y}\left[E_{0,Y(l)}\left[e^{-\int_0^{T-t} \rho(Y(s))ds}\right]\right]
$$

$$
= E_{0,y}\left[E_{0,y}\left[e^{-\int_0^{T-t} \rho(Y(s+l))ds} \mathscr{F}_l\right]\right]
$$

$$
= E_{0,y}\left[e^{-\int_l^{T-t+l} \rho^{(Y(s))ds}}\right]
$$

$$
= E_{0,y}\left[e^{-\int_0^{T-t+l} \rho(Y(s))ds} e^{\int_0^l \rho^{(Y(s))ds}}\right],
$$

(76)

where the second equality in (76) follows from the Markov property of Y (e.g., Proposition 7.9 in [34]). Then, we have

$$
\frac{E_{0,y}[g(T - t, Y(l))] - g(T - t, y)}{l}
$$

$$
= \frac{1}{l} E_{0,y}\left[e^{-\int_0^{T-t+l} \rho(Y(s))ds} \left(e^{\int_0^l \rho(Y(s))ds} - 1\right)\right]
$$

$$
+ \frac{g(T - t + l, y) - g(T - t, y)}{l}.
$$

(77)

Now, by the fundamental theorem of calculus, as $l \downarrow 0$, we a.s. have

$$
e^{-\int_0^{T-t+l} \rho(Y^{0,y}(s))ds} \left\{\frac{1}{l}\left(e^{\int_0^l \rho(Y^{0,y}(s))ds} - 1\right)\right\}
$$

$$
\longrightarrow \rho(y) e^{-\int_0^{T-t} \rho(Y^{0,y}(s))ds}.
$$

(78)

Furthermore, by the mean-value theorem, we have

$$
\frac{1}{l}\left|e^{\int_0^l \rho(Y^{0,y}(s))ds} - 1\right| \leq \sup_{l\in[0,T]}\left|\rho(Y^{0,y}(l)) e^{\int_0^l \rho(Y^{0,y}(s))ds}\right|.
$$

(79)

Since the function in the left-hand side of (78) is uniformly bounded by an integrable function, it follows from the dominated convergence theorem that the right derivative of $g(T - \cdot, y)$ at t exists and satisfies

$$\mathscr{A} g\left(T-t, y\right) = \rho\left(y\right) g\left(T-t\right) + \frac{\partial g}{\partial t}\left(T-t, y\right). \quad (80)$$

Hence, by (72) and (80), we know that $P(t, y)$ satisfies (59). In addition, we have

$$\left| P\left(t, y + z_i \delta_{ij}\right) - P\left(t, y\right)\right|$$
$$\leq K_5 E\left[e^{3 \sum_{j=1}^h (2L_j(\lambda_j T) + z_i \delta_{ij})}\left(e^{\sum_{j=1}^h ((B_\rho/\lambda_j)(2L_j(\lambda_j T) + z_i \delta_{ij}))}\right)\right] z_i, \quad (81)$$

where K_5 is some positive constant. Thus, by Lebesgue's dominated convergence theorem, we can conclude that

$$\int_{z_i > 0} \left| P\left(t, y + z_i e_i\right) - P\left(t, y\right)\right| \nu_i\left(dz_i\right) \quad (82)$$

is continuous in t. Therefore, it follows from (59) that $(\partial P/\partial t)(t, y)$ is continuous in $t \in [0, T)$, which implies that $P \in C^{1,1}([0, T) \times R_c^h, R^1)$. Hence, we complete the proof of Lemma 12. □

Lemma 13. *Let $O(t) \equiv P(t, Y(t))$ defined in (18). Then, under conditions (C1), (C2), and (12), O is a $(0, 1]$-valued semimartingale with $O(T) = 1$. Furthermore, define*

$$K \equiv \mathscr{L}\left(O\right) \equiv \left(\frac{1}{O_-}\right) \cdot O \quad \text{with } K\left(0\right) = 0, \ O_-\left(t\right) \equiv O\left(t^-\right). \quad (83)$$

Then, K is an $\{\mathscr{F}_t\}$-semimartingale and has the following canonical decomposition:

$$dK\left(t\right) \equiv d\mathscr{L}\left(O\right)\left(t\right) = \rho\left(Y\left(t^-\right)\right) dt$$
$$+ \sum_{i=1}^h \int_{z_i > 0} F\left(t, z_i\right) \widetilde{N}_i\left(\lambda_i dt, dz_i\right), \quad (84)$$

where $F(t, z_i, \omega)$ is defined in (25).

Proof. First, we show that O is an $\{\mathscr{F}_t\}$-semimartingale. In fact, it follows from Itô's formula (see, e.g., Theorem 1.14 and Theorem 1.16 in pages 6–9 of [43]) and Lemma 12 that

$$O\left(t\right) = P\left(0, y_0\right) + \int_0^t \rho\left(Y\left(s^-\right)\right) P\left(s, Y\left(s^-\right)\right) ds$$
$$+ \sum_{i=1}^h \int_0^t \int_{z_i > 0} \left(P\left(s, Y\left(s^-\right) + z_i e_i\right) - P\left(s, Y\left(s^-\right)\right)\right)$$
$$\cdot \widetilde{N}_i\left(\lambda_i ds, dz_i\right). \quad (85)$$

Then, by Lemma 12 and the claim in pages 61-62 of Ikeda and Watanabe [40], we know that the third term in the right-hand side of (85) is an $\{\mathscr{F}_t\}$-martingale. Furthermore, by (63) and similar proof as that used for Lemma 10, we know that the second term on the right-hand side of (85) is of finite variation a.s. Hence, we get that O is an $\{\mathscr{F}_t\}$-semimartingale. Thus, it follows from (85) and the definition of $K(t)$ that (84) is true.

Second, M^K defined as follows is an $\{\mathscr{F}_t\}$-martingale:

$$M^K\left(t\right) = \sum_{i=1}^h \int_0^t \int_{z_i > 0} F\left(s, z_i\right) \widetilde{N}_i\left(\lambda_i ds, dz_i\right). \quad (86)$$

In fact, by the mean-value theorem, (63), (41), (12), and the fact that $\nu_i(\cdot)$ ($i \in \{1, \ldots, h\}$) is a σ-finite measure since $\nu_i([\epsilon, \infty)) < \infty$ for any $\epsilon > 0$, we have

$$E\left[\int_0^T \int_{z_i > 0} \left| F\left(t, z_i\right)\right|^2 \nu\left(dz_i\right) dt\right] < \infty. \quad (87)$$

Thus, it follows from (87) and the claims in pages 61-62 of Ikeda and Watanabe [40] that M^K is an $\{\mathscr{F}_t\}$-martingale. Therefore, we can conclude that K is an $\{\mathscr{F}_t\}$-semimartingale. Hence, Lemma 13 is true. □

Lemma 14. *Let b^D and c^D be the drift and the covariance matrix processes associated with D; b^K is the drift process associated with K. Then, under conditions (C1), (C2), and (12), one has*

$$b^K = \left(b^D\right)'\left(c^D\right)^{-1} b^D. \quad (88)$$

Furthermore, the process a defined in (19) satisfies the following relationship:

$$a \equiv \left(c^D\right)^{-1} b^D. \quad (89)$$

Proof. First of all, it follows from Lemmas 10 and 13 that

$$b^D\left(t\right) = \left(D_1\left(t\right)\left(b_1\left(Y\left(t^-\right)\right) - r\right), \ldots,\right.$$
$$\left. D_d\left(t\right)\left(b_d\left(Y\left(t^-\right)\right) - r\right)\right)',$$
$$c^D\left(t\right) = \text{diag}\left(D\left(t\right)\right) \quad (90)$$
$$\cdot \left(\sigma\left(Y\left(t^-\right)\right) \sigma\left(Y\left(t^-\right)\right)'\right) \text{diag}\left(D\left(t\right)\right),$$
$$b^K\left(t\right) = O^{-1}\left(t\right) b^O\left(t\right) = \rho\left(Y\left(t^-\right)\right).$$

Then, by simple calculations, we know that (88) and (89) are true. Hence, we complete the proof of Lemma 14. □

For convenience, we will use $C_{ij}^D \equiv [D_i, D_j]$ to denote the coquadratic variation processes with $i, j \in \{1, \ldots, d\}$ for the process D and write interchangeably $c^{D_i D_j} \equiv c_{ij}^D$ and $c^{D_i} = c_{ii}^D$. Furthermore, similar notations are also used for other processes related in the following discussions.

