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DIRECT AND INVERSE APPROXIMATION THEOREMS FOR THE p-VERSION OF THE FINITE ELEMENT METHOD IN THE FRAMEWORK OF WEIGHTED BESOV SPACES. PART I: APPROXIMABILITY OF FUNCTIONS IN THE WEIGHTED BESOV SPACES*

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Abstract. This is the first of a series devoted to the approximation theory of the p-version of the finite element method in two dimensions in the framework of the Jacobi-weighted Besov spaces, which provides the p-version with a solid mathematical foundation. In this paper, we establish a mathematical framework of the Jacobi-weighted Besov and Sobolev spaces and analyze the approximability of the functions in the framework of these spaces, particularly, singular functions of r^γ-type and $r^\gamma \log^\nu r$-type. These spaces and the corresponding approximation properties are of fundamental importance to the proof of the optimal convergence for the p-version in two dimensions in part II and to various sharp inverse approximation theorems in part III.

Key words. Jacobi-weighted Besov spaces, modified Jacobi-weighted Besov spaces, Jacobi weights, singular function of r^γ-type and $r^\gamma \log^\nu r$-type, approximability, the p-version of the finite element method

AMS subject classifications. 65N30, 65N25, 35D10

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1. Introduction.

Since the late 1970s the p-version of the finite element method, which increases the degree of elements on a fixed mesh to obtain desired accuracy, is widely used in engineering computations. There are several commercial and research codes based on the p- (or h-p) versions of the finite element method, for example, MSC/PROBE, FIESTA, MECHANICA, PHLEX, STRESSCHECK, and STRIPE.

In 1980 it was shown that the p-version in two dimensions converges at least as fast as the traditional h-version with quasi-uniform meshes and that it converges twice as fast as the h-version if the solution has singularity of r^γ-type. In [10] an upper bound $O(p^{-2\gamma+\varepsilon})$ was proven, where p is the degree of elements and $\varepsilon > 0$ is arbitrary. In [8, 9] the ε was removed. The convergence of the p-version in three dimensions was addressed in [14, 15]. A detailed analysis of the p-version in one dimension is available in [17]. The p-version is very close to the spectral method which was independently studied and developed; see, e.g., [12] and references therein.

Although significant progress has been made in the past two decades, several important issues of the p-version in two and three dimensions are still not resolved, for example, the lower bound of error and the optimal convergence rate in energy norm for the solution of practical engineering interest, inverse approximation theorems, and the effective a posteriori error estimation and adaptive selection of polynomial (incomplete) shape functions.

In this paper we analyze the approximation of the functions which has singular behavior of $r^\gamma \log^\nu r$-type. This is a typical singularity occurring in the neighborhood

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of the corners. We proved the optimal rate of convergence depending on γ and ν for γ real or integer.

We have established a framework in two dimensions which can be generalized into three dimensions. Some main ideas and results without detailed proofs were addressed in [3]. In the present paper we elaborate more on these ideas and present detailed proofs which could serve as a starting point for three-dimensional analysis. The main idea, which appeared to be very effective, is the concept of the Jacobi- weighted Besov and Sobolev spaces $B^{s,\beta}$ and $H^{s,\beta}$ with weights depending on s and the order of derivatives under consideration. In contrast to the approximation in L^2-norm, the approximation in the H^1-norm is more complicated because the H^1-norm involves the L^2-norm of derivatives in one variable only. We have shown that a judicious selection of the weight in the spaces under consideration will overcome this difficulty.

In [1, 12] and references cited therein the weighted Sobolev spaces H^s_α with Jacobi-type weights, which are independent of s, are utilized. The application to differential equations are not addressed in details in these references. In contrast, in this paper, we relate the weight function to the sharpest characterization of the smoothness of the solution by using the maximal value of s. Recently, the general Jacobi approximation with nonsymmetric and varying weights was studied in [19, 20, 21, 22] and was applied to singular differential equations with coefficients which degenerate. The analysis in the mentioned papers assumes the regularity of the exact solution and does not address practical important cases that the domain has corners or the boundary conditions are changing the type. In practical computations, singularities of this type are always present, and they can govern the accuracy of the p-version. The optimal error estimates in H^1-norm when the solution has singularity of r^γ- and $r^\gamma \log^\nu r$-type were not addressed in either [1, 12] or [19, 20, 21, 22].

The problem of the approximation by polynomials is a classical problem addressed directly and indirectly in many papers and books; see, e.g., [13, 24] and others. Various abstract results especially related to functional analysis are available; see, e.g., [25]. Nevertheless, the concrete results related to the p-version of the finite element method are not available in the literature.

In this paper we present a mathematical framework and detailed proof of essential theorems for the analysis of the p-version, which will be utilized in the second and third paper of the series. The scope of the paper is as follows. In section 2 we introduce the Jacobi-weighted Besov spaces $B^{s,\beta}(Q)$ and Sobolev space $H^{s,\beta}(Q)$, with $Q = (-1,1)^2$, and analyze the approximability of functions of the r^γ-type, with $\gamma > 0$, in terms of the space $B^{s,\beta}(Q)$. The modified Jacobi-weighted Besov space $B^{s,\beta}_\nu(Q)$ is introduced in section 3 to effectively analyze the approximability of functions of $r^\gamma \log^\nu r$-type. Unlike the space $B^{s,\beta}(Q)$, the space $B^{s,\beta}_\nu(Q)$ is not exact interpolation space, but only a uniform interpolation space according to the definitions of [11]. Various properties, which stand for exact interpolation spaces, have been carefully examined and strictly proved for these modified spaces, in particular, the partial reiteration theorem. Some concluding remarks are given in the last section on the effectiveness of the Sobolev space H^s, the Besov space B^s, and the Jacobi-weighted Besov spaces $B^{s,\beta}$ and $B^{s,\beta}_\nu$ for the analysis of the h-version and the p-version of the finite element method.

2. Jacobi-weighted Besov space $B^{s,\beta}(Q)$ and approximability of singular function of r^γ-type.

We shall introduce Jacobi-weighted Besov and Sobolev spaces $B^{s,\beta}(Q)$ and $H^{k,\beta}(Q)$ and characterize the singularity and analyze the approximability for functions of r^γ-type in the framework of the spaces $B^{s,\beta}(Q)$.

2.1. Sobolev and Besov spaces with Jacobi weights. Let $Q = I^2 = (-1,1)^2$, and let

$$w_{\alpha,\beta}(x) = \prod_{i=1}^{2}(1-x_i^2)^{\alpha_i+\beta_i} \tag{2.1}$$

be a weight function with integer $\alpha_i \geq 0$ and real number $\beta_i > -1$, which is referred to as Jacobi weight. Obviously, the Jacobi polynomials and their derivatives are orthogonal with the weight $w_{\alpha,\beta}(x)$.

The Sobolev space $H^{k,\beta}(Q)$ is defined as a closure of C^∞ functions in the norm with the Jacobi weight

$$||u||^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|=0}^{k} \int_Q |D^\alpha u|^2 \; w_{\alpha,\beta}(x)\, dx, \tag{2.2}$$

where $D^\alpha u = u_{x_1^{\alpha_1} x_2^{\alpha_2}}$, $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$, and $\beta = (\beta_1, \beta_2)$. By $|u|_{H^{k,\beta}(Q)}$ we denote the seminorm,

$$|u|_{H^{k,\beta}(Q)} = \sum_{|\alpha|=k} \int_Q |D^\alpha u|^2 \; \prod_{i=1}^{2}(1-x_i^2)^{\alpha_i+\beta_i}\, dx.$$

Let $\mathcal{B}^{s,\beta}_{2,q}(Q)$ be the interpolation spaces defined by the K-method

$$\Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,q},$$

where $0 < \theta < 1, 1 \leq q \leq \infty, s = (1-\theta)\ell + \theta k$, ℓ and k are integers, $\ell < k$, and

$$||u||_{\mathcal{B}^{s,\beta}_{2,q}(Q)} = \left(\int_0^\infty t^{-q\theta}\, |K(t,u)|^q \frac{dt}{t}\right)^{1/q}, \qquad 1 \leq q < \infty, \tag{2.3a}$$

$$||u||_{\mathcal{B}^{s,\beta}_{2,\infty}(Q)} = \sup_{t>0} t^{-\theta}\, K(t,u), \tag{2.3b}$$

where

$$K(t,u) = \inf_{u=v+w} \Big(||v||_{H^{\ell,\beta}(Q)} + t||w||_{H^{k,\beta}(Q)}\Big). \tag{2.4}$$

In particular, we are interested in the cases $q = 2$ and $q = \infty$. For $q = 2$, we have a theorem on the relation between $\mathcal{B}^{s,\beta}_{2,2}(Q)$ and $H^{m,\beta}(Q)$ if s is an integer m.

THEOREM 2.1. *$\mathcal{B}^{s,\beta}_{2,2}(Q) = H^{m,\beta}(Q)$ for $s = m$, an integer.*

Proof. For the sake of simplicity we shall prove the theorem in one dimension, and the proof in two dimensions is similar.

For any $u \in H^{\ell,\beta}(Q), \ell \geq 0$, there holds

$$u(x) = \sum_{i=0}^{\infty} a_i\, P_i(x,\beta),$$

where

$$P_i(x,\beta) = \frac{(-1)^i}{i!\, 2^i}(1-x^2)^{-\beta} \frac{d^i(1-x^2)^{i+\beta}}{dx^i}$$

is the Jacobi polynomial of degree i. Let $u = v + w$ with

$$w(x) = \sum_{i=0}^{\infty} d_i\, P_i(x,\beta) \in H^{k,\beta}(Q), \qquad k > m,$$

and

$$v(x) = \sum_{i=0}^{\infty} (a_i - d_i)\, P_i(x,\beta) \in H^{\ell,\beta}(Q), \qquad \ell < m,$$

where d_i's are undetermined. By the properties of Jacobi polynomials, we have

$$\begin{aligned} K^2(t,u) &\leq 2\Big(\inf_{u=v+w} ||v||^2_{H^{\ell,\beta}(Q)} + t^2 ||w||^2_{H^{k,\beta}(Q)}\Big) \\ &\leq 2 \inf_{d_0,d_1,\ldots} \sum_{i=0}^{\infty} \frac{(a_i - d_i)^2 i^{2\ell}}{2i+2\beta+1} + \frac{t^2 d_i^2 i^{2k}}{2i+2\beta+1} \\ &= 2 \inf_{d_0,d_1,\ldots} G(d_0, d_1, \ldots) \\ &= 2\tilde{G}(t,u). \end{aligned}$$

It is easy to see that $G(d_0, d_1, \ldots)$ reaches its minimum $\tilde{G}(t,u)$ at $d_i = \frac{a_i}{1+t^2 g^2_{\ell k}(i)}, i = 0, 1, \ldots,$ and

$$\tilde{G}(t,u) = \sum_{i=0}^{\infty} \frac{a_i^2 t^2 g^2_{\ell k}(i) i^{2\ell}}{(2i+2\beta+1)(1+t^2 g^2_{\ell k}(i))},$$

where $g_{\ell k}(i) = i^{(k-\ell)}$. Thus

$$\begin{aligned} \Phi^2_{\theta,2}(u) &= \int_0^{\infty} t^{-2\theta} K^2(t,u) \frac{dt}{t} \\ &\leq 2 \sum_{i=0}^{\infty} \frac{a_i^2 g^2_{\ell k}(i) i^{2\ell}}{(2i+2\beta+1)} \int_0^{\infty} \frac{t^{1-2\theta}}{1+t^2 g^2_{\ell k}(i)} dt. \end{aligned}$$

Noting that

$$\begin{aligned} \int_0^{\infty} \frac{t^{1-2\theta}}{1+t^2 g^2_{\ell k}(i)} dt &\leq \int_0^{\frac{1}{g_{\ell k}(i)}} t^{1-2\theta} dt + \frac{1}{g^2_{\ell k}(i)} \int_{\frac{1}{g_{\ell k}(i)}}^{\infty} t^{-1-2\theta} dt \\ &\leq 2 g^{2(\theta-1)}_{\ell k}(i) \end{aligned}$$

we have

$$\begin{aligned} ll\Phi^2_{\theta,2}(u) &\leq 4 \sum_{i=0}^{\infty} \frac{a_i^2 g^{2\theta}_{\ell k}(i) i^{2\ell}}{2i+2\beta+1} \\ &\leq 4 \sum_{i=0}^{\infty} \frac{a_i^2 i^{2((1-\theta)\ell+\theta k)}}{2i+2\beta+1} \\ &\leq 4\|u\|^2_{H^{m,\beta}(Q)}. \end{aligned} \tag{2.5}$$

On the other hand, there holds

$$K^2(t,u) \geq \tilde{G}(t,u)$$

and

$$\begin{aligned}\int_0^\infty \frac{t^{1-2\theta}}{1+t^2 g_{\ell k}^2(i)}dt &\geq \frac{1}{2}\left(\int_0^{\frac{1}{g_{\ell k}(i)}} t^{1-2\theta}dt + \frac{1}{g_{\ell k}^2(i)}\right)\int_{\frac{1}{g_{\ell k}(i)}}^\infty t^{-1-2\theta}dt \\ &\geq g_{\ell k}^{2(\theta-1)}(i),\end{aligned}$$

which leads to

$$\Phi_{\theta,2}^2(u) \geq C\sum_{i=0}^\infty \frac{a_i^2 i^{2((1-\theta)\ell+\theta k)}}{2i+2\beta+1} \geq \|u\|_{H^{m,\beta}(Q)}^2. \tag{2.6}$$

Combining (2.3), (2.5), and (2.6) we obtain

$$C_1\|u\|_{H^{m,\beta}(Q)} \leq \|u\|_{\mathcal{B}_{2,2}^{m,\beta}(Q)} \leq C_2\|u\|_{H^{m,\beta}(Q)},$$

which completes the proof. $\square$

Remark 2.1. Theorems of this type may follow from the abstract theory presented, for example, in [25]. Nevertheless, we present here a direct and simple proof.

