DISCONNECTED JULIA SETS AS BURIED JULIA COMPONENTS

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Abstract. Let f be a rational map whose Julia set $J(f)$ is disconnected. It is proved that there exists a rational map g such that g has a family of buried Julia components on which g is quasiconformally conjugate to f on $J(f)$ if and only if f has no parabolic basins and rotation domains. This extends the previous result about burying connected Julia sets to the disconnected case. Moreover, the limit behaviors of the conformal dimensions of successively burying Julia sets are also studied.

1. Introduction

1.1. **Backgrounds.** For a given rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the *Fatou set* $F(f)$ of f is defined to be the set of points at which the family of iterations $\{f^{\circ n}\}_{n\geqslant 0}$ of f forms a normal family in the sense of Montel. A connected component of the Fatou set is called a *Fatou component*. According to Sullivan [Sul85], each Fatou component U of f is eventually periodic. That is, there exist two integers $k \geqslant 0$ and $p \geqslant 1$ such that $f^{\circ(k+p)}(U) = f^{\circ k}(U)$. Moreover, based on the dynamical behaviors, the periodic Fatou components of f can be divided into exactly five types: *super-attracting basins*, *attracting basins*, *parabolic basins*, *Siegel disks* and *Herman rings* [MS98]. For the former three types of periodic Fatou components, the points therein are attracted by super-attracting, attracting and parabolic periodic cycles respectively under iterations. For the latter two types, the restriction of f, or some iterate of f therein, are holomorphically conjugate to an irrational rotation of a disk and an annulus respectively. The complement of the Fatou set is called the *Julia set*, which we denote by $J(f)$. Each connected component of $J(f)$ is called a *Julia component*. A point $c \in \widehat{\mathbb{C}}$ is called a *critical point* of f if f is not injective in any neighborhood of c and f has exactly $2\deg(f) - 2$ critical points counted by multiplicity. The dynamical behavior of f is dominated by the forward orbits of critical points (see [Lyu83], [McM94, §3.3] and [Mil06, §§8-11, 15]). For more details on the dynamics of rational maps, one may refer to [Bea91b], [CG93] or [Mil06].

An interesting and important problem in complex dynamics is to describe the topology of the Julia sets. A Julia component (or a point on the Julia set) is called *buried* if it is not on the boundary of any Fatou component. Since the Julia set of any non-linear polynomial coincides with the boundary of the unbounded Fatou component, the buried points cannot occur in polynomial Julia sets. The first example of buried Julia component was constructed by McMullen in [McM88], who considered a family of rational maps which is given by

$$f_\lambda(z) = z^l + \lambda/z^m, \quad \text{where } l \geqslant 2, m \geqslant 1 \text{ and } \lambda \in \mathbb{C}.$$

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McMullen proved that if $1/l + 1/m < 1$ and $\lambda \neq 0$ is small enough, then the Julia set $J(f_\lambda)$ is a *Cantor set of circles* which is homeomorphic to the Cartesian product of the standard middle third Cantor set and the unit circle. In particular, such a Julia set contains infinitely many buried Julia components which are Jordan curves (see also [DLU05]). More such examples can be found in [XQY14] and [QYY15].

As a generalization of Beardon's result [Bea91a], Qiao proved that for a rational map f of degree at least two, $J(f)$ has buried components if and only if $J(f)$ is disconnected and $F(f)$ has no completely invariant component [Qia95]. This provides a criterion to justify which kinds of rational maps contain buried Julia components. In particular, some buried Julia components which are singletons can be found in cubic rational maps. Later, some specific rational maps of degree at least 5 containing buried Julia components which are Jordan curves or singletons were studied further (see [PT00], [BDGR08], [GMR13] and the references therein).

In 2015, Godillon constructed a family of *cubic* rational maps and proved that for suitable parameters, the corresponding Julia sets contain a buried Julia component which is homeomorphic to the Julia set of $z \mapsto 1/(z-1)^2$. In particular, it is neither a Jordan curve nor a singleton [God15]. Godillon's example is optimal in terms of degree since quadratic rational maps cannot contain any buried Julia components (see [Yin92] and [Mil93]). Later, a different method to construct more such kind of examples appeared in [WY20].

1.2. **Main results.** In [McM88, Theorem 3.4], McMullen proved the following result: if a rational map f has an invariant non-singleton Julia component J_0, then there exists a rational map g such that $f : J_0 \to J_0$ is quasiconformally conjugate to $g : J(g) \to J(g)$, i.e., there exists a quasiconformal mapping $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\phi(J_0) = J(g)$ and $\phi \circ f = g \circ \phi$ holds on J_0. For the definitions and properties of quasiconformal mappings, we refer to [LV73], [Ahl06] and [BF14]. McMullen's result implies that one may extract some rational maps with connected Julia sets from the rational maps with disconnected Julia sets. As an inverse procedure of McMullen, the following question was asked in [WY20]:

Question 1.1. Could any *connected* Julia set appear as a buried Julia component of a higher degree rational map?

The following result provides an answer to this question.

Theorem 1.2 (The connected case, [WY20])**.** *Let f be a rational map of degree $d \geqslant 2$ whose Julia set $J(f) \neq \widehat{\mathbb{C}}$ is connected. Then there is a rational map g such that g has a buried Julia component on which g is quasiconformally conjugate to f on $J(f)$ if and only if f has no parabolic basins and Siegel disks. If such g exists, then the degree of g can be chosen such that* $\deg(g) \leqslant 7d - 2$.

The proof of Theorem 1.2 is based on quasiconformal surgery and singular perturbation. In order to control the degree of g as small as possible, the main perturbation in [WY20] is made at critical *values*, but not at critical *points*, which is different from the classical cases. One may also refer to [WZL22] for another proof of Theorem 1.2 which avoids the use of quasiconformal surgery. Based on Theorem 1.2, it is natural to ask:

Question 1.3. Could any *disconnected* Julia set appear as a family of buried Julia components of a higher degree rational map?

A periodic Fatou component is called a *rotation domain* if it is a Siegel disk or a Herman ring. In this paper, we prove the following result.

Theorem A (The disconnected case)**.** *Let f be a rational map whose Julia set $J(f)$ is disconnected. Then there is a rational map g such that g has a family of buried Julia components on which g is quasiconformally conjugate to f on $J(f)$ if and only if f has no parabolic basins and rotation domains. If such g exists, then* $\deg(g)$ *can be chosen such that it is less than a number depending only on* $\deg(f)$.

Theorem 1.2 and Theorem A imply that a rational map can have buried Julia components which are "almost" arbitrary. For example, the rational map f can have a Cantor Julia set, can have a Cremer point and can be infinitely renormalizable. The proof of Theorem A is based on successive perturbations and quasiconformal surgery. See Figure 1 for an example, which illustrates the process of perturbations of the cubic polynomial $f(z) = z^3 - \frac{13}{6}z^2 + \frac{2}{3}$ whose Julia set is disconnected.

Let $\mathscr{F}$ be the family of all rational maps with disconnected Julia sets having no parabolic basins and rotation domains. By Theorem A, for any $f \in \mathscr{F}$, there exists $g \in \mathscr{F}$ having a family of buried Julia components on which g is quasiconformally conjugate to f on $J(f)$. Hence the following set is non-empty and actually consists of infinite elements:

$$\mathrm{Buried}(f) := \left\{ g \in \mathscr{F} \;\middle|\; \begin{array}{l} \exists \text{ a quasiconformal mapping } \varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \\ \text{such that } \varphi(J(f)) \text{ is buried in } J(g) \text{ and} \\ \varphi \circ f|_{J(f)} = g \circ \varphi|_{J(f)} \end{array} \right\}.$$

For each $g \in \mathrm{Buried}(f)$, the set $\mathrm{Buried}(g)$ is also well-defined. Hence such process of burying Julia sets can be repeated infinitely often.

Let (X, d_X) and (Y, d_Y) be two metric spaces. Suppose that there exist two homeomorphisms $h : X \to Y$ and $\psi : [0, +\infty) \to [0, +\infty)$ such that

$$\frac{d_Y(h(x), h(y))}{d_Y(h(x), h(z))} \leqslant \psi\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for any distinct points $x, y, z \in X$. Then (X, d_X) and (Y, d_Y) are said to be *quasisymmetrically equivalent* to each other. The *conformal dimension* $\dim_C(X)$ of a compact set X is the infimum of the Hausdorff dimensions of all metric spaces which are quasisymmetrically equivalent to X. The quasisymmetric geometries of the Julia sets of rational maps, including the quasisymmetric uniformization and conformal dimension etc, have attracted many people's interests. For examples, see [MT10], [HP12], [BLM16], [QYY16], [QYY18], [LM18], [QYZ19], [PT21], [Par22] and the references therein.

According to [Hei01, Theorem 11.14, p. 92], any compact subset X of $\widehat{\mathbb{C}}$ is quasisymmetrically equivalent to $\varphi(X)$ for any quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Hence by definition, for any given $f \in \mathscr{F}$, we have

$$\dim_C(J(f)) \leqslant \dim_C(J(g)), \quad \text{for all } g \in \mathrm{Buried}(f).$$

This implies that if $f_{n+1} \in \mathrm{Buried}(f_n)$ for all $n \geqslant 0$, then $\{\dim_C(J(f_n))\}_{n \geqslant 0}$ forms an increasing sequence and it must have a limit since the conformal dimension of any Julia set in $\widehat{\mathbb{C}}$ is at most 2. A natural question is about the limit of the sequence $\{\dim_C(J(f_n))\}_{n \geqslant 0}$. We give a partial answer to this question.