Lemma 15. *Under conditions (C1), (C2), and (12), $\widehat{Z}$ is an $\{\mathscr{F}_t\}$- and P-martingale.*

Proof. First, we show that $a \in L(D)$. In fact, it follows from the conditions (C1), (41), and (40) that $\|Y(t^-)\| \geq \min\{y_{i0} e^{-\lambda_i T}, \ i = 1, \ldots, h\} > 0$ for any $t \in [0, T]$. Then, for $m, n \in \{1, \ldots, d\}$, we have

$$\overline{\rho}\left(Y\left(t^-\right)\right)$$

$$\equiv \sum_{m=1}^{d} \left(B\left(Y\left(t^-\right)\right)' \left(\sigma\left(Y\left(t^-\right)\right) \sigma\left(Y\left(t^-\right)\right)'\right)^{-1}\right)_m^2$$

$$\cdot \sum_{n=1}^{d} \sigma_{mn}^2 \left(Y\left(t^-\right)\right) \tag{91}$$

$$\leq C_{\overline{\rho}} + \frac{B_b^2 B_\sigma}{b_\sigma^2} \left\|Y\left(t^-\right)\right\|,$$

where $C_{\overline{\rho}}$ is some positive constant. Thus, it follows from the Kunita-Watanabe's inequality (e.g., Theorem 25 in page 69 of [39]) that

$$E\left[\sum_{m=1}^{d}\sum_{n=1}^{d}\left|\int_0^T a_m(t)\,a_n(t)\,d\left[M_m^D, M_n^D\right](t)\right|\right]$$

$$\leq d^2 E\left[\int_0^T \overline{\rho}\left(Y\left(t^-\right)\right) dt\right] < \infty, \tag{92}$$

where a_m and M_m^D with $m \in \{1, \ldots, d\}$ are the mth components of a and M^D, respectively. Furthermore, it follows from (40) that

$$E\left[\sum_{m=1}^{d}\int_0^T a_m(t)\,D_m(t)\,B_m\left(Y\left(t^-\right)\right) dt\right]$$

$$= E\left[\int_0^T \rho\left(Y\left(t^-\right)\right) dt\right] < \infty. \tag{93}$$

Then, by (92)-(93), Definition 6.17 of page 207, Definition 4.3 of page 180, Definition 6.12 of page 206, and Definition 2.6 of page 76 in Jacod and Shiryaev [38], we know that $a \in L(D)$. Thus, $(a \cdot D)(T)$ is well defined.

In addition, it follows from Theorem 4.5(a) in page 180 of Jacod and Shiryaev [38] that, for each $u \in L(D)$, we have

$$(u \cdot D)(t) = \lim_{k \to \infty} \sum_{i=1}^{d} \int_0^t u_i(s)\, I_{\{\|u(s)\| \leq k\}} dM_i^D(s)$$

$$+ \sum_{i=1}^{d} \int_0^t u_i(s)\, dB_i^D(s), \tag{94}$$

where the limit in the first term on the right-hand side of (94) corresponds to the convergence in probability uniformly on every compact set of $[0, T]$. Therefore, by (40), (12), (54)-(55), (94), and the Lebesgue dominated convergence theorem, we know that

$$(a \cdot D)(T) = \sum_{m=1}^{d} \int_0^T a_m(t)\, dD_m(t). \tag{95}$$

Now, it follows from Lemma 13 that O is a semimartingale. Thus, it follows from conditions (C1), (C2), and (95) that $(a \cdot D)$ is also a semimartingale. Then, by Corollary 8.7(b) and (8.19) in pages 135–138 of Jacod and Shiryaev [38], we have that

$$\widehat{Z}(t) = \mathscr{E}\left(K - (a \cdot D) - [K, (a \cdot D)]\right)(t)$$

$$= \mathscr{E}\left(M^K - \left(a \cdot M^D\right) + \left(b^K - a' b^D\right) \cdot A\right)(t) \tag{96}$$

$$= \mathscr{E}\left(G\right)(t),$$

where the second equality follows from the facts that $A(t) = t$ and $K(0) = 0$ and the independence among driving Brownian motions and Lévy processes. The third equality follows from Lemma 14. Furthermore, M^K and M^D are given by (86) and (55), which are $\{\mathscr{F}_t\}$-martingales. Hence,

$$G \equiv M^K - \left(a \cdot M^D\right) \tag{97}$$

is also an $\{\mathscr{F}_t\}$-martingale. Thus, it follows from Theorem 4.61 in page 59 of Jacod and Shiryaev [38] that $\widehat{Z}$ is an $\{\mathscr{F}_t\}$-local martingale.

Second, we prove that $\widehat{Z}$ is of class (D), that is, the set of random variables

$$\left\{\widehat{Z}(\tau), \ \tau \text{ is finite valued } \{\mathscr{F}_t\}\text{-stopping times}\right\} \tag{98}$$

is uniformly integrable (e.g., Definition 1.46 in page 11 of [38]).

In fact, consider an arbitrary finite valued $\{\mathscr{F}_t\}$-stopping time $\tau \leq T$ and an arbitrary constant $\gamma > 0$. Then, we have

$$E\left[\left|\widehat{Z}(\tau)\right| I_{\{|\widehat{Z}(\tau)| \geq \gamma\}}\right]$$

$$\leq \frac{1}{P(0, y_0)} \left(E\left[\left(\mathscr{E}(-a \cdot D)(\tau)\right)^2\right]\right)^{1/2} \tag{99}$$

$$\cdot \left(P\left\{\left|\widehat{Z}(\tau)\right| \geq \gamma\right\}\right)^{1/2},$$

where we have used the facts that $0 < O(\cdot) \leq 1$ and D is continuous. Furthermore, let

$$U_1(t) = \int_0^t \rho\left(Y\left(s^-\right)\right) ds,$$

$$U_2(t) = \sum_{n=1}^{d} \int_0^t \overline{B}_n\left(Y\left(s^-\right)\right) dW_n(s), \tag{100}$$

where $\overline{B}(Y(s^-))$ is defined in (23). Hence, $U_2(t)$ is a continuous $\{\mathscr{F}_t\}$-martingale. Thus,

$$E\left[\left(\mathscr{E}(-a \cdot D)(\tau)\right)^2\right]$$

$$= E\left[e^{\left(-2(U_1(\tau) + U_2(\tau)) - [U_1 + U_2, U_1 + U_2](\tau)\right)}\right]$$

$$\leq E\left[e^{-2U_2(\tau)}\right] \tag{101}$$

$$\leq \left(E\left[e^{8[U_2, U_2](T)}\right]\right)^{1/2} < \infty,$$

where the third inequality follows from the optional sampling theorem, the fact that $e^{-2U_2(t)}$ is a submartingale by Jensen's inequality, and Theorem 39 in page 138 of Protter [39]. The last inequality follows from conditions (C1)-(C2). Therefore, it follows from (101) that $\sup_\tau E[|\widehat{Z}(\tau)|] \leq K_1$, where K_1 is some positive constant. Thus, by Markov's inequality, we have that

$$P\left\{\left|\widehat{Z}(\tau)\right| \geq \gamma\right\} \leq \frac{K_1}{\gamma} \longrightarrow 0 \quad \text{as } \gamma \longrightarrow \infty \qquad (102)$$

for all stopping time $\tau \leq T$. Therefore, it follows from (99)–(102) that $\widehat{Z}$ is of class (D). Hence, it follows from (96) and Proposition 1.47(c) in page 12 of Jacod and Shiryaev [38] that $\widehat{Z}$ is a uniformly integrable $\{\mathcal{F}_t\}$- and P-martingale. $\qquad \square$

Lemma 16. *Under conditions (C1), (C2), and (12), Q^* is an equivalent martingale measure.*

Proof. First, we use P_t to denote the restriction of P to $\mathcal{F}_t$ for each $t \in [0, T]$. Then, we define $dQ_t^* \equiv \widehat{Z}(t)dP_t$ and $dQ^* \equiv \widehat{Z}(T)dP$. Owing to (17)–(20), we know that $\widehat{Z}(t) > 0$ for each $t \in [0, T]$. Furthermore, note that $\widehat{Z}$ is a $\{\mathcal{F}_t\}$- and P-martingale. Hence, it follows from the discussion in page 166 of Jacod and Shiryaev [38] that Q^* is equivalent to P with the density process $\widehat{Z}$.

