DIRECT AND INVERSE APPROXIMATION THEOREMS FOR THE p-VERSION OF THE FINITE ELEMENT METHOD IN THE FRAMEWORK OF WEIGHTED BESOV SPACES

Part II : Optimal Rate of Convergence of the p-version Finite Element Solutions

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Abstract

This is the second of a series devoted to the direct and inverse approximation theorems of the p-version of the finite element method in the framework of the weighted Besov spaces. In this paper, we combine the approximability of singular solutions in the Jacobi-weighted Besov spaces, which were analyzed in the previous paper [4], with the technique of partition of unity in order to prove the optimal rate of convergence of the p-version of the finite element method for elliptic boundary value problems on polygonal domains.

1 Introduction

The p-version, the h-version and the h-p version are the three basic approaches of the finite element method(FEM). The p-version of FEM achieves accuracy by increasing the element degree p on a fixed mesh. The classical Sobolev and Besov spaces are effective tools for the error analysis of the h-version of FEM, but they are not adequate for the analysis of the p-version of FEM. In particular, when exact solutions of boundary value problems on non-smooth domains have singularities, these spaces are not appropriate for proving the optimal convergence of the p-version of FEM.

In order to address this issue, various function spaces have been used since the early 1980's. Very recently the Jacobi-weighted Besov and Sobolev spaces were introduced in [4]. They are appropriate spaces for the analysis of the p-version finite element approximation of singular solutions of $r^\gamma \log^\nu r$-type. Singular solutions of $r^\gamma \log^\nu r$-type arise from elliptic problems on non-smooth domains, for instance, the elasticity problems on domains with piecewise smooth boundary. Hence, the study on the singular solutions of this type and their best finite element approximation in the framework of proper spaces is of great importance theoretically as well as practically.

The weighted Besov spaces $B_\nu^{s,\beta}$ and Sobolev spaces $H^{s,\beta}$ introduced in [4] for two dimensions are furnished with Jacobi weights depending on s. The unique dynamical structure of the Jacobi-weighted Besov and Sobolev spaces leads to extremely important approximation properties such as the uniform projections and sharp inverse inequality. These spaces allow us to show that the polynomial approximation is optimal in the sense of N-width(see [17]), and play a significant role in the proof of the inverse approximation theorems in [5].

The main result of this paper is Theorem 3.4 which gives the optimal upper and lower bounds of the error in the p-version of FEM for elliptic boundary value problems on polygonal domains Ω. More precisely, when the solutions u have singularities of $r^\gamma \log^\nu r$-type in each neighborhood of vertices of Ω, we have shown that, as p increases,

$$C_1 p^{-2\gamma}(1+\log^{\nu^*} p) \leq \|u-u_p\|_{H^1(\Omega)} \leq C_2 p^{-2\gamma}(1+\log^{\nu^*} p),$$

where $\nu^* = \nu - 1$ if γ is an integer > 0, and $\nu^* = \nu$ otherwise.

We now provide a brief history of the approximation theory of the p-version of FEM. The first error analysis for the p-version of FEM was given in [11] in 1981 for solutions $u \in H^k(\Omega), k \geq 0$:

$$\|u-u_p\|_{H^1(\Omega)} \leq C(\epsilon) p^{-k+1+\epsilon} \|u\|_{H^k(\Omega)},$$

and for the solution u with singularity of r^γ-type,

$$\|u-u_p\|_{H^1(\Omega)} \leq C(\epsilon) p^{-2\gamma+\epsilon},$$

with $\epsilon > 0$, arbitrary and constant $C(\epsilon)$ depending on ϵ. The upper bound of the approximation error was not sharp. In 1987 the result was improved in [9] where ϵ was removed. Both papers did not deal with the singularity of $r^\gamma \log^\nu r$-type, and they did not give lower bound of the eroor. The singularity of r^γ-type was treated by special techniques in these papers. The singular solution $u = r^\gamma \log^\nu r$ was studied by [10] in 1987. In the h-p setting with quasi-uniform mesh. It was shown that

$$\|u-u_p\|_{H^1(\Omega)} \leq C p^{-2\gamma}(1+\log^\nu p).$$

where there was no distinction between non-integer γ and integer γ. The upper bound of the error, as given above, is not optimal for integer γ. Also the lower bound of the error was not addressed in [10]. Since then there had been no progress on the approximation theory of the p-version of FEM in two dimensions until the very late 1990's.

The p-version of FEM is closely related to the spectral methods. But the singularities of $r^\gamma \log^\nu r$-type and the related best approximation were not addressed in [1, 13] and [18, 19, 20, 21], where Sobolev spaces with a fixed Jacobi-weight was introduced and was used to analyze the upper bound of the error in the spectral method for singular equations with degenerate coefficients in differential operators.

Using the framework of the Jacobi-weighted Besov spaces and Sobolev spaces, which are furnished with dynamic Jacobi-weights, we derived both upper and lower bounds of the error measured in H^1-norm for elliptic boundary value problems on non-smooth domains. Moreover, we showed that the two bounds are of the same order. Thus, the issue on the optimal rate of convergence in the p-version finite element approximation in two dimensions is completely resolved. For the first time in the context of singular solutions, the lower bound of the error is proved in this paper. The proof for the upper bound of the error in this paper is quite different from those in the previous papers [9, 10, 11]. Namely, we combine the technique of partition of unity with the approximability of singular solutions in the framework of the Jacobi-weighted Besov spaces. It not only simplifies the proof, but also make it more robust to analysis for all dimensions. A simple version of the theorem was addressedd in [3] without complete and detailed proof incorporating all types of boundary conditions.

This is the second paper of a series devoted to the direct and inverse theorems of the p-version of FEM in two dimensions. In Section 2 we shall quote the notations and definitions of Jacobi-weighted Besov spaces $B^{s,\beta}$ and $B^{s,\beta}_{\nu}$, and major theorems on approximability of functions in these spaces, which were proved in [4]. The upper and lower bounds of the approximation error in the p-version of FEM are analyzed in Section 3, which proves the optimality of the error estimate. Some concluding remarks are given in the last section.

2 Preliminaries

2.1 Weighted Besov spaces $B^{s,\beta}(Q)$ and $B_\nu^{s,\beta}(Q)$

Let $Q = I^2 = (-1,1)^2$, and let

$$w_{\alpha,\beta}(x) = \prod_{i=1}^{2}(1-x_i^2)^{\alpha_i+\beta_i} \tag{2.1}$$

be a weight function with integer $\alpha_i \geq 0$ and real number $\beta_i > -1$, which is refereed to as Jacobi weight.

The weighted Sobolev space $H^{k,\beta}(Q)$ is defined as a closure of C^∞ functions in the norm with the Jacobi weight

$$||u||^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|=0}^{k} \int_Q |D^\alpha u|^2 \, w_{\alpha,\beta}(x)\, dx \tag{2.2}$$

where $D^\alpha u = u_{x_1^{\alpha_1}, x_2^{\alpha_2}}$, $\alpha = (\alpha_1, \alpha_2), |\alpha| = \alpha_1 + \alpha_2$, and $\beta = (\beta_1, \beta_2)$. By $|u|_{H^{k,\beta}(Q)}$ we denote the semi-norm,

$$|u|_{H^{k,\beta}(Q)} = \sum_{|\alpha|=k} \int_Q |D^\alpha u|^2 \, w_{\alpha,\beta}(x)\, dx$$

The weighted Besov space $B^{s,\beta}(Q)$ is defined as an interpolation space by the K-method, i.e.

$$B^{s,\beta}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty}$$

where $0 < \theta < 1, s = (1-\theta)\ell + \theta k$, ℓ and k are integers, $\ell < k$, and

$$||u||_{B^{s,\beta}(Q)} = \sup_{t>0} t^{-\theta}\, K(t,u) \tag{2.3}$$

where

$$K(t,u) = \inf_{u=v+w} \Big(||v||_{H^{\ell,\beta}(Q)} + t||w||_{H^{k,\beta}(Q)}\Big) \tag{2.4}$$

The modified weighted Besov space $B_\nu^{s,\beta}(Q)$ with $\nu \geq 0$ is defined as an interpolation space

$$B_\nu^{s,\beta}(Q) = \Big(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\Big)_{\theta,\infty,\nu}$$

with a modified norm

$$||u||_{B_{\nu}^{s,\beta}(Q)} = \sup_{t>0} K(t,u)\, \frac{t^{-\theta}}{(1+|\log t|)^{\nu}} \tag{2.5}$$

Remark 2.1. The space $B_{\nu}^{s,\beta}(Q)$ with $\nu > 0$ is only a uniform interpolation space, and $B_{0}^{s,\beta}(Q) = B^{s,\beta}(Q)$ is a standard exact interpolation space of exponent θ. For the definition and properties of exact interpolation spaces of exponent θ we refer to [12], and for the proof of various properties of uniform interpolation space $B_{\nu}^{s,\beta}(Q)$ with integer $\nu > 0$ we refer to [4].

Let $P_p(Q)$ be the set of all polynomials of degree (separate) $\leq p$. Then we have the following approximation property:

Theorem 2.1. (Theorem 2.2 of [4]) Let $u \in H^{k,\beta}(Q)$ with integer $k \geq 1$, $\beta_i > -1$, $1 \leq i \leq 2$, and u_p be its $H^{0,\beta}(Q)$-projection on $P_p(Q)$. Then we have for integer $\ell \leq k \leq p+1$

$$|u-u_p|_{H^{\ell,\beta}(Q)} \leq C \left(\frac{1}{p}\right)^{k-\ell} |u|_{H^{k,\beta}(Q)}\,. \tag{2.6}$$

$\square$

Remark 2.2. For $u \in H^{k,\beta}(Q)$ with integer $k \geq 0$, the $H^{0,\beta}(Q)$-projection of u on $P_p(Q)$ is $H^{\ell,\beta}(Q)$-projection on $P_p(Q)$ for any $0 \leq \ell \leq k$.

Using the properties of interpolation spaces, we have approximation theorems for functions in Besov spaces $B^{s,\beta}(Q)$ and $B_{\nu}^{s,\beta}(Q)$.

Theorem 2.2. (Theorem 2.3 of [4]) Let $u \in B^{s,\beta}(Q)$, $s > 0$, $\beta_i = (\beta_1, \beta_2)$, $\beta_i > -1$, $1 \leq i \leq 2$, and let u_p be the $H^{0,\beta}(Q)$- projection of u on $P_p(Q)$. Then for any integer $\ell < s$ there holds

$$||u-u_p||_{H^{\ell,\beta}(Q)} \leq C \left(\frac{1}{p}\right)^{s-\ell} ||u||_{B^{s,\beta}(Q)} \tag{2.7}$$

with constant C independent of p. $\square$

Theorem 2.3 (Theorem 3.7 of [4]) Let $u \in B_{\nu}^{s,\beta}(Q)$, $s > 0$, $\beta_i = (\beta_1, \beta_2)$, $\beta_i > -1$, $1 \leq i \leq 2$, integer $\nu \geq 0$, and let u_p be the $H^{0,\beta}(Q)$- projection of u on $P_p(Q)$.

Then for any integer $\ell < s$ there holds

$$||u - u_p||_{H^{\ell,\beta}(Q)} \leq C \left(\frac{1}{p}\right)^{s-\ell} (1 + \log p)^{\nu} \, ||u||_{B_{\nu}^{s,\beta}(Q)} \tag{2.8}$$

with constant C independent of p. $\square$

2.2 Approximability of singular functions of r^γ-type and $r^\gamma \log^\nu r$-type

Let (r, θ) be the polar coordinates with respect to the vertex $(-1,-1)$, where $r = \{(x_1+1)^2 + (x_2+1)^2\}^{1/2}$, $\theta = \arctan\left(\frac{x_2+1}{x_1+1}\right)$, and let for $\gamma > 0$ and integer $\nu \geq 0$

$$u(x) = r^\gamma \, \chi(r) \, \Phi(\theta) \tag{2.9}$$

and

$$v(x) = r^\gamma \log^\nu r \, \chi(r) \, \Phi(\theta) \tag{2.10}$$

are functions defined on $Q = (-1,1)^2$, where $\chi(r)$ and $\Phi(\theta)$ are C^∞ functions such that for $0 < r_0 < 2$

$$\chi(r) = \begin{cases} 1 & \text{for} \quad 0 < r \leq \frac{r_0}{2} \\ 0 & \text{for} \quad r \geq r_0 \end{cases} \tag{2.11}$$

and for $\theta_0 \in (0, \pi/2)$

$$\Phi(\theta) = 0 \qquad \text{for} \qquad \theta \notin (\theta_0, \pi/2 - \theta_0). \tag{2.12}$$

Therefore, $u(x)$ has a support $R_0 = R_{r_0,\theta_0}$ with

$$R_{r_0,\theta_0} = \left\{ x \in Q \;\middle|\; r < r_0, \quad \theta_0 < \theta < \pi/2 - \theta_0 \right\} \tag{2.13}$$

which is shown in Fig. 2.1. For $x \in R_0$ we have $0 < 1 - r_0 < (1 - x_i) < 2$, and

$$\frac{1}{\kappa_0} \leq \frac{1 + x_2}{1 + x_1} \leq \kappa_0 = \tan\theta_0. \tag{2.15}$$

Now we characterize the singularity of $u(x)$ and $v(x)$ in terms of the weighted Besov spaces $B^{s,\beta}(Q)$ and $B_\nu^{s,\beta}(Q)$.