Remark 2.2. Due to Theorem 2.1 we shall write for $s \geq 0$ and $q = 2$

$$H^{s,\beta}(Q) = \mathcal{B}_{2,2}^{s,\beta}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,2}$$

with $0 < \theta < 1$ and $s = (1-\theta)\ell + \theta k$. This space is called Jacobi-weighted space with fractional order if s is not an integer, and it coincides with $H^{m,\beta}(Q)$ if s is an integer m. The above definition of $H^{s,\beta}(Q)$ is independent of the selection of ℓ and k. Furthermore, ℓ and k do not have to be integers.

Remark 2.3. According to the arguments of the proof, we can introduce an equivalent norm in discrete form for functions in $H^{s,\beta}(Q)$:

$$|||u|||_{H^{s,\beta}(Q)}^2 = \sum_{0\leq m_1,m_2<\infty} |c_{m_1,m_2}|^2 \prod_{i=1}^2 \frac{2^{2\beta_i+1}}{2m_i+2\beta_i+1}\left(1+\sum_{i=1}^2 m_i^{2s}\right),$$

where c_{m_1,m_2} are the coefficients of the Jacobi–Fourier expansion of u,

$$u = \sum_{0\leq m_1,m_2\leq\infty} c_{m_1,m_2}\prod_{i=1}^2 P_{m_i}(x_i,\beta_i).$$

Similarly, the Jacobi-weighted Besov spaces can be introduced as interpolations between spaces $H^{\ell,\beta}(Q)$ and $H^{k,\beta}(Q)$ as in (2.4), but furnished with the above discrete norms in (2.4), instead of $\|u\|_{H^{\ell,\beta}(Q)}$ and $\|u\|_{H^{m,\beta}(Q)}$.

Remark 2.4. For $q = \infty$, we shall write

$$B^{s,\beta}(Q) = \mathcal{B}_{2,\infty}^{s,\beta}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty},$$

which are referred to as the Jacobi-weighted Besov spaces. It is an exact interpolation space according to [11], and all properties of exact interpolation spaces stand for the space $B^{s,\beta}(Q)$. It can be defined as an interpolation between two fractional order Jacobi-weighted Sobolev spaces.

Remark 2.5. The Jacobi-weighted Besov spaces $B^{s,\beta}(Q)$ and Sobolev spaces $H^{k,\beta}(Q)$ can be generalized with nonsymmetric weights

$$w_{\alpha,\beta}(x) = \prod_{i=1}^{2}(1-x_i)^{\alpha_i+\beta_{i,1}}(1+x_i)^{\alpha_i+\beta_{i,2}}$$

with $\beta = (\beta_1, \beta_2), \beta_i = (\beta_{i,1}, \beta_{i,2}), \beta_{i,j} > -1, i, j = 1, 2$. All properties proved the Jacobi-weighted Besov and Sobolev spaces with symmetric weights will stand for those with nonsymmetric weights.

Let $P_p(Q)$ be set of all polynomials of (separate) degree $\leq p$, and let u_p be the $H^{0,\beta}(Q)$-projection of a function $u(x) \in H^{k,\beta}(Q)$ on $P_p(Q)$. Then we have the following approximation property.

THEOREM 2.2. *Let $u \in H^{k,\beta}(Q)$ with integer $k \geq 1$, $\beta_i > -1, i = 1, 2$, and let u_p be its $H^{0,\beta}(Q)$-projection on $P_p(Q)$. Then we have for integer $\ell \leq k \leq p+1$*

$$|u - u_p|_{H^{\ell,\beta}(Q)} \leq C\, p^{-(k-\ell)}\, |u|_{H^{k,\beta}(Q)}. \tag{2.7}$$

Proof. Let $P_m(\xi, \nu, \mu) = \frac{(-1)^m}{m!\,2^m}(1-\xi)^{-\nu}(1+\xi)^{-\mu}\frac{d^m(1-\xi)^{\nu+m}(1+\xi)^{\mu+m}}{d\xi^m}$, with $\nu, \mu > -1$, be the Jacobi polynomial of degree m, and let $P_m(\xi, \nu) = P_m(\xi, \nu, \nu)$. For $u \in H^{k,\beta}(Q)$, $k \geq 0$, we have the Jacobi–Fourier expansion

$$u(x) = \sum_{i,j=0}^{\infty} C_{ij}\, P_i(x_1, \beta_1)\, P_j(x_2, \beta_2).$$

Then

$$u_p(x) = \sum_{i,j=0}^{p} C_{ij}\, P_i(x_1, \beta_1)\, P_j(x_2, \beta_2)$$

is the projection of $u(x)$ on $P_p(Q)$ in $H^{0,\beta}(Q)$, and

$$u - u_p = \left(\sum_{i=0}^{\infty}\sum_{j=p+1}^{\infty} + \sum_{i=p+1}^{\infty}\sum_{j=0}^{p}\right) C_{ij}\, P_i(x_1, \beta_1)\, P_j(x_2, \beta_2) = U + V.$$

By the property of the Jacobi polynomials (see [16]), we have for $|\alpha| \leq \ell \leq k \leq p+1$

$$\begin{aligned}
&\int_Q |D^\alpha U|^2\, (1-x_1^2)^{\alpha_1+\beta_1}\, (1-x_2^2)^{\alpha_2+\beta_2}\, dx \\
&= \sum_{i=\alpha_1}^{\infty}\sum_{j=p+1}^{\infty} |C_{ij}|^2\, 2^{2(\beta_1+\beta_2+1)}\, \frac{\Big[(i+2\beta_1+1)\cdots(i+2\beta_1+\alpha_1)\Big]^2\,\Gamma^2(i+\beta_1+1)}{(i-\alpha_1)!\,(2i+2\beta_1+1)\Gamma(i+\alpha_1+2\beta_1+1)} \\
&\quad\times \frac{\Big[(j+2\beta_2+1)\cdots(j+2\beta_2+\alpha_2)\Big]^2\,\Gamma^2(j+\beta_2+1)}{(j-\alpha_2)!\,(2j+2\beta_2+1)\Gamma(j+\alpha_2+2\beta_2+1)}.
\end{aligned}$$

On the other hand, we have, by noting that $\alpha_2 + k - \ell \leq p+1$ for $|\alpha| = \ell \leq k \leq p+1$,

$$\int_Q |U_{x_1^{\alpha_1} x_2^{\alpha_2+k-\ell}}|^2 \,(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+k-\ell+\beta_2}\,dx$$

$$\geq \sum_{i=\alpha_1}^{\infty}\sum_{j=p+1}^{\infty} |C_{ij}|^2\, 2^{2(\beta_1+\beta_2+1)}\, \frac{\Big[(i+2\beta_1+1)\cdots(i+2\beta_1+\alpha_1)\Big]^2\,\Gamma^2(i+\beta_1+1)}{(i-\alpha_1)!\,(2i+2\beta_1+1)\Gamma(i+\alpha_1+2\beta_1+1)}$$

$$\times \frac{\Big[(j+2\beta_2+1)\cdots(j+2\beta_2+\alpha_2)\Big]^2\,\Gamma^2(j+\beta_2+1)}{(j-\alpha_2)!\,(2j+2\beta_2+1)\Gamma(j+\alpha_2+2\beta_2+1)}\Psi(j),$$

where

$$\Psi(j) = \frac{\Big[(j+2\beta_2+\alpha_2+1)\cdots(j+2\beta_2+k-\ell)\Big]^2\,(j-\alpha_2)!\,\Gamma(j+\alpha_2+2\beta_2+1)}{(j-\alpha_2-(k-\ell))!\,\Gamma(j+\alpha_2+(k-\ell)+2\beta_2+1)}.$$

It is easy to verify that for $j \geq p+1$

$$\Psi(j) \geq C\,p^{2(k-\ell)},$$

which leads to

(2.8)
$$\begin{aligned}\int_Q |D^\alpha U|^2\,(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2}\,dx \\ &\leq C\,p^{-2(k-\ell)} \int_Q |u_{x_1^{\alpha_1} x_2^{\alpha_2+k-\ell}}|^2\,(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2+k-\ell}\,dx \\ &\leq C\,p^{-2(k-\ell)}\,|u|^2_{H^{k,\beta}(Q)}.\end{aligned}$$

Similarly, we have for $|\alpha| = \ell \leq k \leq p+1$

(2.9)
$$\begin{aligned}\int_Q |D^\alpha V|^2\,(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2}\,dx \\ &\leq C\,p^{-2(k-\ell)} \int_Q |u_{x_1^{\alpha_1+k-\ell} x_2^{\alpha_2}}|^2\,(1-x_1^2)^{\alpha_1+k-\ell+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2}\,dx \\ &\leq C\,p^{-2(k-\ell)}\,|u|^2_{H^{k,\beta}(Q)}.\end{aligned}$$

It follows from (2.8) and (2.9) that for $|\alpha| = \ell \leq k \leq p+1$

$$\int_Q |D^\alpha(u-u_p)|^2\,(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2}\,dx \leq C\,p^{-2(k-\ell)}\,|u|^2_{H^{k,\beta}(Q)},$$

which yields (2.7), and completes the proof. □

Remark 2.6. The above proof shows that if φ is the Jacobi projection of $u \in H^{k,\beta}(Q)$ on $P_p(Q)$ in $H^{0,\beta}(Q)$, then φ is the Jacobi projection of u on $P_p(Q)$ in $H^{\ell,\beta}(Q)$ for all $0 \leq \ell \leq k$, and

$$|\varphi|^2_{H^{\ell,\beta}(Q)} + |u-\varphi|^2_{H^{\ell,\beta}(Q)} = |u|^2_{H^{\ell,\beta}(Q)}.$$

This is a very important and special property of the Jacobi projection.

By a standard argument of interpolation spaces, we are able to generalize Theorem 2.2 to an approximation theorem for functions in the Jacobi-weighted Besov spaces $B^{s,\beta}(Q)$.

THEOREM 2.3. *Let $u \in B^{s,\beta}(Q)$, $s > 0$ with $\beta_i > -1, i = 1, 2$, and let u_p be the Jacobi projection of u on $P_p(Q)$ with $p + 1 \geq s$. Then for any integer $\ell < s$ there holds*

$$||u - u_p||_{H^{\ell,\beta}(Q)} \leq C\, p^{-(s-\ell)}\, ||u||_{B^{s,\beta}(Q)} \tag{2.10}$$

with constant C independent of p.

2.2. Characterization of functions of r^γ-type in terms of weighted Besov spaces $B^{s,\beta}(Q)$. Let (r, θ) be the polar coordinates with respect to the vertex $(-1, -1)$, where $r = \{(x_1 + 1)^2 + (x_2 + 1)^2\}^{1/2}$, $\theta = \arctan(\frac{x_2+1}{x_1+1})$, and let for $\gamma > 0$

$$u(x) = r^\gamma\, \chi(r)\, \Phi(\theta) \tag{2.11}$$

be a function defined on $Q = (-1, 1)^2$, where $\chi(r)$ and $\Phi(\theta)$ are C^∞ functions such that for $0 < r_0 < 2$

$$\chi(r) = \begin{cases} 1 & \text{for} \quad 0 < r \leq \frac{r_0}{2}, \\ 0 & \text{for} \quad r \geq r_0, \end{cases} \tag{2.12}$$

and for $\theta_0 \in (0, \pi/2)$

$$\Phi(\theta) = 0 \qquad \text{for} \quad \theta \notin (\theta_0, \pi/2 - \theta_0). \tag{2.13}$$

Therefore, $u(x)$ has a support $R_0 = R_{r_0,\theta_0}$ with

$$R_{r_0,\theta_0} = \left\{ x \in Q \;\middle|\; r < r_0, \quad \theta_0 < \theta < \pi/2 - \theta_0 \right\}, \tag{2.14}$$

which is shown in Figure 2.1. For $x \in R_0$, there hold

$$1 - r_0 < (1 - x_i) < 2 \qquad \text{for} \quad i = 1, 2 \tag{2.15a}$$

and

$$\frac{1}{\kappa_0} \leq \frac{1 + x_2}{1 + x_1} \leq \kappa_0 = \tan\theta_0. \tag{2.15b}$$

We now characterize the singularity of $u(x)$ in terms of Jacobi-weighted Sobolev spaces $H^{k,\beta}(Q)$ and Besov spaces $B^{s,\beta}(Q)$.