Next, we show that D is Q^*-martingale. In fact, since D is an P-semimartingale with the decomposition given in (40), it follows from Girsanov-Meyer Theorem (e.g., Theorem 35 in page 132 of [39]) that D is also Q^*-semimartingale with the decomposition $D = \widetilde{D} + \overline{D}$. The process $\overline{D}$ is Q^*-finite variation process. For each $m \in \{1, \ldots, d\}$,

$$\widetilde{D}_m(t)$$

$$= M_m^D(t) - \int_0^t \frac{1}{\widehat{Z}(s)} d\left[\widehat{Z}, M_m^D\right](s)$$

$$= M_m^D(t) - \sum_{n=1}^d \int_0^t \frac{D_m(s)\,\sigma_{mn}\left(Y\left(s^-\right)\right)}{\widehat{Z}(s)\,O_0}$$

$$\qquad\qquad \cdot d\left[O\mathcal{E}\left(-a\cdot D\right), W_n\right]^c(s)$$

$$= M_m^D(t)$$

$$\quad - \sum_{n=1}^d \int_0^t \frac{D_m(s)\,\sigma_{mn}\left(Y\left(s^-\right)\right)}{\widehat{Z}(s)\,O_0}\mathcal{E}\left(-a\cdot D(s)\right)$$

$$\qquad \cdot \Big(d\left[O, W_n\right]^c(s) + O(s)\,d\left[U, W_n\right]^c(s)$$

$$\qquad\qquad + \frac{1}{2}d\left[[O, U]^c, W_n\right]^c(s)\Big)$$

$$= M_m^D(t) - \sum_{n=1}^d \int_0^t D_m(s)\,\sigma_{mn}\left(Y\left(s^-\right)\right)d\left[U, W_n\right]^c(s)$$

$$= M_m^D(t)$$

$$+ \sum_{r=1}^d \sum_{n=1}^d \int_0^t D_m(s)\,\sigma_{mn}\left(Y\left(s^-\right)\right)a_r(s)\,d\left[D_r, W_n\right]^c(s)$$

$$= M_m^D(t) + \int_0^t \sum_{n=1}^d D_m(s)\,\sigma_{mn}\left(Y\left(s^-\right)\right)\overline{B}_n\left(Y\left(s^-\right)\right)ds, \tag{103}$$

where $\overline{B}_n(Y(s^-))$ is defined in (24). The second equality in (103) follows from Theorem 29 in page 75 of Protter [39], the proof of Corollary in page 83 of Protter [39], the fact that W is continuous, Theorem 4.52 in page 55 of Jacod and Shiryaev [38], and the explanation in page 70 of Protter [39]. The third equality in (103) follows from Itô's formula for multidimensional semimartingales (e.g., Theorem 33 in pages 81-82 of [39]), and the associated function f is taken to be $f(O, U) = Oe^U$. Furthermore, a_r is the rth component of a, and U is defined by $U(t) \equiv -a \cdot D(t) - (1/2)[a \cdot D, a \cdot D](t)$. Thus, we have

$$\overline{D}(t) = D(t) - \widetilde{D}(t) = \overline{s} \equiv (s_1, \ldots, s_d)' \quad \text{or}$$
$$D(t) = \widetilde{D}(t) + \overline{s}, \tag{104}$$

where s_i for each $i \in \{1, \ldots, d\}$ is the initial price as given in (4).

Therefore, to show that D is Q^*-martingale, it suffices to show that $\widetilde{D}$ is Q^*-martingale. More precisely, by the last equation in the proof of Theorem 35 in pages 132-133 of Protter [39], we have that

$$\widetilde{D}_m(t) = \left(M_m^D(t) - \frac{1}{\widehat{Z}(t)}\left[\widehat{Z}, M_m^D\right](t)\right)$$
$$\quad + \int_0^t \left[\widehat{Z}, M_m^D\right](s^-)\,d\left[\frac{1}{\widehat{Z}}\right](s). \tag{105}$$

Then, we can show that both terms on the right-hand side of (105) are Q^*-martingales.

For the first term on the right-hand side of (105), it follows from integration by parts (e.g., equations $(*)$ and $(**)$ in page 132 of [39]), Itô's formula (e.g., Theorem 1.14 and Theorem 1.16 in pages 6–9 of [43]), and Lemma 12 that

$$\left(M_m^D(t) - \frac{1}{\widehat{Z}(t)}\left[\widehat{Z}, M_m^D\right](t)\right)\widehat{Z}(t)$$

$$= \int_0^t \widehat{Z}(s^-)\,dM_m^D(s) + \int_0^t M_m^D(s)\,d\widehat{Z}(s)$$

$$= \int_0^t \widehat{Z}(s^-)\,dM_m^D(s)$$

$$\quad - \sum_{n=1}^d \int_0^t M_m^D(s)\,\widehat{Z}(s^-)\,\overline{B}_n\left(Y\left(s^-\right)\right)dW_n(s)$$

$$\quad + \sum_{i=1}^h \int_0^t \int_{z_i>0} \frac{M_m^D(s)\,\mathcal{E}\left((-a\cdot D)(s)\right)}{O_0}$$

$$\qquad\qquad \cdot \left(P\left(s, Y\left(s^-\right)+z_i e_i\right) - P\left(s, Y\left(s^-\right)\right)\right)$$

$$\qquad\qquad \cdot \widetilde{N}_i\left(\lambda_i ds, dz_i\right), \tag{106}$$

where $\overline{B}_n(Y(s^-))$ is defined in (24). The second equality follows from (86)–(55) and the fact that

$$d\widehat{Z}(t) = \widehat{Z}(t^-) dG(t) \tag{107}$$

owing to (96)-(97), the definition of Doléans-Dade exponential, and Theorem 37 in pages 84-85 of Protter [39].

Then, we can show that each of the three terms on the right-hand side of (106) is Q^*-martingale.

The claim that the first term on the right-hand side of (106) is Q^*-martingale can be proved as follows. First, it follows from similar argument as used in (109) that M^D is a square-integrable P-martingale. Second, by Tonelli's Theorem (e.g., Theorem 20 in page 309 of [44]) and Hölder's inequality, we have

$$E\left[\int_0^T \widehat{Z}^2(s) d\left[M_m^D, M_m^D\right](s)\right]$$

$$\leq \overline{K} \int_0^T \left(E\left[O^8(s)\right]\right)^{1/2} \left(E\left[(\mathscr{E}(-a \cdot D)(s))^{16}\right]\right)^{1/4} \tag{108}$$

$$\cdot \left(E\left[D_m^{16}(s)\right]\right)^{1/4} ds < \infty,$$

where $\overline{K}$ is some positive constant. The last inequality in (108) follows from the same arguments as in (101) and (53). Thus, it follows from Theorem 4.40(b) in page 48 of Jacod and Shiryaev [38] that the first term on the right-hand side of (106) is an $\{\mathscr{F}_t\}$- and P-martingale.