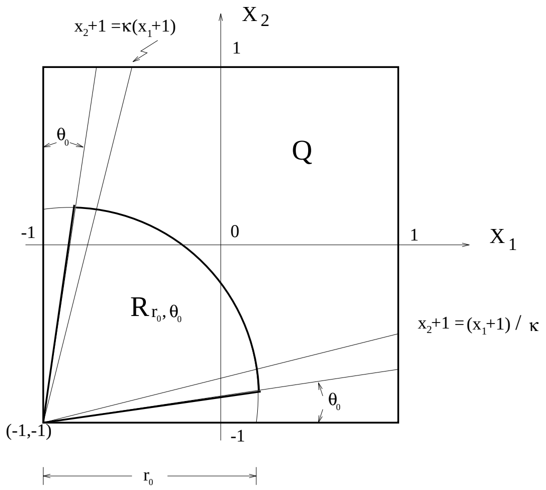


Fig. 2.1 Square Domain Q and subregion R_{r_0,θ_0}

Theorem 2.4 (Theorem 2.4 of [4]) Let $u(x) = r^\gamma\,\chi(r)\,\Phi(\theta)$ be given by (2.9) with $\gamma > 0$. Then $u \in B^{s,\beta}(Q)$ with $s = 1 + 2\gamma$, $\beta = (-1/2, -1/2)$. $\square$

Theorem 2.5. (Theorem 3.4-3.5 of [4]) Let $v = r^\gamma \log^\nu r\,\chi(r)\,\Phi(\theta)$ given in (2.10) with $\gamma > 0$ and integer $\nu \geq 0$. Then $v \in B^{s,\beta}_{\nu^*}(Q)$ with $s = 1+2\gamma$, $\beta = (-1/2, -1/2)$, and

$$\nu^* = \begin{cases} \nu & \text{if } \gamma \text{ is not an integer,} \\ \nu - 1 & \text{if } \gamma \text{ is an integer and } \quad \nu > 0, \\ 0 & \text{if } \quad \nu = 0. \end{cases} \tag{2.14}$$

$\square$

Combination of Theorems 2.4-2.5 and Theorems 2.2-2.3 leads to the approximabilities of the singular functions of r^γ-type and $r^\gamma \log^\nu r$-type.

Theorem 2.6. (Theorem 2.6-2.8 of [4]) Let $u(x) = r^\gamma\,\chi(r)\,\,\Phi(\theta)$ with $\gamma > 0$ as given in (2.9). Then:

(i) There exists a polynomial $\varphi \in P_p(Q)$ such that

$$||u - \varphi||_{H^1(R_0)} \leq C \left(\frac{1}{p}\right)^{2\gamma} \tag{2.15}$$

where R_0 is given in (2.13);

(ii) If $u|_{\theta=\theta_\kappa,\theta_{1/\kappa}} = 0, \theta_\kappa = tan^{-1}\kappa \geq \theta_0$, there exists a polynomial $\varphi \in P_p(Q)$ such that φ vanishes on the lines $\theta = \theta_k$ and $\theta = \theta_{1/k}$ and satisfies (2.15);

(iii) If $u|_{\theta=\theta_\kappa} = 0, \theta_\kappa = tan^{-1}\kappa \geq \theta_0$, there exists a polynomial $\varphi \in P_p(Q)$ such that φ vanishes on the line $\theta = \theta_k$ and satisfies (2.15). $\square$

Theorem 2.7. (Theorem 3.8 of [4]) Let $u(x) = r^\gamma \log^\nu r\ \chi(r)\ \Phi(\theta)$ with $\gamma > 0$ and integer $\nu \geq 0$ be given in (2.10). Then:

(i) There exists a polynomial $\varphi \in P_p(Q)$ such that

$$||u-\varphi||_{H^1(R_0)} \leq C \left(\frac{1}{p}\right)^{2\gamma} (1+\log p)^{\nu^*} \tag{2.16}$$

where R_0 is given in (2.13) and ν^* is given in (2.14);

(ii) If $u|_{\theta=\theta_\kappa,\theta_{1/\kappa}} = 0, \theta_\kappa = tan^{-1}\kappa \geq \theta_0$, there exists a polynomial $\varphi \in P_p(Q)$ such that φ vanishes on the lines $\theta = \theta_k$ and $\theta = \theta_{1/k}$ and satisfies (2.16);

(iii) If $u|_{\theta=\theta_\kappa} = 0, \theta_\kappa = tan^{-1}\kappa \geq \theta_0$, there exists a polynomial $\varphi \in P_p(Q)$ such that φ vanishes on the line $\theta = \theta_k$ and satisfies (2.16). $\square$

3 Rate of convergence of the p-version of the finite element method

3.1 A model problem in a polygonal domain

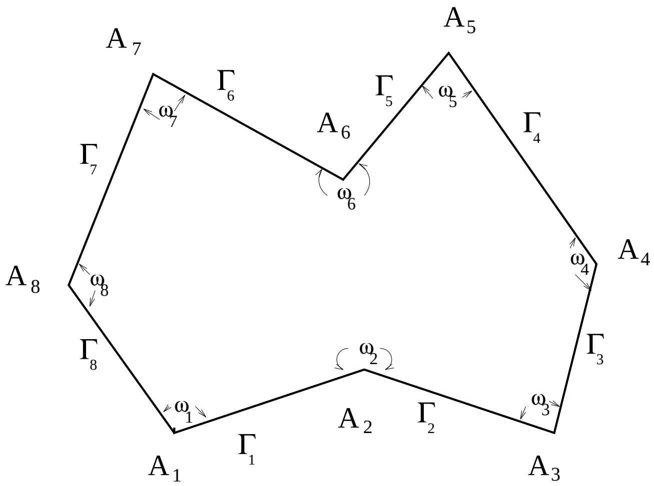


Fig. 3.1 Polygonal domain Ω

Let Ω be a polygon, shown in Fig. 3.1, with vertices A_i, $1 \le i \le M$ ($A_{M+1} = A_1$), and (open) edges Γ_i connecting the vertices A_i and A_{i+1}. By ω_i we denote the internal angle between Γ_i and Γ_{i+1}. Let $\mathcal{D}$ be a subset of $\mathcal{M} = \{1, 2, \ldots, M\}$, and $\mathcal{N} = \mathcal{M}|\mathcal{D}$. We refer to $\Gamma_{\mathrm{D}} = \bigcup\limits_{i \in \mathcal{D}} \overline{\Gamma}_i$ as the Dirichlet boundary and to $\Gamma_{\mathrm{N}} = \bigcup\limits_{i \in \mathcal{N}} \Gamma_i$ as the Neumann boundary. We allow also polygons with internal angle 2π, which is important in applications.

Consider a boundary value problem:

$$\begin{aligned} -\Delta u + u &= f \qquad\qquad \text{in} \;\; \Omega \\ u\big|_{\Gamma_{\mathrm{D}}} &= 0 \\ \frac{\partial u}{\partial n}\bigg|_{\Gamma_{\mathrm{N}}} &= g. \end{aligned} \tag{3.1}$$

By this simple model problem we shall show how to derive the lower and upper bounds of the approximation error of the p-version, which applies general elliptic problem on no n-smooth domains. By $H^k(\Omega)$, $k \ge 0$ integer, we denote the usual Sobolev space and $H^1_{\mathrm{D}}(\Omega) = \left\{ u \in H^1(\Omega) \;\middle|\; u\big|_{\Gamma_{\mathrm{D}}} = 0 \right\}$. The variational form of (3.1) is to seek $u(x) \in H^1_{\mathrm{D}}(\Omega)$ such that

$$B(u, v) = F(v), \qquad\qquad \forall\, v \in H^1_{\mathrm{D}}(\Omega) \tag{3.2}$$

where B is a bilinear form on $H^1_{\mathrm{D}}(\Omega) \times H^1_{\mathrm{D}}(\Omega)$ and F is a linear functional on $H^1_{\mathrm{D}}(\Omega)$, given by

$$B(u, v) = \int_\Omega (\nabla u \cdot \nabla v + u\, v)\, dx \tag{3.3}$$

and

$$F(v) = \int_\Omega f\, v\, dx + \int_{\Gamma_{\mathrm{N}}} g\, v\, ds. \tag{3.4}$$

Let $S_{\delta_i} = \left\{ x \in \Omega \;\; \mathrm{dist}(x, A_i) < \delta_i \right\}$ be a neighborhood of the vertices A_i, shown in Fig. 3.2, with $\delta_i \in (0, 1)$. δ_i is selected such that $S_{\delta_i} \cap S_{\delta_j} = \emptyset$ for $i \ne j$. $\Omega_0 = \Omega | \bigcup\limits_{i \in \mathcal{M}} S_{\delta_i/2}$ contains no vertices of Ω, and $\Omega_0 \cap S_{\delta_i} \ne \emptyset$ for $i \in \mathcal{M}$. Ω_0 is called the regular part of Ω.

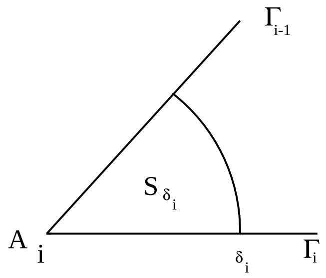


Fig. 3.2 A neighborhood of the vertex A_i

We assume that f and g are such that the solution u of (3.1) is in $H^k(\Omega_0), k \geq 1$, and in each neighborhood S_{δ_i}, u have an expansion in terms of singular functions of $r^\gamma \log^\nu r$ -type

$$u = \sum_{0<\gamma_m^{[i]} \leq k-1} C_m^{[i]}\, r_i^{\gamma_m^{[i]}}\, |\log r_i|^{\nu_m^{[i]}} \Phi_m^{[i]}(\theta_i)\, \chi(r_i) + u_0^{[i]} \tag{3.5}$$

where (r_i, θ_i) are polar coordinates with the vertex A_i, $u_0^{[i]} \in H^k(S_{\delta_i})$ is the smooth part of u, $\gamma_m^{[i]} > 0$, and $\nu_m^{[i]} \geq 0$ are integers. We assume that $\nu_m^{[i]} < \nu_{m+1}^{[i]}$ and $\gamma_m^{[i]} \leq \gamma_{m+1}^{[i]}$, $\chi(r_i)$ and $\Phi_m^{[i]}(\theta_i)$ are C^∞ functions, $\chi(r_i) = 1$ for $0 < r_i < \delta_i < \frac{1}{2}$, $\chi(r_i) = 0$ for $r_i > \delta_i$. Let

$$\gamma = \min_i \gamma_1^{[i]} = \gamma_1^{[i_0]}, \qquad \nu_\gamma = \nu_1^{[i_0]} \tag{3.6a}$$

and

$$\nu_\gamma^* = \begin{cases} \nu_\gamma \ \text{ if } \gamma \text{ is not an integer or } \ \nu_\gamma = 0, \\ \\ \nu_\gamma - 1 \ \text{ if } \gamma \text{ is an integer and } \ \nu_\gamma \geq 1. \end{cases} \qquad (3.6b).$$

3.2 Upper bound of approximation error in the p-version finite element solutions

Let $\Delta = \left\{ \Omega_j,\, 1 \leq j \leq J \right\}$ be a partition of the domain Ω. The elements Ω_i are triangles or parallelograms. We shall assume that $\overline{\Omega}_i \cap \overline{\Omega}_j$ is either the empty set, or an entire side, or a vertex of Ω_i and Ω_j, and assume that all vertices of Ω are vertices of some Ω_i.

By $P_p(\Omega)$ (or $P_p(\Omega_i)$) we denote the space of all polynomials of total degree $\leq p$ defined on Ω (or Ω_i), and let $S^p = S^p(\Omega;\Delta) = \Big\{u \in H^1(\Omega) \;\Big|\; u \in P_p(\Omega_j),$ $j = 1,2,\ldots,J\Big\}$ and $S^p_D = S^p_D(\Omega;\Delta) = S^p(\Omega;\Delta) \cap H^1_D(\Omega)$.

The p-version finite element solution $u_p \in S^p_D(\Omega;\Delta)$ is such that

$$B(u_p, v) = F(v), \qquad \forall\, v \in S^p_D(\Omega;\Delta). \tag{3.7}$$

Using the coercivity of the bilinear form (3.3), one can show that

$$||u - u_p||_{H^1(\Omega)} \leq C \inf_{w \in S^p_D} ||u - w||_{H^1(\Omega)}. \tag{3.8}$$

Below, we will analyze the asymptotic rate of convergence of the p-version of FEM. We will use at various places approximations by polynomials of degree p separately in two directions. These polynomials are polynomial of total degree $2p$, and hence we will not distinguish between polynomials of total degree and degree in two directions, which will only influence the value of the constants in the estimates.

Theorem 3.1. Let $\Delta = \Big\{\Omega_j,\ 1 \leq j \leq J\Big\}$ be a partition of Ω containing triangular and parallelogram elements and $S^p_D(\Omega;\Delta)$ be the finite element space defined as above. The data functions f and g are assumed such that the solution u of (3.1) is in $H^k(\Omega_0)$ with $k \geq \max\{2,\ 1{+}2\gamma\}$, and u have the expansion (3.5) with $u_0^{[i]} \in H^k(S_{\delta_i})$ in each neighborhood S_{δ_i}. Then the finite element solution $u_p \in S^p_D(\Omega;\Delta)$ for the problem (3.1) satisfies

$$||u - u_p||_{H^1(\Omega)} \leq C\, p^{-2\gamma}\, (1 + \log p)^{\nu^*_\gamma} \tag{3.9}$$

with constant C depending on u, k, γ and ν^*_γ, but not on p, where γ and ν^*_γ are given in (3.6).

To prove Theorem 3.1 we need several lemmas.

Lemma 3.2. Let $Q = (-1,1)^2$ with four sides $\Gamma_i, 1 \leq i \leq 4$ lying on the lines $x_1 = -1$, $x_2 = -1$, $x_1 = 1$, and $x_2 = 1$, respectively, and let $u \in H^k_\Gamma(Q) = \Big\{u \in$

$H^k(Q)|\ u|_\gamma = 0\Big\}$ with $k \geq 2$, where $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$. Then there exists a polynomial $\varphi \in P_p(Q)$, $p \geq k-1$ such that $\varphi|_\Gamma = 0$, and

$$||u - \varphi||_{H^1(Q)} \leq C\, p^{-(k-1)}\, |u|_{H^k(Q)} \tag{3.10}$$

with constant C independent of p.