LEMMA 2.4. *Let $u(x) = r^\gamma\, \chi(r)\, \Phi(\theta)$ be given by (2.11), and let $\beta = (\beta_1, \beta_2)$ with $\beta_1, \beta_2 > -1$. Then $u \in H^{\ell,\beta}(Q)$ for all integers $\ell \leq [2 + \beta_1 + \beta_2 + 2\gamma]$. Hereafter $[a]$ denotes the largest integer $< a$.*

Proof. It is easy to see that for any α

$$|D^\alpha u| \leq C\, r^{\gamma - |\alpha|} \tag{2.16}$$

and for $|\alpha| < 2\gamma + 2 + \beta_1 + \beta_2$

$$\int_Q |D^\alpha u|^2\, (1 - x_1^2)^{\alpha_1+\beta_1}\, (1 - x_2^2)^{\alpha_2+\beta_2}\, dx \leq C \int_0^\rho r^{2\gamma+\beta_1+\beta_2+1-|\alpha|}\, dr < \infty,$$

which proves the lemma. $\square$

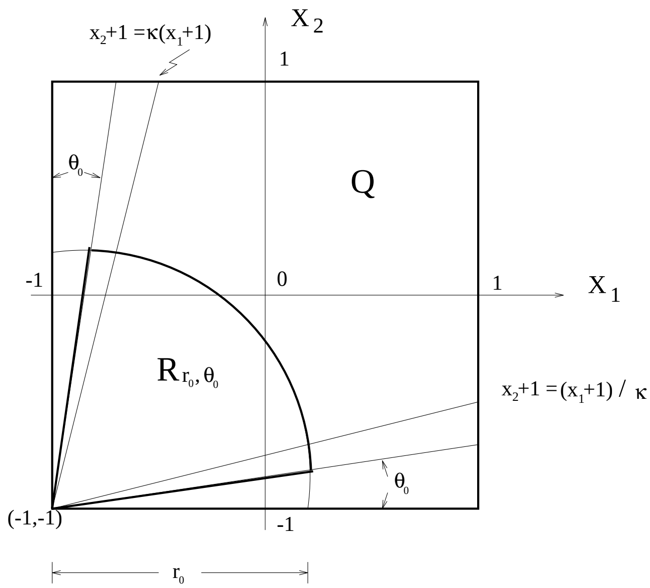

FIG. 2.1. *Square domain Q and subregion R_{r_0,θ_0}.*

THEOREM 2.5. *Let $u = r^\gamma \chi(r)\,\Phi(\theta)$ be given in (2.11) with $\gamma > 0$. Then $u \in B^{s,\beta}(Q)$ with $s = 2+2\gamma+\beta_1+\beta_2$ and $\beta_1, \beta_2 > -1$.*

Proof. Let $\chi_\delta(r) = \chi(\frac{r}{\delta})$ with $0 < \delta \leq r_0$ undetermined, and $u = v + w$ with $v = \chi_\delta(r)\,u$ and $w = (1-\chi_\delta(r))u$. Then $v \in H^{\ell,\beta}(Q)$, $\ell < 2+2\gamma+\beta_1+\beta_2$, by Lemma 2.4, and $w \in H^{k,\beta}(Q)$, $k > 2+2\gamma+\beta_1+\beta_2$. Note that for $|\alpha| = \ell' \leq \ell$ and $r < \delta$

$$|D^\alpha v| \leq C \sum_{t=0}^{\ell'} \delta^{-(\ell'-t)}\, r^{\gamma-t}$$

and it vanishes for $r > \delta$. It follows that

$$\begin{aligned}\int_Q |D^\alpha v|^2\,(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2}\,dx &\leq\ C\sum_{t=0}^{\ell'}\delta^{-2(\ell'-t)}\int_{R_0} r^{2(\gamma-t)+\ell'+\beta_1+\beta_2}dx\\ &\leq\ C\sum_{t=0}^{\ell'}\delta^{-2(\ell'-t)}\int_0^\delta r^{2(\gamma-t)+\beta_1+\beta_2+1-\ell'}dr\\ &\leq\ C\,\delta^{2\gamma+\beta_1+\beta_2+2-\ell'},\end{aligned}$$

which yields, for $\ell < 2+2\gamma+\beta_1+\beta_2$,

$$||v||^2_{H^{\ell,\beta}(Q)} \leq C\,\delta^{2\gamma+\beta_1+\beta_2+2-\ell}. \tag{2.17}$$

Note that for $|\alpha| = \ell' \leq k$

$$|D^\alpha w| \leq C\Big\{(1-\chi_\delta(r))r^{\gamma-\ell'} + \sum_{t=0}^{\ell'-1} r^{\gamma-t}\frac{d^{\ell'-t}}{dr^{\ell'-t}}(1-\chi_\delta(r))\Big\}. \tag{2.18}$$

Since the first term on the right-hand side of (2.18) vanishes for $r < \delta/2$ and the second term vanishes for $r < \delta/2$ and $r > \delta$, there holds

$$\begin{aligned}
&\int_Q |D^\alpha w|^2 (1-x_1^2)^{\alpha_1+\beta_1} (1-x_2^2)^{\alpha_2++\beta_2}\, dx \\
\leq\ & C\Big\{ \int_{\delta/2}^{\rho} r^{2\gamma+\beta_1+\beta_2+1-\ell'} dr + \sum_{t=0}^{\ell'-1} \delta^{-2(\ell'-t)} \int_{\delta/2}^{\delta} r^{2\gamma-2t+\beta_1+\beta_2+1-\ell'} dr \Big\} \\
\leq\ & C \begin{cases} \delta^{2\gamma-\ell'+2+\beta_1+\beta_2} + 1 & \text{if} \quad \ell' \neq 2\gamma+2+\beta_1+\beta_2, \\ \ln\delta + 1 & \text{if} \quad \ell' = 2\gamma+2+\beta_1+\beta_2. \end{cases}
\end{aligned}$$

Therefore we always have for $k > 2+2\gamma+\beta_1+\beta_2$

$$||w||^2_{H^{k,\beta}(Q)} \leq C\delta^{2\gamma-k+2+\beta_1+\beta_2}. \tag{2.19}$$

For $0 < t < 1$, we have by (2.18) and (2.19)

$$\begin{aligned}
K(t,u) =\ & \inf_{u=u_1+u_2} \Big(||u_1||_{H^{\ell,\beta}(Q)} + t\, ||u_2||_{H^{k,\beta}(Q)} \Big) \\
\leq\ & ||v||_{H^{\ell,\beta}(Q)} + t\, ||w||_{H^{k,\beta}(Q)} \\
\leq\ & C(\delta^{\gamma+1+(\beta_1+\beta_2-\ell)/2} + t\, \delta^{\gamma+1+(\beta_1+\beta_2-k)/2}).
\end{aligned}$$

By selecting $\delta = t^{\frac{2}{k-\ell}}$ and $\theta = \frac{2\gamma+2+\beta_1+\beta_2-\ell}{k-\ell}$, we have for $0 < t < 1$

$$K(t,u) \leq C\, \delta^{\gamma+1+(\beta_1+\beta_2-\ell)/2} \leq C\, t^{\frac{2\gamma+2+\beta_1+\beta_2-\ell}{k-\ell}}$$

and

$$\sup_{0<t<1} t^{-\theta} K(t,u) \leq C,$$

and for $t \geq 1$ we always have

$$\sup_{t\geq 1} t^{-\theta} K(t,u) \leq ||u||_{H^{\ell,\beta}(Q)} \leq C,$$

which implies that $u = (H^{\ell,\beta}(Q), H^{k,\beta}(Q))_{-}\theta, \infty = B^{s,\beta}(Q)$, with $\theta = \frac{2\gamma+2+\beta_1+\beta_2-\ell}{k-\ell}$ and $s = 2+2\gamma+\beta_1+\beta_2$. ∎

Remark 2.7. Although the singular function $u(x) = r^\gamma\, \chi(r)\, \Phi(\theta) \in B^{s,\beta}(Q)$ with $s = 2+2\gamma+\beta_1+\beta_2$ for general β with $\beta_i > -1, i = 1,2$, the results with $\beta_i = -1/2$ are much more interesting to us for approximation errors measured in H^1-norm (energy norm) without weights. In that case, $u \in B^{s,\beta}(Q)$ with $s = 1+\gamma$ and $\beta_1 = \beta_2 = -1/2$.

2.3. Approximation of the functions of r^γ-type. We will prove approximation theorems for singular functions of r^γ-type in the framework of the Jacobi-weighted Besov and Sobolev spaces.

THEOREM 2.6. *Let $u(x) = r^\gamma\, \chi(r)\, \Phi(\theta)$ given in* (2.11). *Then there exists a polynomial $\varphi(x) \in P_p(Q)$ with $p \geq 2\gamma$ such that*

$$||u-\varphi||_{H^1(R_0)} \leq C \left(\frac{1}{p}\right)^{2\gamma} \tag{2.20}$$

with constant C independent of p, where R_0 is the support of u given in (2.14).

Proof. By Theorem 2.5 and Remark 2.7, $u \in B^{s,\beta}(Q)$ with $s = 1 + 2\gamma$ and $\beta = (-1/2, -1/2)$. Due to Theorem 2.3, there exists a polynomial $\varphi \in P_p(Q)$ such that

$$||u - \varphi||_{H^{1,\beta}(Q)} \leq C\, p^{-2\gamma}\, ||u||_{B^{s,\beta}(Q)}. \tag{2.21}$$

Obviously

$$||u - \varphi||_{L^2(Q)} \leq C\, ||u - \varphi||_{H^{0,\beta}(Q)} \leq C\, ||u - \varphi||_{H^{1,\beta}(Q)}. \tag{2.22}$$

Due to (2.15) it always holds for $\alpha = (1,0)$ or $(0,1)$ and for $x \in R_0$ that

$$C_1 \leq (1 + x_1)^{\alpha_1 - 1/2}\, (1 + x_2)^{\alpha_2 - 1/2} \leq C_2.$$

Then, we have for α with $|\alpha| = 1$

$$\begin{aligned}
\int_{R_0} \Big| D^\alpha (u - \varphi) \Big|^2 dx \leq &\quad C \int_{R_0} \Big| D^\alpha (u - \varphi) \Big|^2 (1 + x_1)^{\alpha_1 - 1/2}\, (1 + x_2)^{\alpha_2 - 1/2}\, dx \\
\leq &\quad C \int_{R_0} \Big| D^\alpha (u - \varphi) \Big|^2 (1 - x_1^2)^{\alpha_1 - 1/2}\, (1 - x_2^2)^{\alpha_2 - 1/2}\, dx,
\end{aligned}$$

which together with (2.21) and (2.22) leads to (2.20). □

In the applications to boundary value problems on polygonal domains, the singular solution of r^α-type may satisfy homogeneous Dirichlet boundary condition on one or two edges of the polygon around each vertex. Hence we have to enforce the same condition on approximation polynomials. Let us consider the approximation of the singular functions by polynomials vanishing on one or two lines lying in the supports R_0.

If $u(x) = 0$ on the line $x_2 + 1 = \kappa(x_1 + 1)$ for $1 < \kappa \leq \kappa_0 = \tan\theta_0$, we introduce

$$\zeta_1(x) = x_2 + 1 - \kappa(x_1 + 1) \tag{2.23a}$$

and

$$u_1(x) = \frac{u(x)}{\zeta_1(x)} = r^{\gamma - 1}\, \chi(r)\, \psi_1(\theta), \tag{2.23b}$$

where $\psi_1(\theta) = \Phi(\theta)/(\sin\theta - \kappa\cos\theta)$ is a C^∞ function on $[0, \frac{\pi}{2}]$.

If $u(x) = 0$ on the lines $x_2 + 1 = \kappa(x_1 + 1)$ and $x_2 + 1 = (x_1 + 1)/\kappa$, we similarly introduce

$$\zeta_2(x) = \Big(x_2 + 1 - \kappa(x_1 + 1)\Big)\Big(x_2 + 1 - (x_1 + 1)/\kappa\Big) \tag{2.24a}$$

and

$$u_2(x) = \frac{u(x)}{\zeta_0(x)} = r^{\gamma - 2}\, \chi(r)\, \psi_2(\theta), \tag{2.24b}$$

where

$$\psi_2(\theta) = \frac{\Phi(\theta)}{(\sin\theta - \kappa\cos\theta)(\sin\theta - \frac{1}{\kappa}\cos\theta)}$$

is a C^∞ function on $[0, \frac{\pi}{2}]$.

It is easy to verify that there are positive constants C_1 and C_2 such that for $\ell = 1, 2$

$$|D^{\alpha} u| \leq C_1\, r^{\ell} \sum_{|\alpha'| \leq |\alpha|} |D^{\alpha'} u_\ell| \tag{2.25a}$$

and

$$|D^{\alpha} u_\ell| \leq C_2\, r^{-\ell} \sum_{|\alpha'| \leq |\alpha|} |D^{\alpha'} u|, \tag{2.25b}$$

which lead immediately to the following theorem.

THEOREM 2.7. *$u_\ell(x) \in B^{2+2\gamma+\beta_1+\beta_2,\beta^{[\ell]}}(Q)$ with $\beta^{[\ell]} = \beta+\ell = (\beta_1+\ell, \beta_2+\ell), \ell = 1, 2$, and there are constants C_1 and C_2 such that*

(2.26)
$$C_1\, ||u(x)||_{B^{2+2\gamma+\beta_1+\beta_2,\beta}(Q)} \leq ||u_\ell(x)||_{B^{2+2\gamma+\beta_1+\beta_2,\beta^{[\ell]}}(Q)} \leq C_2\, ||u(x)||_{B^{2+2\gamma+\beta_1+\beta_2,\beta}(Q)}.$$

In particular, $u_\ell(x) \in B^{1+2\gamma,\beta^{[\ell]}}(Q)$ with $\beta^{[\ell]} = \ell - 1/2 = (\ell - 1/2, \ell - 1/2)$.