The claim that the second term on the right-hand side of (106) is Q^*-martingale can be proved as follows. It follows from (53) and Exercise 3.25 in page 163 of Karatzas and Shreve [45] that

$$E\left[\int_0^t \left(M_m^D(s)\right)^{16} ds\right] < \infty. \tag{109}$$

Then, by (109), Hölder's inequality, and similar method as used in (108), we know that the second term on the right-hand side of (106) is an $\{\mathscr{F}_t\}$- and P-martingale.

The claim that the third term on the right-hand side of (106) is Q^*-martingale can be proved as follows. It follows from Tonelli's Theorem (e.g., Theorem 20 in page 309 of [44]) that

$$E\left[\int_0^T \int_{z_i>0} \frac{|M_m^D(t)|}{O_0} \left|(P(t, Y(t^-) + z_i e_i) - P(t, Y(t^-)))\right|\right.$$

$$\left. \cdot \mathscr{E}((-a \cdot D)(t)) \right| \nu_i(dz_i) dt\right]$$

$$\leq K_1 \left(E\left[\int_0^T \int_{z_i>0} \sup_{\xi(Y(t^-))\in[0,z_i]} \left|\frac{\partial P(t, Y(t^-) + \xi(Y(t^-))e_i)}{\partial y_i}\right|^2\right.\right.$$

$$\left.\left. \cdot z_i \nu_i(dz_i) dt\right]\right)^{1/2} < \infty, \tag{110}$$

where K_1 is some positive constant. The inequalities in (110) follow from the same proofs as used in (101), (109), Hölder's inequality, the proof of (71), and the fact that

$$\int_{z_i>0} z_i \nu(dz_i) \leq \int_{0<z_i<1} z_i \nu_i(dz_i) + \int_{z_i\geq 1} (e^{z_i} - 1) Y_i(d_{2i})$$

$$+ \int_{z_i\geq 1} \nu_i(dz_i) < \infty. \tag{111}$$

Then, it follows from (110) and the argument in pages 61-62 in Ikeda and Watanabe [40] that the third term on the right-hand side of (106) is also an $\{\mathscr{F}_t\}$- and P-martingale.

Therefore, by summarizing the discussions for the three terms on the right-hand side of (106), we know that the process given by (106) is an $\{\mathscr{F}_t\}$- and P-martingale. Moreover, by applying Proposition 3.8(a) in page 168 of Jacod and Shiryaev [38], we can conclude that the first term on the right-hand side of (105) is Q^*-martingale.

For the second term on the right-hand side of (105), we can show that it is also an $\{\mathscr{F}_t\}$- and Q^*-martingale. In fact, since $\widehat{Z}$ is a density process of Q^* in terms of P and $(1/\widehat{Z})\widehat{Z} = 1$ (that is P-martingale), it follows from Proposition 3.8(a) in page 168 of Jacod and Shiryaev [38] that $1/\widehat{Z}$ is Q^*-martingale. Furthermore, it follows from Itô's formula (e.g., Theorem 32 in page 78 of [39]), (107), and the calculation of $d\widehat{Z}(t)$ in the last equality in (106) that

$$d\left(\frac{1}{\widehat{Z}(t)}\right) = \frac{1}{\widehat{Z}(t^-)} \sum_{n=1}^d \left(\overline{B}_n(Y(t^-))\right)^2 dt$$

$$- \frac{1}{\widehat{Z}(t^-)} \sum_{n=1}^d \overline{B}_n(Y(t^-)) dW_n(t) \tag{112}$$

$$- \sum_{i=1}^h \int_{z_i>0} \frac{F(t, z_i)}{\widehat{Z}(t)} \widetilde{N}_i(\lambda_i dz_i, dt),$$

where $\overline{B}(Y(t^-))$ is defined in (24). Thus, it follows from (112) that $1/\widehat{Z}$ is squarely integrable under Q^*; that is,

$$E_{Q^*}\left[\left(\frac{1}{\widehat{Z}(t)}\right)^2\right] \leq E_{Q^*}\left[\sup_{0\leq s\leq T} \frac{1}{\widehat{Z}^2(s)}\right]$$

$$\leq 4 E_{Q^*}\left[\frac{1}{\widehat{Z}^2(T)}\right]$$

$$\leq 4 \left(E\left[\widehat{Z}^2(T)\right]\right)^{1/2} \left(E\left[\frac{1}{\widehat{Z}^4(T)}\right]\right)^{1/2}$$

$$= \frac{4}{O_0} \left(E\left[(\mathscr{E}((-a \cdot D)(T))^2\right]\right)^{1/2}$$

$$\cdot \left(E\left[\frac{1}{(\mathscr{E}((-a \cdot D)(T)))^4}\right]\right)^{1/2} < \infty, \tag{113}$$

where the second inequality in (113) follows from Doob's martingale inequality (e.g., Theorem 2.1.5 in page 74 of [31]) since $1/\widehat{Z}$ is Q^*-martingale. The last inequality of (113) follows from the same argument as in (101).

Therefore, to show that the second term on the right-hand side of (105) is Q^*-martingale, it suffices to show that the following expectation under Q^* is finite owing to (113) and Theorem 4.40(b) in page 48 of Jacod and Shiryaev [38]:

$$E_{Q^*}\left[\int_0^T \left(\left[\widehat{Z}, M_m^D\right](s^-)\right)^2 d\left[\frac{1}{\widehat{Z}}, \frac{1}{\widehat{Z}}\right](s)\right]$$

$$= E_{Q^*}\left[\int_0^T \left(\left[\widehat{Z}, M_m^D\right]^c(s^-)\frac{1}{\widehat{Z}(s^-)}\right)^2 \sum_{n=1}^d \left(\overline{B}_n\left(Y\left(s^-\right)\right)\right)^2 ds\right]$$

$$+ \sum_{i=1}^h E_{Q^*}\left[\int_0^T \int_{z_i>0} \left(\left[\widehat{Z}, M_m^D\right]^c(s^-)\frac{F(s,z_i)}{\widehat{Z}(s)}\right)^2\right.$$

$$\left. \cdot \lambda_i \nu_i (dz_i) \, ds\right].$$

$$(114)$$

The first term on the right-hand side of (114) is finite since

$$E_{Q^*}\left[\int_0^T \frac{1}{\widehat{Z}^2(s^-)}\left(\left[\widehat{Z}, M_m^D\right]^c(s^-)\right)^2 \overline{\rho}\left(Y\left(s^-\right)\right) ds\right]$$

$$\leq K_1 \left(\frac{4}{3}E\left[\left(\frac{1}{\mathscr{E}(-a\cdot D)(T)}\right)^3\right]\right)^{1/2}$$

$$\cdot \left(\frac{20}{19}E\left[\left(\mathscr{E}(-a\cdot D)(T)\right)^{20}\right]\right)^{1/8}$$

$$\cdot \left(\int_0^T E\left[D_m^8(s)\right] ds\right)^{1/8} < \infty,$$

$$(115)$$

where K_1 is some positive constant. The first inequality in (115) follows from Doob's martingale inequality (e.g., page 74 of [31]). The second inequality in (115) follows from the same arguments as in (99) and (53). Similarly, the second term on the right-hand side of (114) is also finite, which can be proved along the line of the discussion as in (115).