Proof: Since $u \in H^k_\Gamma(Q)$, $k \geq 2$, $u_{x_1 x_2}$ has a Legendre-Fourier expansion:

$$u_{x_1 x_2} = \sum_{i,j=0}^{\infty} a_{ij}\, L_i(x_1)\, L_j(x_2) \tag{3.11a}$$

$$\begin{aligned} u_{x_1} &= \sum_{i,j=0}^{\infty} a_{ij}\, L_i(x_1) \int_{-1}^{x_2} L_j(\xi_2)\, d\xi_2 = \sum_{i,j=0}^{\infty} a_{ij}\, L_i(x_1)\, \frac{L_{j+1}(x_2) - L_{j-1}(x_2)}{2j+1}, \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} b_{ij}\, L_i(x)\, L_j(x), \end{aligned} \tag{3.11b}$$

$$\begin{aligned} u_{x_2} &= \sum_{i,j=0}^{\infty} a_{ij}\, L_j(x_2) \int_{-1}^{x_1} L_i(\xi_1)\, d\xi_1 = \sum_{i,j=0}^{\infty} a_{ij}\, L_j(x_2)\, \frac{L_{i+1}(x_1) - L_{i-1}(x_1)}{2i+1}, \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} d_{ij}\, L_i(x_1)\, L_j(x_2) \end{aligned} \tag{3.11c}$$

$$\begin{aligned} u &= \sum_{i,j=0}^{\infty} a_{ij} \int_{-1}^{x_1} L_i(\xi_1)\, d\xi_1 \int_{-1}^{x_2} L_j(\xi_2)\, d\xi_2 \\ &= \sum_{i,j=0}^{\infty} a_{ij}\, \frac{L_{i+1}(x_1) - L_{i-1}(x_1)}{2i+1}\, \frac{L_{j+1}(x_2) - L_{j-1}(x_2)}{2j+1} \end{aligned} \tag{3.11d}$$

where $b_{ij} = \dfrac{a_{i,j-1}}{2j-1} - \dfrac{a_{i,j+1}}{2j+3}$, $d_{ij} = \dfrac{a_{i-1,j}}{2i-1} - \dfrac{a_{i+1,j}}{2i+3}$ with $a_{i,-1} = a_{i,0}$ and $a_{-1,j} = a_{0,j}$, and $L_i(x_1)$ is the Legendre polynomial of degree i, etc. Define

$$\varphi = \sum_{i,j=0}^{p-1} a_{ij} \int_{-1}^{x_1} L_i(\xi_1)\, d\xi_1 \int_{-1}^{x_2} L_j(\xi_2)\, d\xi_2, \tag{3.12}$$

and we have $\varphi|_{x_1=-1} = \varphi|_{x_2=-1} = 0$ and

$$u - \varphi = \left(\sum_{i,j=0}^{\infty} - \sum_{i,j=0}^{p-1} \right) a_{ij} \int_{-1}^{x_1} L_i(\xi_1)\, d\xi_1 \int_{-1}^{x_2} L_j(\xi_2)\, d\xi_2;$$

$$(u - \varphi)_{x_1} = \left(\sum_{i,j=0}^{\infty} - \sum_{i,j=0}^{p-1} \right) b_{ij}\, L_i(x_1)\, L_j(x_2);$$

$$(u-\varphi)_{x_2} = \left(\sum_{i,j=0}^{\infty} - \sum_{i,j=0}^{p-1}\right) d_{ij}\; L_i(x_1)\; L_j(x_2).$$

By the properties of Legendre polynomials ([15]):

$$\int_{-1}^{1} L_k^{(\ell)}(t)\, L_m^{(\ell)}(t)(1-t^2)^\ell\, dt = \begin{cases} \dfrac{2}{2k+1}\,\dfrac{(k+\ell)!}{(k-\ell)!} & \text{for } \; k=m \geq \ell, \\ \\ 0 & \text{for } k \neq m \text{ or } \; \min(k,m) < \ell. \end{cases}$$

we have the following estimates:

$$||u-\varphi||^2_{L^2(Q)} \leq E_1 + E_2 \leq C\, \frac{1}{p^{2(k-1)}}\, |u|^2_{H^k(Q)} \tag{3.13}$$

with

$$\begin{aligned} E_1 = &\; \sum_{i=0}^{\infty}\sum_{j=p}^{\infty} |a_{ij}|^2\, \frac{2}{2i+1}\, \frac{2}{2j+1}\, \frac{1}{(2i+1)^2}\, \frac{1}{(2j+1)^2} \\ \leq &\; C\, \frac{1}{p^2}\, \frac{(p-k+2)!}{(p+k-2)!} \sum_{i=0}^{\infty}\sum_{j=p}^{\infty} |a_{ij}|^2\, \frac{2}{2i+1}\, \frac{2}{2j+1}\, \frac{(j+k-2)!}{(j-k+2)!} \\ \leq &\; C\, \frac{1}{p^{2(k-1)}} \int_Q |u_{x_1 x_2^{k-1}}|^2 (1-x_2^2)^{k-2}\, dx_1 dx_2 \leq C\, \frac{1}{p^{2(k-1)}}\, |u|^2_{H^k(Q)} \end{aligned}$$

and

$$\begin{aligned} E_2 = &\; \sum_{i=p}^{\infty}\sum_{j=0}^{p} |a_{ij}|^2\, \frac{2}{2i+1}\, \frac{2}{2j+1}\, \frac{1}{(2i+1)^2}\, \frac{1}{(2j+1)^2} \\ \leq &\; C\, \frac{1}{p^2}\, \frac{(p-k+2)!}{(p+k-2)!} \sum_{i=p}^{\infty}\sum_{j=0}^{p} |a_{ij}|^2\, \frac{2}{2i+1}\, \frac{2}{2j+1}\, \frac{(i+k-2)!}{(i-k+2)!} \\ \leq &\; C\, \frac{1}{p^{2(k-1)}} \int_Q |u_{x_1^{k-1} x_2}|^2 (1-x_1^2)^{k-2}\, dx_1 dx_2 \leq C\, \frac{1}{p^{2(k-1)}}\, |u|^2_{H^k(Q)}\,. \end{aligned}$$

Similarly we have

$$||(u-\varphi)_{x_1}||^2_{L^2(Q)} \leq F_1 + F_2 \leq C\, \frac{1}{p^{2(k-1)}}\, |u|^2_{H^k(Q)} \tag{3.14}$$

with

$$
\begin{aligned}
F_1 = &\sum_{i=0}^{\infty}\sum_{j=p}^{\infty}|b_{ij}|^2\,\frac{2}{2i+1}\,\frac{2}{2j+1}\\
\leq\ & C\,\frac{(j-k+1)!}{(j+k-1)!}\sum_{i=0}^{\infty}\sum_{j=p}^{\infty}|b_{ij}|^2\,\frac{2}{2i+1}\,\frac{2}{2j+1}\,\frac{(j+k-1)!}{(j-k+1)!}\\
\leq\ & C\,\frac{1}{p^{2(k-1)}}\int_Q|u_{x_1x_2^{k-1}}|^2(1-x_2^2)^{k-2}\,dx_1dx_2 \leq C\,\frac{1}{p^{2(k-1)}}\,|u|^2_{H^k(Q)}
\end{aligned}
$$

and

$$
\begin{aligned}
F_2 = &\sum_{i=p}^{\infty}\sum_{j=0}^{p}|b_{ij}|^2\,\frac{2}{2i+1}\,\frac{2}{2j+1}\\
\leq\ & C\,\frac{(p-k+1)!}{(p+k-1)!}\sum_{i=p}^{\infty}\sum_{j=0}^{p}|b_{ij}|^2\,\frac{2}{2i+1}\,\frac{2}{2j+1}\,\frac{(i+k-1)!}{(i-k+1)!}\\
\leq\ & C\,\frac{1}{p^{2(k-1)}}\int_Q|u_{x_1^k}|^2(1-x_1^2)^{k-1}\,dx_1dx_2 \leq C\,\frac{1}{p^{2(k-1)}}\,|u|^2_{H^k(Q)}\,.
\end{aligned}
$$

It can be proved in the same way that

$$
||(u-\varphi)_{x_2}||^2_{L^2(Q)} \leq C\,\frac{1}{p^{2(k-1)}}\,|u|^2_{H^k(Q)}\,. \tag{3.15}
$$

The combination of (3.13)–(3.15) yields (3.10). □

Lemma 3.3. Let $Q=(-1,1)^2$ and $u\in H^k_\Gamma(Q)=\left\{v\in H^k(Q)\;\middle|\; v|_\Gamma=0\right\}$, $k\geq 2$, where Γ is a part of ∂Q consisting of the entire sides of Q or the whole of ∂Q. Then there exists a polynomial of φ of degree p, $p\geq k-1$ such that $\varphi|_\Gamma=0$, and

$$
||u-\varphi||_{H^1(Q)} \leq C\left(\frac{1}{p}\right)^{k-1}||u||_{H^k(Q)} \tag{3.16}
$$

with constant C independent of p.

Proof: Let Γ_i, $1\leq i\leq 4$ be four sides of Q as in the previous lemma. Since the result for $\Gamma=\Gamma_1\cup\Gamma_2$ has been proved in Lemma 3.1, we need only to analyze the following five cases: (1) $\Gamma=\bigcup_{i=1}^{3}\bar{\Gamma}_i$; (2) $\Gamma=\bigcup_{i=1}^{4}\bar{\Gamma}_i$; (3) $\Gamma=\bar{\Gamma}_1\bigcup\bar{\Gamma}_3$; (4) $|\Gamma|=0$; (5) $\Gamma=\Gamma_1$.

Case (1): Let φ be the polynomial defined in (3.12). It suffices to show $\varphi|_{\Gamma_3} = 0$. By (3.11c)

$$u_{x_2}(x_1, x_2) = \sum_{i,j=0}^{\infty} a_{ij}\, L_j(x_2) \int_{-1}^{x_1} L_i(\xi_1)\, d\xi_1 .$$

Since $u_{x_2}(1, x_2) = 0$, we have

$$0 = \sum_{i,j=0}^{\infty} a_{ij} L_j(x_2) \int_{-1}^{1} L_i(\xi_1)\, d\xi_1 = \sum_{i=0}^{\infty} a_{0j} L_j(x_2)$$

which implies $a_{0j} = 0$ for all j. Therefore

$$\varphi(1, x_2) = \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} a_{ij} \int_{-1}^{1} L_i(\xi_1)\, d\xi_1 \int_{1}^{x_2} L_j(\xi_2)\, d\xi_2 = 0.$$

Case (2): The arguments in Case (1) can be carried over to $\varphi|_{\Gamma_3}$ as well as to $\varphi|_{\Gamma_4}$, we have

$$\varphi(x_1, x_2) = \sum_{i,j=1}^{p} a_{ij} \int_{-1}^{x_1} L_i(\xi_1)\, d\xi_1 \int_{-1}^{x_2} L_j(\xi_2)\, d\xi_2$$

which implies that $\varphi|_{\Gamma_i} = 0$, $1 \le i \le 4$.

Case (3): Let $\chi_1(x_2)$ and $\chi_2(x_2)$ be C^∞ functions such that $\chi_1(x_2) = 1$ for $-1 < x_2 < 0$ and $\chi_1(x_2) = 0$ for $x_2 > 1/2$; $\chi_2(x_2) = 1$ for $0 < x_2 < 1$ and $\chi_2(x_2) = 0$ for $x_2 < -1/2$, and let $u_j = \chi_i(x_2)\, u$, $j = 1, 2$, and $u_3 = \Big(1 - \chi_1(x_2) - \chi_2(x_2)\Big) u$. Hence u_1 vanishes on $\Gamma_1 \cup \Gamma_3 \cup \Gamma_4$, u_2 vanishes on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and u_3 vanishes on $\bigcup_{i=1}^{4} \Gamma_i$. Due to the results in Cases (2) and (3), there are polynomials φ_j, $1 \le j \le 3$ of degree p such that φ_j vanishes on the corresponding sides of Q and

$$||u_j - \varphi_j||_{H^1(Q)} \le \left(\frac{1}{p}\right)^{k-1} ||u_j||_{H^k(Q)} .$$

Let $\varphi = \sum_{i=1} \varphi_i$. Then φ vanishes on $\Gamma_1 \cup \Gamma_3$, and (3.16) holds.

Case (4): First we extend u to $\tilde{Q} = (-2, 2)^2$. Let $\chi(t)$ be a C^∞ functions such that $\chi(t) = 1$ for $-1 \le t \le 1$ and $\chi(t) = 0$ for $|t| \ge 2$, and let $v = \chi(x_1)\, \chi(x_2)\, u$. Then $v \in H^k(\tilde{Q})$ and $v|_{\partial \tilde{Q}} = 0$. By the result of Case (2), there is a polynomial φ of degree p such that $\varphi|_{\partial \tilde{Q}} = 0$

$$||v - \varphi||_{H^1(\tilde{Q})} \le C \left(\frac{1}{p}\right)^{k-1} |v|_{H^k(\tilde{Q})} \le C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(Q)}$$

which implies (3.16).