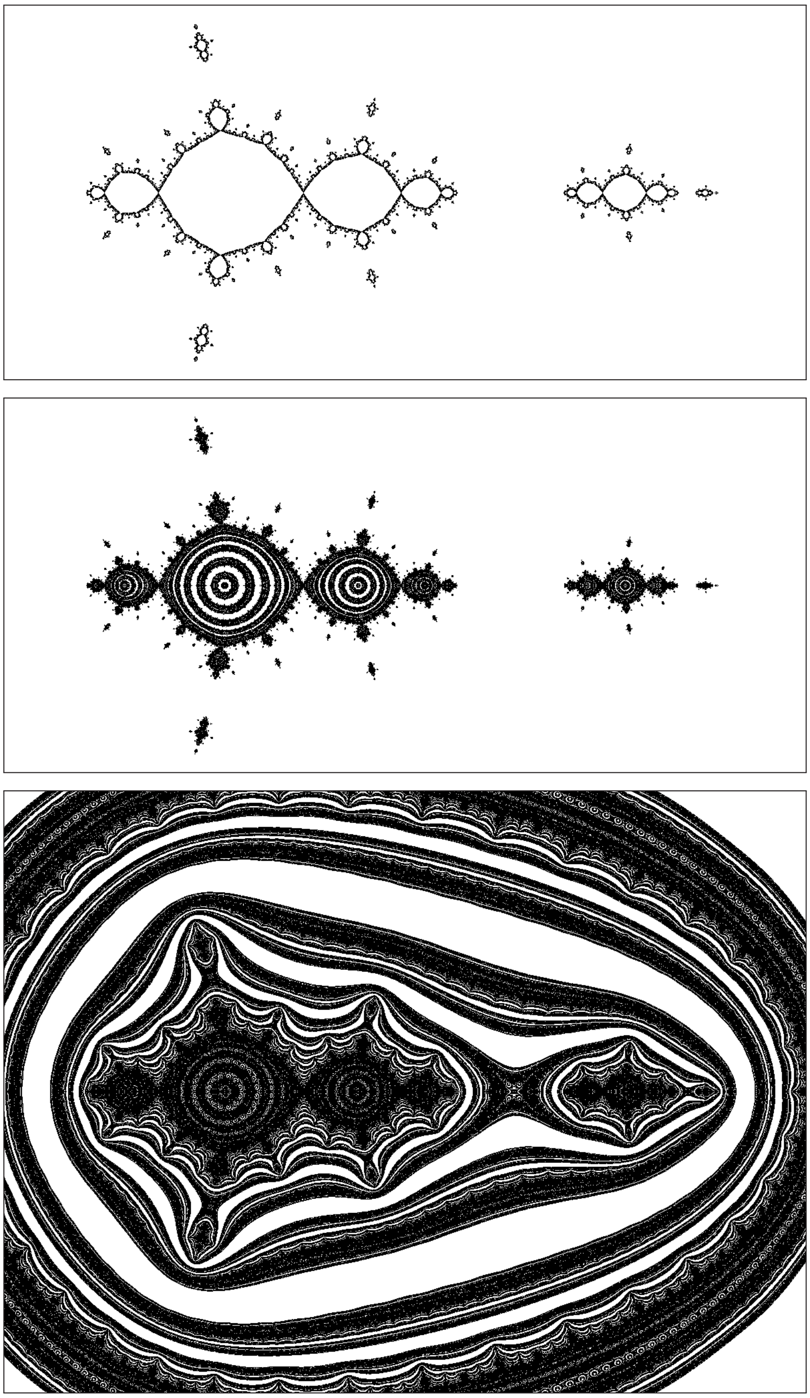

Figure 1: The Julia sets of

$$f(z) = z^3 - \frac{13}{6}z^2 + \frac{2}{3},\ h(z) = f(z) + \frac{\lambda}{(z-2/3)^2} \text{ and } g(z) = 1/\Big(\frac{1}{h(z)} + \mu z^4\Big)$$

(from the top down), where $\lambda = \mu = 10^{-9}$. The middle and bottom Julia sets, respectively, contain a family of semi-buried and buried components which are homeomorphic to the top Julia set.

Theorem B. *There exist two sequences of hyperbolic rational maps $(f_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 0}$ in $\mathscr{F}$ satisfying $f_{n+1}\in \mathrm{Buried}(f_n)$ and $g_{n+1}\in \mathrm{Buried}(g_n)$ for all $n\geqslant 0$ with $f_0=g_0$ and $\dim_C(J(f_0))\in(1,2)$ such that*

$$\lim_{n\to\infty}\dim_C(J(f_n))=2 \quad \textit{and} \quad \lim_{n\to\infty}\dim_C(J(g_n))<2.$$

The rational maps in Theorem B will be chosen in Cantor circle hyperbolic components since their dynamics can be characterized clearly and the conformal dimensions of the corresponding Julia sets can be calculated precisely.

1.3. **Sketch of the proofs.** The proof of Theorem A is inspired by Theorem 1.2. Since the necessity part is similar, we only consider the sufficiency. Let f be a rational map with a disconnected Julia set. We use singular perturbations and quasiconformal surgery to obtain two rational maps h and g, such that the Julia set of h has a family of "semi-buried" Julia components (see §3) and g contains a family of "fully buried" Julia components (see §4) which are homeomorphic copies of $J(f)$ respectively. Note that all Fatou components of f in Theorem 1.2 are simply connected while some attracting Fatou components of f in this paper are infinitely connected.

Compared to Theorem 1.2, there are three main differences in the proof of Theorem A. The first difference is: we need to use quasiconformal surgery to transfer infinitely connected attracting Fatou components to super-attracting ones (The surgery is different from the simply connected case, see §2). The second difference is: we need to consider more complicated combinations of holomorphic coverings while showing that h has a family of "semi-buried" Julia components and g contains a family of "fully buried" Julia components (in Theorem 1.2 we only need to consider the annulus-to-annulus combination). The third difference is: we need to consider Cantor Julia sets and find Julia sets of higher degree rational maps containing them as buried Julia components and this case is quite different (see §4).

For Theorem B, we first consider a hyperbolic rational map with Cantor circle Julia set. Then we use another Cantor circle Julia set to bury the previous one, and this process can be repeated infinitely many times. It was known that the conformal dimension of the Cantor circle Julia sets of hyperbolic rational maps depend only on some combinatorial information which can be calculated precisely. Theorem B will be proved by applying quasiconformal surgery and arranging the combinatorial information on Cantor circle Julia sets suitably (see §5).

Notations. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, be the set of natural, real and complex numbers. For $a\in\mathbb{C}$ and $r>0$, we denote $\mathbb{D}(a,r):=\{z\in\mathbb{C}:|z-a|<r\}$, $\mathbb{D}_r:=\mathbb{D}(0,r)$, $\mathbb{T}_r:=\partial\mathbb{D}_r$, $\mathbb{D}:=\mathbb{D}_1$ and $\mathbb{T}:=\mathbb{T}_1$. For $0<r<1$, we denote $\mathbb{A}_r:=\{z\in\mathbb{C}:r<|z|<1\}$ and its conformal modulus is $\mathrm{mod}(\mathbb{A}_r)=\frac{1}{2\pi}\log\frac{1}{r}$.

For a Jordan curve $\gamma\subset\widehat{\mathbb{C}}\setminus\{\infty\}$, we use $D(\gamma)$ to denote the connected component of $\widehat{\mathbb{C}}\setminus\gamma$ which does not contain ∞. For two disjoint connected compact subsets γ_1 and γ_2 in $\widehat{\mathbb{C}}$ which are not singletons, we use $A(\gamma_1,\gamma_2)$ (or $A(\gamma_2,\gamma_1)$) to denote the unique annular component of $\widehat{\mathbb{C}}\setminus(\gamma_1\cup\gamma_2)$.

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2. Attracting to super-attracting

Since singular perturbations are generally carried out at super-attracting periodic points, this requires us to transfer the geometric attracting cycles to super-attracting ones by quasiconformal surgery. In fact, such a surgery is well known for simply connected periodic attracting Fatou components. See [CG93, Theorem 5.1, p. 106] and [BF14, Exercise 4.2.2, p. 168]. In this section, we show that such kind of surgery can be performed in multiply connected attracting basins (actually infinitely connected by [Mil06, Theorem 8.9, p. 83]).

A map $F : U \to \widehat{\mathbb{C}}$ is called *quasi-regular* if it can be written as $F = G \circ \phi$, where $\phi : U \to \phi(U)$ is a quasiconformal mapping defined in the open set U and $G : \phi(U) \to \widehat{\mathbb{C}}$ is holomorphic. For other equivalent characterizations of quasi-regular maps, see [BF14, §1.6]. The following lemma is very useful when performing quasiconformal surgery.

Lemma 2.1 ([Shi87])**.** *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasi-regular map. Suppose there exist an open set $E \subset \widehat{\mathbb{C}}$ and an integer $N \geqslant 0$ satisfying the following two conditions:*

- *$F(E) \subset E$; and*
- *$\partial F/\partial \overline{z} = 0$ holds in E and on $\widehat{\mathbb{C}} \setminus F^{-N}(E)$ a.e.*

Then there is a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\varphi \circ F \circ \varphi^{-1}$ is rational.

Lemma 2.1 was established by Shishikura although its original statement is more general. One may refer to [Shi87, §3] or [BF14, Proposition 5.2] for more details.