Thus, it follows from the finiteness of (114) that the second term on the right-hand side of (105) is Q^*-martingale. Therefore, by combining this fact with (105) and (106), we know that $D = \widetilde{D} + \overline{s}$ displayed in (104) is Q^*-martingale (i.e., Q^* is an equivalent martingale measure). Finally, by applying the same discussion as that used in (108), we conclude that $dQ^*/dP \in L^2(P)$, which implies that $Q^* \in \mathscr{U}_2^e(D)$. □

Lemma 17. *Under conditions (C1), (C2), and (12), Q^* is the VOMM.*

Proof. It suffices to justify that all conditions stated in Theorem 3.25 of Černý and Kallsen [8] are satisfied. First of all, for any stopping time τ, we can show that

$$u^\tau(t) \equiv a(t) I_{(\tau,T]}(t) \mathscr{E}\left(\left(-aI_{(\tau,T]}\right)\cdot D\right)(t^-) \in \overline{\Theta}(D). \quad (116)$$

In fact, it follows from the proof of Lemma 16 that $\mathscr{U}_2^e(D)$ is nonempty. Furthermore, since D is a continuous P-semimartingale, it is sufficient to prove that the three equivalent conditions stated in Theorem 2.1 of Černý and Kallsen

[25] are satisfied for (116), which can be done by tedious computations similarly to before. In addition, we can show that $O\mathscr{E}((-aI_{(\tau,T]})\cdot D)$ is of class (D). Therefore, by combining this claim with Lemma 14, (116), and Theorem 3.25 in Černý and Kallsen [8], we know that O and a are the opportunity and adjustment processes in the sense defined in Section 3 of [8]. Thus, it follows from Proposition 3.13 in Černý and Kallsen [8] that Q^* is the VOMM. Hence, we complete the proof of Proposition 11. □

5.3. The Unique Existence of Solution to a Type of BSDEs. Consider the following q-dimensional BSDE with jumps and a terminal condition H:

$$V(t) = H - \int_t^T g\left(s, V(s^-), \overline{V}(s), \widetilde{V}(s,\cdot), Y(s^-)\right) ds$$

$$- \int_t^T \sum_{i=1}^d \overline{V}_i(s) dW_i(s)$$

$$- \int_t^T \sum_{i=1}^h \int_{z_i>0} \widetilde{V}_i(s,z_i) \widetilde{N}_i(\lambda_i ds, dz_i),$$

$$(117)$$

where $H \in L^2_{\mathscr{F}_T}(\Omega, R^q, P)$, $\overline{V} = (\overline{V}_1, \ldots, \overline{V}_d) \in R^{q\times d}$, $\widetilde{V} = (\widetilde{V}_1, \ldots, \widetilde{V}_h) \in R^{q\times h}$, g is the random function $[0,T] \times R^q \times R^{q\times d} \times L^2_\nu(R_+^h, R^{q\times h}) \times R^h \times \Omega \to R^h$, and

$$L^2_\nu\left(R_+^h, R^{q\times h}\right)$$

$$\equiv \left\{\widetilde{\nu}: R_+^h \longrightarrow R^{q\times h}, \sum_{i=1}^h \int_{z_i>0} |\widetilde{\nu}_i(z_i)|^2 \nu_i(dz_i) < \infty\right\}.$$

$$(118)$$

Furthermore, for any $\widetilde{\nu} \in L^2_\nu(R_+^h, R^{q\times h})$, the associated norm is defined by

$$\|\widetilde{\nu}\|_\nu \equiv \left(\sum_{i=1}^h \int_{z_i>0} |\widetilde{\nu}_i(z_i)|^2 \lambda_i \nu_i(dz_i)\right)^{1/2}. \quad (119)$$

Proposition 18. *Replace $H \in L^4_{\mathscr{F}_T}(\Omega, R, P)$ by $H \in L^2_{\mathscr{F}_T}(\Omega, R, P)$ in Assumption 4. Suppose that $g(t, v, \overline{v}, \widetilde{v}, Y(t^-))$ is $\{\mathscr{F}_t\}$-adapted for any given $(v, \overline{v}, \widetilde{v}) \in R^q \times R^{q\times d} \times L^2_\nu(R_+^h, R^{q\times h})$ with*

$$g\left(\cdot, 0, 0, 0, Y\left(\cdot^-\right)\right) \in L^2_{\mathscr{F}}\left([0,T], R^q\right) \quad (120)$$

such that

$$\left\|\left(g\left(t, v, \overline{v}, \widetilde{v}, Y\left(t^-\right)\right) - g\left(t, u, \overline{u}, \widetilde{u}, Y\left(t^-\right)\right)\right) I_{\{t\leq \tau_n\}}\right\|$$

$$\leq K_n\left(\|u - v\| + \|\overline{u} - \overline{v}\| + \|\widetilde{u} - \widetilde{v}\|_\nu\right)$$

$$(121)$$

for any $(u, \overline{u}, \widetilde{u})$ and $(v, \overline{v}, \widetilde{v}) \in R^q \times R^{q\times d} \times L^2_\nu(R_+^h, R^{q\times h})$, where K_n depending on n are positive constants. Then, the BSDE in (117) has a unique solution

$$\left(V, \overline{V}, \widetilde{V}\right) \in L^2_{\mathscr{F}}\left([0,T], R^q, P\right) \times L^2_{\mathscr{F},p}\left([0,T], R^{q\times d}, P\right)$$

$$\times L^2_P\left([0,T], R^{q\times h}, P\right),$$

$$(122)$$

where V is a Càdlàg process. The uniqueness is in the sense that if there exists another solution $(U, \overline{U}, \widetilde{U})$ as required, then

$$E\left[\int_0^T \left(\|U(t) - V(t)\|^2 + \|\overline{U}(t) - \overline{V}(t)\|^2 \right. \right.$$
$$\left. \left. + \|\widetilde{U}(t, \cdot) - \widetilde{V}(t, \cdot)\|_\nu^2\right) dt\right] = 0. \tag{123}$$

Proof. First, for each $n \in \{1, 2, \ldots\}$, we define

$$\tau_n \equiv \inf\{t > 0, \|L(\lambda t)\| > n\}. \tag{124}$$

Then, it follows from Theorem 3 in page 4 of Protter [39] and condition (12) that $\{\tau_n\}$ is a sequence of nondecreasing $\{\mathcal{F}_t\}$-stopping times and satisfies $\tau_n \to \infty$ a.s. as $n \to \infty$ since

$$P\{\tau_n \le t\} = P\{\|L(\lambda t)\| > n\}$$
$$\le \frac{E\left[\|L(\lambda t)\|^2\right]}{n^2} \longrightarrow 0 \tag{125}$$

as $n \to \infty$ for any given $t \in [0, \infty)$, where we have used (12), (49), (47), and the fact that $L(\lambda t)$ is h-dimensional nonnegative and nondecreasing Càdlàg process.

Second, for each n, consider the following BSDE with a random terminal time $\sigma_n \equiv T \wedge \tau_n$ and a terminal condition H_{τ_n}:

$$V(t) = H_{\tau_n} - \int_{t \wedge \sigma_n}^{\sigma_n} g\left(s, V\left(s^-\right), \overline{V}(s), \widetilde{V}(s, \cdot), Y\left(s^-\right)\right) ds$$
$$- \int_{t \wedge \sigma_n}^{\sigma_n} \sum_{i=1}^d \overline{V}_i(s) \, dW_i(s)$$
$$- \int_{t \wedge \sigma_n}^{\sigma_n} \sum_{i=1}^h \int_{z_i > 0} \widetilde{V}_i(s, z_i) \, \widetilde{N}_i(\lambda_i ds, dz_i). \tag{126}$$

Then, by slightly generalizing the discussion as in Yong and Zhou [46] and Tang and Li [47] (see also [48–50] for related discussions), we know that (126) has a unique adapted solution as required over $[0, \sigma_n]$.