Case (5): Let $\chi(x_1)$ be a C^∞ function such that $\chi(x_1) = 1$ for $-1 \leq x_1 \leq 0$ and $\chi(x_1) = 0$ for $x_1 > 1/2$, $u = \chi(x_1)\, u + \Big(1 - \chi(x_1)\Big) u = u_1 + u_2$. Hence $u_1 \in H^k(Q)$ and $u_1|_{\Gamma_1 \cup \Gamma_3} = 0$. By the result of Case (3) there is a polynomial $\varphi_1 \in P_p(Q)$ such that $\varphi_1|_{\Gamma_1 \cup \Gamma_3} = 0$ and

$$||u_1 - \varphi_1||_{H^1(Q)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u_1||_{H^k(Q)}.$$

Since $u_2 = 0$ for $x_1 \leq 0$, $\tilde{u}_2 = \dfrac{u_2}{1 + x_1} \in H^k(Q)$ and

$$||\tilde{u}_2||_{H^k(Q)} \leq C\, ||u_2||_{H^k(Q)} \leq C\, ||u||_{H^k(Q)}.$$

From the result in Case (4), there exists a polynomial $\tilde{\varphi}_2 \in P_{p-1}(Q)$ such that

$$||\tilde{u}_2 - \tilde{\varphi}_2||_{H^1(Q)} \leq C \left(\frac{1}{p-1}\right)^{k-1} ||u_2||_{H^k(Q)} \leq C \left(\frac{1}{p-1}\right)^{k-1} ||u||_{H^k(Q)}.$$

Let $\varphi_2 = (1 + x_1)\tilde{\varphi}_2$. Then $\varphi_2|_{\Gamma_1} = 0$, and

$$||u_2 - \varphi_2||_{H^1(Q)} \leq C\, ||\tilde{u}_2 - \tilde{\varphi}_2||_{H^1(Q)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(Q)}$$

Define $\varphi = \varphi_1 + \varphi_2$, then $\varphi|_{\Gamma_1} = 0$, and satisfies (3.16). □

Lemma 3.4 Let $u \in H^k(\Omega), k \geq 2$ and $u = 0$ on Γ_i, where Γ_i is an edge of a convex polygonal domain Ω lying on the x_1 axis with one endpoint A_i. Then there exists a polynomial $\varphi_p \in P_p(\Omega)$, $p \geq k - 1$ such that $\varphi_p|_{\Gamma_i} = 0$ and

$$||u - \varphi_p||_{H^1(\Omega)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)}. \tag{3.17}$$

Proof: Let T be a trapezoid $\supset \Omega$, whose boundary is surrounded by Γ_i, the extensions $\tilde{\Gamma}_{i-1}$ of Γ_{i-1} and the extension $\tilde{\Gamma}_{i+1}$ of Γ_{i+1}, and a line parallel to Γ_i with distance H, shown in Fig. 3.3. The trapezoid T exists because Ω is convex. We now extend u in T with preserving the H^k- norm. We may assume without loss of the generality that $\Gamma_i = \{(x_1, 0)|\, 0 < x_1 < 1\ \}$ and $\omega_i = \frac{\pi}{2}$. If $\omega_{i+1} = \frac{\pi}{2}$

then T is a rectangle $(0,1)\times(0,H)$, shown in Fig. 3.3(a). By Lemma 3.2 there is a polynomial $\varphi_p \in P_p(T)$ such that $\varphi_p|_{\Gamma_i} = 0$ and

$$||u-\varphi_p||_{H^1(T)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)}.$$

Then (3.17) follows immediately.

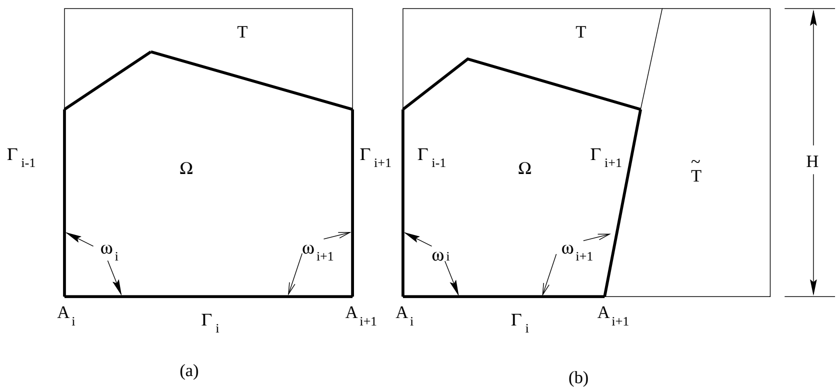


Fig. 3.3. A polygonal domain Ω contained in a trapezoid T
(a) $\omega_i = \omega_{i+1} = \frac{\pi}{2}$; (b) $\omega_{i+1} \neq \omega_i = \frac{\pi}{2}$.

If $\omega_{i+1} \neq \frac{\pi}{2}$, then we extend u in $\tilde{T} \supset T$ by the Nikolskij-Babič extension (see [22])such that $u \in H^k(\tilde{T})$ and $u|_{\tilde{\Gamma}_i} = 0$, where $\tilde{\Gamma}_i$ is the extension of Γ_i, and $\tilde{T}$ is a rectangle with $\tilde{\Gamma}_i$ and $\tilde{\Gamma}_{i-1}$ as its edges shown in Fig. 4.3(b). By Lemma 3.2 there is a polynomial $\varphi_p \in P_p(\tilde{T})$ such that $\varphi_p|_{\tilde{\Gamma}_i} = 0$ and

$$||u-\varphi_p||_{H^1(\tilde{T})} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)}.$$

Then φ_p vanishes on Γ_i and satisfies (3.17). □

Lemma 3.5. Let $u \in H^k(\Omega), k \geq 2$ where Ω is a convex polygon and $u = 0$ on Γ_{i-1} and Γ_i, where Γ_{i-1} and Γ_i are two sides of Ω with a common endpoint (vertex) A_i. Then there exists a polynomial $\varphi_p \in P_p(\Omega)$, $p \geq k-1$ such that $\varphi_p|_{\Gamma_i \cup \Gamma_{i-1}} = 0$ and

$$||u-\varphi_p||_{H^1(\Omega)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)}. \tag{3.18}$$

Proof: Let $T \supset \Omega$ be a trapezoid with height H as before, and assume that $\omega_i = \frac{\pi}{2}$ and that Γ_i and Γ_{i-1} with unit length lie in the x_1 and x_2 axis, respectively. Let

$\chi_1(x_1)$ and $\chi_2(x_2)$ be C^∞ functions such that $\chi_i(x_i) = 1$ for $0 < x_i < \delta$ and $\chi_i(x_i) = 0$ for $x_i > 2\delta, i = 1, 2$. $\delta \in (0, \frac{1}{2})$ is selected such that $(0, 2\delta) \times (0, 2\delta) \subset \Omega$, and the line $x_2 = 2\delta$ intercepts only Γ_{i-1} and Γ_{i+1}, and the line $x_1 = 2\delta$ intercepts only Γ_{i-2} and Γ_i. Set

$$u_1 = u\,\chi_2(x_2), \qquad u_2 = u\,(1 - \chi_2(x_2))$$

and

$$u_{11} = u_1\,\chi_1(x_1), \qquad u_{12} = u_1\,(1 - \chi_1(x_1))$$

Then $u = u_1 + u_2 = u_{11} + u_{12} + u_2$. Note that $u_{11} = 0$ on $\Gamma_i \cup \Gamma_{i-1}$ and $u_{11} \equiv 0$ for $x \notin [0, 2\delta] \times [0, 2\delta] \subset \Omega$; $u_{12} = 0$ on Γ_i and $u_{12} \equiv 0$ for $x \in \Omega$ with $x_1 < \delta$, $u_2 = 0$ on Γ_{i-1} and $u_2 \equiv 0$ for $x \in \Omega$ with $x_2 < \delta$. We extend u_{11} by zero extension outside $(0, 2\delta) \times (0, 2\delta)$ in a square $T = (0, H) \times (0, H) \supset \Omega$ with a suitable $H > 1$. Then by Lemma 3.2, we have a polynomial $\varphi_{11} \in P_p(T)$ vanishing on Γ_i and Γ_{i-1} such that

$$||u_{11} - \varphi_{11}||_{H^1(T)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)}. \tag{3.19}$$

Since $u_2 \equiv 0$ for $x_2 < \delta$, set $\tilde{u}_2 = \frac{u_2}{x_2} \in H^1(\Omega)$ and vanishes on Γ_{i-1}. Due to Lemma 3.3, there exists a polynomial $\tilde{\varphi}_2 \in P_{p-1}(\Omega)$ such that $\tilde{\varphi}_2 = 0$ on Γ_{i-1} and

$$||\tilde{u}_2 - \tilde{\varphi}_2||_{H^1(\Omega)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)}.$$

Let $\varphi_2 = \tilde{\varphi}_2\, x_2$. Then $\varphi_2 \in P_p(\Omega)$ and vanishes on Γ_{i-1} and Γ_i, and there holds

$$||u_2 - \varphi_2||_{H^1(\Omega)} \leq C\, ||\tilde{u}_2 - \tilde{\varphi}_2||_{H^1(\Omega)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)}. \tag{3.20}$$

Similarly we can find a polynomial $\varphi_{12} \in P_p(\Omega)$ vanishing on Γ_{i-1} and Γ_i, and

$$||u_{12} - \varphi_{12}||_{H^1(\Omega)} \leq C \left(\frac{1}{p}\right)^{k-1} ||u||_{H^k(\Omega)} \tag{3.21}$$

Let $\varphi_p = \varphi_{11} + \varphi_{12} + \varphi_2 \in P_p(\Omega)$, which vanishes on Γ_{i-1} and Γ_i, and (3.18) follows from (3.19) $\sim$ (3.21). $\square$

We now prove the main theorem of this section.

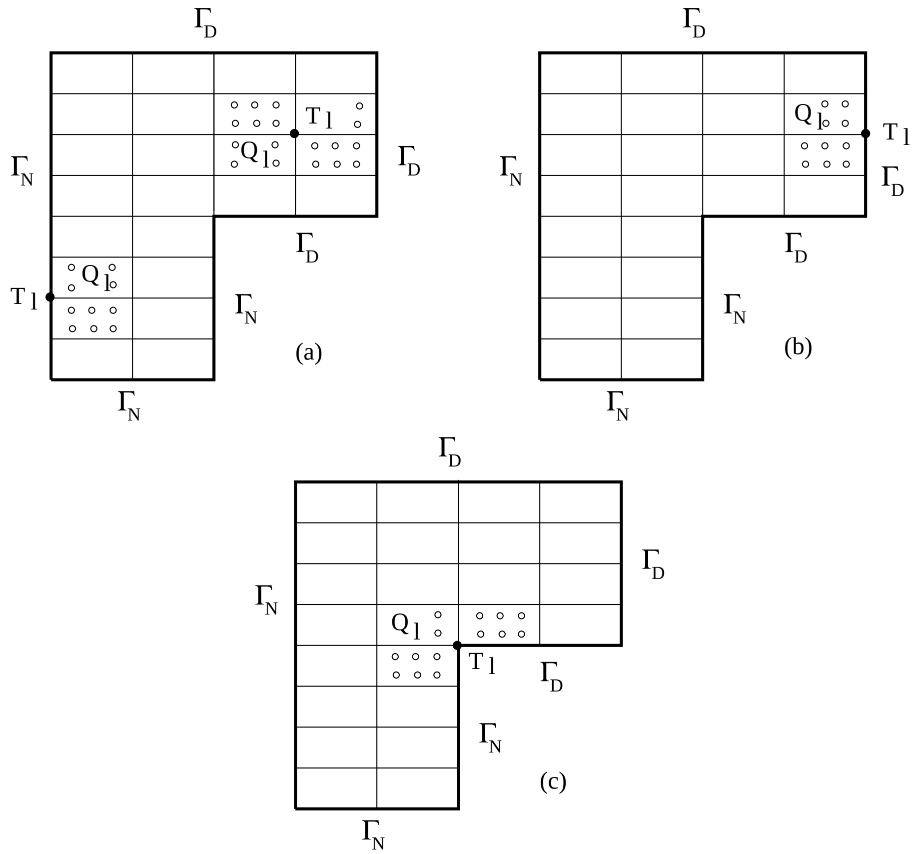


Fig. 3.4. Patches Q_ℓ and centers T_ℓ

(a) Q_ℓ contains no vertex and $T_\ell \notin \Gamma_D$;

(b) Q_ℓ contains no vertex and $T_\ell \in \Gamma_D$;

(c) Q_ℓ contains vertices of Ω.

Proof of Theorem 3.1: We may assume that $p \geq k$. Due to (3.8), it suffices to construct a piecewise polynomial $\phi_p \in S^p_D(\Omega;\Delta)$ such that

$$||u - \varphi_p||_{H^1(\Omega)} \leq C\, p^{-2\gamma}(1+\log p)^{\nu^*_\gamma}. \tag{3.22}$$

By T_ℓ, $1 \leq \ell \leq L$ we denote vertices of Ω_j, $1 \leq j \leq J$, and by Q_ℓ we denote a union of elements which have T_ℓ as a vertex, $Q_\ell = \bigcup_{T_\ell \in \bar{\Omega}_j} \Omega_j$. T_ℓ and Q_ℓ are called a node and a patch centered at T_ℓ, respectively. Let $\phi_\ell \in S^1(\Omega;\Delta)$ such that $\phi_\ell(T_\ell) = 1$ and $\phi_\ell(T_m) = 0$ for $m \neq \ell$. Note that the support of ϕ_ℓ is the patch Q_ℓ, and $\sum_{\ell=1}^{L} \phi_\ell \equiv 1$. Therefore $\{\phi_\ell\}_{\ell=1}^L$ is a partition of unity for Ω subordinate to $\{Q_\ell\}_{\ell=1}^L$. Let $\varphi_p = \sum_{\ell=1}^{L} \phi_\ell \varphi_{p-1}^{[\ell]}$ with $\varphi_{p-1}^{[\ell]} \in P_{p-1}(Q_\ell)$. Then $\varphi_p \in S^p(\Omega;\Delta)$ and $u - \varphi_p = \sum_{\ell=1}^{L} \phi_\ell (u - \varphi_{p-1}^{[\ell]})$.