For approximation to the singular functions of r^γ-type vanishing on one or two lines, we need the following lemmas.

LEMMA 2.8. *Let $r^2\, |D^1 v| \in L^2(Q)$ and $v = 0$ for $r > \rho > 0$. Then, $r\, v \in L^2(Q)$, and*

$$||r\, v||_{L^2(Q)} \leq C\, ||r^2\, D^1 v||_{L^2(Q)}. \tag{2.27}$$

Proof. We quote the Hardy inequality 330 from [24, p. 245]

$$\int_0^\infty x^{-s}\, F^p\, dx < \left(\frac{p}{|s-1|}\right)^p \int_0^\infty x^{-s}\, (x\, f)^p\, dx, \tag{2.28}$$

where $F(x) = \int_x^\infty f(t)\, dt$ for $s < 1$, and $F(x) = \int_0^x f(t)\, dt$ for $s > 1$. The application of (2.28) with $p = 2$, $s = -3$, $f = \frac{\partial v}{\partial r}$, and $F = -v$ leads to

$$\int_0^\rho r^3\, |v|^2\, dr < C \int_0^\rho r^5 \left|\frac{\partial v}{\partial r}\right|^2 dr,$$

which implies (2.27). $\square$

LEMMA 2.9. *Let $r\, |D^1 v| \in L^2(Q)$, and $v = 0$ for $r > \rho > 0$. Then $v \in L^2(Q)$, and*

$$||v||_{L^2(Q)} \leq C\, ||r\, D^1 v||_{L^2(Q)}. \tag{2.29}$$

Proof. The application of (2.28) with $p = 2$, $s = -1$, $f = \frac{\partial v}{\partial r}$, and $F = -v$ yields (2.29) immediately. $\square$

These lemmas above and Theorem 2.7 lead to the following theorem.

THEOREM 2.10. *Let $u(x) = r^\gamma\, \chi(r)\, \Phi(\theta)$ given in (2.11), and $u = 0$ on both lines $\theta = \theta_\kappa = \arctan(\kappa)$ and $\theta = \theta_{1/\kappa} = \arctan(1/\kappa)$ (resp., $u = 0$ on one line $\theta = \theta_\kappa$) with $1 \geq \kappa \geq \kappa_0$, with κ_0 given in (2.15). Then there exists a polynomial $\varphi(x) \in P_p(Q)$, $p \geq \max\{2, 2\gamma\}$ (resp., $p \geq \max\{1, 2\gamma\}$) such that $\varphi|_{\theta=\theta_\kappa} = \varphi|_{\theta=\theta_{1/\kappa}} = 0$ (resp., $\varphi|_{\theta=\theta_\kappa} = 0$), and*

$$||u - \varphi(x)||_{H^1(R_0)} \leq C\, p^{-2\gamma} \tag{2.30}$$

with constant C independent of p, where R_0 is the support of u given in (2.14).

Proof. We would prove the theorem for u vanishing on both lines; the proof for u vanishing on one line is similar. Let $u_2(x) = u(x)/\zeta_2(x)$, given by (2.24). By Theorem 2.7, $u_2 \in B^{s,\beta^{[2]}}(Q)$ with $s = 1+2\gamma$ and $\beta^{[2]} = (3/2, 3/2)$. Due to Theorem 2.3 and Theorem 2.7, there exists a polynomial $\phi(x) \in P_{p-2}(Q)$, $p \geq \max\{2, 2\gamma\}$, such that for $\ell \leq s$

$$||u_2 - \phi(x)||_{H^{\ell,\beta^{[2]}}(Q)} \leq C\, p^{-(s-\ell)}\, ||u_2||_{B^{s,\beta^{[2]}}(Q)} \leq C\, p^{-(2\gamma+1-\ell)}\, ||u||_{B^{s,\beta}(Q)}. \tag{2.31}$$

Now let $\varphi(x) = \zeta_2(x)\,\phi(x)$. Then $\varphi(x) = 0$ on the lines $\theta = \theta_\kappa$ and $\theta = \theta_{1/\kappa}$, and

$$\begin{aligned}\int_{R_0} |u(x) - \varphi(x)|^2\,dx = &\int_{R_0} |\zeta_0|^2\,|u_2(x) - \phi(x)|^2\,dx \leq C\int_{R_0} |u_2(x) - \phi(x)|^2\,r^4\,dx \\ \leq\ & C\int_{R_0} |u_2(x) - \phi(x)|^2\,(1-x_1^2)^{3/2}\,(1-x_2^2)^{3/2}\,dx \\ \leq\ & C\,||u_2 - \phi||^2_{H^{0,\beta^{[2]}}(Q)},\end{aligned}$$

which together with (2.31) implies

$$\int_{R_0} |u(x) - \varphi(x)|^2\,dx \leq C\,p^{-2(2\gamma+1)}\,||u||^2_{B^{s,\beta}(Q)}. \tag{2.32}$$

Letting $\chi(r)$ be the C^∞ function given in (2.12), we have

$$\begin{aligned}&\int_{R_0} \left|D^1\Big(u(x) - \varphi(x)\Big)\right|^2 dx \\ &\quad\leq C\left(\int_{R_0}\left|\zeta_2^2\right| D^1\Big(\chi(r)(u_0 - \varphi_0)\Big)\Big|^2 dx + \int_{R_0} |D^1\zeta_2|\,\left|\chi(r)(u_2 - \phi)\right|^2 dx\right).\end{aligned} \tag{2.33}$$

For $\alpha = (1,0)$ or $\alpha = (0,1)$ and for $x \in R_0$, we have from (2.15)

$$\tilde{C}_1 r \leq (1 - x_i^2) \leq \tilde{C}_2 r \quad \text{for} \quad i = 1, 2.$$

Therefore,

$$\begin{aligned}\int_{R_0} |\zeta_2|^2 &\left|D^\alpha\Big(\chi(r)(u_2 - \phi)\Big)\right|^2 dx \leq \int_{R_0} r^4 \left|D^\alpha\Big(\chi(r)(u_2 - \phi)\Big)\right|^2 dx \\ \leq\ & \int_{R_0}\left|D^\alpha\Big(\chi(r)(u_2 - \phi)\Big)\right|^2 (1-x_1^2)^{\alpha_1+3/2}\,(1-x_2^2)^{\alpha_2+3/2}\,dx \\ \leq\ & C\Bigg(\int_Q \left|D^\alpha(u_2 - \phi)\right|^2 (1-x_1^2)^{\alpha_1+3/2}\,(1-x_2^2)^{\alpha_2+3/2}\,dx \\ &\quad + \int_Q |u_2 - \phi|^2\,(1-x_1^2)^{3/2}\,(1-x_2^2)^{3/2}\,dx\Bigg) \\ =\ & C\,||u_2 - \phi||^2_{H^{1,\beta}(Q)},\end{aligned}$$

which together with (2.31) leads to

$$\int_{R_0} |\zeta_2|^2 \left|D^\alpha\Big(\chi(r)(u_0 - \varphi_0)\Big)\right|^2 dx \leq C\,p^{-4\gamma}\,||u||^2_{B^{s,\beta^{[2]}}(Q)}. \tag{2.34}$$

For the second term on the right-hand side of (2.31) we have

$$\int_{R_0} |D^1\zeta_2|^2 \left|\chi(r)(u_2-\phi)\right|^2 dx \le C \int_{R_{\rho,k_0}} r^2 \left|\chi(r)(u_2-\phi)\right|^2 dx,$$

by Lemma 2.8

$$\le C \int_{R_0} r^4 \Big|D^1\Big(\chi(r)(u_0-\varphi_0)\Big)\Big|^2 dx,$$

by (2.33)

$$\le C\, p^{-4\gamma}\, ||u||^2_{B^{s,\beta}(Q)}. \tag{2.35}$$

The combination of (2.32)–(2.35) yields (2.30). ∎

The proof above can be carried out for u vanishing on one line, except replacing ζ_2 by ζ_1, $B^{s,\beta^{[2]}}$ by $B^{s,\beta^{[1]}}$, and using Lemma 2.9 for (2.34) instead of Lemma 2.8. Thus we have completed the proof of the theorem.

3. Modified Jacobi-weighted Besov spaces $B^{s,\beta}_\nu(Q)$ and approximation of the functions of $r^\gamma \log^\nu r$-type. It can be shown that functions of $r^\gamma \log^\nu r$-type, $\nu > 0$, belong to the weighted Besov spaces $B^{s,\beta}(Q)$ with $s = 1+2\gamma-\varepsilon$, $\varepsilon > 0$ arbitrary, which will lead to a loss of $(1/p)^\varepsilon$ in rate of convergence for the p-version of the finite element method. Hence we have to sharpen the mathematical tool for a better characterization of the singularity of functions of $r^\gamma \log^\nu r$-type and a better approximation by the high-order polynomial. This sharpened tool is the modified Jacobi-weighted Besov space $B^{s,\beta}_\nu(Q)$.

3.1. Modified interpolation spaces $(A_0, A_1)_{\theta,\infty,\nu}$. We shall modify general interpolation space defined by the K-method and the J-method, then apply it to the modified Jacobi-weighted Besov spaces . Let $\bar{A} = (A_0, A_1)$ be a couple of compatible Banach spaces. For the sake of simplicity we assume that $A_0 \subseteq A_1$ or $A_1 \subseteq A_0$, and this assumption holds in practical applications. We introduce for $a \in \bar{A}$

$$\Phi_{\theta,\infty,\nu}(K(t,a)) = \sup_{t>0} K(t,a)\, \frac{t^{-\theta}}{(1+|\log t|)^\nu}, \tag{3.1}$$

where $K(t,a)$ is defined as usual by

$$K(t,a) = \inf_{a=a_0+a_1} \Big(||a_0||_{A_0} + t||a_1||_{A_1}\Big), \tag{3.2}$$

and $\nu \ge 0$, $0 < \theta < 1$. If $\nu = 0$, $\Phi_{\theta,\infty,0}(K(t,a)) = \Phi_{\theta,\infty}(K(t,a))$. It is easy to verify that $\Phi_{\theta,\infty,\nu}(K(t,a))$ is a norm on $\bar{A}$. By $\bar{A}_{\theta,\infty,\nu}$ we denote the interpolation space $(A_0, A_1)_{\theta,\infty,\nu}$ with the norm $||a||_{\theta,\infty,\nu} = \Phi_{\theta,\infty,\nu}(K(t,a))$.

$(A_0, A_1)_{\theta,\infty,\nu}$ and $\Phi_{\theta,\infty,\nu}$, $\nu > 0$, preserve many properties of $(A_0, A_1)_{\theta,\infty}$ and $\Phi_{\theta,\infty}$. It is trivial to verify by definition the following properties:

(P1) $(A_0, A_1)_{\theta,\infty,\nu} = (A_1, A_0)_{1-\theta,\infty,\nu}$;
(P2) $(A_0, A_0)_{\theta,\infty,\nu} = A_0$;
(P3) $(A_0, A_1)_{\theta_2,\infty,\nu} \subset (A_0, A_1)_{\theta_1,\infty,\nu}$ if $\theta_1 < \theta_2$;
(P4) $(A_0, A_1)_{\theta_1,\infty,\nu} \cap (A_0, A_1)_{\theta_2,\infty,\nu} \subset (A_0, A_1)_{\theta,\infty,\nu}$ if $\theta_1 < \theta < \theta_2$;
(P5) $(A_0, A_1)_{\theta,\infty} \subset (A_0, A_1)_{\theta,\infty,\nu_1} \subset (A_0, A_1)_{\theta,\infty,\nu_2}$ for $\nu_2 > \nu_1 > 0$;
(P6) $(A_0, A_1)_{\theta-\varepsilon,\infty,\nu} \subset (A_0, A_1)_{\theta,\infty}$ for $\varepsilon \in (0,\theta)$ arbitrary.

Some properties of $\Phi_{\theta,\infty}$ and $(A_0, A_1)_{\theta,\infty}$ may hold, but not obviously, for $\Phi_{\theta,\infty,\nu}$ and $(A_0, A_1)_{\theta,\infty,\nu}$ with $\nu > 0$. We have to argue precisely; for instance, the partial reiteration theorems (Theorem 3.4) needed to be proved for the space $(A_0, A_1)_{\theta,\infty,\nu}$. Some properties of $\Phi_{\theta,\infty}$ and $(A_0, A_1)_{\theta,\infty}$ may no longer stand for $\Phi_{\theta,\infty,\nu}$ and $(A_0, A_1)_{\theta,\infty,\nu}$, which we need to modify substantially. For instance, unlike $(A_0, A_1)_{\theta,\infty}$, the space $(A_0, A_1)_{\theta,\infty,\nu}$ with $\nu > 0$ is not exact of exponent θ; instead, it is only a uniform interpolation space. We elaborate it in Theorem 3.2, which is very essential to the theory of interpolation spaces and analysis of the approximation errors for functions of $r^\gamma \log^\nu r$-type.