Lemma 2.2. *Let f be a rational map having a p-cycle of attracting Fatou components $\{B_i : 1 \leqslant i \leqslant p\}$. Then there exists a rational map g and a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that*

- *$\varphi \circ f = g \circ \varphi$ holds in a neighborhood of $J(f)$ and $J(g) = \varphi(J(f))$; and*
- *$\{\varphi(B_i) : 1 \leqslant i \leqslant p\}$ is a p-cycle of super-attracting Fatou components of g.*

Proof. Without loss of generality, we assume that $p = 1$ and B is a fixed attracting Fatou component of f which contains the fixed point 0 with multiplier $\lambda \in \mathbb{D} \setminus \{0\}$. Let Ω be the maximal linearizable domain of f in B containing 0. Then there exists a conformal map $\phi : \Omega \to \mathbb{D}$ such that $\phi(f(z)) = \lambda\phi(z)$ for all $z \in \Omega$ and $\partial\Omega$ contains at least one critical point c of f. Note that $f(\partial\Omega) = \phi^{-1}(\mathbb{T}_{|\lambda|})$ is a smooth Jordan curve and $f : \Omega \to f(\Omega)$ is conformal. By [Pil96, Proposition 2.8], Ω is a Jordan domain and $f : \overline{\Omega} \to f(\overline{\Omega})$ is a homeomorphism.

We choose two Jordan domains V_1, V_2 in B with smooth boundaries such that $\overline{\Omega} \subset V_1$, $\overline{V}_1 \subset V_2$ and $\overline{V}_2 \setminus \overline{\Omega}$ is disjoint with the critical orbits of f. Let U_i be the connected component of $f^{-1}(V_i)$ containing $\overline{\Omega}$, where $i = 1, 2$. We conclude that $\overline{U}_1 \subset U_2$, each connected component of ∂U_i is a smooth Jordan curve and each component of $f^{-1}(V_2 \setminus \overline{V}_1)$ is an annulus containing no critical orbits. Without loss of generality, we assume that the boundaries of V_1 and V_2 are sufficiently close to $\partial\Omega$ such that $\overline{V}_2 \subset U_1$. See Figure 2.

Without loss of generality, we assume that $\infty \notin U_2$. Then there exists a small $\varepsilon > 0$ such that $g_\varepsilon(U_1)$ is a Jordan domain contained in V_1, where

$$g_\varepsilon(z) := \varepsilon(f(z) - f(c)) + c.$$

Let A be a connected component of $f^{-1}(V_2 \setminus \overline{V}_1)$ in $U_2 \setminus \overline{U}_1$. Then A is an annulus containing no critical orbits whose boundary components $\partial_1 A$ and $\partial_2 A$ are smooth,

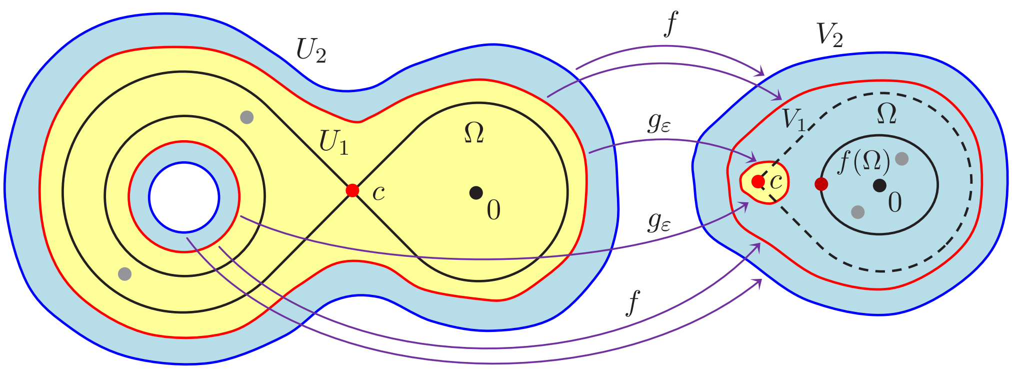

Figure 2: A sketch of the surgery construction which transfers an attracting fixed point to a super-attracting one. Some special curves and points are marked.

where $\partial_i A$ is the boundary component of A contained in ∂U_i for $i = 1, 2$. Hence there exists an integer $d_A \geqslant 1$ such that

$$\deg(g_\varepsilon|_{\partial_1 A}) = \deg(f|_{\partial_2 A}) = d_A.$$

Therefore, there exists a continuous map $h : \overline{A} \to \overline{V}_2 \setminus g_\varepsilon(U_1)$ such that

- $h|_A : A \to V_2 \setminus \overline{g_\varepsilon(U_1)}$ is a quasi-regular covering map of degree d_A; and
- $h|_{\partial_1 A} = g_\varepsilon$ and $h|_{\partial_2 A} = f$.

For quasi-regular interpolation in annuli, we refer to [BF14, §2.3.2]. Define

$$F(z) := \begin{cases} f(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus U_2, \\ g_\varepsilon(z) & \text{if } z \in U_1, \\ h(z) & \text{otherwise.} \end{cases}$$

Then $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map whose degree is $\deg(f)$.

Define $E := U_1$. Since $\overline{V}_2 \subset U_1$, we have $F(E) \subset V_1 \subset V_2 \subset E$ and $\partial F / \partial \overline{z} = 0$ holds in E and on $\widehat{\mathbb{C}} \setminus F^{-1}(E)$ a.e. By Lemma 2.1, there exists a quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $g := \varphi \circ F \circ \varphi^{-1}$ is a rational map. Note that $F(c) = g_\varepsilon(c) = c$. It follows that $\varphi(c)$ is a fixed super-attracting fixed point of g. Since the surgery is performed only in Fatou components, the rest statements hold. □

Remark. The proof of Lemma 2.2 is valid for any attracting Fatou component B, no matter B is simply connected or infinitely connected. If B is simply connected, then as mentioned before, one can use surgery to make the dynamics in B to be unicritical. However, because of the Riemann-Hurwitz formula, one cannot pinch all critical points into only one if B is infinitely connected.

3. Burying boundaries of Fatou components

In this section, we perform the quasiconformal surgery on a given rational map f with disconnected Julia set to obtain a new rational map h such that the homeomorphic image of the Julia set $J(f)$ is "semi-buried" by the Julia components of h. Then we perform another quasiconformal surgery on h to obtain a rational map g such that the homeomorphic image of $J(f)$ is fully buried by the Julia components of g.

3.1. **Basic classifications and definitions.** By [Bea91b, Theorem 5.6.2], the number of Fatou components of a rational map is either 0, 1, 2 or ∞. Since $J(f)$ is disconnected, we don't need to consider the first case. The following result shows that we don't need to consider the third case either.

Lemma 3.1. *Let f be a rational map whose Julia set is disconnected. Then*
(a) *f has either exactly one or infinitely many Fatou components;*
(b) *If f has infinitely many Fatou components, then it has at most one attracting or parabolic basin consisting of finitely many Fatou components, and in particular, if such a basin exists then it consists of only one Fatou component which is completely invariant.*

Proof. (a) It is sufficient to prove that if U_1 and U_2 are the only Fatou components of f, then $J(f)$ is connected. Iterating f twice if necessarily, we assume that U_1 and U_2 are completely invariant. Then $\partial U_1 = J(f) = \partial U_2$. Therefore, U_1 and U_2 are simply connected and $J(f)$ is connected.

(b) Suppose f has two attracting or parabolic basins consisting of finitely many Fatou components. Iterating f finitely many times if necessary, we conclude that f has two completely invariant Fatou components. By a similar argument to Part (a), it follows that $J(f)$ is connected, which is a contradiction. Hence f has at most one attracting or parabolic basin consisting of finitely many Fatou components U_1, $\cdots$, U_n. Iterating f finitely many times if necessary, we assume that $U_1, \cdots, U_n$ are completely invariant. Similarly as above, we must have $n = 1$ and U_1 is completely invariant under f. □

Based on Lemma 3.1(a), we consider the following two cases:
- Case I: f has infinitely many Fatou components;
- Case II: f has exactly one Fatou component which is completely invariant.

In the rest place of this section, we consider Case I and the second case will be discussed in §4.

Let f be a rational map with disconnected Julia set which does not contain parabolic basins and rotation domains. Then all periodic Fatou components of f are attracting or super-attracting. By Lemma 2.2, without loss of generality, we assume in the following that the periodic Fatou components f are all super-attracting.

Suppose f has infinitely many Fatou components and has degree $d \geqslant 2$. Then there exists a p-cycle of super-attracting periodic Fatou components $O = \{B_1, \cdots, B_p\}$ of f such that the grand orbit of O has infinitely many components, where $p \geqslant 1$ and

$$f(B_j) = B_{j+1} \text{ for } 1 \leqslant j \leqslant p-1 \quad \text{and} \quad f(B_p) = B_1.$$

According to Böttcher's theorem (see [Mil06, Theorem 9.3]), there exist simply connected domains $U_j \subset B_j$ and conformal maps $\phi_j : U_j \to \mathbb{D}_{r_j}$ with $r_j \in (0,1]$, where $1 \leqslant j \leqslant p$, such that

$$\phi_j \circ f^{\circ p}(z) = (\phi_j(z))^{d_0}, \quad \text{for } z \in U_j,$$

where $d_0 = \prod_{j=1}^{p} d_j \geqslant 2$ and $d_j = \deg(f|_{U_j}) \geqslant 1$ for $1 \leqslant j \leqslant p$. Moreover, either $U_j = B_j$ and $r_j = 1$ for all $1 \leqslant j \leqslant p$, or ∂U_j is a subset of B_j containing a critical point of f for some $1 \leqslant j \leqslant p$ and $0 < r_j < 1$ for all $1 \leqslant j \leqslant p$. Note that the *potential* function $z \mapsto |\phi_j(z)|$ for $z \in U_j$ can be extended continuously to B_j by

$$L_j(z) := |\phi_j(f^{\circ kp}(z))|^{1/d_0^k},$$

where $k \geqslant 0$ is the minimal integer such that $f^{\circ kp}(z) \in U_j$.