We need to construct a polynomial $\varphi_{p-1}^{[\ell]} \in S^{p-1}(\Omega;\Delta)$ which vanishes on $\partial Q_\ell \cap \Gamma_D$ and satisfies

$$||u - \varphi_{p-1}^{[\ell]}||_{H^1(Q_\ell)} \leq C\, p^{-(k-1)} \tag{3.23}$$

if $\bar{Q}_\ell$ contains no vertices of Ω, and

$$||u - \varphi_{p-1}^{[\ell]}||_{H^1(Q_\ell)} \leq C\, p^{-2\gamma}(1+\log p)^{\nu_\gamma^*} \tag{3.24}$$

if $\bar{Q}_\ell$ contains vertices of Ω.

We shall construct $\varphi_{p-1}^{[\ell]}$ on each Q_ℓ according to their locations. There are three cases shown in Fig. 3.4:

Case A : Q_ℓ contains no vertex of Ω and $T_\ell \notin \Gamma_D$;

Case B : Q_ℓ contains no vertex of Ω and $T_\ell \in \Gamma_D$;

Case C : Q_ℓ contains vertices of Ω.

Case A. Since Q_ℓ contains no vertex of Ω, $u \in H^k(Q_\ell)$. By M_ℓ we denote an affine mapping which maps Q_ℓ onto $M_\ell(Q_\ell) \subset Q = (-1,1)^2$. Then we extend $\tilde{u} = u \circ M_\ell$ to whole Q such that the H^k-norm is preserved . By Lemma 3.2 there exists a polynomial $\tilde{\varphi}_{p-1} \in P_{p-1}(Q)$ such that

$$||u \circ M_\ell - \tilde{\varphi}_{p-1}||_{H^1(Q)} \leq C\, p^{-(k-1)}.$$

Then, $\varphi_{p-1} = \tilde{\varphi}_{p-1} \circ M_\ell \in P_{p-1}(Q_\ell)$, and satisfies (3.23).

Case B. Let Q_ℓ be a patch containing no vertices of Ω with the center $T_\ell \in \Gamma_D$ shown in Fig. 3.5. To fix the ideas, suppose that Q_ℓ has five edges $\tilde{\gamma}_i, 1 \leq i \leq 5$ and that the edge $\tilde{\gamma}_1$, lying on the x_1 axis, is a portion of Γ_D. Then the angle θ_2 between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, and θ_1 between $\tilde{\gamma}_5$ and $\tilde{\gamma}_1$ are less than π. We may assume without loss of generality that Q_ℓ is convex. If Q_ℓ is not convex, we can extend u in a convex polygon containing Q_ℓ and having $\tilde{\gamma}_1$ as its one whole edge. Therefore Q_ℓ can be contained in a trapezoid T between the extension of $\tilde{\gamma}_5$ and $\tilde{\gamma}_2$. Due to Lemma 3.3, we have a polynomial $\varphi_{p-1} \in P_{p-1}(Q_\ell)$ such that $\varphi_{p-1}^{[\ell]} = 0$ on $\tilde{\gamma}_1$ satisfies (3.23).

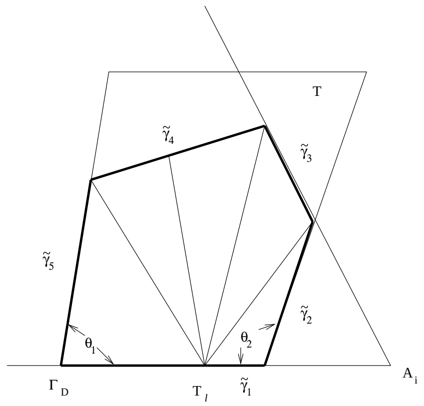


Fig. 3.5.A patch Q_ℓ containing no vertices of Ω with center T_ℓ on Γ_D

Case C. We will construct $\varphi_{p-1}^{[\ell]}$ in Q_ℓ whose center T_ℓ is a vertex A_i of Ω. The construction of $\varphi_{p-1}^{[\ell]}$ on Q_ℓ with the center T_ℓ, which is not a vertex of the domain, is similar to what follows.

We assume that A_i is the center T_ℓ of Q_ℓ, located at the origin and that Γ_i and Γ_{i-1} are on the lines $\theta = 0$ and $\theta = \omega_i$ respectively, where ω_i is the internal angle between Γ_i and Γ_{i-1}. In Q_ℓ, the solution u has the asymptotic expansion:

$$u = \sum_{\substack{\ell \geq 1 \\ \gamma_\ell^{[i]} \leq k-1}} C_\ell \, r^{\gamma_\ell^{[i]}} (\log r)^{\nu_\ell^{[i]}} \chi(r) \, \Phi(\theta) + u_0 = v + u_0$$

with $k \geq \max\{2, 2\gamma + 1\}$, $0 < \gamma_\ell^{[i]} < \gamma_{\ell+1}^{[i]}$, and $\nu_\ell^{[i]} \geq 0$. Here $u_0 \in H^k(Q_\ell)$ is the smooth part of the solution, $\Phi(0) = 0$ if $\Gamma_i \subset \Gamma_D$, $\Phi(\omega_i) = 0$ if $\Gamma_{i-1} \subset \Gamma_D$. We may assume that $\gamma_1^{[i]} = \gamma$ and $\nu_1^{[i]} = \nu_\gamma$.

Note that $u_0 \in H^k(Q_\ell)$. If $\Gamma_i \not\subset \Gamma_D$ and $\Gamma_{i-1} \not\subset \Gamma_D$, a polynomial $\varphi_{p-1}^{[0]} \in P_{p-1}(Q_\ell)$ can be constructed as in **Case A** such that

$$||u_0 - \varphi_{p-1}^{[0]}||_{H^1(Q_\ell)} \leq C \left(\frac{1}{p}\right)^{k-1}. \tag{3.25}$$

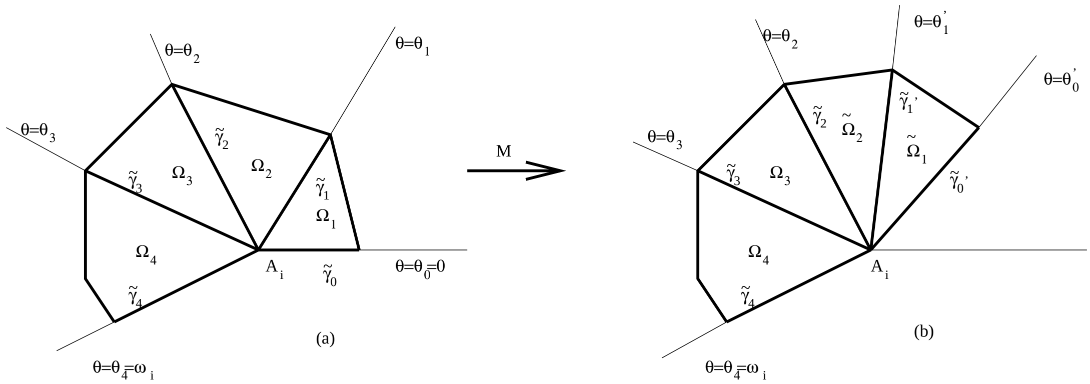


Fig. 3.6. A patch Q_ℓ centered at the vertex A_i with $\omega_i \geq \pi$ and $\omega_i - \pi < \theta_2 < \pi$.

If $\Gamma_i \subset \Gamma_D$ and the internal angle $\omega_i < \pi$, we have $\varphi_{p-1}^{[0]} \in P_{p-1}(Q_\ell)$ by Lemma 3.3 such that $\varphi_{p-1}^{[0]} = 0$ on Γ_i and (3.25) holds.

If $\Gamma_i \subset \Gamma_D$ and $\omega_i \geq \pi$, there are several elements Ω_m, $1 \leq m \leq s$ around the vertex A_i. These elements Ω_m are between the lines $\theta = \theta_m$ and $\theta = \theta_{m-1}$, where we write $\theta_0 = 0$ and $\theta_s = \omega_i$. Assume now that there is θ_{m_0} among θ_m's such that $\theta_{m_0} - \theta_0 = \theta_{m_0} < \pi$ and $\theta_s - \theta_{m_0} = \omega_i - \theta_{m_0} < \pi$ (see Fig. 3.6 where $m_0 = 2$). By $\tilde{\gamma}_m$ we denote the element edges lying on the line θ_m, $0 \leq m \leq s$. Let M_{m_0} be a linear mapping which maps the $\tilde{\gamma}_{m_0}$ onto itself and maps $\tilde{\gamma}_0$ onto $\tilde{\gamma}_0'$ such that $\omega_i - \theta_0' < \pi$, and let $\tilde{u}_0 = u_0 \circ M_\ell \in H^k(\tilde{Q}_\ell^1)$, where $\tilde{Q}_\ell^1 = Q_\ell^1 \circ M_{m_0}$, and $Q_\ell^1 = \bigcup\limits_{1 \leq m \leq m_0} \bar{\Omega}_m$. Then $\tilde{u}_0|_{\tilde{\gamma}_0'} = 0$, and $\tilde{u}_0$ can be extended into $Q_\ell^2 = Q_\ell \setminus Q_\ell^1 = \bigcup\limits_{m_0 < m \leq s} \bar{\Omega}_M$. Note that $\tilde{u}_0 = 0$ on $\tilde{\gamma}_0'$ and $u_0 - \tilde{u}_0 = 0$ on $\tilde{\gamma}_{m_0}$. By the previous arguments for internal angle $< \pi$, there is a polynomial $\tilde{\varphi} \in P_{p-1}(\tilde{Q}_\ell)$, where $\tilde{Q}_\ell = \tilde{Q}_\ell^1 \cup Q_\ell^2$ such that

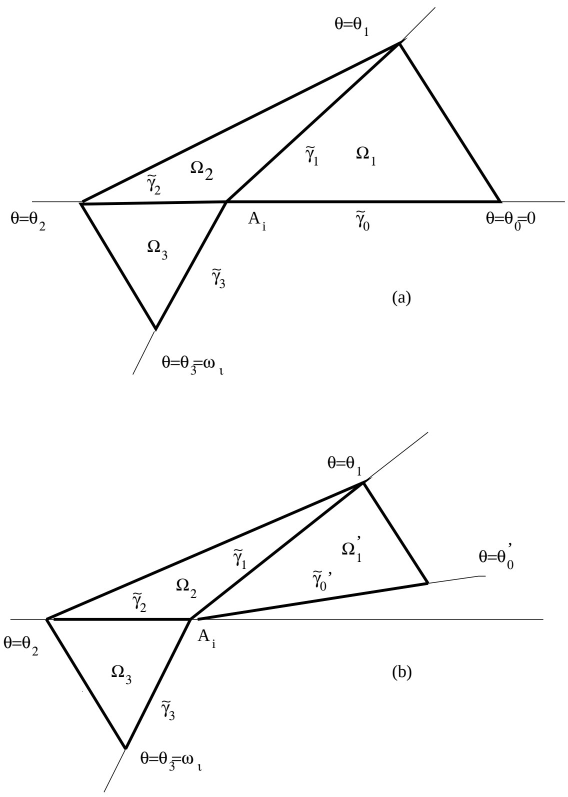


Fig. 3.7. A patch Q_ℓ centered at the vertex A_i with $\omega_i > \pi$, $\theta_1 < \pi$ and $\theta_2 > \pi$.

$\tilde{\varphi}|_{\tilde{\gamma}'_0} = 0$ and

$$||\tilde{u}_0 - \tilde{\varphi}||_{H^1(\tilde{Q}_\ell)} \leq C\left(\frac{1}{p}\right)^{k-1}, \tag{3.26}$$

and there is a polynomial $\phi \in P_{p-1}(Q_\ell^2)$ such that $\phi|_{\tilde{\gamma}_{m_0}} = 0$ and

$$||(u_0 - \tilde{u}_0) - \phi||_{H^1(Q_\ell^2)} \leq C\left(\frac{1}{p}\right)^{k-1}. \tag{3.27}$$

We extend ϕ into Q_ℓ^1 by a zero extension, and set

$$\varphi_{p-1}^{[0]} = \begin{cases} (\tilde{\varphi} + \phi) \circ M_{m_0}^{-1} & \text{in } Q_\ell^1, \\ \\ \tilde{\varphi} + \phi & \text{in } Q_\ell^2. \end{cases} \tag{3.28}$$

Then $\varphi_{[0]}^{p-1} \in S_D^{p-1}(\Omega;\Delta)$, and there holds

$$||u_0 - \varphi_{p-1}^{[0]}||_{H^1(Q_\ell)} \leq C\left\{||\tilde{u}_0 - \tilde{\varphi}||_{H^1(\tilde{Q}_\ell)} + ||(u_0 - \tilde{u}_0) - \phi||_{H^1(Q_\ell^2)}\right\}$$

which together with (3.26) and (3.27) yields (3.25).