Third, for each $n \in \{1, 2, \ldots\}$, let $\Omega_n = \{\omega \in \Omega : \sigma_n(\omega) = T\}$. Since σ_n is a sequence of nondecreasing stopping times and $\sigma_n \to T$ a.s. as $n \to \infty$, we have that $\Omega = \cup_{n=1}^\infty \Omega_n$ and $\Omega_l \subseteq \Omega_n$ whenever $l \le n$. Now, we use $\Pi^n(t, z) \equiv (V^n(t), \overline{V}^n(t), \widetilde{V}^n(t, z))$ for $t \le \sigma_n$ and $z \in R_+^h$ to denote the unique solution to (126) for each n. Since $H_{\tau_n}(\omega) = H(\omega)$ for all $\omega \in \{\omega : \tau_n(\omega) \ge T\}$, we know that $\Pi^n(t, z) = \Pi^{n-1}(t, z) = \cdots = \Pi^l(t, z)$ for all $t \le \sigma_l(\omega)$, a.s. $\omega \in \Omega_l$, and any $z \in R_+^h$. By the continuity of probability, we know that, for any given $\epsilon > 0$, there exists a sufficiently large $n_0 > 0$ such that $P\{\Omega_n\} > 1 - \epsilon$ when $n > n_0$. Thus, for any given $\delta > 0$ and for all $n, l > n_0$, we have

$$P\left\{\sup_{0 \le t \le T, z \in R_+^h} \|\Pi^n(t \wedge \sigma_n, z) - \Pi^l(t \wedge \sigma_l, z)\| > \delta\right\} < \epsilon; \tag{127}$$

that is, $\{\Pi^n(\cdot \wedge \sigma_n, \cdot), n \in \{1, 2, \ldots\}\}$ is uniformly Cauchy in probability. Thus, it is uniformly convergent in probability to a process $\Pi = \{\Pi(t, z), t \in [0, T], z \in R_+^h\}$. Therefore, we can extract a subsequence from $\{\Pi^n(\cdot \wedge \sigma_n, \cdot), n \in \{1, 2, \ldots\}\}$ such that the convergence holds uniformly a.s. Hence, we can conclude that Π is a solution to (117) and have all the properties as stated in the proposition. Furthermore, assume that $\Pi' = \{\Pi'(t, z), t \in [0, T], z \in R_+^h\}$ is another solution to (117). Then, we can conclude that, for all $n \ge l$, $\Pi'(t, z, \omega) = \Pi^l(t, z, \omega)$ for all $t \in [0, T]$, $z \in R_+^h$, and almost all $\omega \in \Omega_l$. In fact, if the claim fails to be true for some $n \ge l$, define $\Pi_n''(t, z, \omega) = \Pi'(t, z, \omega)$ for $\omega \in \Omega_l$ and $\Pi_n''(t, z, \omega) = \Pi^n(t, z, \omega)$ for $\omega \in \Omega_l^c$. Then, Π_n'' and Π^n are distinct solutions to (126) with the same terminal condition H_{τ_n}, which contradicts the uniqueness of solution to (126). Then, $P\{\Pi(t, z) = \Pi'(t, z)$ for all $t \in [0, T], z \in R_+^h\} = 1$ follows from a straightforward limiting argument as above. Furthermore, by applying the same argument as that used for Definition 2.4 and its associated remark in page 57 of Ikeda and Watanabe [40], we know that Π is the unique solution to (117) (interested readers are also referred to pages 309-310 of [31] for some related discussion). Hence, we complete the proof of Proposition 18. □

5.4. Remaining Proof of Theorem 5.

First of all, by Hölder's inequality and the same calculation as for (101), we have that

$$E\left[(H\mathscr{E}(-a \cdot D)(T))^2\right]$$
$$\le \left(E\left[H^4\right]\right)^{1/2} \left(E\left[(\mathscr{E}(-a \cdot D)(T))^4\right]\right)^{1/2} < \infty. \tag{128}$$

Thus, it follows from Jensen's inequality that the process $X = \{X(t), t \in [0, T]\}$ with

$$X(t) \equiv E\left[H\mathscr{E}(-a \cdot D)(T) \mid \mathcal{F}_t\right] \tag{129}$$

is a square-integrable martingale. Thus, by the martingale representation theorem (e.g., Lemma 2.3 in [47]), we have

$$X(t) = X(0) + \sum_{j=1}^d \int_0^t \overline{X}_j(s) \, dW_j(s)$$
$$+ \sum_{i=1}^h \int_0^t \int_{z_i > 0} \widetilde{X}_i(s, z_i) \, \widetilde{N}_i(\lambda_i ds, dz_i) \tag{130}$$

with $\overline{X} = (\overline{X}_1, \ldots, \overline{X}_d)' \in L^2_{\mathcal{F}, P}([0, T], R^d, P)$ and $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_h)' \in L^2_P([0, T], R^h, P)$. Furthermore, it follows from Bayes' rule (e.g., Lemma 8.6.2 in page 160 of [41]) and Proposition 11 that

$$X(t) = O_0 E\left[H\widehat{Z}(T) \mid \mathcal{F}_t\right] = O_0 \widehat{Z}(t) V(t), \tag{131}$$

where $V(t)$ is defined in (34). Thus, by the integration by parts formula (e.g., Corollary 2 in page 68 of [39]) and (130)-(131), we have

$$
dV(t) = \frac{1}{O_0} \left(X(t^-) d\left(\frac{1}{\widehat{Z}(t)} \right) + \frac{1}{\widehat{Z}(t^-)} dX(t) \right.
$$
$$
\left. + d\left[X, \frac{1}{\widehat{Z}} \right](t) \right)
$$
$$
= g\left(t, V(t^-), \overline{V}(t), \widetilde{V}(t, \cdot), Y(t^-) \right) dt \qquad (132)
$$
$$
+ \sum_{i=1}^{d} \overline{V}_i(t) \, dW_i(t)
$$
$$
+ \sum_{i=1}^{h} \int_{z_i > 0} \widetilde{V}_i(t, z_i) \, \widetilde{N}_i(\lambda_i dz_i, dt),
$$

where g is defined in (26) and

$$
\overline{V}_i(t) = -V(t^-) \overline{B}_i(Y(t^-)) + \frac{\overline{X}_i(t)}{O_0 \widehat{Z}(t^-)},
$$
$$
\text{for } i = 1, \dots, d,
$$
$$
\widetilde{V}_i(t, z_i) = -V(t^-) F(t, z_i) \overline{Z}(t) + \frac{\overline{X}_i(t, z_i)}{O_0 \widehat{Z}(t^-)},
$$
$$
\text{for } i = 1, \dots, h, \qquad (133)
$$

with $\overline{Z}$ given by (23). Hence, by (132), we know that V satisfies the BSDE (27).

Next, we check that $g(t, v, \overline{v}, \widetilde{v}, Y(t^-))$ defined in (26) satisfies the conditions as stated in Proposition 18. In fact, from (26), we see that $g(t, v, \overline{v}, \widetilde{v}, Y(t^-))$ is $\mathscr{F}_t$-adapted for any given $(v, \overline{v}, \widetilde{v}) \in R \times R^{1 \times d} \times L^2_\nu(R^h_+, R^{1 \times h})$ with $g(t, 0, 0, 0, Y(t^-)) \equiv 0 \in L^2_{\mathscr{F}}([0, T], R, P)$. Furthermore, for the sequence of nondecreasing stopping times $\{\tau_n, n = 1, 2, \dots\}$ as defined in (124), we have

$$
\left| \overline{Z}(t) \right| I_{\{t \le \tau_n\}} \le \overline{K}_n e^{\sum_{i=1}^{h}(2B_\rho/\lambda_i)\|L(\lambda t)\|} I_{\{t \le \tau_n\}} \le \widetilde{K}_n, \qquad (134)
$$

where $\overline{K}_n$ and $\widetilde{K}_n$ are positive constants depending on n. In addition, it follows from the proof of (87) that

$$
\left(\int_{z_i > 0} (F(t, z_i))^2 \, \nu_i(dz_i) \right) I_{\{t \le \tau_n\}}
$$
$$
\le \overline{L} e^{\sum_{i=1}^{h}(6 + 4B_\rho/\lambda_i)\|L(\lambda t)\|} I_{\{t \le \sigma_n\}} \le \widetilde{L}_n, \qquad (135)
$$

where $\overline{L}$ is some positive constant and $\widetilde{L}_n$ is a positive constant depending on n. Therefore, for any $(u, \overline{u}, \widetilde{u})$, $(v, \overline{v}, \widetilde{v}) \in R \times R^{1 \times d} \times L^2_\nu(R^h_+, R^{1 \times h})$, we have