Definition. An *equipotential curve* γ in B_j is a connected component of $L_j(z) = C$ for some constant $C \in (0,1)$. We call that γ has *potential* $L_j(\gamma) = C$.

Since the grand orbit of $O = \{B_1, \cdots, B_p\}$ has infinitely many components, there exists $1 \leqslant j \leqslant p$ such that $f^{-1}(B_j) \setminus B_{j-1} \neq \emptyset$ (resp., $f^{-1}(B_1) \setminus B_p \neq \emptyset$ if $j = 1$). Without loss of generality, we assume that $f^{-1}(B_1) \setminus B_p \neq \emptyset$ and $\infty \in f^{-1}(B_1) \setminus B_p$ with $f(\infty) = 0$, where $0 \in U_1 \subset B_1$ is a p-periodic point. Let V_1 be the connected component of $f^{-1}(B_1)$ containing ∞ and let $m_1 \geqslant 1$ be the local degree of f at ∞.

3.2. **Holomorphic coverings I.** For a Jordan curve $\gamma \subset \widehat{\mathbb{C}} \setminus \{\infty\}$, we use $D(\gamma)$ to denote the connected component of $\widehat{\mathbb{C}} \setminus \gamma$ which does not contain ∞. The following lemma is slightly similar to [WY20, Lemma 3.1].

Lemma 3.2 (holomorphic covering between disks I, see Figure 3)**.** *Let $\ell \geqslant 1$ be an integer satisfying $\ell > (d_1 + \frac{d_0}{m_1})/(d_0^{1/(2d-2)} - 1)$. Then there exist equipotential curves γ_1, α, β, $\gamma_{p+1} \subset U_1$ with $L_1(\gamma_1) > L_1(\alpha) > L_1(\beta) > L_1(\gamma_{p+1})$, equipotential curve*[1] *$\gamma_2 \subset U_2$ and a holomorphic branched covering map $F : D(\alpha) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_2)$ satisfying the following conditions:*

(a) *$F(0) = \infty$ and $F : D(\alpha) \setminus \{0\} \to \mathbb{C} \setminus \overline{D}(\gamma_2)$ is a degree ℓ covering map;*
(b) *$F(\alpha) = \gamma_2$ and $F(\beta) = \eta$, where η is a real-analytic Jordan curve in V_1 separating ∞ from ∂V_1 such that $f(\widehat{\mathbb{C}} \setminus D(\eta)) \subset D(\gamma_{p+1})$; and*
(c) *The closed annulus $\overline{A}(\beta, \gamma_1)$ is disjoint with the critical grand orbits of f.*

Proof. For small $r \in (0, r_1)$, let γ_1, γ_{p+1} be the equipotential curves in U_1 such that $L_1(\gamma_1) = r$ and $L_1(\gamma_{p+1}) = r^{d_0}$. Note that f has exactly $2d-2$ critical points counted by multiplicity, where $d = \deg(f) \geqslant 2$. Besides the super-attracting periodic points in $O = \{B_1, \cdots, B_p\}$, the attracting basin of O contains at most other $2d-3$ critical orbits. According to the local dynamics of $f^{\circ p}$ near the super-attracting point 0, the equipotential curves passing through critical orbits divide the annulus $A(\gamma_1, \gamma_{p+1})$ into at most $2d-2$ subannuli and each of them is disjoint with the critical grand orbit of f. There exists an equipotential curve $\beta \subset U_1$ and the small $r \in (0, r_1)$ can be chosen such that the closed annulus $\overline{A}(\beta, \gamma_1)$ is disjoint with the critical grand orbits of f and moreover,

$$L_1(\beta) = r^{d_0^C} \in (r^{d_0}, r), \qquad \text{where } C = 1/(2d-2).$$

Let α be an equipotential curve in U_1 such that $L_1(\alpha) = s \in (r^{d_0^C}, r)$. Then we have $L_1(\gamma_1) > L_1(\alpha) > L_1(\beta) > L_1(\gamma_{p+1})$. For $2 \leqslant j \leqslant p$, we denote $\gamma_j := f^{\circ(j-1)}(\gamma_1)$. Then γ_j is an equipotential curve in U_j for all $1 \leqslant j \leqslant p$. See Figure 3.

Note that $\phi_1 : D(\alpha) \to \mathbb{D}(0, s)$ is the restriction of the Böttcher map. For $\ell \geqslant 1$, we define $Q_\ell(z) = z^\ell / s^\ell : \mathbb{D}(0, s) \to \mathbb{D}$. Let $\psi_1 : \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_2) \to \mathbb{D}$ be a conformal map such that $\psi_1(\infty) = 0$. Define

$$F := \psi_1^{-1} \circ Q_\ell \circ \phi_1.$$

Then $F : D(\alpha) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_2)$ is a holomorphic branched mapping with degree ℓ, $F(0) = \infty$ and 0 is the unique possible critical point. Since

$$Q_\ell \circ \phi_1(\beta) = \{w \in \mathbb{C} : |w| = r^{d_0^C \ell} / s^\ell\} \subset \mathbb{D}, \tag{3.1}$$

[1] If $p = 1$, then $U_2 = U_1$ and γ_2 is regarded as an equipotential in U_1.

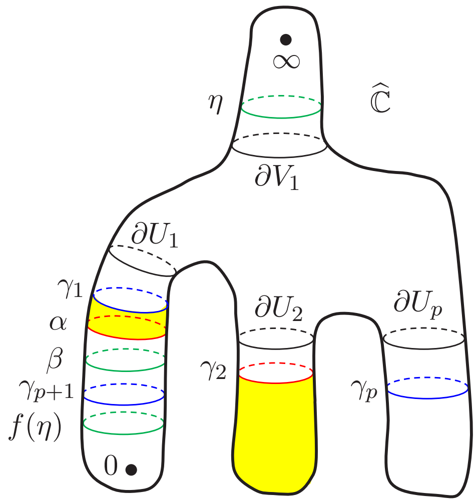

Figure 3: In the dynamical plane of f, some curves in the definition of $F : D(\alpha) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_2)$ are marked.

it follows that $\eta := F(\beta) = \psi_1^{-1}(\{w \in \mathbb{C} : |w| = r^{d_0^C \ell}/s^\ell\})$ is a real-analytic Jordan curve separating ∞ from γ_2.

We need to find a sufficient condition to guarantee that η is contained in V_1 and $f(\widehat{\mathbb{C}} \setminus D(\eta)) \subset D(\gamma_{p+1})$. Since the degree of the restriction of f on U_1 is d_1, the map f can be written near the origin as

$$f(z) = f(0) + b_1 z^{d_1} + O(z^{d_1+1}),$$

where $b_1 \neq 0$ is a constant depending only on f. If $r > 0$ is sufficiently small, then there exists a constant $C_1 > 0$ independent of r such that $D(\gamma_2) = f(D(\gamma_1))$ is a Jordan disk centered at $f(0)$ with Euclidean radius about $C_1 r^{d_1}$. More specifically, r can be chosen small enough such that

$$\mathbb{D}(f(0), C_1 r^{d_1}/2) \subset D(\gamma_2) \subset \mathbb{D}(f(0), 2C_1 r^{d_1}).$$

Take a large $R > 1$ such that the round circle $\mathbb{T}_R \subset V_1$ separates ∞ from ∂V_1. There exists a constant $C_2 > 0$ depending on C_1 but independent of the large $R > 1$ and small $r > 0$ such that the conformal modulus of $\mathrm{mod}(\mathbb{D} \setminus \overline{D}(\psi_1(\mathbb{T}_R))) = \mathrm{mod}(A(\gamma_2, \mathbb{T}_R))$ satisfies

$$\frac{1}{2\pi} \log \frac{R}{r^{d_1}} - C_2 \leqslant \mathrm{mod}(\mathbb{D} \setminus \overline{D}(\psi_1(\mathbb{T}_R))) \leqslant \frac{1}{2\pi} \log \frac{R}{r^{d_1}} + C_2, \tag{3.2}$$

where $A(\gamma_2, \mathbb{T}_R)$ is the annulus in $\widehat{\mathbb{C}}$ bounded by γ_2 and $\mathbb{T}_R$. By the Koebe distortion theorem (see [Pom75, p. 21] or [Dur83, p. 33]), there exists a constant $C_3 \geqslant 1$ independent of large $R > 1$ and small $r > 0$ such that

$$C_3^{-1} |w_2| \leqslant |w_1| \leqslant C_3 |w_2|, \quad \text{for all } w_1, w_2 \in \psi_1(\mathbb{T}_R).$$

Since $\psi_1(\mathbb{T}_R)$ is a Jordan curve in $\mathbb{D}$ separating $\partial\mathbb{D}$ from 0, it follows from (3.2) that there exists a constant $C_4 > 0$ independent of large $R > 1$ and small $r > 0$ such that

$$\log |w| \geqslant \log \frac{r^{d_1}}{R} - C_4, \quad \text{for all } w \in \psi_1(\mathbb{T}_R).$$