In general, such θ_{m_0}, satisfying $\omega_i - \pi < \theta_{m_0} < \pi$, may not exist, but there always exists a m_0 such that $\theta_{m_0-1} < \pi$ and $\theta_{m_0} \geq \pi$, shown in Fig. 3.7 where $s = 3, m_0 = 2, \theta_2 > \pi$ and $\theta_1 < \omega_i - \pi$. Let M_ℓ be a linear mapping which maps $\tilde{\gamma}_{m_0-1}$ onto itself and maps $\tilde{\gamma}_0$ onto $\tilde{\gamma}_0'$ such that $\theta_{m_0} - \theta_0' < \pi$. By $\tilde{u}_0 = u_0 \circ M_\ell$, we denote the transformed function in $\tilde{Q}_\ell^1 = Q_\ell^1 \circ M_\ell$ where $Q_\ell^1 = \bigcup_{1 \leq m \leq m_0 - 1} \bar{\Omega}_m$, and we further extend $\tilde{u}_0$ into $Q_\ell^2 = Q_\ell \setminus Q_\ell^1$ such that the H^k-norm is preserved. Then $\tilde{u}_0 \in H^k(\tilde{Q}_\ell)$ where $\tilde{Q}_\ell = \tilde{Q}_\ell^1 \cup Q_\ell^2$, and vanishes on $\tilde{\gamma}_0'$. It is exactly the previous case that the internal angle $\geq \pi$ with $\theta_{m_0} - \theta_0' < \pi$ and $\theta_s - \theta_{m_0} = \omega_i - \theta_{m_0} < \pi$. Therefore, there exist a polynomial $\tilde{\varphi} \in S_D^{p-1}(\Omega; \Delta)$ such that $\tilde{\varphi}|_{\tilde{\gamma}_0'} = 0$ and satisfies (3.26). Note that $u_0 - \tilde{u}_0 = 0$ on $\tilde{\gamma}_{m_0}$, and that $\theta_{m_0} - \theta_{m_0-1} < \pi$ and $\theta_s - \theta_{m_0} < \pi$. Again, it is exactly same as the case above. Hence, there is a polynomial $\phi \in S_D^{p-1}(\Omega; \Delta)$ such that $\phi|_{\tilde{\gamma}_{m_0}} = 0$ and satisfies (3.27). We extend ϕ into Q_ℓ^1 by a zero extension, and define $\varphi_{p-1}^{[0]}$ as in (3.28). Then $\varphi_{p-1}^{[0]} \in S_D^{p-1}(\Omega; \Delta)$ and satisfies (3.25).

We next discuss the situation in which $\Gamma_i \cup \Gamma_{i-1} \subset \Gamma_D$. If $\omega_i < \pi$, the polynomial $\varphi_{p-1}^{[0]} \in S_D^{p-1}(\Omega; \Delta)$ satisfying (3.25) can be constructed due to Lemma 3.5.

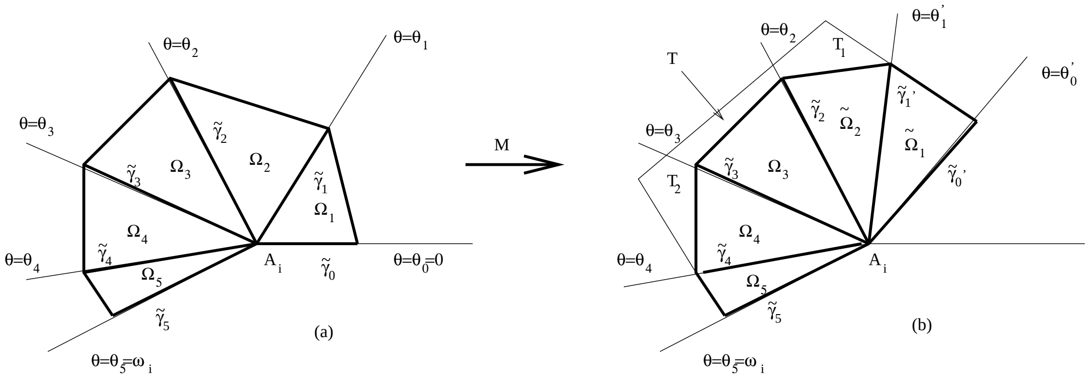


Fig. 3.8. A patch Q_ℓ centered at the vertex A_i with $\Gamma_i \cup \Gamma_{i-1} \subset \Gamma_D$ and with $\omega_i \geq \pi$ and $\omega_i - \pi < \theta_2 < \pi$.

If $\omega_i > \pi$ and there is a m_0 such that $\theta_{m_0} - \theta_0 = \theta_{m_0} < \pi$ and $\theta_s - \theta_{m_0} =$

$\omega_i - \theta_{m_0} < \pi$, we introduce a linear mapping M_ℓ which maps the $\tilde{\gamma}_{m_0}$ onto itself and maps $\tilde{\gamma}_0$ onto $\tilde{\gamma}_0'$ such that $\tilde{\gamma}_0'$ lies on the extension of $\tilde{\gamma}_s$. By T we denote a trapezoid with $\tilde{\gamma}_0' \cup \tilde{\gamma}_s$ and the extensions of the edges neighboring to $\tilde{\gamma}_0' \cup \tilde{\gamma}_s$ as its edges. T is divided by $\tilde{\gamma}_m$ and its extension into T_1 and T_2, shown in Fig. 3.8. Let $\tilde{u}_0 = u_0 \circ M_\ell$ in $\tilde{Q}_\ell^1$, where $\tilde{Q}_\ell^1 = Q_\ell^1 \circ M_{m_0}$, $Q_\ell^1 = \bigcup_{1 \le m \le m_0} \bar{\Omega}_M$. We first extend $\tilde{u}_0$ into T_1, then extend it into T_2 by an extension of the Nikolskij-Babič -type (see [22]), denoted again by $\tilde{u}_0$. The extension leads to $\tilde{u}_0|_{\tilde{\gamma}_0'} = \tilde{u}_0|_{\tilde{\gamma}_s} = 0$. By Lemma 3.3, there exists a polynomial $\tilde{\varphi} \in P_{p-1}(T)$ such that $\tilde{\varphi}|_{\tilde{\gamma}_s \cup \tilde{\gamma}_0'} = 0$ and

$$||\tilde{u}_0 - \tilde{\varphi}||_{H^1(T)} \le C \left(\frac{1}{p}\right)^{k-1}. \tag{3.29}$$

Let

$$\varphi = \begin{cases} \tilde{\varphi} \circ M_\ell^{-1} & \text{in } Q_\ell^1, \\ \tilde{\varphi} & \text{in } Q_\ell^2 = Q \setminus Q_\ell^1. \end{cases}$$

Hence $\varphi \in S_D^{p-1}(\Omega; \Delta)$ and

$$||u_0 - \varphi||_{H^1(Q_\ell^1)} \le C \left(\frac{1}{p}\right)^{k-1}. \tag{3.30}$$

Let $w = u_0 - \tilde{u}_0$ in Q_ℓ^2. Note that $w_0|_{\tilde{\gamma}_{m_0}} = w|_{\tilde{\gamma}_s} = 0$. By the previous result, there is a polynomial $\phi \in S_D^{p-1}(\Omega; \Delta)$ such that $\phi|_{\tilde{\gamma}_{m_0}} = \phi|_{\tilde{\gamma}_s} = 0$, and

$$||w - \phi||_{H^1(Q_\ell^2)} \le C \left(\frac{1}{p}\right)^{k-1}. \tag{3.31}$$

We extend ϕ into Q_ℓ^1 by a zero extension, and set $\varphi_{p-1}^{[0]} = \varphi + \phi$. Then $\varphi_{p-1}^{[0]} \in S_D^{p-1}(\Omega; \Delta)$ and satisfies (3.25) due to (3.29)-(3.31).

If $\omega_i > \pi$ and there exists no m_0 such that $\omega_i - \pi < \theta_{m_0} < \pi$, we have to apply the extension of the Nikolskij-Babič -type twice in order to construct the desired $\varphi_{p-1}^{[0]}$. For the sake of the simplicity, we assume that $s = 3, \theta_1 < \omega_i - \pi$ and $\theta_2 > \pi$, as shown in Fig. 3.9. Let M_1 be a linear mapping of Ω_1 onto Ω_1', which maps $\tilde{\gamma}_1$ onto itself and $\tilde{\gamma}_0$ onto $\tilde{\gamma}_0'$ such that $\tilde{\gamma}_0'$ lies in the opposite direction of $\tilde{\gamma}_2$. By u_1 we denote the transformed function $u_0 \circ M_1$ in Ω_1', and by $\tilde{u}_1$ the Nikolskij-Babič extension of u_1 in $\Omega_1' \cup \Omega_2$ such that $\tilde{u}_1|_{\tilde{\gamma}_0'} = \tilde{u}_1|_{\tilde{\gamma}_2} = 0$. By Lemma 3.5, there exists a polynomial $\tilde{\varphi}_1$ of degree $(p-1)$ in $\Omega_1' \cup \Omega_2$ such that $\tilde{\varphi}_1|_{\tilde{\gamma}_0'} = \tilde{\varphi}_1|_{\tilde{\gamma}_2} = 0$ and

$$||\tilde{u}_1 - \tilde{\varphi}_1||_{H^1(\Omega_1' \cup \Omega_2)} \leq C \left(\frac{1}{p}\right)^{k-1}. \tag{3.32}$$

Let

$$\varphi_1 = \begin{cases} \tilde{\varphi}_1 \circ M_1^{-1} & \text{in } \ \Omega_1, \\ \tilde{\varphi}_1 & \text{in } \ \Omega_2. \end{cases}$$

Then

$$||u_0 - \varphi_1||_{H^1(\Omega_1)} \leq C \, ||\tilde{u}_1 - \tilde{\varphi}_1||_{H^1(\Omega_1')} \leq C \left(\frac{1}{p}\right)^{k-1}. \tag{3.33}$$

Similarly we introduce a linear mapping M_3 of Ω_3 onto Ω_3', which maps $\tilde{\gamma}_2$ onto itself and $\tilde{\gamma}_3$ onto $\tilde{\gamma}_3'$, such that $\tilde{\gamma}_3'$ lies in the opposite direction of $\tilde{\gamma}_1$. Let $u_3 = u_0 \circ M_1$ in Ω_3', and let $\tilde{u}_3$ be the Nikolskij-Babič extension of u_3 in $\Omega_3' \cup \Omega_2$ such that $\tilde{u}_3|_{\tilde{\gamma}_3'} = \tilde{u}_3|_{\tilde{\gamma}_2} = 0$. By Lemma 3.5, there exists a polynomial $\tilde{\varphi}_3$ of degree $(p-1)$ in $\Omega_3' \cup \Omega_2$ such that $\tilde{\varphi}_3|_{\tilde{\gamma}_3'} = \tilde{\varphi}_1|_{\tilde{\gamma}_2} = 0$ and

$$||\tilde{u}_3 - \tilde{\varphi}_3||_{H^1(\Omega_3' \cup \Omega_2)} \leq C \left(\frac{1}{p}\right)^{k-1}. \tag{3.34}$$

Let

$$\varphi_3 = \begin{cases} \tilde{\varphi}_3 \circ M_3^{-1} & \text{in } \ \Omega_3, \\ \tilde{\varphi}_3 & \text{in } \ \Omega_2. \end{cases}$$

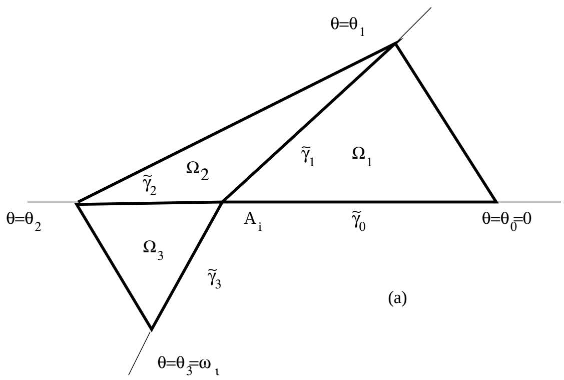
Then

$$||u_0 - \varphi_3||_{H^1(\Omega_3)} \leq C \, ||\tilde{u}_3 - \tilde{\varphi}_3||_{H^1(\Omega_3')} \leq C \left(\frac{1}{p}\right)^{k-1}. \qquad (3.35).$$

Let $w = u_0 - \tilde{u}_1 - \tilde{u}_3$ in Ω_2. Note that $w|_{\tilde{\gamma}_1} = w|_{\tilde{\gamma}_2} = 0$. By the previous result, there is a polynomial $\varphi_2 \in P_{p-1}(\Omega_2)$ such that $\varphi_2|_{\tilde{\gamma}_1} = \varphi_2|_{\tilde{\gamma}_2} = 0$ and

$$||w - \varphi_2||_{H^1(\Omega_2)} \leq C \left(\frac{1}{p}\right)^{k-1}. \tag{3.36}$$

We extend φ_1 into Ω_3, φ_3 into Ω_1 and φ_2 into $\Omega_1 \cup \Omega_3$ by zero extensions, respectively, and let $\varphi_{p-1}^{[0]} = \varphi_1 + \varphi_2 + \varphi_3$. Then $\varphi_{p-1}^{[0]} \in S_D^{p-1}(\Omega; \Delta)$ and satisfies (3.25) due to (3.32)-(3.36).



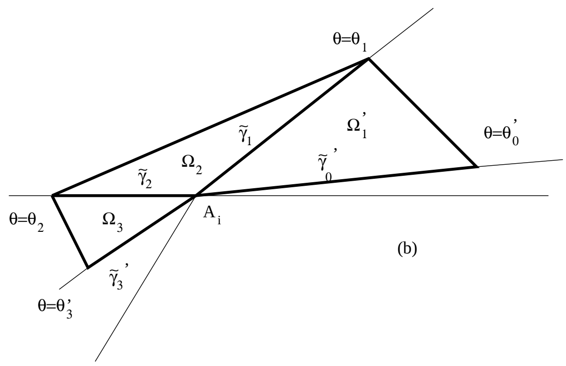


Fig. 3.9. A patch Q_ℓ centered at the vertex A_i with $\Gamma_i \cup \Gamma_{i-1} \subset \Gamma_D$ and with $\omega_i > \pi$, $\theta_1 < \pi$ and $\theta_2 > \pi$.