LEMMA 3.1. *Let $a \in (A_0, A_1)$, $0 < \theta < 1$, $\nu \geq 0$, and $t, s > 0$. Then*

$$\Phi_{\theta,\infty,\nu}(K(t/s, a)) \leq s^{-\theta}\,(1 + |\log s|)^\nu\, \Phi_{\theta,\infty,\nu}\,(K(t, a)). \tag{3.3}$$

Proof. For $\nu = 0$ it is a standard argument for Besov spaces; we refer to [11]. We now assume that $\nu > 0$. By the definition (3.2) we have

$$\Phi_{\theta,\infty,\nu}(K(t/s, a)) = \sup_{t>0} \frac{t^{-\theta} K(t/s, a)}{(1 + |\log t|)^\nu} = s^{-\theta} \sup_{t>0}\, g(t)^\nu\, \frac{t^{-\theta} K(t, a)}{(1 + |\log t|)^\nu}, \tag{3.4}$$

where $g(t) = (1 + |\log t|)/(1 + |\log t + \log s|)$. We estimate $g(t)$ for $s > 1$ and $s \leq 1$. In both cases we can easily prove that $g(t)$ has maximum value at $t = 1/s$. Hence,

$$g(t) \leq g(1/s) = 1 + |\log s|. \tag{3.5}$$

The combination of (3.4) and (3.5) yields (3.3) immediately. $\square$

Remark 3.1. The inequality (3.3) becomes an equality if $\nu = 0$.

THEOREM 3.2. *Let T be an operator $\bar{A} = (A_0, A_1) \longrightarrow \bar{B} = (B_0, B_1)$ and let $||T||_i$, $i = 0, 1$, be the norm of the operator $A_i \longrightarrow B_i$. Then T is an operator $\bar{A}_{\theta,\infty,\nu} \longrightarrow \bar{B}_{\theta,\infty,\nu}$ for $0 < \theta < 1$, $\nu \geq 0$, and*

$$||T|| = ||T||_{\bar{A}_{\theta,\infty,\nu} \longrightarrow \bar{B}_{\theta,\infty,\nu}} \leq \left(1 + \left|\log \frac{||T||_1}{||T||_0}\right|\right)^\nu ||T||_0^{1-\theta}\, ||T||_1^\theta. \tag{3.6}$$

Proof. For any $a = a_0 + a_1 \in \bar{A}$ with $a_i \in A_i$, $i = 0, 1$,

$$\begin{aligned} K(t, T\,a) = & \inf_{a=a_0+a_1} \Big(||T\,a_0||_{B_0} + t\,||T\,a_1||_{B_1}\Big) \\ \leq & \inf_{a=a_0+a_1} \Big(||T||_0\,||a_0||_{A_0} + t\,||T||_1\,||a_1||_{A_1}\Big) \\ \leq & \; ||T||_0 \inf_{a=a_0+a_1} \Big(||a_0||_{A_0} + \tfrac{t}{s}\,||a_1||_{A_1}\Big) = ||T||_0\, K(\tfrac{t}{s}, a) \end{aligned}$$

with $s = ||T||_0/||T||_1$. By Lemma 3.1

$$\begin{aligned} ||T\,a||_{\theta,\infty,\nu} = & \sup_{t>0} \frac{t^{-\theta}}{(1 + |\log t|)^\nu}\, K(t, T\,a) \leq ||T||_0 \sup_{t>0} \frac{t^{-\theta}}{(1 + |\log t|)^\nu}\, K(\tfrac{t}{s}, a) \\ \leq & \; ||T||_0\, s^{-\theta}\,(1 + |\log s|)^\nu \sup_{t>0} \frac{t^{-\theta} K(t, a)}{(1 + |\log t|)^\nu} \\ \leq & \; ||T||_0^{1-\theta}\, ||T||_1^\theta \left(1 + \left|\log \frac{||T||_1}{||T||_0}\right|\right)^\nu ||a||_{\theta,\infty,\nu}, \end{aligned}$$

which leads to (3.6) immediately. $\square$

COROLLARY 3.3. *$(A_0, A_1)_{\theta,\infty,\nu}$, $\nu \geq 0$, is a uniform interpolation space, and it is exact of exponent θ if $\nu = 0$.*

Proof. Due to (3.6) it holds that for $\nu \geq 0$

$$||T|| \leq C \ \max(||T||_0, ||T||_1).$$

According to the definitions (for the definitions and properties of uniform and exact interpolation spaces, we refer to, e.g., [11]), $(A_0, A_1)_{\theta,\infty,\nu}$, $\nu > 0$, is a uniform interpolation space. For $\nu = 0$, it is true by a standard argument (see, e.g., [11]) that

$$||T|| \leq ||T||_0^{1-\theta} \ ||T||_1^{\theta},$$

which implies that $(A_0, A_1)_{\theta,\infty,0}$ is exact of exponent θ. ◻

THEOREM 3.4 (partial reiteration). *Let $X_i = (A_0, A_1)_{\theta_i,\infty}$ with $\theta_i \in (0,1), \ell = 1,2$. Then for $\eta \in (0,1)$, $\nu \geq 0$ and $\theta = (1-\eta)\theta_0 + \eta\,\theta_1$*

$$(X_0, X_1)_{\eta,\infty,\nu} = (A_0, A_1)_{\theta,\infty,\nu}. \tag{3.7}$$

Remark 3.2. The reiteration theorem for modified interpolation spaces does not stand in general; i.e., if $X_i = (A_0, A_1)_{\theta_i,\infty,\nu}$ with $\nu > 0$, then (3.7) may not be true. The theorem holds only for the special case that $X_i, i = 1,2$, are the exact interpolation space $(A_0, A_1)_{\theta_i,\infty}$. We call it the partial reiteration theorem, which is sufficient for the need of our approximation purpose.

The proof of the theorem is not trivial; we need to introduce the modified interpolation space defined by the J-method. For the J-method we refer to [11].

Let $a \in \Delta(\bar{A}) = A_0 \cap A_1$ and define

$$J(t,a) = J(t,a,\bar{A}) = \max\Big(||a||_{A_0}, t\,||a||_{A_1}\Big).$$

By $\bar{A}_{\theta,\infty,\nu,J}$ we denote the spaces defined by the J-method; meanwhile, by $\bar{A}_{\theta,\infty,\nu,K}$ or $A_{\theta,\infty,\nu}$ we denote those defined by the K-method. The space $\bar{A}_{\theta,\infty,\nu,J}$ is now defined as follows: The elements a in $\bar{A}_{\theta,\infty,\nu,J}$ are those in $\sum(\bar{A}) = A_1 \cup A_2$ which can be represented by

$$a = \int_0^\infty u(s)\,ds, \tag{3.8}$$

where $u \in \Delta(\bar{A})$ is measurable with values in $\Delta(\bar{A})$, and

$$\Phi_{\theta,\infty,\nu}\Big(J(t,u(t))\Big) = \sup_{t>0} \frac{t^{-\theta}\,J(t,u(t))}{(1+|\log t|)^{\nu}} < \infty \tag{3.9}$$

for $0 < \theta < 1$. The norm of $\bar{A}_{\theta,\infty,\nu,J}$ is defined as

$$||a||_{\theta,\infty,\nu,J} = \inf_u \Phi_{\theta,\infty,\nu}\Big(J(t,u(t))\Big), \tag{3.10}$$

where the infimum is taken over all u satisfying (3.8).

It was proven in Lemma 3.1 that $\Phi_{\theta,\infty,\nu}$, $\nu > 0$, is not an exact functor of exponent θ for $K(t,a)$. The next lemma will show that it is not an exact functor of exponent θ for $J(t,u(t))$ either.

LEMMA 3.5. *For $u \in \Delta(\bar{A})$, $\theta \in (0,1)$, $\nu \geq 0$ integer, and $t, s > 0$, there holds*

$$\Phi_{\theta,\infty,\nu}\Big(J(t/s, u(t/s))\Big) \leq s^{-\theta}\,(1+|\log s|)^{\nu}\ \Phi_{\theta,\infty,\nu}\Big(J(t, u(t))\Big). \tag{3.11}$$

Proof. Due to the definition (3.1),

$$\Phi_{\theta,\infty,\nu}\Big(J(t/s, u(t/s))\Big) = \sup_{t>0} \frac{t^{-\theta}\,J(t/s, u(t/s))}{(1+|\log t|)^{\nu}} = s^{-\theta} \sup_{t'>0} \frac{t'^{-\theta}\,J(t', u(t'))}{(1+|\log t' + \log s|)^{\nu}}$$

$$= s^{-\theta} \sup_{t'>0} (g(t'))^{\nu} \frac{t'^{-\theta}\,J(t', u(t'))}{(1+|\log t'|)^{\nu}},$$

where $g(t') = \frac{(1+|\log t'|)}{(1+|\log t' + \log s|)}$. It has been shown in Lemma 3.1 that for all $t' \in (0,\infty)$ and $s \in (0,\infty)$

$$|g(t')| \leq (1+|\log s|)^{\nu},$$

which leads to (3.11) immediately. $\square$

It is well known that the spaces $\bar{A}_{\theta,\infty,\nu,K}$ and $\bar{A}_{\theta,\infty,\nu,J}$ are equivalent for $\nu = 0$. The equivalence theorem can be proved for integers $\nu > 0$ by proceeding the arguments for $\nu = 0$ with some very straightforward modifications, which we will not elaborate on here; instead we refer to, e.g., [11, Theorem 3.3].

THEOREM 3.6. *If $0 < \theta < 1$, then the spaces $\bar{A}_{\theta,\infty,\nu,K} = \bar{A}_{\theta,\infty,\nu,J}$ with equivalent norms.*

We now are able to prove Theorem 3.4.

Proof of Theorem 3.4. (1) Prove

$$(X_0, X_1)_{\eta,\infty,\nu} \subset (A_0, A_1)_{\theta,\infty,\nu}. \tag{3.12}$$

Let $a \in (X_0, X_1)_{\eta,\infty,\nu}$. Then $a = a_0 + a_1$ with $a_i \in X_i$, $i = 0, 1$, and

$$K(t, a, \bar{A}) \leq C\, t^{\theta_i}\, ||a_i||_{X_i}, \qquad i = 0, 1.$$

Since $K(t, a, \bar{A})$ is a norm on $\sum(\bar{A})$,

$$\begin{aligned} K(t, a; \bar{A}) &\leq K(t, a_0; \bar{A}) + K(t, a_1; \bar{A}) \leq C\Big(t^{\theta_0}\, ||a_0||_{X_0} + t^{\theta_1}\, ||a_1||_{X_1}\Big) \\ &= C\, t^{\theta_0} \Big(||a_0||_{X_0} + t^{\theta_1 - \theta_0}\, ||a_1||_{X_1}\Big). \end{aligned}$$

Here we may assume without losing generality that $\theta_0 < \theta_1$. Therefore

$$\sup_{t>0} \frac{t^{-\theta}\, K(t, a, \bar{A})}{(1+|\log t|)^{\nu}} \leq C \sup_{t>0} t^{-(\theta-\theta_0)} \frac{\Big(||a_0||_{X_0} + t^{\theta_1-\theta_0}\, ||a_1||_{X_1}\Big)}{(1+|\log t|)^{\nu}}.$$

Setting $s = t^{\theta_1 - \theta_0}$ and noting that $0 < \theta_1 - \theta_0 < 1$ and $\eta = (\theta - \theta_0)/(\theta_1 - \theta_0)$, we have

$$||a||_{\bar{A}_{\theta,\infty,\nu}} \leq C \sup_{s>0} \frac{s^{-\eta}\Big(||a_0||_{X_0} + s\, ||a_1||_{X_1}\Big)}{(1+|\log s|)^{\nu}} = C\, ||a||_{\bar{X}_{\eta,\infty,\nu}},$$

which implies (3.12).

(2) Prove

$$(A_0, A_1)_{\theta,\infty,\nu} \subset (X_0, X_1)_{\eta,\infty,\nu}. \tag{3.13}$$

Assume that $a \in (A_0, A_1)_{\theta,\infty,\nu}$ with a representation $a = \int_0^\infty u(s)\ \frac{ds}{s}$, $u(s) \in \Delta(\bar{A})$. Consider

$$||a||_{\bar{X}_{\eta,\infty,\nu}} = \sup_{s>0} \frac{s^{-\eta}\, K(s, a; \bar{X})}{(1 + |\log s|)^{\nu}},$$

and we have, by changing variable $s = t^{\theta_1 - \theta_0}$,

$$||a||_{\bar{X}_{\eta,\infty,\nu}} \le \sup_{t>0} \frac{t^{-(\theta-\theta_0)}\, K(t^{\theta_1-\theta_0}, a; \bar{X})}{(1 + |\log t|)^{\nu}}.$$

Due to Lemma 3.2.1 of [11]

$$\begin{aligned} t^{\theta_0}\ K(t^{\theta_1-\theta_0}, a; \bar{X}) &\le \int_0^\infty t^{\theta_0}\ K(t^{\theta_1-\theta_0}, u(s); \bar{X})\ \frac{ds}{s} \\ &\le \int_0^\infty \min\left(1, \left(\frac{t}{s}\right)^{\theta_1-\theta_0}\right) t^{\theta_0}\ J(s^{\theta_1-\theta_0}, u(s); \bar{X})\ \frac{ds}{s}. \end{aligned}$$

Since $X_i = (A_0, A_1)_{\theta_i,\infty}$, there holds (see [8, Theorem 3.2.2])

$$||a||_{X_i} \le C_i\ s^{-\theta_i}\ J(s, a, \bar{A}),$$

which implies

$$\begin{aligned} J(s^{\theta_1-\theta_0}, u(s), \bar{X}) \quad &= \max\nolimits_{s>0}\left(||u(s)||_{X_0}, s^{\theta_1-\theta_0}\ ||u(s)||_{X_1}\right) \\ &\le \max(C_0, C_1) s^{-\theta_0}\ J(s, u(s), \bar{A}) \end{aligned}$$

and

$$t^{\theta_0}\ K(t^{\theta_1-\theta_0}, a; \bar{X}) \le C \int_0^\infty \min\left(\left(\frac{t}{s}\right)^{\theta_0}, \left(\frac{t}{s}\right)^{\theta_1}\right) J(s, u(s), \bar{A})\ \frac{ds}{s}.$$

Changing variable $s = \sigma\, t$ for fixed $t > 0$, we have

$$t^{\theta_0}\ K(t^{\theta_1-\theta_0}, a; \bar{X}) \le C \int_0^\infty \min(\sigma^{-\theta_0}, \sigma^{-\theta_1}) J(\sigma\, t, u(\sigma\, t), \bar{A})\ \frac{d\sigma}{\sigma},$$

and by Lemma 3.5

$$\begin{aligned} \sup_{t>0} \frac{t^{-(\theta+\theta_0)}\, K(t^{\theta_1-\theta_0}, a; \bar{X})}{(1 + |\log t|)^{\nu}} &\le C \int_0^\infty \min(\sigma^{-\theta_0}, \sigma^{-\theta_1})\ \Phi_{\theta,\infty,\nu}\ (J(\sigma\, t, u(\sigma\, t), \bar{A})\ \frac{d\sigma}{\sigma}, \\ &\le\ C \int_0^\infty \min(\sigma^{-\theta_0}, \sigma^{-\theta_1}) \sigma^{\theta} (1 + |\log(1/\sigma)|)^{\nu}\ \frac{d\sigma}{\sigma}\ \Phi_{\theta,\infty,\nu}\ J(t, u(t), \bar{A}), \end{aligned}$$

where the above integral is finite for $\theta = (1-\eta)\theta_0 + \eta\, \theta_1$, $\theta_0 \ne \theta_1$, $\eta \in (0, 1)$, and $\nu \ge 0$.