In order to guarantee that $\eta \subset V_1$, by (3.1), it is sufficient to obtain the inequality

$$\log \frac{r^{d_0^C \ell}}{s^\ell} \leqslant \log \frac{r^{d_1}}{R} - C_4. \tag{3.3}$$

Since the local degree of f at ∞ is m_1 and $f(\infty) = 0$, it implies that f can be written near ∞ as

$$f(z) = b_0/z^{m_1} + O(1/z^{m_1+1}),$$

where $b_0 \neq 0$ is a constant depending only on f. In order to guarantee $f(\widehat{\mathbb{C}} \setminus D(\eta)) \subset D(\gamma_{p+1})$, it is sufficient to obtain the inequality

$$\log \frac{1}{R^{m_1}} < \log r^{d_0} - C_5, \tag{3.4}$$

where $C_5 > 0$ is a constant depending on f but independent of large R and small r. Note that d_0, d_1, m_1, ℓ are positive integers and $s \in (r^{d_0^C}, r)$. Since $r > 0$ can be arbitrarily small (and hence $R > 1$ should be sufficiently large) and s can be arbitrarily close to r, by (3.3) and (3.4), it is sufficient to guarantee that

$$\frac{r^{d_0^C \ell}}{r^\ell} < \frac{r^{d_1}}{R} \quad \text{and} \quad \frac{1}{R} < r^{\frac{d_0}{m_1}}.$$

This is equivalent to $(d_0^C - 1)\ell - d_1 > \frac{d_0}{m_1}$, i.e., $\ell > (d_1 + \frac{d_0}{m_1})/(d_0^C - 1)$, as desired. □

Remark. If one can use a surgery to reduce the number of critical orbits in the attracting basin of O, then the number $C \in (0, 1]$ can be magnified and the bound of ℓ can be improved. In particular, if B_1 is simply connected, then one can choose $C = 1$ after performing a surgery on f. See [WY20, Lemma 3.1].

Based on Lemma 3.2, we define:

$$H(z) := \begin{cases} f(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_1), \\ F(z) & \text{if } z \in D(\alpha), \\ \zeta(z) & \text{if } z \in \overline{A}(\alpha, \gamma_1), \end{cases}$$

where $\zeta : \overline{A}(\alpha, \gamma_1) \to \overline{D}(\gamma_2)$ is a continuous map satisfying

- $\zeta : A(\alpha, \gamma_1) \to D(\gamma_2)$ is a quasi-regular and branched covering map of degree $d_1 + \ell$; and
- $\zeta|_{\gamma_1} = f$ and $\zeta|_\alpha = F$.

Hence $H : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map of degree $d + \ell$. For the existence of ζ, i.e., the annulus-to-disk quasi-regular interpolation, see [PT99, Lemma 2.1] and [BF14, Lemma 7.47].

Corollary 3.3. *There exists a quasiconformal mapping $\varphi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing* 0, 1 *and* ∞ *such that $h := \varphi_1 \circ H \circ \varphi_1^{-1}$ is a rational map of degree $d + \ell$ satisfying*

- *h has a 2-cycle of super-attracting Fatou components containing $\varphi_1(D(\beta)) \cup \varphi_1(\widehat{\mathbb{C}} \setminus \overline{D}(\eta))$ and the 2-cycle $\{0, \infty\}$; and*
- *The closed annulus $\varphi_1(\overline{A}(\beta, \gamma_1))$ is disjoint with the forward orbits of the critical values of h.*

Proof. We define an open set $E_1 := D(\beta) \cup (\widehat{\mathbb{C}} \setminus \overline{D}(\eta))$. By Lemma 3.2, $H(E_1) \subset E_1$ and H is holomorphic in $E_1 \cup (\widehat{\mathbb{C}} \setminus H^{-p}(\overline{E}_1))$. By Lemma 2.1, there exists a quasiconformal mapping $\varphi_1 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0, 1 and ∞ such that $h = \varphi_1 \circ H \circ \varphi_1^{-1}$ is a rational map of degree $d + \ell$ having a 2-cycle of super-attracting Fatou components containing $\varphi_1(E_1)$ and the 2-cycle $\{0, \infty\}$. By Lemma 3.2(c) and the quasi-regular interpolation ζ in the definition of H, the annulus $\varphi_1(A(\beta,\gamma_1))$ is disjoint with the forward orbits of the critical values of h. $\square$

3.3. **Semi-buried property.** Let J' be a Julia component of h and W a connected component of $\widehat{\mathbb{C}}\setminus J'$. If J' is disjoint with the boundary of any Fatou component of h in W, then we call that J' is *semi-buried* from W under h. Let φ_1 be the quasiconformal mapping introduced in Corollary 3.3, and $\mathcal{A}$ the super-attracting basin of f containing $O=\{B_1,\cdots,B_p\}$.

Proposition 3.4. *Every component of $\varphi_1(J(f))$ is a Julia component of h which is semi-buried from every component of $\varphi_1(\mathcal{A})$ under h.*

Proof. We first prove that $\varphi_1(J(f))\subset J(h)$. Indeed, since $\partial\mathcal{A}=J(f)$, for any $z_0\in\varphi_1(J(f))$ and any open neighborhood U of z_0, there exists $z_1\in U$ and a minimal $\ell\geqslant 0$ such that $h^{\circ\ell}(z_1)\in\varphi_1(D(\beta))$. Since z_1 will be attracted by the 2-cycle $\{0,\infty\}$ under h eventually while the orbit of z_0 is contained in $\varphi_1(J(f))$, this implies that $\{h^{\circ k}\}_{k\in\mathbb{N}}$ is not equi-continuous in U. Hence $z_0\in J(h)$ and $\varphi_1(J(f))\subset J(h)$.

We now prove that for each component W of $\mathcal{A}$ and each component J_* of ∂W, there exist sequences of Julia components $\{J_k'\}_{k\in\mathbb{N}}$ and Fatou components $\{U_k'\}_{k\in\mathbb{N}}$ of h in $\varphi_1(W)$ which converge to $\varphi_1(J_*)$ in the Hausdorff metric. Note that

$$B_1=\bigcup_{k\geqslant 0} f^{-kp}(D(\alpha))\cap B_1=\bigcup_{k\geqslant 0} f^{-kp}(D(\beta))\cap B_1,$$

where B_1 is the Fatou component of f containing 0. By Lemma 3.2(c), for each $k\geqslant 0$, every component W_α^k of $f^{-kp}(D(\alpha))\cap B_1$ and every component W_β^k of $f^{-kp}(D(\beta))\cap B_1$, are bounded by finitely many Jordan equipotential curves with the following potentials respectively:

$$(L_1(\alpha))^{1/d_0^k} \quad\text{and}\quad (L_1(\beta))^{1/d_0^k}.$$

Let J_* be a component of ∂B_1. For any $k\geqslant 0$, there exist a component W_α^k of $f^{-kp}(D(\alpha))\cap B_1$ and a component W_β^k of $f^{-kp}(D(\beta))\cap B_1$ such that

- Every component of $W_\alpha^k\setminus\overline{W_\beta^k}$ is an annulus whose closure is disjoint with the critical orbit of f;
- J_* is contained in a component X_α^k of $\widehat{\mathbb{C}}\setminus\overline{W_\alpha^k}$ and a component X_β^k of $\widehat{\mathbb{C}}\setminus\overline{W_\beta^k}$, where $X_\beta^k\setminus\overline{X_\alpha^k}$ is a component of $W_\alpha^k\setminus\overline{W_\beta^k}$; and
- $f^{\circ kp}: X_\beta^k\setminus\overline{X_\alpha^k}\to A(\alpha,\beta)$ and $h^{\circ kp}:\varphi_1(X_\beta^k\setminus\overline{X_\alpha^k})\to\varphi_1(A(\alpha,\beta))$ are holomorphic covering maps between annuli.

By Lemma 3.2 and Corollary 3.3, $\varphi_1(\alpha)$ and $\varphi_1(\beta)$ are contained in the Fatou set of h. Since $\varphi_1(J(f))\subset J(h)$, it follows that $\varphi_1(A(\alpha,\beta))$ contains a Julia component J_0' of h separating $\varphi_1(\alpha)$ from $\varphi_1(\beta)$. Therefore, $\varphi_1(X_\beta^k\setminus\overline{X_\alpha^k})$ contains a Julia component J_k' of h separating $\varphi_1(\partial X_\alpha^k)$ from $\varphi_1(\partial X_\beta^k)$ and

$$(L_1(\beta))^{1/d_0^k}<L_1(z)<(L_1(\alpha))^{1/d_0^k} \quad \text{for any } z\in\varphi_1^{-1}(J_k'). \tag{3.5}$$

This implies that the Hausdorff distance between J_* and $\varphi_1^{-1}(J_k')$ tends to zero as $k\to\infty$. Equivalently, the sequence of Julia components $\{J_k'\}_{k\in\mathbb{N}}$ of h in $\varphi_1(B_1)$ converges to $\varphi_1(J_*)$ in the Hausdorff metric.

Note that $\varphi_1(\partial X_\alpha^k)$ is contained in the Fatou set of h for each $k\geqslant 0$. Let U_k' be the Fatou component of h containing $\varphi_1(\partial X_\alpha^k)$. Then by a completely similar argument as above, $\{U_k'\}_{k\in\mathbb{N}}$ converges to $\varphi_1(J_*)$ in the Hausdorff metric. By considering the preimages of $\varphi_1(B_1)$ under h, it follows that for each component W of $\mathcal{A}$ and each component J_* of ∂W, $\varphi_1(J_*)$ is semi-buried from $\varphi_1(W)$ under h.