We now construct a polynomial $\varphi_{p-1}^{[v]} \in P_{p-1}(Q_\ell)$ to approximate the singular solution v in two different cases: **Case C1**: $\omega_i < \pi$ and **Case C2**: $\omega_i \geq \pi$.

Case C1: Since $\omega_i < \pi$, there is $\sigma > 0$ such that $\omega_i + 2\sigma < \pi$. We extend $\Phi(\theta)$ to R^1 such that $\Phi(\theta) = 0$ for $\theta > \omega_i + \sigma$ or $\theta < -\sigma$. Thus v is extended to Q_ℓ^*, which contains Q_ℓ and is between the lines $\theta = \omega_i + \sigma$ and $\theta = -\sigma$. We introduce a linear mapping M_ℓ which maps Q_ℓ^* onto $\tilde{Q}_\ell^* \subset R_0 = R_{r_0,\theta_0} \subset Q = (-1,1)^2$, where R_0 is defined in (2.14). Then $\tilde{v} = v \circ M_\ell \in B_\nu^{s,\beta}(Q)$ with $s = 1 + 2\gamma$, $\nu = \nu_\gamma^*$. By Theorem 2.8 for $\nu = 0$, or by Theorem 3.8 for $\nu > 0$, there exists a polynomial $\tilde{\varphi}(\tilde{x}) \in P_{p-1}(Q)$ such that

$$||\tilde{v} - \tilde{\varphi}||_{H^1(R_0)} \leq C \left(\frac{1}{p}\right)^{2\gamma} (1 + \log p)^{\nu_\gamma^*}. \tag{3.37}$$

Then $\varphi_{p-1}^{[v]} = \tilde{\varphi} \circ M_\ell^{-1} \in P_{p-1}(Q_\ell)$, and satisfies

$$||v - \varphi_{p-1}^{[v]}||_{H^1(Q_\ell)} \le C \left(\frac{1}{p}\right)^{2\gamma} (1 + \log p)^{\nu_\gamma^*}. \tag{3.38}$$

If $\bar{\Gamma}_i \cup \bar{\Gamma}_{i-1} \subset \Gamma_D$ (resp. $\Gamma_i \subset \Gamma_D$), we should apply Theorem 2.6 (ii) (resp. (iii)) or Theorem 2.7 (ii) (resp. (iii)) to construct a polynomial $\tilde{\varphi}(\tilde{x}) \in P_{p-1}(Q)$ vanishing on the lines $M(\Gamma_i \cup \Gamma_{i-1})$ (resp. $M(\Gamma_i)$) and satisfying (3.20). Then $\varphi_{p-1}^{[v]} \in P_{p-1}(Q_\ell)$ and $\varphi_{p-1}^{[v]} = 0$ on $\partial Q_\ell \cap \Gamma_D$.

Let $\varphi_{p-1}^{[\ell]} = \varphi_{p-1}^{[v]} + \varphi_{p-1}^{[0]} \in P_{p-1}(Q_\ell)$ (or $S_D^{p-1}(\Omega; \Delta)$). Then $\varphi_{p-1}^{[\ell]} = 0$ on $\partial Q_\ell \cap \Gamma_D$ and satisfies (3.24).

Case C2: $\omega_i \ge \pi$. We extend $\Phi(\theta)$ to $(-\sigma, \omega_i + \sigma)$, $\sigma > 0$ (if $\omega_i = 2\pi$, it is understood that the non-periodic function $\Phi(\theta)$ is defined on $(-\sigma, \omega_i + \sigma)$) such that $\theta_1 + \sigma < \pi$ and $(\theta_s - \theta_{s-1}) + \sigma < \pi$. Let $\theta_{-1} = -\sigma$, $\theta_{s+1} = \omega_i + \sigma$. Two additional elements, which are between the lines $\theta = \theta_{-1}$ and $\theta = \theta_0$, and lines $\theta = \theta_s$ and $\theta = \theta_{s+1}$, respectively, are denoted by Ω_0 and Ω_{s+1}.

We now introduce a partition of unity $\{\psi_m(\theta),\ m = 0, 1, 2, \ldots, s\}$ such that $\sum_{m=0}^{s} \psi_m(\theta) = 1$ on $[0, \omega_i]$, and each $\psi_m(\theta)$ is a C^∞ function with compact support $S_m = (\theta_{m-1} + \sigma_m, \theta_{m+1} - \sigma_m), 0 \le m \le s$ which form a cover of $[0, \omega_i]$. Let $v_m = \psi_m v$, which has the compact support S_m. If $(\theta_{m+1} - \theta_{m-1}) < \pi$ for all m. It becomes the **Case C1** for each v_m, namely, there exists a polynomial $\varphi^{[m]} \in P_{p-1}(\bar{\Omega}_m \cup \bar{\Omega}_{m+1})$, $1 \le m \le s$, such that $\varphi^{[m]}$ vanishes on the line $\theta = \theta_{m-1}$ and $\theta = \theta_{m+1}$, and

$$||v_m - \varphi^{[m]}||_{H^1(\bar{\Omega}_m \cup \bar{\Omega}_{m+1})} \le C \left(\frac{1}{p}\right)^{2\gamma} (1 + \log p)^{\nu_\gamma^*}. \tag{3.39}$$

We next extend $\varphi^{[m]}$ by zero extension outside of S_m, and let $\varphi_{p-1}^{[v]} = \sum_{m=1}^{s} \varphi^{[m]}$. Then $\varphi_{p-1}^{[v]} \in S^{p-1}(\Omega; \Delta)$ and

$$||v - \varphi_{p-1}^{[v]}||_{H^1(Q_\ell)} \le C \sum_{m=1}^{s} ||v_m - \varphi^{[m]}||_{H^1(\bar{\Omega}_m \cup \bar{\Omega}_{m+1})} \le C \left(\frac{1}{p}\right)^{2\gamma} (1 + \log p)^{\nu_\gamma^*}. \tag{3.40}$$

Thus (3.24) is satisfied.

If $\Gamma_i \subset \Gamma_D$, $\varphi^{[1]}$ can be constructed such that $\varphi^{[1]} \in \mathcal{P}_{p-1}(\bar{\Omega}_0 \cup \bar{\Omega}_1)$ and vanish on the line $\theta = \theta_{-1}$ and $\theta = \theta_0$. If $\Gamma_{i-1} \subset \Gamma_D$, $\varphi^{[m]}$ should be constructed in a similar way.

If $(\theta_{m+1} - \theta_{m-1}) > \pi$ for some m, we introduce a linear mapping M of Ω_m onto Ω'_m which maps the line $\theta = \theta_m$ onto itself, and the line $\theta = \theta_{m-1}$ onto a line $\theta = \theta'_{m-1}$ such that $\theta_{m+1} - \theta'_{m-1} < \pi$. By $\tilde{v}_m$ we denote the transformed function of $v_m \circ M^{-1}$ on Ω'_m, and we extend $\tilde{v}_m$ to Ω_{m+1} such that the extension vanishes on the line $\theta = \theta_{m+1}$. The construction of approximation polynomial $\tilde{\varphi}^{[m]}$ to $\tilde{v}_m$ on $\Omega_{m+1} \cup \Omega'_m$ is one of previous case since $\theta_{m+1} - \theta'_{m-1} < \pi$. Note that $(\tilde{v}_m - v_m)|_{\theta=\theta_m} = 0$ and $\tilde{v}_m|_{\theta=\theta_{m+1}} = v_m|_{\theta=\theta'_{m+1}} = 0$. Therefore there exists a polynomial $\tilde{\phi}^{[m]} \in P_{p-1}(\Omega'_m)$ to approximate $(v_m - \tilde{v}_m)$ on Ω_{m+1}, which vanishes on the lines $\theta = \theta_m$ and $\theta = \theta_{m+1}$. Define

$$\varphi^{[m]}_{p-1} = \begin{cases} \tilde{\varphi}^{[m]} \circ M^{-1} & \text{in } \Omega_m, \\ \tilde{\varphi}^{[m]} + \phi & \text{in } \Omega_{m+1}. \end{cases}$$

and extend $\varphi^{[m]}_{p-1}$ by zero extension outside $\bar{\Omega}_{m+1} \cup \bar{\Omega}_m$. Then $\varphi^{[m]}_{p-1} \in S^{p-1}_D(\Omega, \Delta)$ and satisfies (3.30).

Combining all the cases discussed above, we complete the proof of this theorem. □

The assumption of Theorem 3.1, namely, that the elements are triangles or parallelograms is not substantial. The theorem can be proved for quadrilateral elements. In order to generalize Theorem 3.1. to the mesh containing quadrilateral elements we introduce the finite element spaces of pull-back polynomial type as usual :

$$S^p = S^p(\Omega; \Delta) = \left\{ \varphi \in H^1(\Omega) \,\middle|\, \varphi = \phi \circ M_j^{-1}, \phi \in P_p(Q), j = 1, 2, \cdots, J \right\} \quad (3.41)$$

and

$$S^p_D = S^p_D(\Omega; \Delta) = S^p(\Omega; \Delta) \cap H^1_D(\Omega). \quad (3.42)$$

where M_j is a mapping of the reference element Q onto the element Ω_j, which is bilinear if Ω_j is a quadrilateral element, and linear if Ω_j is a triangle or parallelogram, respectively.

On each patch Q_ℓ, which may contain quadrilateral elements, with the center denoted by $T_\ell, \ell = 1, 2 \cdots L$, we define a piecewise bilinear function ϕ_ℓ such that $\phi_\ell(T_\ell) = 1$ and $\phi_\ell(T_m) = 0$ for $m \neq \ell$. Note that $\phi_\ell \in S^2(\Omega; \Delta)$ and $\{\phi\}_{\ell=1}^L$ is a partition of unity for Ω subordinate to $\{Q_\ell\}_{\ell=1}^L$. Let $\varphi = \sum_{\ell=1}^{L} \phi_\ell \, \varphi^{[\ell]}$, where $\varphi^{[\ell]}$ is a polynomial of (separate) degree $(p-1)$ and belongs to $S^{2(p-1)}(\Omega; \Delta)$. Then $\varphi \in S^{2p}(\Omega; \Delta)$ and $u - \varphi = \sum_{\ell=1}^{L} \phi_\ell (u - \varphi^{[\ell]})$. By γ_ℓ we denote the edges of Q_ℓ such that $T_\ell \in \gamma_\ell$. We need to construct a polynomial $\varphi^{[\ell]} \in S^{2(p-1)}(\Omega; \Delta)$ which vanishes on $\gamma_\ell \cap \Gamma_D$ and satisfies

$$||u - \varphi^{[\ell]}||_{H^1(Q_\ell)} \leq C\, p^{-(k-1)} ||u||_{H^k(Q_\ell)} \tag{3.43}$$

if $\bar{Q}_\ell$ contains no vertices of Ω, and

$$||u - \varphi^{[\ell]}||_{H^1(Q_\ell)} \leq C\, p^{-2\gamma} (1 + \log p)^{\nu_\gamma^*} \tag{3.44}$$

if $\bar{Q}_\ell$ contains vertices of Ω.

The construction of $\varphi_{p-1}^{[\ell]}$ on each patch Q_ℓ in the proof of Theorem 3.1 can be carried over for the construction of $\varphi^{[\ell]}$. Hence we have proved the following theorem for general partition Δ containing triangular and quadrilateral elements.

Theorem 3.6. Let $\Delta = \left\{ \Omega_j, \; 1 \leq j \leq J \right\}$ be a quasi-uniform partition of Ω containing triangular and quadrilateral elements, and let $S_D^p(\Omega, \Delta)$ be the finite element space defined in (3.41)$\sim$ (3.42). The data functions f and g are smooth so that the solution u of the boundary value problem (3.1) is in $H^k(\Omega_0)$ and has the expansion (3.5) with $u_0^{[i]} \in H^k(S_{\delta_i})$ in each neighborhood S_{δ_i} with $k \geq \max\{2, \; 1 + 2\gamma\}$. Then the finite element solution $u_p \in S_D^p(\Omega; \Delta)$ for the problem (3.1) satisfies

$$||u - u_p||_{H^1(\Omega)} \leq C\, p^{-2\gamma} \, (1 + \log p)^{\nu_\gamma^*} \tag{3.45}$$

with constant C depending on k, γ and ν_γ^*, but not on p, where γ and ν_γ^* are given in (3.6). $\square$

Remark 3.1. The arguments above can be generalized further for meshes containing curvilinear elements where high-order mapping are used. However, we will not elaborate on it here.

Remark 3.2. The theorem is also valid for the data functions f and g which may have singularity at the vertices of the domain, i.e. $f \in \mathbf{C}_\beta^{k,0}(\Omega)$ or $H_\beta^{k,0}(\Omega)$ and $g \in \mathbf{C}_\beta^{k-3/2,1/2}(\Gamma)$ or $H_\beta^{k-3/2,1/2}(\Gamma_N)$. For the definitions of these weighted Sobolev spaces with Babuška-Guo weight, we refer to [2, 8].

3.3 Lower bound of approximation error in the p-version finite element solutions

We now study the lower bound of the approximation error in the p-version finite element solutions in two dimensional setting. The main theorem of this section is the following :

Theorem 3.7. Let u be the solution of the problem (3.1) with the data functions f and g, which are assumed such that the solution u is in $H^k(\Omega_0)$ with $k \geq \max\{2,\ 1+2\gamma\}$, and u has the expansion (3.5) with $u_0^{[i]} \in H^k(S_{\delta_i})$ in each neighborhood S_{δ_i}. and let $u_p \in S_D^p(\Omega;\Delta)$ be the finite element solution of the problem (3.7). Then

$$\|u - u_p\| \geq Cp^{-2\gamma}(1+\log p)^{\nu_\gamma^*}, \tag{3.46}$$

where C is independent of p, γ and ν_γ^* are given in (3.6). $\square$

The proof of the theorem needs several lemmas.