$$
\left\| \left(g\left(t, u, \overline{u}, \widetilde{u}, Y(t^-) \right) - g\left(t, v, \overline{v}, \widetilde{v}, Y(t^-) \right) \right) I_{\{t \le \tau_n\}} \right\|
$$
$$
\le h \widetilde{K}_n^2 \widetilde{L}_n \|u - v\| + \|\overline{u} - \overline{v}\| \left(\frac{1}{2} \left(\rho\left(Y(t^-) \right) + d \right) \right) I_{\{t \le \tau_n\}}
$$

$$
+ h \lambda_i \widetilde{K}_n \left(\widetilde{L}_n \right)^{1/2} \|\widetilde{u} - \widetilde{v}\|_\nu
$$
$$
\le K_n \left(\|u - v\| + \|\overline{u} - \overline{v}\| + \|\widetilde{u} - \widetilde{v}\|_\nu \right), \qquad (136)
$$

where K_n is some positive constant depending on n and in the last inequality, we have used (63). Thus, all conditions stated in Proposition 18 are satisfied, which implies that (27) has a unique adapted solution.

Now, for each $t \in [0, T]$ and $B^K(t) = \int_0^t \rho(Y(s^-)) ds$, we define the density process

$$
Z^{P^*}(t) \equiv \frac{O(t)}{O_0 \mathscr{E}(B^K)(t)}. \qquad (137)
$$

Then, the corresponding probability $P^* \sim P$. Thus, it is the opportunity-neutral probability measure in the sense of Definition 3.16 in Cěrný and Kallsen [8]. Furthermore, by Corollary 8.7(b) and (8.19) in pages 135–138 of Jacod and Shiryaev [38], we can rewrite Z^{P^*} in (137) as

$$
Z^{P^*}(t) = \mathscr{E}(K)(t) \mathscr{E}(-B^K)(t) = \mathscr{E}(M^K)(t) \qquad (138)
$$

for each $t \in [0, T]$, where K is defined in (83) and M^K is defined in (86). Then, by a similar method to that used in the proof of Proposition 11 (2), we know that Z^{P^*} is a bounded positive martingale. Thus, for each pair of $i, j \in \{1, \dots, d\}$ and $t \in [0, T]$, we have

$$
\langle D_i, D_j \rangle^{P^*}(t) = \left[D_i, D_j \right]^{P^*}(t) = \left[D_i, D_j \right](t)
$$
$$
= \int_0^t \widetilde{c}_{ij}^{D^*}(s) \, ds, \qquad (139)
$$

where the first equality in (139) is owing to the continuity of D, Theorem 5.52 in page 55 of Jacod and Shiryaev [38], Theorem 4.47(c) in page 52 of Jacod and Shiryaev [38], the equivalence between P^* and P, and Girsanov-Meyer Theorem in page 132 of Protter [39]. The second equality follows from Theorem 4.47(a) in page 52 of Jacod and Shiryaev [38] and Girsanov-Meyer Theorem in page 132 of Protter [39]. Furthermore, $\widetilde{c}_{ij}^{D^*}$ in the last equality is defined in (31).

Now, note that D is continuous. Then, by Theorem 4.52 in page 55 of Jacod and Shiryaev [38] (or the proof of Corollary in page 83 of [39]), we know that $[D_i, V](t)$ and $[D_i, V]^c(t)$ for each $i \in \{1, \dots, d\}$ under P or P^* have the same compensator. Hence, we have

$$
\langle D_i, V \rangle^{P^*}(t) = \left(\langle D_i, V \rangle^c \right)^{P^*}(t) = \left([D_i, V]^c \right)^{P^*}(t)
$$
$$
= [D_i, V]^c(t) = \int_0^t \widetilde{c}_i^{DV^*}(s) \, ds, \qquad (140)
$$

where $\tilde{c}_i^{DV^*}$ is defined in (32). The last equality of (140) follows from Theorem 4.47(a) in page 52 of Jacod and Shiryaev [38] and the fact that

$$
\begin{aligned}
V(t) = V(0) + &\int_0^t g\left(s, V\left(s^-\right), \overline{V}(s), \widetilde{V}(s, \cdot), Y\left(s^-\right)\right) ds \\
&+ \int_0^t \sum_{i=1}^d \overline{V}_i(s) \, dW_i(s) \\
&+ \int_0^t \sum_{i=1}^h \int_{z_i>0} \widetilde{V}_i(s, z_i) \, \widetilde{N}_i\left(\lambda_i dz_i, ds\right).
\end{aligned}
$$

(141)

Then, it follows from (139)-(140), Definition 4.6, and (4.8) in Cěrný and Kallsen [8] that (30) is true.

Finally, the unique existence of solution to (33) is owing to Theorem 6.8 in Jacod [51] and the proofs of Lemma 4.9 and Theorem 4.10 in Cěrný and Kallsen [8]. Thus, by Theorem 4.10 in Cěrný and Kallsen [8], we know that the mean-variance hedge strategy is given by (29). Hence, we complete the proof of Theorem 5.

6. Conclusion

In this paper, we prove the global risk optimality of the hedging strategy explicitly constructed for an incomplete financial market. Owing to the discussions in Pigorsch and Stelzer [52] and references therein, our discussion in this paper can be extended to the cases that the external risk factors in (5) are correlated in certain manners. For the simplicity of notation, we keep the presentation of the paper in the current way. Furthermore, our study in this paper establishes the connection between our financial system and existing general semimartingale based study in Cěrný and Kallsen [8] since we can overcome the difficulties in Cěrný and Kallsen [8] by explicitly constructing the process N and the VOMM Q^*. In addition, our objective and discussion in this paper are different from the recent study of Jeanblanc et al. [10] since the authors in Jeanblanc et al. [10] did not aim to derive any concrete expression. Nevertheless, interested readers may make an attempt to extend the study in Jeanblanc et al. [10] and apply it to our financial market model to construct the corresponding explicit results. Finally, unlike the studies in Hubalek et al. [29] and Kallsen and Vierthauer [18], our option H is generally related to a multivariate terminal function and hence a BSDE involved approach is employed. Interested readers may take an attempt to study whether the Laplace transform related method developed in Hubalek et al. [29] and Kallsen and Vierthauer [18] for single-variate terminal function can be extended to our general multivariate case.

Disclosure

Partial results and graphs are briefly summarized and reported in 2012 Spring World Congress of Engineering and Technology. This enhanced version with extension and complete proofs of results is a journal version of the short conference report.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

References

[1] O. E. Barndorff-Nielsen and N. Shephard, "Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics," *Journal of the Royal Statistical Society. Series B. Statistical Methodology*, vol. 63, no. 2, pp. 167–241, 2001.

[2] F. E. Benth, K. H. Karlsen, and K. Reikvam, "Merton's portfolio optimization problem in a Black and Scholes market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type," *Mathematical Finance*, vol. 13, no. 2, pp. 215–244, 2003.

[3] F. E. Benth and T. Meyer-Brandis, "The density process of the minimal entropy martingale measure in a stochastic volatility model with jumps," *Finance and Stochastics*, vol. 9, no. 4, pp. 563–575, 2005.

[4] C. Lindberg, "News-generated dependence and optimal portfolios for n stocks in a market of Barndorff-Nielsen and Shephard type," *Mathematical Finance*, vol. 16, pp. 549–568, 2006.

[5] Ł. Delong and C. Klüppelberg, "Optimal investment and consumption in a Black-Scholes market with Lévy-driven stochastic coefficients," *The Annals of Applied Probability*, vol. 18, no. 3, pp. 879–908, 2008.

[6] F. Delbaen and W. Schachermayer, "A general version of the fundamental theorem of asset pricing," *Mathematische Annalen*, vol. 300, no. 3, pp. 463–520, 1994.