Let J_0 be a Julia component of f. In the following we prove that $\varphi_1(J_0)$ is a Julia component of h. We have proved that $\varphi_1(J_0) \subset J(h)$. On the other hand, if $\varphi_1(J_0)$ is a proper subset of a Julia component J' of h, then there exists $z \in J' \setminus \varphi_1(J_0)$ such that $\varphi_1^{-1}(z) \in \mathcal{A}$. However, by (3.5) this is impossible. Hence $\varphi_1(J_0)$ is a Julia component of h which is semi-buried under h from every component of $\varphi_1(\mathcal{A})$. □

3.4. **From semi-buried to buried.** We have assumed that f has infinitely many Fatou components. If each attracting basin of f has infinitely many components, one can perform the same surgery as above in each attracting basin and Theorem A can be proved directly in this special case (this will be discussed in a moment). However, we also needs to consider the case that some attracting basin consists of only finite many components. By Lemma 3.1(b), we assume that f has infinitely many Fatou components and also a completely invariant Fatou component which is super-attracting.

Let h be the rational map obtained in Corollary 3.3. In this section, we perform the surgery on h to obtain a rational map g and one can see that the semi-buried Julia component of h can be transferred to a fully buried Julia component of g.

By changing coordinates if necessary, we assume that 1 is the super-attracting fixed point in the completely invariant Fatou component of f. By Corollary 3.3, h has a completely invariant Fatou component $\widetilde{B}$ which is infinitely connected and contains the super-attracting fixed point 1. According to Böttcher's theorem, there exist a simply connected domains $\widetilde{U} \subset \widetilde{B}$ and a conformal map $\widetilde{\phi} : \widetilde{U} \to \mathbb{D}_{\widetilde{r}}$ with $\widetilde{r} \in (0,1)$, such that

$$\widetilde{\phi} \circ h(z) = (\widetilde{\phi}(z))^{\widetilde{d}}, \quad \text{for } z \in \widetilde{U},$$

where $\widetilde{d} = \deg(h|_{\widetilde{U}}) \geqslant 2$. Moreover, $\partial \widetilde{U}$ is a subset of $\widetilde{B}$ containing a critical point of h. Similar to §3.1, the potential function $\widetilde{L}$ in $\widetilde{B}$ and the equipotential curves in $\widetilde{B}$ can be defined accordingly. Let $\widetilde{V}$ be the Fatou component of h containing ∞.

Lemma 3.5 (holomorphic covering between disks II, see Figure 4)**.** *Let $\widetilde{m}$ be an integer satisfying $\widetilde{m} > \widetilde{d}/(\widetilde{d}^{1/(2d-2)} - 1)$. Then there exist equipotential curves $\widetilde{\gamma}_1$, $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}_2 \subset \widetilde{U}$ surrounding* 1 *with $\widetilde{L}(\widetilde{\gamma}_1) > \widetilde{L}(\widetilde{\alpha}) > \widetilde{L}(\widetilde{\beta}) > L(\widetilde{\gamma}_2)$ and a holomorphic branched covering map $\widetilde{H} : D(\widetilde{\alpha}) \to \widehat{\mathbb{C}} \setminus \overline{D}(\widetilde{\gamma}_2)$ satisfying the following conditions:*

(a) *$\widetilde{H}(1) = \infty$ and $\widetilde{H} : D(\widetilde{\alpha}) \setminus \{1\} \to \mathbb{C} \setminus \overline{D}(\widetilde{\gamma}_2)$ is a degree $\widetilde{m}$ covering map;*
(b) *$\widetilde{H}(\widetilde{\alpha}) = \widetilde{\gamma}_2$ and $\widetilde{H}(\widetilde{\beta}) = \widetilde{\eta}$, where $\widetilde{\eta}$ is a real-analytic Jordan curve in $\widetilde{V}$ separating ∞ from $\partial \widetilde{V}$ such that $\widehat{\mathbb{C}} \setminus \overline{D}(\widetilde{\eta})$ is contained in $\widetilde{V}$; and*
(c) *The closed annulus $\overline{A}(\widetilde{\beta}, \widetilde{\gamma}_1)$ is disjoint with the critical grand orbits of h.*

The proof of Lemma 3.5 is completely similar to Lemma 3.2 by setting $\widetilde{\gamma}_2 = h(\widetilde{\gamma}_1)$ (see also [WY20, Lemma 2.1]). We omit the details.

Based on Lemma 3.5, we define:

$$G(z) := \begin{cases} h(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus \overline{D}(\widetilde{\gamma}_1), \\ \widetilde{H}(z) & \text{if } z \in D(\widetilde{\alpha}), \\ \widetilde{\zeta}(z) & \text{if } z \in \overline{A}(\widetilde{\alpha}, \widetilde{\gamma}_1), \end{cases}$$

where $\widetilde{\zeta} : \overline{A}(\widetilde{\alpha}, \widetilde{\gamma}_1) \to \overline{D}(\widetilde{\gamma}_2)$ is a continuous map satisfying

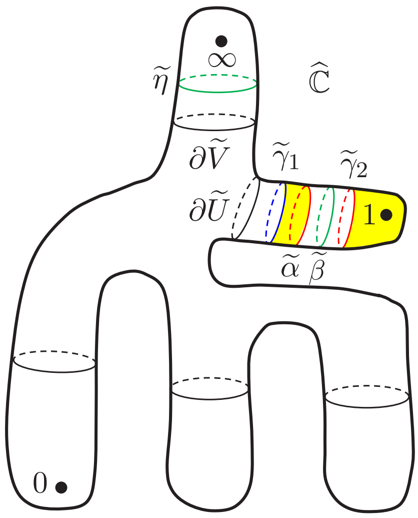

Figure 4: In the dynamical plane of h, some curves in the definition of $\widetilde{H} : D(\widetilde{\alpha}) \to \widehat{\mathbb{C}} \setminus \overline{D}(\widetilde{\gamma}_2)$ are marked.

- $\widetilde{\zeta} : A(\widetilde{\alpha}, \widetilde{\gamma}_1) \to D(\widetilde{\gamma}_2)$ is a quasi-regular and branched covering map of degree $\widetilde{d} + \widetilde{m}$; and
- $\widetilde{\zeta}|_{\widetilde{\gamma}_1} = h$ and $\widetilde{\zeta}|_{\widetilde{\alpha}} = \widetilde{H}$.

Then $G : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map of degree $d + \ell + \widetilde{m}$. As before, the existence of $\widetilde{\zeta}$ is also guaranteed by the annulus-to-disk quasi-regular interpolation.

Corollary 3.6. *There exists a quasiconformal mapping $\varphi_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0, 1 and ∞ such that $g := \varphi_2 \circ G \circ \varphi_2^{-1}$ is a rational map of degree $d + \ell + \widetilde{m}$ satisfying*

- *g has a 2-cycle of super-attracting Fatou components containing the 2-cycle $\{0, \infty\}$; and*
- *The closed annulus $\varphi_2(\overline{A}(\widetilde{\gamma}_1, \widetilde{\beta}))$ is disjoint with the forward orbits of the critical values of g.*

Proof. Define the open set $E_2 := \widetilde{V} \cup h(\widetilde{V})$. By Lemma 3.5, $G(E_2) \subset E_2$ and G is holomorphic in $E_2 \cup (\widehat{\mathbb{C}} \setminus G^{-2}(\overline{E}_2))$. Then the result follows by Lemmas 2.1 and 3.5. □

Based on Lemma 3.5 and Corollary 3.6, the following result can be proved completely similar to Proposition 3.4.

Proposition 3.7. *Every component of $\varphi_2(\partial\widetilde{B})$ is a Julia component of g which is semi-buried from $\varphi_2(\widetilde{B})$ under g.*

Now we can give a proof of Theorem A in Case I, i.e., when f has infinitely many Fatou components.

Proof of Theorem A in Case I. The proof of the necessity, i.e., if f has a parabolic basin or a rotation domain, then one cannot find a rational map g such that g has a family of buried Julia components on which g is quasiconformally conjugate to f on $J(f)$, is based on the analysis of local dynamics near parabolic fixed point and the boundary of rotation domains. For details, see [WY20, p. 7307]. For the sufficiency, suppose that f is a rational map of degree $d \geqslant 2$ having no parabolic basins and rotation domains whose Julia set $J(f)$ is disconnected.

Case I.1. We first consider the case that each attracting basin of f has infinitely many components. By Lemma 2.2, we assume that each attracting basin is super-attracting. Then by Lemma 3.2, Corollary 3.3 and Proposition 3.4, one can perform the same surgery in *each* super-attracting basin of f to obtain a quasiconformal mapping $\widehat{\varphi}:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ and a rational map $\widehat{h}$, such that

- For each Fatou component W of f and each component J_* of ∂W, $\widehat{\varphi}(J_*)$ is semi-buried from $\widehat{\varphi}(W)$ under $\widehat{h}$, and in particular, every component of $\widehat{\varphi}(J(f))$ is a buried Julia component of $\widehat{h}$; and
- $f : J(f) \to J(f)$ is conjugate to $\widehat{h} : \widehat{\varphi}(J(f)) \to \widehat{\varphi}(J(f))$ by a restriction of $\widehat{\varphi}:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$.