Lemma 3.8. Let $F(t)$ be a non-increasing function on $[0,\infty)$, and $\lim\limits_{t\to\infty} F(t) = 0$. Then, there is a function $G(t)$ on $(0,\infty)$ with the following properties:

(P1) $G(t) \geq F(t)$ for $t \in (0,\infty)$,
(P2) $G(t)$ is non-increasing,
(P3) $\lim_{t\to\infty} G(t) = 0$,
(P4) $\frac{G(t^k)}{G(t)} > \frac{1}{2}$ for $t \in (1,\infty)$ and integer $k \geq 1$.

Proof. We refer to Lemma 2.2 of [3]. □

We now introduce a weighted Besov space associated with the function G satisfying (P1)–(P4) :

$$B_G^{s,\beta}(Q) = \left(H^{\ell,\beta}(Q), H^{k,\beta}(Q)\right)_{\theta,\infty,G}$$

with norm

$$\|u\|_{B_G^{s,\beta}(Q)} = \sup_{t>0} \frac{t^{-\theta}K(t,u)}{G(1/t)},$$

where $0 < \theta < 1$, $s = (1-\theta)\ell + \theta k$.

Lemma 3.9 If for all $p \geq 1$ there holds

$$\inf_{\varphi \in P_p(Q)} \|u - \varphi\|_{L^2(Q)} \leq Cp^{-s}G(p)$$

where $G(t)$ satisfies (P1)–(P4), then $u \in B_G^{s,\beta'}(Q)$ with $\beta' = (0,0)$.

Proof. Refer to the proof of Lemma 2.3 in [3]. □.

Letting $Q_h = (-1, -1+2h)^2$, we introduce Sobolev spaces $H^{k,\beta}(Q_h)$ with Jacobi-weighted norm

$$\|u\|_{H^{k,\beta}(Q_h)} = \sum_{|\alpha| \leq k} \int_{Q_h} |D^\alpha u|^2 \prod_{i=1}^{2} (1-x_i^2)^{\alpha_i+\beta_i}\, dx,$$

where $k \geq 0$, integer and $\beta = (\beta_1, \beta_2), \beta_i > -1, i = 1, 2$. The semi-norm $|u|_{H^{k,\beta}(Q_h)}$ involves only the k-th derivatives of u.

The weighted Besov space $B_G^{s,\beta}(Q_h)$ is defined as an interpolation space by the K-method, i.e.

$$B_G^{s,\beta}(Q_h) = \Big(H^{\ell,\beta}(Q_h), H^{k,\beta}(Q_h)\Big)_{\theta,\infty,G}$$

where $0 < \theta < 1, s = (1-\theta)\ell + \theta k$, ℓ and k are integers with norm

$$\|u\|_{B_G^{s,\beta}(Q_h)} = \sup_{t>0} \frac{t^{-\theta}K(t,u)}{G(1/t)},$$

with $K(t,u)$ given in (2.4) and G satisfying (P1)–(P4).

Obviously, the spaces $H^{k,\beta}(Q_h)$ and $B_G^{s,\beta}(Q_h)$ are the restrictions of the spaces $H^{k,\beta}(Q)$ and $B_G^{s,\beta}(Q)$ on $Q_h = (-1,-1+2h)^2$, respectively, and

$$\|u\|_{H^{k,\beta}(Q_h)} \leq \|u\|_{H^{k,\beta}(Q)}, \; \|u\|_{B^{sk,\beta}(Q_h)} \leq \|u\|_{B^{s,\beta}(Q)}.$$

Lemma 3.10. Let $u(x) \in H^{k,\beta}(Q_h)$ with $\beta_i \geq 0$, and let $\tilde{u}(\tilde{x}) = u(h(1+\tilde{x})-1)$. Then $\tilde{u}(\tilde{x}) \in H^{k,\beta}(Q)$, and for integer $\ell \leq k$

$$|\tilde{u}|^2_{H^{\ell,\beta}(Q)} \leq C\, h^{\ell-2-\beta_1-\beta_2}\, |u|^2_{H^{\ell,\beta}(Q_h)}.$$

Proof. The mapping : $x+1 = h(\tilde{x}+1)$ maps $Q = (-1,1)^2$ onto Q_h. Then $\tilde{u}(\tilde{x})$ is defined on Q. Noting that $1 \leq (1-x_i) \leq 2$ for $x \in Q_h$, we have

$$\begin{aligned}
|\tilde{u}|^2_{H^{\ell,\beta}(Q)} &= \textstyle\sum_{|\alpha|=\ell} \int_Q |D^\alpha \tilde{u}|^2 \; \prod_{i=1}^2 (1-\tilde{x}_i^2)^{\alpha_i+\beta_i}\, \tilde{d}x \\
&\leq C \textstyle\sum_{|\alpha|=\ell} h^{\ell-2-\beta_1-\beta_2} \int_{Q_h} |D^\alpha u|^2 \; \prod_{i=1}^2 (1+x_i)^{\alpha_i+\beta_i}\, dx \\
&\leq C h^{\ell-2-\beta_1-\beta_2} |u|^2_{H^{\ell,\beta}(Q_h)}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.11. Let $u(x) \in H^{k,\beta}(Q_h)$ with $\beta_1 = \beta_2 = 0$. Then there exists a polynomial $\varphi_p \in P_p(Q_h)$ with $1+p \geq k$ such that

$$|u - \varphi_p|^2_{H^{0,\beta}(Q_h)} \leq C\, h^k p^{-2k}\, |u|^2_{H^{k,\beta}(Q_h)}. \tag{3.47}$$

Proof. By Lemma 3.10, $\tilde{u}(\tilde{x}) = u(h(1+\tilde{x})-1) \in H^{k,\beta}(Q)$. Due to Theorem 2.1 , there exist a polynomial $\tilde{\varphi}_p(\tilde{x}) \in P_p(Q_h)$ with $1+p \geq k$, which is the Jacobi projection of $\tilde{u}(\tilde{x})$ on $P_p(Q_h)$, such that

$$|\tilde{u} - \tilde{\varphi}_p|^2_{H^{0,\beta}(Q)} \leq C p^{-2k} |\tilde{u}|^2_{H^{k,\beta}(Q)}.$$

Let $\varphi_p(x) = \tilde{\varphi}_p(\dfrac{1+x}{h} - 1) \in P_p(Q_h)$, and by Lemma 3.10 and scaling we have

$$\begin{aligned}
|u - \varphi|^2_{H^{0,\beta}(Q_h)} &\leq C \textstyle\int_{Q_h} |u - \varphi|^2\, dx \\
&\leq C h^2 \textstyle\int_Q |\tilde{u} - \tilde{\varphi}_p|^2\, dx \\
&\leq C h^2 p^{-2k} |\tilde{u}|^2_{H^{k,\beta}(Q)} \\
&\leq C h^k p^{-2k} |u|^2_{H^{k,\beta}(Q_h)}.
\end{aligned}$$

$\square$.

Lemma 3.12. Let $u(x) \in B_G^{s,\beta}(Q_h)$ with $\beta_1 = \beta_2 = 0$. Then there exists $\varphi \in P_p(Q_h)$ with $p+1 > s$ such that

$$|u - \varphi_p|_{H^{0,\beta}(Q_h)} \leq C\, h^{s/2} G(1/h^{s/2})\, \|u\|_{B_G^{s,\beta}(Q_h)}. \tag{3.48}$$

where the constant C depends on p but not on h.

Proof. Due to the definition, there are $v \in H^{0,\beta}(Q_h)$ and $w \in H^{k,\beta}(Q_h)$ such that for all $t > 0$

$$\|v\|_{H^{0,\beta}(Q_h)} + t\|w\|_{H^{k,\beta}(Q_h)} \leq Ct^{\theta} G(1/t)\|u\|_{B_G^{s,\beta}(Q)} \tag{3.49}$$

where $0 < s < k$, $\theta = \dfrac{s}{k}$. By Lemma 3.11 there exists a polynomial $\varphi_p \in P_p(Q_h)$ satisfying

$$\|w - \varphi_p\|_{H^{0,\beta}(Q_h)} \leq Ch^{k/2}\|w\|_{H^{k,\beta}(Q_h)}.$$

Therefore

$$\begin{aligned} \|u - \varphi_p\|_{H^{0,\beta}(Q_h)} &\leq \|v\|_{H^{0,\beta}(Q_h)} + \|w - w_p\|_{H^{0,\beta}(Q_h)} \\ &\leq \|v\|_{H^{0,\beta}(Q_h)} + Ch^{k/2}\|w\|_{H^{k,\beta}(Q_h)} \\ &\leq C\left(\|v\|_{H^{0,\beta}(Q_h)} + h^{k/2}\|w\|_{H^{k,\beta}(Q_h)}\right). \end{aligned} \tag{3.50}$$

Letting $t = h^{k/2}$ and combining (3.49) and (3.50), we have

$$\begin{aligned} \|u - \varphi_p\|_{H^{0,\beta}(Q_h)} &\leq C\left(\|v\|_{H^{0,\beta}(Q_h)} + t\|w\|_{H^{k,\beta}(Q_h)}\right) \\ &\leq Ct^{\theta} G(1/t)\|u\|_{B_G^{s,\beta}(Q_h)} \\ &\leq Ch^{s/2} G(1/h^{s/2})\|u\|_{B^{s,\beta}(Q_h)}. \end{aligned}$$

Thus the proof of Lemma 3.12 is completed. $\square$

We now are ready to prove the main theorem in this section.

Proof of Theorem 3.7. We shall prove (3.46) for $\nu_\gamma^* = 0$. The proof for $\nu_\gamma^* > 0$ is similar to what follows. We assume that Ω_1 is the element containing the vertex A_{i_0} where the strongest singularity occurs. It suffices to prove for some α with $|\alpha| = 1$

$$\|D^{\alpha}(u - u_p)\|_{L^2(\Omega_1)} \geq Cp^{-2\gamma}. \tag{3.51}$$

We may assume that Ω_1 is a quadrilateral without loss of generality. If Ω_1 is a triangle, we may consider a quadrilateral with the vertices at A_{i_0} and middle points of three edges of Ω_1. By M_1 we denote a linear mapping which maps $Q=(-1,1)^2$ onto Ω_1 and A_{i_0} to the vertex $(-1,-1)$. Let $v=u\circ M_1$ and $v_p=u_p\circ M_1$. Then $v_p\in P_p(Q)$ and we need to show that

$$\|D^\alpha(v-v_p)\|_{L^2(\Omega_1)}\geq Cp^{-2\gamma}$$

If it is false, then there exists a function F(t) such that $\lim_{t\to\infty}F(t)=0$, and

$$\|D^\alpha(v-v_p)\|_{L^2(\Omega_1)}\leq CF(p)\,p^{-2\gamma}$$

By Lemma 3.8 we can construct a function G satisfying (P1)-(P4), and by Lemma 3.9 $D^\alpha v\in B_G^{2\gamma,\beta'}(Q)$ with $\beta_1'=\beta_2'=0$ and $|\alpha|=1$. Then $v\in B_G^{2\gamma,\beta'}(Q_h)$, and due to Lemma 3.12, there is $\varphi\in P_p(Q_h)$ with $p+1\geq k$ such that

$$|D^\alpha v-\varphi|_{L^2(Q_h)}\leq C\,h^\gamma G(1/h^\gamma)\,|v|_{B_G^{2\gamma,\beta'}(Q_h)}\leq C\,h^\gamma G(1/h^\gamma)\,|v|_{B_G^{2\gamma,\beta'}(Q)}.\ (3.52).$$

On the other hand, v has singularity of r^γ-type at A_{i_0} it is known [14, 23] that for $|\alpha|=1$

$$\inf_{\phi\in P_1(Q_h)}\|D^\alpha(r^\gamma\chi(r)\Phi(\theta))-\phi\|_{L^2(Q_h)}\geq C\,h^\gamma,$$

which contradicts (3.52). Thus (3.51) holds.

For $\nu_\gamma^*>0$, we need only to introduce the space $B^{s,\beta}_{G,\nu_\gamma^*}(Q)$, instead of $B_G^{s,\beta}(Q)$, and its norm :

$$\|u\|_{B_G^{s,\beta}(Q)}=\sup_{t>0}\frac{t^{-\theta}K(t,u)}{(1+|\log t|)^{\nu_\gamma^*}G(1/t)}.$$

Then all arguments above can be carrid out for $\nu_\gamma^*>0$. Hence the theorem is proved. $\square$

3.4 Optimal convergence

Combining the estimates on the lower and upper bounds of the approximation error in the p-version finite element solution, we conclude with the optimal convergence of the p-version.

Theorem 3.13. Let u be the solution of the problem (3.1) on polygonal domain Ω with the assumption on the data functions f and g as in Theorem 3.1-3.3, and let u_p be the finite element solution of the p-version in $S_D^p(\Omega;\Delta)$, respectively. Then $p^{-2\gamma}(1+\log p)^{\nu^*}$ is the optimal rate of convergence for the finite element solution u_p, i.e. there are constants C_1 and C_2 independent of p such that

$$C_1 p^{-2\gamma}(1+\log p)^{\nu^*} \leq \|u-u_p\|_{H^1(\Omega)} \leq C_2 p^{-2\gamma}(1+\log p)^{\nu^*},$$

where γ and ν_γ^* are given in (3.6), which represent the strongest singularity of the solution of the problem (3.1). $\square$

Remark 3.4. It has been shown that the constants C_1 and C_2 in Theorem 3.13 are asymptotically the same in one dimension [16]. Whether C_1 and C_2 in two dimensions are asymptotically the same remains to be answered yet. Nevertheless, the same order on the the upper and lower bound of errors allows us to develop a-posteriori error estimators by extrapolation of computational solutions, which will be reasonably reliable in practice if the difference between C_1 and C_2 is not too large.

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