By Theorem 3.6, we have

$$||a||_{\bar{X}_{\eta,\infty,\nu}} \le \sup_{t>0} \frac{t^{-(\theta-\theta_0)}\, K(t^{\theta_1-\theta_0}, a; \bar{X})}{(1 + |\log t|)^{\nu}} \le C\ ||a||_{\bar{A}_{\theta,\infty,\nu,J}} \le C\ ||a||_{\bar{A}_{\theta,\infty,\nu}},$$

which implies (3.13) and completes the proof.

3.2. Characterization of functions of $r^\gamma \log^\nu r$-type in terms of modified weighted Besov spaces $B_\nu^{s,\beta}(Q)$. In the framework of modified interpolation spaces $(A_0, A_1)_{\theta,\infty,\nu}$, we now introduce the modified weighted Besov space $B_\nu^{s,\beta}(Q)$, $\nu \geq 0$:

$$B_\nu^{s,\beta}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty,\nu} \tag{3.14}$$

with $\theta \in (0,1)$ and $s = (1-\theta)\ell + \theta\, k$, $\beta_i > -1, i = 1,2$. The singular functions of $r^\gamma \log^\nu r$-type can be precisely characterized by this space.

Remark 3.3. Due to Theorem 3.4 (partial reiteration), the definition of $B_\nu^{s,\beta}(Q)$ is independent of the individual values of ℓ, k, and θ. Namely,

$$\Big(H^{\ell',\beta}(Q), H^{k',\beta}(Q)\Big)_{\theta',\infty,\nu} = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty,\nu} = B_\nu^{s,\beta}(Q)$$

with $\theta' = \frac{s-\ell'}{k'-\ell'}$ and $\theta = \frac{s-\ell}{k-\ell}$, where $0 \leq \ell, \ell' < s < k, k'$.

We now consider functions of $r^\gamma \log^\nu r$-type. Let

$$u(x) = r^\gamma \log^\nu r\, \chi(r)\, \Phi(\theta) \tag{3.15}$$

be defined on $Q = (-1,1)^2$ with $\gamma > 0$ and integer $\nu \geq 0$ where $\chi(r)$ and $\Phi(\theta)$ are smooth functions satisfying (2.12) and (2.13), respectively. Obviously supp. $u \subset R_0 = R_{r_0,\theta_0} \subset Q, R_0$ is given in (2.14). Arguing as we did in the proof of Lemma 2.4, we can prove a similar lemma.

LEMMA 3.7. *Let $u = r^\gamma \log^\nu r\ \chi(r)\, \Phi(\theta)$ with $\gamma > 0$ and integer $\nu \geq 0$. Then $u(x) \in H^{k,\beta}(Q)$ with $\beta_1, \beta_2 > -1$ and $k = [2 + 2\gamma + \beta_1 + \beta_2]$.*

Remark 3.4. Lemmas 3.7 and 2.4 indicate that the singular function of $r^\gamma \log^\nu r$-type belong to $H^{k,\beta}(Q)$ with $k = [2 + 2\gamma + \beta_1 + \beta_2]$ for $\nu \geq 0$. The factor $\log^\nu r$ with $\nu > 0$ does not affect the maximum value of k, but it affects the value of s for the space $B^{s,\beta}(Q)$ which $u(x)$ belongs to, namely, $u \in B^{2+2\gamma+\beta_1+\beta_2-\varepsilon,\beta}(Q)$, $\varepsilon > 0$ arbitrary. The loss can be recovered only if we use the space $B_\nu^{s,\beta}(Q)$, instead of $B^{s,\beta}(Q)$, to characterize the singularity of $r^\gamma \log^\nu r$-type.

THEOREM 3.8. *Let $u = r^\gamma \log^\nu r\ \chi(r)\, \Phi(\theta)$ be given in (3.15) with $\gamma > 0$ and integer $\nu \geq 0$. Then $u \in B_\nu^{s,\beta}(Q)$ with $\beta_1, \beta_2 > -1$ and $s = 2 + 2\gamma + \beta_1 + \beta_2$.*

Proof. Let $\chi_\delta(r) = \chi(\frac{r}{\delta})$ with $0 < \delta < r_0$, and $u = v + w$ with $v = \chi_\delta(r)\, u$ and $w = (1 - \chi_\delta(r))u$. Then $v \in H^{\ell,\beta}(Q)$ with $\ell < 2 + 2\gamma + \beta_1 + \beta_2$ due to Lemma 3.7, and $w \in H^{k,\beta}(Q)$ for $k > 2 + 2\gamma + \beta_1 + \beta_2$. Note that for $|\alpha| = \ell' \leq \ell$

$$|D^\alpha v| \leq C \sum_{t=0}^{\ell'} \delta^{-(\ell'-t)}\, r^{\gamma-t}\, |\log r|^\nu \qquad \text{for} \quad x \in R_0$$

and

$$D^\alpha v = 0 \qquad \text{for} \quad x \notin R_0.$$

Therefore, for $|\alpha| = \ell' < \ell$

$$\begin{aligned}
&\int_Q |D^\alpha v|^2 (1-x_1^2)^{\alpha_1+\beta_1} (1-x_2^2)^{\alpha_2+\beta_2}\, dx \\
&\qquad \leq C \sum_{t=0}^{\ell'} \delta^{-2(\ell'-t)} \int_{R_{\delta,\theta_0}} r^{2(\gamma-t)+\ell'+\beta_1+\beta_2}\, |\log r|^{2\nu} dx \\
&\qquad \leq C \sum_{t=0}^{\ell'} \delta^{-2(\ell'-t)} \int_0^\delta r^{2\gamma-2t+\ell'+1+\beta_1+\beta_2}\, |\log r|^{2\nu} dr.
\end{aligned}$$

Noting that ν is an integer, we have by integration by part

$$\begin{aligned}
&\int_0^\delta r^{2\gamma-2t+\ell'+1+\beta_1+\beta_2}|\log r|^{2\nu}dr \\
&\quad= \delta^{2\gamma-2t+\ell'+2+\beta_1+\beta_2}\sum_{m=0}^{2\nu}\frac{|\log\delta|^{2\nu-m}\Gamma(2\nu+1)}{(2\gamma-2t+\ell'+2+\beta_1+\beta_2)^{m+1}\Gamma(2\nu-m-1)} \\
&\qquad+ \frac{\Gamma(2\nu+1)}{(2\gamma-2t+\ell'+2+\beta_1+\beta_2)^{2\nu}}\int_0^\delta r^{2\gamma-2t+\ell'+1+\beta_1+\beta_2}\,dr \\
&\quad\le C\,\delta^{2\gamma-2t+\ell'+2+\beta_1+\beta_2}\,|\log\delta|^{2\nu},
\end{aligned}$$

which implies, for $|\alpha|\le\ell<2+2\gamma+\beta_1+\beta_2$,

$$||v||^2_{H^{\ell,\beta}(Q)}\le C\,\delta^{2\gamma-\ell+2+\beta_1+\beta_2}\,|\log\delta|^{2\nu}. \tag{3.16}$$

Note that for $|\alpha|=\ell'\le k$

$$|D^\alpha w|\le C\left\{(1-\chi_\delta)r^{\gamma-\ell'}+\sum_{t=0}^{\ell'-1}\frac{d^{\ell'-t}(1-\chi_\delta(r))}{dr^{\ell'-t}}r^{\gamma-t}\right\}; \tag{3.17}$$

the first term on the right hand of (3.17) vanishes for $r<\delta/2$, and the second term vanishes for $r<\delta/2$ and $r>\delta$. This implies

$$\begin{aligned}
&\int_Q|D^\alpha w|^2(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2}\,dx \\
&\quad\le C\left\{\sum_{t=0}^{\ell'-1}\delta^{2(\ell-t)}\int_{\frac{\delta}{2}}^{\delta}r^{2\gamma-2t+\ell'+1+\beta_1+\beta_2}\,|\log r|^{2\nu}dr\right. \\
&\qquad\left.+\int_{\frac{\delta}{2}}^{r_0}r^{2\gamma-\ell'+1+\beta_1+\beta_2}\,|\log r|^{2\nu}dr\right\} \\
&\quad\le\ C\,|\log\delta|^{2\nu}\Big\{\delta^{2\gamma-\ell'+2+\beta_1+\beta_2}+(r_0^{2\gamma-\ell'+2+\beta_1+\beta_2}+\delta^{2\gamma-\ell'+2+\beta_1+\beta_2})\Big\}.
\end{aligned}$$

For $\ell'<2+2\gamma+\beta_1+\beta_2$

$$(r_0^{2\gamma-\ell'+1}+\delta^{2\gamma-\ell'+2+\beta_1+\beta_2})\le C,$$

and for $2+2\gamma+\beta_1+\beta_2\le\ell'\le k$

$$(\rho^{2\gamma-\ell'+2+\beta_1+\beta_2}+\delta^{2\gamma-\ell'+2+\beta_1+\beta_2})\le C\,\delta^{2\gamma-k+2+\beta_1+\beta_2},$$

which yields for $|\alpha|\le k$

$$\int_Q|D^\alpha w|^2\,(1-x_1^2)^{\alpha_1+\beta_1}\,(1-x_2^2)^{\alpha_2+\beta_2}\,dx\le C\,\delta^{2\gamma-k+2+\beta_1+\beta_2}\,|\log\delta|^{2\nu}. \tag{3.18}$$

We now have by (3.16) and (3.18) that for $0<t<1$

$$K(u,t)=\inf_{u=u_0+u_1}\Big(||u_0||_{H^{\ell,\beta}(Q)}+t\,||u_1||_{H^{k,\beta}(Q)}\Big)\le\Big(||v||_{H^{\ell,\beta}(Q)}+t\,||w||_{H^{k,\beta}(Q)}\Big)$$

$$\leq\ C\ |\log\delta|^{\nu}\Big(\delta^{\gamma+1+\frac{\beta_1+\beta_2-\ell}{2}}+t\ \delta^{\gamma+1=\frac{\beta_1+\beta_2-k}{2}}\Big).$$

Selecting $\delta=t^{\frac{2}{k-\ell}}$ we have for $0\leq t<1$

$$K(u,t)\leq C\ |\log\delta|^{\nu}\delta^{\gamma+1+\frac{\beta_1+\beta_2-\ell}{2}}\leq C\ |\log t|^{\nu}t^{\frac{2\gamma+2+\beta_1+\beta_2-\ell}{k-\ell}},$$

and selecting $\theta=\frac{2\gamma+2+\beta_1+\beta_2-\ell}{k-\ell}\in(0,1)$ we get

$$\sup_{0<t<1}\frac{t^{-\theta}K(u,t)}{(1+|\log t|)^{\nu}}\leq C. \tag{3.19}$$

For $t\geq 1$ and θ selected as above, we have

$$\sup_{t>1}\frac{t^{-\theta}K(u,t)}{(1+|\log t|)^{\nu}}\leq ||u||_{H^{\ell,\beta}(Q)}\leq C. \tag{3.20}$$

Equations (3.19) and (3.20) imply $u\in(H^{\ell,\beta}(Q),H^{k,\beta}(Q))_{\theta,\infty,\nu}$ for any integers ℓ and k such that $0<\ell<2+2\gamma+\beta_1+\beta_2<k$ with $\theta=\frac{2\gamma+2+\beta_1+\beta_2-\ell}{k-\ell}$. By Theorem 3.4 (partial reiteration theorem) these spaces are the same. We denoted $B^{s,\beta}_{\nu}(Q)$ with $s=2+2\gamma+\beta_1+\beta_2$, which depends on the value of s but not on the individual value of ℓ,k,β, and θ. ∎

Theorem 3.8 stands for all $\gamma>0$ and integer $\nu\geq 0$. If γ is an integer and $\nu>0$, a sharper result should be expected.