Note that every attracting basin contains at least one critical point, and that if an attracting basin has components which are not simply connected then it contains at least 2 critical points. Hence f has at most $2d-3$ attracting basins since f has $2d-2$ critical points counted by multiplicity and $J(f)$ is disconnected. Thus by Lemma 3.2, $\deg(\widehat{h})$ can be chosen such that it is less than a number depending only on d.

Case I.2. Next we consider the case that each attracting basin $\mathcal{A}_1, \cdots, \mathcal{A}_n$ of f has infinitely many components except a completely invariant attracting Fatou component $\widetilde{\mathcal{A}}$. By Lemma 2.2, we assume that each attracting basin is super-attracting. By Lemma 3.2, Corollary 3.3 and Proposition 3.4, one can perform the same surgery in *each* $\mathcal{A}_1, \cdots, \mathcal{A}_n$ to obtain a quasiconformal mapping $\widehat{\varphi}:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ and a rational map $\widehat{h}$, and then by Lemma 3.5, Corollary 3.6 and Proposition 3.7, we perform the surgery in $\widetilde{\varphi}(\widetilde{\mathcal{A}})$ to obtain a quasiconformal mapping $\varphi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ and a rational map g such that

- For each Fatou component W of f and each component J_* of ∂W, $\varphi(J_*)$ is semi-buried from $\varphi(W)$ under g, and in particular, every component of $\varphi(J(f))$ is a buried Julia component of g; and
- $f : J(f) \to J(f)$ is conjugate to $g : \varphi(J(f)) \to \varphi(J(f))$ by a restriction of $\varphi:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$.

Similarly, f has at most $2d-3$ attracting basins. By Lemmas 3.2 and 3.5, $\deg(g)$ can be chosen such that it is less than a number depending only on d. □

Remark. By changing the order of perturbations, the above Case I.2 can be proved in another way, i.e., one first maps the super-attracting periodic points in $\mathcal{A}_i$ to $\widetilde{\mathcal{A}}$ and then maps the super-attracting fixed point in $\widetilde{\mathcal{A}}$ back to $\mathcal{A}_1$. Figure 1 is generated by this idea.

4. Burying Julia sets with only one Fatou component

In this section, we assume that f is a rational map of Case II, i.e., it has exactly one Fatou component B which is completely invariant, and moreover, B is an attracting basin whose boundary is disconnected. By Lemma 2.2, without loss of generality, we assume that B is super-attracting. We will perform a surgery on f to obtain a rational map which is considered in the previous section.

Since $J(f) = \partial B$ is disconnected, up to changing coordinates, we assume that f has a critical point ∞ in B whose forward orbit is infinite:

$$\infty \mapsto 0 \mapsto 1 \mapsto \cdots .$$

Moreover, there exists a small Jordan disk Ω_1 in B containing 1 with smooth boundary such that

- $\overline{\Omega}_1 \setminus \{1\}$ is disjoint with the critical orbits of f; and
- $\overline{\Omega}_0 \cap \overline{\Omega}_1 = \emptyset$, where Ω_0 is the component of $f^{-1}(\Omega_1)$ that contains 0.

Let Ω_∞ be the component of $f^{-1}(\Omega_0)$ that contains ∞. Then $f : \Omega_\infty \setminus \{\infty\} \to \Omega_0 \setminus \{0\}$ and $f : \Omega_0 \setminus \{0\} \to \Omega_1 \setminus \{1\}$ are covering maps. Moreover, Ω_1 can be chosen small enough such that

$$f^{\circ m}(\overline{\Omega}_\infty) \cap f^{\circ n}(\overline{\Omega}_\infty) = \emptyset, \quad \text{for any different } m, n \geqslant 0.$$

Lemma 4.1 (see Figure 5)**.** *There exist smooth Jordan curves γ_1, α, β in $\Omega_0 \setminus \{0\}$ separating 0 from $\partial\Omega_0$ such that γ_1 separates α from $\partial\Omega_0$, α separates β from γ_1 and β separates 0 from α, a Jordan curve γ_2 in $\Omega_1 \setminus \{1\}$ separating 1 from $\partial\Omega_1$, a Jordan curve η in $\Omega_\infty \setminus \{\infty\}$ separating ∞ from $\partial\Omega_\infty$, and a conformal map $F : D(\alpha) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_2)$ such that $F(0) = \infty$ and $F(\beta) = \eta$.*

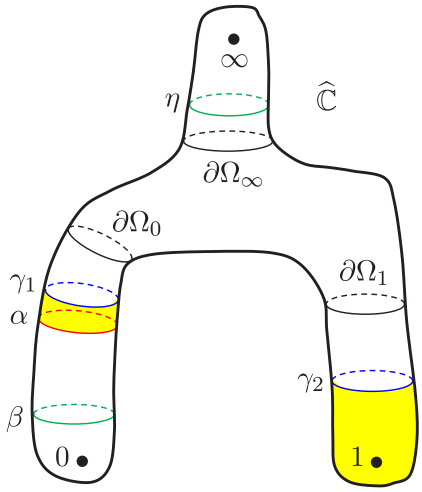

Figure 5: In the dynamical plane of f, some curves in the definition of $F : D(\alpha) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_2)$ are marked.

Proof. Let γ_2 be a smooth Jordan curve in Ω_1 separating 1 from $\partial\Omega_1$. Let γ_1 be the component of $f^{-1}(\gamma_2)$ in Ω_0. Let α and β be two smooth Jordan curves in $D(\gamma_1)$ such that α separates 0 from γ_1 and β separates 0 from α. Let $F : D(\alpha) \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_2)$ be a conformal map such that $F(0) = \infty$. Then $\eta = F(\beta)$ is a smooth Jordan curve separating ∞ from γ_2. In fact, we can choose β sufficiently close to 0 such that η is a Jordan curve in Ω_∞. $\square$

Based on Lemma 4.1, similar to §3.2 we define:

$$H(z) := \begin{cases} f(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_1), \\ F(z) & \text{if } z \in D(\alpha), \\ \zeta(z) & \text{if } z \in \overline{A}(\alpha, \gamma_1), \end{cases}$$

where $\zeta : \overline{A}(\alpha, \gamma_1) \to \overline{D}(\gamma_2)$ is a continuous map satisfying

- $\zeta : A(\alpha, \gamma_1) \to D(\gamma_2)$ is a quasi-regular and branched covering map of degree $\deg(f|_0) + 1$, where $\deg(f|_0)$ is the local degree of f at 0; and
- $\zeta|_{\gamma_1} = f$ and $\zeta|_\alpha = F$.

Then $H : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map of degree $d + 1$. As before, the existence of ζ is also guaranteed by the annulus-to-disk quasi-regular interpolation.

Corollary 4.2. *There exists a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0, 1 and ∞ such that $h := \varphi \circ H \circ \varphi^{-1}$ is a rational map of degree $d + 1$ satisfying*

- *h has a 2-cycle of super-attracting points $\{0, \infty\}$ whose attracting basin consists of infinitely many Fatou components; and*
- *Each component of $\varphi(J(f))$ is a Julia component of h.*

Proof. Let z_0 be the super-attracting fixed point in B. There exists a small Jordan disk U_0 containing z_0 such that $f(\overline{U}_0) \subset U_0$ and $\overline{U}_0$ is disjoint with $\overline{\Omega}_1$. We define an open set

$$E := \bigcup_{n \geqslant 0} f^{\circ n}(D(\gamma_2)) \cup U_0.$$

By Lemma 4.1, $H(E) \subset E$ and H is holomorphic in $E \cup (\widehat{\mathbb{C}} \setminus H^{-1}(\overline{E}))$. By Lemma 2.1, there exists a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0, 1 and ∞ such that $h := \varphi \circ H \circ \varphi^{-1}$ is rational map of degree $d+1$ having a 2-cycle of super-attracting points $\{0, \infty\}$.

Let B_0 and B_∞ be the Fatou components of h containing 0 and ∞ respectively. Note that $\varphi(A(\alpha, \gamma_1))$ is contained in the super-attracting basin of 1. It follows that $B_0 \subset \varphi(D(\alpha))$ and $B_\infty \subset \varphi(\Omega_\infty)$. Denote $X := \bigcup_{n \geqslant 0} f^{-n}(\Omega_0)$. Since $\overline{\Omega}_1 \setminus \{1\}$ is disjoint with the critical orbits of f, we conclude that X consists of infinitely many disjoint Jordan disks and each component of $\varphi(X)$ contains a Fatou component which is the preimage of B_0. This implies that the attracting basin of $\{0, \infty\}$ consists of infinitely many Fatou components.

For any Julia component J_0 of f and any neighborhood U of $\varphi(J_0)$, there exists $z \in U$ whose forward orbit is attracted by the super-attracting fixed point $\varphi(z_0)$ of h. This implies that $\varphi(J_0)$ is contained in the Julia set of h. Note that $J(h) \subset \varphi(J(f)) \cup \varphi(X)$. Since $\Omega_1 \subset B$, we have $X \subset B$ and hence $\varphi(J(f)) \cap \varphi(X) = \emptyset$. Let J' be the Julia component of h containing $\varphi(J_0)$. Then $J' \subset \varphi(J(f))$ or $J' \subset \varphi(X)$. But the latter case is impossible. Hence $J' \subset \varphi(J(f))$ and we have $J' = \varphi(J_0)$, i.e., each component of $\varphi(J(f))$ is a Julia component of h. □

Proof of Theorem A in Case II. We only consider the sufficiency. Suppose that f is a rational map of degree $d \geqslant 2$ having exactly one Fatou component B which is completely invariant, and moreover, B is an attracting basin whose boundary is disconnected. By Corollary 4.2, we obtain a rational map h of degree $d+1$ and a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

- Every component of $\varphi(J(f))$ is a Julia component of h;
- $f : J(f) \to J(f)$ is conjugate to $h : \varphi(J(f)) \to \varphi(J(f))$ by the restriction of φ; and
- h has infinitely many Fatou components.