[7] J. M. Harrison and D. M. Kreps, "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, vol. 20, no. 3, pp. 381–408, 1979.

[8] A. Cěrný and J. Kallsen, "On the structure of general mean-variance hedging strategies," *The Annals of Probability*, vol. 35, no. 4, pp. 1479–1531, 2007.

[9] W. Dai, "Optimal hedging and its performance based on a Lévy driven volatility model," in *Proceedings of International Conference on Applied Mathematics and Sustainable Development—Special Track within 2012 Spring World Congress of Engineering and Technology*, pp. 44–49, Scientific Research Publishing, 2012.

[10] M. Jeanblanc, M. Mania, M. Santacroce, and M. Schweizer, "Mean-variance hedging via stochastic control and BSDEs for general semimartingales," *The Annals of Applied Probability*, vol. 22, no. 6, pp. 2388–2428, 2012.

[11] H. Föllmer and D. Sondermann, "Hedging of non-redundant contingent claims," in *Contributions to Mathematical Economics*, M. Mas-Colell and W. Hildebrand, Eds., pp. 205–223, North Holland, Amsterdam, The Netherlands, 1986.

[12] H. Föllmer and M. Schweizer, "Hedging of contingent claims under incomplete information," in *Applied Stochastic Analysis (London 1989)*, vol. 5 of *Stochastics Monogr*, pp. 389–414, Gordon and Breach, New York, NY, USA, 1991.

[13] M. Schweizer, "Option hedging for semimartingales," *Stochastic Processes and Their Applications*, vol. 37, no. 2, pp. 339–363, 1991.

[14] M. Schweizer, "Mean-variance hedging for general claims," *The Annals of Applied Probability*, vol. 2, no. 1, pp. 171–179, 1992.

[15] H. Fäollmer and M. Schweizer, "The minimal martingale measure," in *Encyclopedia of Quantitative Finance*, R. Cont, Ed., pp. 1200–1204, John Wiley & Sons, 2010.

[16] M. Schweizer, "A guided tour through quadratic hedging approaches," in *Option Pricing, Interest Rates and Risk Management*, E. Jouini, J. Cvitanic, and M. Musiela, Eds., pp. 538–574, Cambridge University Press, 2001.

[17] H. Pham, "On quadratic hedging in continuous time," *Mathematical Methods of Operations Research*, vol. 51, no. 2, pp. 315–339, 2000.

[18] J. Kallsen and R. Vierthauer, "Quadratic hedging in affine stochastic volatility models," *Review of Derivatives Research*, vol. 12, no. 1, pp. 3–27, 2009.

[19] T. Arai, "An extension of mean-variance hedging to the discountinuous case," *Finance & Stochastics*, vol. 9, no. 1, pp. 129–139, 2005.

[20] T. Chan, J. Kollar, and A. Wiese, "The variance-optimal martingale measure in Lévy models with stochastic volatility," *International Journal of Theoretical and Applied Finance*. In press.

[21] D. Duffie and H. R. Richardson, "Mean-variance hedging in continuous time," *Annals of Applied Probability*, vol. 1, no. 1, pp. 1–15, 1991.

[22] C. Gourieroux, J. P. Laurent, and H. Pham, "Mean-variance hedging and numérair," *Mathematical Finance*, vol. 8, no. 3, pp. 179–200, 1998.

[23] D. Heath, E. Platen, and M. Schweizer, "A comparison of two quadratic approaches to hedging in incomplete markets," *Mathematical Finance*, vol. 11, no. 4, pp. 385–413, 2001.

[24] J. P. Laurent and H. Pham, "Dynamic programming and mean-variance hedging," *Finance and Stochastics*, vol. 3, no. 1, pp. 83–110, 1999.

[25] A. Cěrný and J. Kallsen, "Mean-variance hedging and optimal investment in Heston's model with correlation," *Mathematical Finance*, vol. 18, no. 3, pp. 473–492, 2008.

[26] D. Heath and M. Schweizer, "Martingales versus PDEs in finance: an equivalence result with examples," *Journal of Applied Probability*, vol. 37, no. 4, pp. 947–957, 2000.

[27] A. F. Roch, "Viscosity solutions and American option pricing in a stochastic volatility model of the Ornstein-Uhlenbeck type," *Journal of Probability and Statistics*, vol. 2010, Article ID 863585, 18 pages, 2010.

[28] W. Dai, "Numerical methods and analysis via random field based Malliavin calculus for backward stochastic PDEs," http://arxiv.org/abs/1306.6770.

[29] F. Hubalek, J. Kallsen, and L. Krawczyk, "Variance-optimal hedging for processes with stationary independent increments," *The Annals of Applied Probability*, vol. 16, no. 2, pp. 853–885, 2006.

[30] W. Dai, "Mean-variance portfolio selection based on a generalized BNS stochastic volatility model," *International Journal of Computer Mathematics*, vol. 88, no. 16, pp. 3521–3534, 2011.

[31] D. Applebaum, *Lévy Processes and Stochastic Calculus*, vol. 93 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 2004.

[32] J. Bertoin, *Lévy Processes*, Cambridge University Press, Cambridge, UK, 1996.

[33] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, vol. 68 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, Cambridge, UK, 1999.

[34] O. Kallenberg, *Foundations of Modern Probability*, Springer, New York, NY, USA, 1997.

[35] W. Dai, "On the conflict of truncated random variable vs. heavy-tail and long range dependence in computer and network simulation," *Journal of Computational Information Systems*, vol. 7, no. 5, pp. 1488–1499, 2011.

[36] W. Dai, "Heavy traffic limit theorems for a queue with Poisson ON/OFF long-range dependence sources and general service time distribution," *Acta Mathematicae Applicatae Sinica*, vol. 28, no. 4, pp. 807–822, 2012.

[37] W. Dai, "Optimal rate scheduling via utility-maximization for *J*-user MIMO Markov fading wireless channels with cooperation," *Operations Research*, vol. 61, no. 6, pp. 1450–1462, 2013.

[38] J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer, Berlin, Germany, 2nd edition, 2003.

[39] P. E. Protter, *Stochastic Integration and Differential Equations*, vol. 21 of *Applications of Mathematics*, Springer, New York, NY, USA, 2nd edition, 2004.

[40] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Kodansha/North-Holland, 2nd edition, 1989.

[41] B. Øksendal, *Stochastic Differential Equations*, Springer, New York, NY, USA, 6th edition, 2005.

[42] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, New York, NY, USA, 1984.

[43] B. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Springer, Berlin, Germany, 2005.

[44] H. L. Royden, *Real Analysis*, Macmillan Publishing Company, New York, NY, USA, 3rd edition, 1988.

[45] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, New York, NY, USA, 2nd edition, 2000.

[46] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, vol. 43, Springer, New York, NY, USA, 1999.

[47] S. J. Tang and X. J. Li, "Necessary conditions for optimal control of stochastic systems with random jumps," *SIAM Journal on Control and Optimization*, vol. 32, no. 5, pp. 1447–1475, 1994.

[48] N. El Karoui, S. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Mathematical Finance*, vol. 7, no. 1, pp. 1–71, 1997.

[49] S. Rong, "On solutions of backward stochastic differential equations with jumps and applications," *Stochastic Processes and Their Applications*, vol. 66, no. 2, pp. 209–236, 1997.

[50] J. Yin and X. Mao, "The adapted solution and comparison theorem for backward stochastic differential equations with Poisson jumps and applications," *Journal of Mathematical Analysis and Applications*, vol. 346, no. 2, pp. 345–358, 2008.

[51] J. Jacod, *Calcul Stochastique et Problèmes de Martingales*, vol. 714 of *Lecture Notes in Mathematics*, Springer, Berlin, 1979.

[52] C. Pigorsch and R. Stelzer, "On the definition, stationary distribution and second order structure of positive semidefinite Ornstein-Uhlenbeck type processes," *Bernoulli*, vol. 15, no. 3, pp. 754–773, 2009.