THEOREM 3.9. *If $\gamma>0$ is an integer and integer $\nu>0$, then $u=r^{\gamma}\log^{\nu}r\ \chi(r)\,\Phi(\theta)\in B^{s,\beta}_{\nu-1}(Q)$ with $s=2+2\gamma+\beta_1+\beta_2$ and $\beta_i>-1,i=1,2$.*

Proof. Let $\delta\in(0,r_0)$, and let $u=v+w$ with

$$\begin{aligned} w\ &=r^{\gamma}\log^{\nu}(r+\delta)\ \chi(r)\,\Phi(\theta)\\ &=\sum_{\ell=0}^{\gamma}\begin{pmatrix}\ell\\ \gamma\end{pmatrix}(r+\delta)^{\ell}(-\delta)^{\gamma-\ell}\log^{\nu}(r+\delta)\ \chi(r)\,\Phi(\theta)\end{aligned}$$

and

$$\begin{aligned} v\ &=r^{\gamma}\Big(\log^{\nu}r-\log^{\nu}(r+\delta)\Big)\ \chi(r)\,\Phi(\theta)\\ &=r^{\gamma}\ \Big(\log r-\log(r+\delta)\Big)\sum_{\ell=0}^{\nu-1}\log^{\nu-1-\ell}r\ \ \log^{\ell}(r+\delta).\end{aligned}$$

Note that for $m>\ell$ and $\nu\geq 1$

$$\left|\frac{d^m}{dr^m}(r+\delta)^{\ell}\ \log^{\nu}(r+\delta)\right|\leq C\ (r+\delta)^{\ell-m}\ \log^{\nu-1}(r+\delta)$$

and for $m<\ell$

$$\left|\frac{d^m}{dr^m}(r+\delta)^{\ell}\ \log^{\nu}(r+\delta)\right|\leq C\ (r+\delta)^{\ell-m}\ \log^{\nu}(r+\delta),$$

which implies that for $m > \gamma$

$$
\begin{aligned}
|w|^2_{H^{m,\beta}(Q)} \leq\ & C \sum_{\ell=0}^{\gamma} \delta^{2(\gamma-\ell)} \int_0^1 (r+\delta)^{2(\ell-m)} \,|\log(r+\delta)|^{2(\nu-1)} r^{m+1+\beta_1+\beta_2}\, dr \\
\leq\ & C \sum_{\ell=0}^{\gamma} \delta^{2(\gamma-\ell)} \delta^{2\ell-m+2+\beta_1+\beta_2}\, |\log\delta|^{2(\nu-1)} \\
\leq\ & C\, \delta^{2\gamma-m+2+\beta_1+\beta_2}\, |\log\delta|^{2(\nu-1)}
\end{aligned}
$$

and for $m \leq \gamma$

$$
\begin{aligned}
|w|^2_{H^{m,\beta}(Q)} \leq\ & C \sum_{\ell=0}^{m} \delta^{2(\gamma-\ell)} \int_0^1 (r+\delta)^{2(\ell-m)} \,|\log(r+\delta)|^{2\nu} r^{m+1+\beta_1+\beta_2}\, dr \\
\leq\ & C \sum_{\ell=0}^{m} \delta^{2(\gamma-\ell)} \delta^{2\ell-m+2+\beta_1+\beta_2}\, |\log\delta|^{2\nu} \\
\leq\ & C\, \delta^{2\gamma-m+2+\beta_1+\beta_2}\, |\log\delta|^{2\nu}.
\end{aligned}
$$

Therefore, there holds for $k > 2\gamma + 2 + \beta_1 + \beta_2$

$$
||w||^2_{H^{k,\beta}(Q)} \leq C\, \delta^{2\gamma-k+2+\beta_1+\beta_2}\, |\log\delta|^{2(\nu-1)}. \tag{3.21}
$$

For $0 < r < \delta$ we have

$$
\begin{aligned}
|v| =\ & r^\gamma \log\left(1+\frac{\delta}{r}\right) \left|\sum_{\ell=0}^{\nu-1} \log^{\nu-1-\ell} r\ \log^\ell(r+\delta)\right| \\
\leq\ & C\, r^\gamma \log\frac{\delta}{r} \sum_{\ell=0}^{\nu-1} |\log r|^{\nu-1-\ell}\, |\log(r+\delta)|^\ell \\
\leq\ & C\, r^\gamma \log\frac{\delta}{r} \sum_{\ell=0}^{\nu-1} |\log r|^{\nu-1-\ell}\, |\log\delta|^\ell
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^\delta |v|^2 (1-x_1^2)^{\beta_1} (1-x_2^2)^{\beta_2}\, dx \\
&\quad\leq C \int_0^\delta r^{2\gamma+1+\beta_1+\beta_2} |\log\frac{\delta}{r}|^2 \sum_{\ell=0}^{\nu-1} |\log r|^{2(\nu-1-\ell)}\, |\log\delta|^{2\ell} dr \\
&\quad\leq C\, \delta^{2\gamma+2+\beta_1+\beta_2} |\log\delta|^{2(\nu-1)} \int_0^1 z^{2\gamma} |\log z|^2 \sum_{\ell=0}^{\nu-1} (|\log z| + |\log\delta|)^{2(\nu-1-\ell)}\, dz \\
&\quad\leq C\, \delta^{2\gamma+2+\beta_1+\beta_2} |\log\delta|^{2(\nu-1)}.
\end{aligned} \tag{3.22}
$$

Also we have for $\delta < r < r_0$

$$|v| \leq r^\gamma \left| \log\left(1+\frac{\delta}{r}\right)\right| \sum_{\ell=0}^{\nu-1} |\log r|^{\nu-1-\ell}\, |\log\delta|^\ell \leq C\,\delta\, r^{\gamma-1} \sum_{\ell=0}^{\nu-1} |\log r|^{\nu-1-\ell}\, |\log\delta|^\ell$$

and

$$\begin{aligned}\int_\delta^1 |v|^2(1-x_1^2)^{\beta_1}(1-x_2^2)^{\beta_2}\, dx &\leq C\,\delta^2 \int_\delta^1 r^{2\gamma-1+\beta_1+\beta_2} \sum_{\ell=0}^{\nu-1} |\log r|^{2(\nu-1-\ell)}\, |\log\delta|^{2\ell} dr \\ &\leq C\ \delta^{2\gamma+2+\beta_1+\beta_2} |\log\delta|^{2(\nu-1)},\end{aligned}$$

which together with (3.22) yields

$$||v||^2_{H^{0,\beta}(Q)} \leq C\ \delta^{2\gamma+2+\beta_1+\beta_2} |\log\delta|^{2(\nu-1)}. \tag{3.23}$$

Combining (3.21) and (3.22) we have

$$\begin{aligned} K(t,u) = &\ \inf_{u=u_1+u_2}\Big(||u_1||_{H^{0,\beta}(Q)} + t\ ||u_2||_{H^{k,\beta}(Q)}\Big) \\ \leq &\ ||v||_{H^{0,\beta}(Q)} + t\ ||w||_{H^{k,\beta}(Q)} \\ \leq &\ C\ |\log\delta|^{\nu-1}\Big(\delta^{\gamma+1+\frac{\beta_1+\beta_2}{2}} + t\ \delta^{\gamma+1+\frac{(\beta_1+\beta_2-k)}{2}}\Big). \end{aligned}$$

Selecting $\delta = t^{2/k}$ and $\theta = (2\gamma+2+\beta_1+\beta_2)/k$ we have for $0 < t < 1$

$$\frac{t^{-\theta}K(t,u)}{(1+|\log t|)^{\nu-1}} \leq C$$

and for $t \geq 1$

$$\frac{t^{-\theta}K(t,u)}{(1+|\log t|)^{\nu-1}} \leq C\ ||u||_{H^{0,\beta}(Q)},$$

which leads to

$$\Phi_{\theta,\infty,\nu-1}(u) \leq C$$

and $u \in B^{1+2\gamma,\beta}_{\nu-1}(Q)$.

Combining Theorems 3.8 and 3.9 we have the following theorem. ☐

THEOREM 3.10. *Let $u = r^\gamma \log^\nu r\ \chi(r)\,\Phi(\theta)$ be given in* (3.15). *Then $u \in B^{s,\beta}_{\nu^*}(Q)$ with $s = 2+2\gamma+\beta_1+\beta_2$ and*

$$\nu^* = \begin{cases} \nu & \text{if } \gamma \text{ is not an integer,} \\ \nu-1 & \text{if } \gamma \text{ is an integer and } \ \nu > 0, \\ 0 & \text{if } \ \nu = 0. \end{cases} \tag{3.24}$$

We are interested in the special case that $\beta = (-1/2,-1/2)$ for the approximability of the singular functions in the H^1- or energy norm without weight.

COROLLARY 3.11. *Let $u = r^\gamma \log^\nu r\ \chi(r)\,\Phi(\theta)$. Then $u \in B^{s,\beta}_{\nu^*}(Q)$ with $s = 1+2\gamma, \beta = (-1/2,-1/2)$, and ν^* given by* (3.24).

3.3. Approximation of the functions of $r^\gamma \log^\nu r$-type based on the spaces $B^{s,\beta}_\nu(Q)$. We now are able to establish sharp approximation theorems for singular functions of $r^\gamma \log^\nu r$-type in the frame of modified Jacobi-weighted Besov space $B^{s,\beta}_\nu(Q)$, $\nu > 0$.

THEOREM 3.12. *Let $u \in B^{s,\beta}_\nu(Q)$ with $s > 0$, and $\beta_i > -1$, $i = 1, 2$. Then the Jacobi -projection u_p of u on $P_p(Q)$ with $p+1 \geq s$ satisfies for $\ell < s$*

$$||u - u_p||_{H^{\ell,\beta}(Q)} \leq C\, p^{-(s-\ell)}(1+\log p)^\nu ||u||_{B^{s,\beta}_\nu(Q)}. \tag{3.25}$$

Proof. We write $B^{s,\beta}_\nu(Q) = (H^{\ell,\beta}(Q), H^{k,\beta}(Q))_{\theta,\infty,\nu}$ for $k > s > \ell$, with $\theta = \frac{s-\ell}{k-\ell}$ and let $E\,u = u - u_p$. Arguing as in the proof of Theorem 2.3, we have

$$||E\,u||_{H^{\ell,\beta}(Q)} \leq C\, ||u||_{H^{\ell,\beta}(Q)}$$

and

$$||E\,u||_{H^{\ell,\beta}(Q)} \leq C\, p^{-(k-\ell)}\, ||u||_{H^{k,\beta}(Q)}.$$

Due to the property (P2) of the modified interpolation spaces,

$$\Big(H^{\ell,\beta}(Q), H^{\ell,\beta}(Q)\Big)_{\theta,\infty,\nu} = H^{\ell,\beta}(Q),$$

and by Theorem 3.4 E is an operator $B^{s,\beta}_\nu(Q) \longrightarrow H^{\ell,\beta}(Q)$, and

$$\begin{aligned} ||E\,u||_{H^{\ell,\beta}(Q)} &\leq C\Big(1 + (k-\ell)\log p\Big)^\nu p^{-\theta(k-\ell)}\, ||u||_{B^{s,\beta}_\nu(Q)} \\ &\leq C\,(1+\log p)^\nu\; p^{-(s-\ell)}||u||_{B^{s,\beta}_\nu(Q)}, \end{aligned}$$

which is (3.25). $\square$

COROLLARY 3.13. *Let $u(x) = r^\gamma \log^\nu r\ \chi(r)\ \Phi(\theta)$ with $\gamma > 0$. Then there exists a polynomial $\varphi \in P_p(Q)$ with $p \geq 1 + 2\gamma + \beta_1 + \beta_2 - \ell$ such that for $\ell < 2 + 2\gamma + \beta_1 + \beta_2, \beta_i > -1, i = 1, 2,$*

$$||u - \varphi||_{H^{\ell,\beta}(Q)} \leq C\, p^{-(2+2\gamma+\beta_1+\beta_2-\ell)}(1+\log p)^{\nu^*}||u||_{B^{s,\beta}_{\nu^*}(Q)}. \tag{3.26}$$

The approximabilities of functions of $r^\gamma \log^\nu r$-type for general β, given in Corollary 3.13, does not give us the estimation of approximation error in H^1-norm. Combining Corollary 3.11 and Theorem 3.12 and using the property (2.15), we will have a desired result.

THEOREM 3.14. *Let $u(x) = r^\gamma \log^\nu r\ \chi(r)\ \Phi(\theta)$ with $\gamma > 0$. Then there exists a polynomial $\varphi \in P_p(Q)$ with $p \geq 2\gamma$ such that*

$$||u - \varphi||_{H^1(R_0)} \leq C\, p^{-2\gamma}(1+\log p)^{\nu^*}, \tag{3.27}$$

where R_0 is given in (2.14) and ν^* is given in (3.24).

Proof. By Theorem 3.10 and Corollary 3.11, $u \in B^{s,\beta}_{\nu^*}(Q)$, $s = 1 + 2\gamma$, $\beta = (-1/2, -1/2)$. The arguments used in the proof of Theorem 2.6 can be carried over except using the estimate (3.25) of Theorem 3.12 instead of (2.10) of Theorem 2.3. $\square$

As mentioned in the previous section, for applications to boundary value problems on polygonal domains, the singular solution of $r^\gamma \log^\nu r$-type may satisfy a homogeneous Dirichlet condition. We should approximate such singular functions by polynomials vanishing on one or two lines lying in the support R_0.

THEOREM 3.15. *Let* $u(x) = r^\gamma \log^\nu r\, \chi(r)\, \Phi(\theta)$ *given in* (2.11), *and* $u = 0$ *on both lines* $\theta = \theta_\kappa = \arctan(\kappa)$ *and* $\theta = \theta_{1/\kappa} = \arctan(1/\kappa)$ *(resp.,* $u = 0$ *on one line* $\theta = \theta_\kappa$*) with* $1 > \kappa \geq \kappa_0$, *with* κ_0 *given in* (2.15). *Then there exists a polynomial* $\varphi(x) \in P_p(Q)$, $p \geq \max\{2, 2\gamma\}$ *(resp.,* $p \geq \max\{1, 2\gamma\}$*) such that* $\varphi|_{\theta=\theta_\kappa} = \varphi|_{\theta=\theta_{1/\kappa}} = 0$ *(resp.,* $\varphi|_{\theta=\theta_\kappa} = 0$*) and*

$$||u - \varphi(x)||_{H^1(R_0)} \leq C\, p^{-2\gamma}(1 + \log p)^{\nu^*} ||u||_{B_{\nu^*}^{s,\beta}(Q)} \tag{3.28}$$

with constant C *independent of* p, *where* R_0 *is the support of* u *given in* (2.14) *and* ν^* *is given in* (3.24).

Proof. Let $u_2 = \frac{u(x)}{\zeta_2(x)}$ if u vanishes on both lines where $\zeta_2(x)$ is given in (2.24a). Due to the property (2.15), $u_2 \in B_{\nu^*}^{s,\beta^{[2]}}(Q)$ with $s = 1+2\gamma$ and $\beta^{[2]} = (3/2, 3/2), \beta = (-1/2, -1/2)$, and

$$||u_2(x)||_{B_{\nu^*}^{1+2\gamma,\beta^{[2]}}(Q)} \leq C\, ||u(x)||_{B_{\nu^*}^{1+2\gamma,\beta}(Q)}.$$

By Theorem 3.12, there is a polynomial $\phi \in P_{p-2}(Q)$ with $p \geq \max\{2, 2\gamma\}$ such that

$$||u_2 - \phi||_{H^{1,\beta^{[2]}}(Q)} \leq C\, p^{-2\gamma}(1 + \log p)^{\nu^*} ||u_2(x)||_{B_{\nu^*}^{1+2\gamma,\beta^{[2]}}(Q)}.$$

Let $\varphi(x) = \zeta_2(x)\phi(x)$, which vanishes on both lines. Then, using Lemma 2.8 and the property (2.15) and arguing as in the proof of Theorem 2.10, we have (3.28). The above arguments can be carried out for u vanishing on one line, by replacing u_2 and $\zeta_2(x)$ with u_1 and $\zeta_1(x)$, respectively, using Lemma 2.9 instead of Lemma 2.8. □

3.4. The extreme values of the indices s and ν' of weighted Besov space $B_{\nu'}^{s,\beta}(Q)$ for the functions of $r^\gamma \log^\nu r$-type. In previous sections we have shown that the functions of $r^\gamma \log^\nu r$-type with $\gamma > 0$ and integer $\nu \geq 0$ belong to the spaces $B_{\nu^*}^{s,\beta}(Q)$ with $s = 1 + 2\gamma$, $\beta = (-1/2, -1/2)$, and ν^* given in (3.24). Is $(1 + 2\gamma)$ the maximum value of s, and is ν^* the minimum value of ν' for $B_{\nu'}^{s,\beta}(Q)$ which contains these functions? This question is equivalent to asking whether $p^{-2\gamma}(1+\log p)^{\nu^*}$ is the optimal rate of the approximation error to these functions. The following theorem tells that $1 + 2\gamma$ and ν^* are the optimal values of s and ν' for functions of this type.

THEOREM 3.16. *Let* $u = r^\gamma \log^\nu r\ \chi(r)\ \Phi(\theta)$, $\gamma > 0$, $\nu \geq 0$, *be given in* (3.15). *Then* $u \bar{\in} B_{\nu'}^{s,\beta}(Q)$ *with* $\beta = (-1/2, -1/2)$ *for any* $s > 1 + 2\gamma$ *or for any* $\nu' < \nu^*$ *if* $s = 1 + 2\gamma$, *where* ν^* *is given in* (3.24).

Proof. Suppose that $u \in B_{\nu'}^{s,\beta}(Q)$ with $s > 1 + 2\gamma$ or $s = 1 + 2\gamma$, $\nu' < \nu^*$. Then by Theorem 3.14

$$\inf_{\varphi \in P_p(Q)} ||u - \varphi||_{H^1(R_0)} \leq C\, p^{-(s-1)}(1 + \log p)^{\nu'}. \tag{3.29}$$

On the other hand, it has been proved in [3, Theorems 2.9–2.11] that

$$\inf_{\varphi \in P_p(Q)} ||u - \varphi||_{H^1(R_0)} \geq C\, p^{-2\gamma}(1 + \log p)^{\nu^*}. \tag{3.30}$$

The contradiction between (3.29) and (3.30) implies the assertion of the theorem. □

TABLE 4.1

The value of k and s in Sobolev, Besov, and weighted Besov spaces for functions of r^γ-type and $r^\gamma \log^\nu r$-type.

Space	$H^k(Q)$	$H^{k,\beta}(Q)$	$H^s(Q)$	$B^s(Q)$	$B^{s,\beta}(Q)$	$B^{s,\beta}_{\nu *}(Q)$
r^γ	$1+[\gamma]$	$1+[2\gamma]$	$1+\gamma-\varepsilon$	$1+\gamma$	$1+2\gamma$	$1+2\gamma$
$r^\gamma \log^\nu r$	$1+[\gamma]$	$1+[2\gamma]$	$1+\gamma-\varepsilon$	$1+\gamma-\varepsilon$	$1+2\gamma-\varepsilon$	$1+2\gamma$

TABLE 4.2

Accuracy of approximation of the h- and p-version to functions of r^γ- and $r^\gamma \log^\nu r$-type based on Sobolev, Besov, and weighted Besov spaces.

	h-version		p-version			
Space	$H^s(Q)$	$B^s(Q)$	$H^s(Q)$	$B^s(Q)$	$B^{s,\beta}(Q)$	$B^{s,\beta}_{\nu *}(Q)$
r^γ	$h^{\gamma-\varepsilon}$	h^γ	$p^{-(\gamma-\varepsilon)}$	$p^{-\gamma}$	$p^{-2\gamma}$	$p^{-2\gamma}$
$r^\gamma \log^\nu r$	$h^{\gamma-\varepsilon}$	$h^{\gamma-\varepsilon}$	$p^{-(\gamma-\varepsilon)}$	$p^{-(\gamma-\varepsilon)}$	$p^{-(2\gamma-\varepsilon)}$	$p^{-2\gamma}\log^{\nu^*} p$

4. Concluding remarks. In order to effectively analyze approximability of functions including singular functions of r^γ-type and $r^\gamma \log^\nu r$-type, we successfully introduced the weighted Besov spaces $B^{s,\beta}$ and $B^{s,\beta}_\nu$, respectively. These two spaces could lead to the optimal convergence rate and to the sharp direct and inverse theorems for the p-version of the finite element method (see [4, 5]).

As many function spaces with and without weights, these two spaces characterize the singularities of solutions of problems in nonsmooth domains and to serve various mathematical interests, such as approximation. The selection of function spaces, to best serve our numerical analysis, depends on not only the characters of the solutions but also the numerical methods selected to approximate the solutions. The values of k and s in Sobolev spaces and Besov spaces with and without weights for functions of r^γ-type and $r^\gamma \log^\nu r$-type are listed in Table 4.1, where $\gamma > 0$ and $\nu \geq 0$, k is an integer, s is real, $[\gamma]$ denotes the largest integer $< \gamma$, etc. These values directly yield the rates of convergence given in Table 4.2 for the h- and p-versions of the finite element method, with ν^* given in (3.24), and the error is measured in the H^1-norm.

For the elliptic problems on smooth domains, the regularity results in Sobolev space $H^s(Q)$ may very adequately serve numerical analysis. For the linear elliptic problems on nonsmooth domain, the regularity in usual Besov space $B^s(Q)$ can lead to the optimal rate of convergence for the h-version of the finite element method, and it has been shown in this paper that the Jacobi-weighted Besov spaces $B^{s,\beta}(Q)$ and $B^{s,\beta}_\nu(Q)$, $\nu > 0$, are the best mathematical tools to analyze the convergence rate of the p-version of the finite element method. It is well known that the regularity theory, which can lead to the exponential convergence for the h-p version associated with geometric meshes, is the one in the framework of countably normed spaces $\mathcal{B}^\ell_\beta(Q)$ (see [2, 7, 18]), instead of the Jacobi-weighted Besov spaces $B^{s,\beta}_\nu(Q)$.

The Jacobi-weighted Besov spaces can be introduced for analysis of approximability of singular functions in three dimensions. Since the singular functions have different characters in different singular neighborhoods, the Jacobi-weighted Besov spaces should be furnished with different weight functions accordingly. We refer to

[6] for the details.

REFERENCES

[1] M. Azaiez, C. Bernardi, M. Dauge, and Y. Maday, *Spectral Methods for Axisymmetric Domains*, Ser. Appl. Math. 3, Gauthier-Villars & North-Holland, Paris, 1999.

[2] I. Babuška and B.Q. Guo, *Regularity of the solution of elliptic problems with piecewise analytic data. Part* I. *Boundary value problems for linear elliptic equation of second order*, SIAM J. Math. Anal., 19 (1988), pp. 172–203.

[3] I. Babuška and B.Q. Guo, *Optimal estimates for lower and upper bounds of approximation errors in the p-version of the finite element method in two dimensions*, Numer. Math., 85 (2000), pp. 219–255.

[4] I. Babuška and B.Q. Guo, *Direct and inverse approximation theorems of the p-version of the finite element method in the framework of weighted Besov spaces, part* 2*: Optimal convergence of the p-version of the finite element method*, Math. Models Methods Appl. Sci., to appear.

[5] I. Babuška and B.Q. Guo, *Direct and Inverse Approximation Theorems of the p-Version of the Finite Element Method in the Framework of Weighted Besov Spaces, Part* 3*: Inverse Approximation Theorems*, TICAM report 99-32, University of Texas, Austin, TX, 1999.

[6] I. Babuška and B.Q. Guo, *Direct and Inverse Approximation Theorems of the p-Version of the Finite Element Method in Three Dimensions, Part* 1: *Approximability of Functions in the Framework of Weighted Besov Spaces*, in preparation.

[7] I. Babuška and B.Q. Guo, *The h-p version of the finite element method for domains with curved boundaries*, SIAM J. Numer. Anal., 25 (1988), pp. 837–861.

[8] I. Babuška and M. Suri, *The optimal convergence rate of the p-version of the finite element method*, SIAM J. Numer. Anal., 25 (1987), pp. 750–776.

[9] I. Babuška and M. Suri, *The h-p version of the finite element method with quasi-uniform meshes*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 199–238.

[10] I. Babuška, M. Szabó, and N. Katz, *The p-version of the finite element method*, SIAM J. Numer. Anal., 18 (1981), pp. 515–545.

[11] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin, New York, 1976.

[12] C. Bernardi and Y. Maday, *Spectral methods*, in Handbook of Numerical Analysis, Vol. V, Part 2, P.G. Ciarlet and J.L. Lions, eds., North-Holland, Amsterdam, 1997, pp. 209–475.

[13] P. Borwein and T. Erdelyi, *Polynomial and Polynomial Inequalities*, Springer-Verlag, New York, 1995.

[14] M.R. Dorr, *The approximation theory for the p-version of the finite element method*, SIAM J. Numer. Anal., 21 (1984), pp. 1181–1207.

[15] M.R. Dorr, *The approximation solutions of elliptic boundary-value problems via the p-version of the finite element method*, SIAM J. Numer. Anal., 23 (1986), pp. 58–77.

[16] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1975.

[17] W. Gui and I. Babuška, *The h, p, and h-p versions of the finite element method in* 1 *dimension, Part* 1. *The error analysis of the p-version*, Numer. Math., 49 (1986), pp. 577–612.

[18] B.Q. Guo and I. Babuška, *The h-p version of the finite element method, Part* 1: *The basic approximation results, Part* 2 : *The general results and application*, Comput. Mech., 1 (1986), pp. 22–41 and pp. 203–220.

[19] B.Y. Guo, *Jacobi approximation in certain Hilbert spaces and their applications to singular differential equations*, J. Math. Anal. Appl., 243 (2000), pp. 373–408.

[20] B.Y. Guo, *Jacobi spectral approximations to differential equations on the half line*, J. Comput. Math., 18 (2000), pp. 95–112.

[21] B.Y. Guo, *Gegenbauer approximation in certain Hilbert spaces and its application to differential equations on the whole line*, J. Math. Anal. Appl., 226 (1998), pp. 180–206.

[22] B.-Y. Guo, *Gegenbauer approximation in certain Hilbert spaces and its application to singular differential equations*, SIAM J. Numer. Anal., 37 (2000), pp. 621–645.

[23] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, UK, 1959.

[24] A. Pinkus, *N-Width in Approximation Theory*, Springer-Verlag, Berlin, 1985.

[25] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.