Therefore, one can perform the same surgery in the previous section to h and obtain a rational map g and a quasiconformal mapping $\widetilde{\varphi} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

- $f : J(f) \to J(f)$ is conjugate to $g : \widetilde{\varphi}(J(f)) \to \widetilde{\varphi}(J(f))$ by the restriction of $\widetilde{\varphi}$; and

- $\widetilde{\varphi}(J(f))$ are buried Julia components of g.

Since h has degree $d+1$, it follows that the degree of g can be chosen such that it is less than a number depending only on d. $\square$

Remark. If f has exactly one Fatou component which is completely invariant and $J(f)$ has a non-singleton component, then one can also use the same method as in [WY20, §5] to prove Theorem A. However, if $J(f)$ is a Cantor set, then the method there cannot work.

5. Proof of Theorem B

In this section, we study the conformal dimension of Cantor circle Julia sets and prove Theorem B.

5.1. **Geometries of Cantor circle Julia sets.** The dynamics of rational maps with Cantor circle Julia sets has been studied extensively. See [McM88], [DLU05], [HP12], [QYY15], [QYY16], [QY21], [LQY22] and the references therein. In this subsection we recall all possible combinations of the rational maps whose Julia sets are Cantor circles. For details, see [QYY15] and [QY21].

Let f be a hyperbolic rational map of degree $d \geqslant 2$ whose Julia set is a Cantor set of circles. Note that the complement of any Cantor circle Julia set (i.e., the Fatou set) consists of two simply connected components and countably many nested doubly connected components. Without of loss generality, we assume that the two simply connected Fatou components of f, denoted by D_0 and D_∞, contain 0 and ∞ respectively.

Note that all the doubly connected Fatou components of f are iterated to D_0 or D_∞ eventually. For $n \geqslant 2$, let $D_1, \cdots, D_{n-1}$ be the annular components such that $f^{-1}(D_0 \cup D_\infty) = D_0 \cup D_\infty \cup \bigcup_{i=1}^{n-1} D_i$, where $\{D_i\}_{1\leqslant i \leqslant n-1}$ are labeled such that D_i separates $D_{i'}$ from $D_{i''}$ for all $0 \leqslant i' < i < i'' \leqslant n-1$. Let A_i be the annulus between D_{i-1} and D_i, where $1 \leqslant i \leqslant n-1$ and A_n the annulus between D_{n-1} and D_∞. Then $f^{-1}(A) = \bigcup_{i=1}^{n} A_i$, where $A = \widehat{\mathbb{C}} \setminus (\overline{D}_0 \cup \overline{D}_\infty)$. See Figure 6.

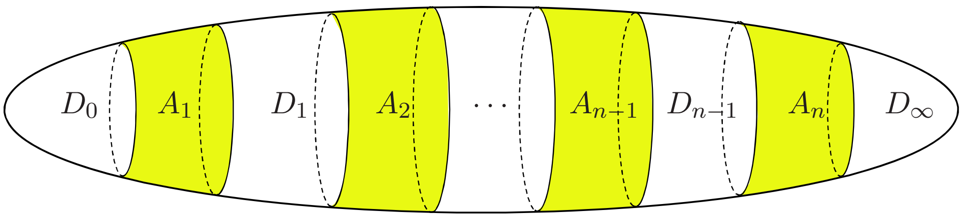

Figure 6: The structure of the Cantor circle Julia sets on the Riemann sphere.

Note that $f|_{A_i} : A_i \to A$ is a covering map and we suppose that $\deg(f|_{A_i} : A_i \to A) = d_i$, where $1 \leqslant i \leqslant n$. Then $\deg(f|_{D_i} : D_i \to D_0 \text{ or } D_\infty) = d_i + d_{i+1}$, where $1 \leqslant i \leqslant n-1$. Moreover, $\deg(f|_{D_0}) = d_1$ and $\deg(f|_{D_\infty}) = d_n$. Up to a conformal conjugacy, the map f belongs to one of the following three types.

Type I: $f(D_0) = D_\infty$, $f(D_\infty) = D_\infty$ and $n \geqslant 2$ is even. Moreover,

$$f^{-1}(D_0) = \bigcup_{i=1}^{n/2} D_{2i-1} \text{ and } f^{-1}(D_\infty) = D_0 \cup D_\infty \cup \bigcup_{i=1}^{(n-2)/2} D_{2i}.$$

Type II: $f(D_0) = D_0$, $f(D_\infty) = D_\infty$ and $n \geqslant 3$ is odd. Moreover,

$$f^{-1}(D_0) = D_0 \cup \bigcup_{i=1}^{(n-1)/2} D_{2i} \text{ and } f^{-1}(D_\infty) = D_\infty \cup \bigcup_{i=1}^{(n-1)/2} D_{2i-1}.$$

Type III: $f(D_0) = D_\infty$, $f(D_\infty) = D_0$ and $n \geqslant 3$ is odd. Moreover,

$$f^{-1}(D_0) = D_\infty \cup \bigcup_{i=1}^{(n-1)/2} D_{2i-1} \text{ and } f^{-1}(D_\infty) = D_0 \cup \bigcup_{i=1}^{(n-1)/2} D_{2i}.$$

Note that $f^{-1}(A) = \bigcup_{i=1}^{n} A_i$ and each A_i is essentially contained in A. It follows from Grötzsch's module inequality that

$$\sum_{i=1}^{n} d_i = d \quad \text{and} \quad \sum_{i=1}^{n} \frac{1}{d_i} < 1. \tag{5.1}$$

It is easy to check that for any $n \geqslant 2$, the equation (5.1) has a solution $(d_1, \cdots, d_n)$ if and only if $d \geqslant 5$.

Definition (Combinations of Cantor circles)**.** Let $\mathscr{C}$ be the collection of all the *combinations* with the form $\mathcal{C} = (\kappa; d_1, \cdots, d_n)$, where $\kappa \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ is the *type*, the array of positive integers $(d_1, \cdots, d_n)$ satisfies (5.1), and

$$n \geqslant 2 \text{ is } \begin{cases} \text{even} & \text{if } \kappa = \mathrm{I}, \\ \text{odd} & \text{if } \kappa = \mathrm{II} \text{ or } \mathrm{III}. \end{cases}$$

For a hyperbolic rational map f with Cantor circle Julia set, there exists at least one combinatorial data $\mathcal{C}(f) = (\kappa; d_1, \cdots, d_n) \in \mathscr{C}$ corresponding to f. Moreover, each combination $\mathcal{C} = (\kappa; d_1, \cdots, d_n) \in \mathscr{C}$ can be realized by a hyperbolic rational map [QYY15] (see also [HP12]). Let $\mathcal{H}_C$ be the set of all hyperbolic rational maps whose Julia sets are Cantor circles. The following result was proved in [QY21, §5.1] (see also [HP12, Proposition 1.1]).

Lemma 5.1. *If $f \in \mathcal{H}_C$ has the combination $\mathcal{C} = (\kappa; d_1, \cdots, d_n) \in \mathscr{C}$, then the conformal dimension of $J(f)$ is*

$$\dim_C(J(f)) = 1 + \alpha_{d_1,\cdots,d_n},$$

where $\alpha = \alpha_{d_1,\cdots,d_n} \in (0,1)$ is the unique positive root of

$$\sum_{i=1}^{n} \Big(\frac{1}{d_i}\Big)^{\alpha} = 1.$$

5.2. **Buried Cantor circle Julia sets.** Let f be a hyperbolic rational map of degree $d \geqslant 5$ whose Julia set is a Cantor set of circles. It is easy to see that a Julia component J_0 of f is buried if and only if J_0 is not a preimage of ∂D_0 and ∂D_∞. The following result implies that one may extract subsystem of Cantor circles from the rational maps with "big" Cantor circle Julia sets.

Lemma 5.2. *If $f \in \mathcal{H}_C$ has the combination $\mathcal{C} = (\kappa; d_1, \cdots, d_n) \in \mathscr{C}$, where $n \geqslant 4$, then there exists $g \in \mathcal{H}_C$ with the combination $\mathcal{C}' = (\kappa'; d_1', \cdots, d_m') \in \mathscr{C}$, where $2 \leqslant m \leqslant n-1$, and a quasiconformal mapping $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\phi(J(g))$ are buried Julia components of f and $g : J(g) \to J(g)$ is conjugate to $f : \phi(J(g)) \to \phi(J(g))$ by the restriction of ϕ. In particular, g can be chosen such that*

- *If $\kappa = \mathrm{I}$, then $\mathcal{C}' = (\mathrm{I}; d_{n-1}, \cdots, d_2)$;*

- *If* $\kappa = \mathrm{II}$, *then* $\mathcal{C}' = (\mathrm{III}; d_2, \cdots, d_{n-1})$; *and*
- *If* $\kappa = \mathrm{III}$, *then* $\mathcal{C}' = (\mathrm{II}; d_2, \cdots, d_{n-1})$.

Proof. To clarify the construction, we only prove the case that $\kappa = \mathrm{II}$ since the rest cases are completely similar. Let γ_0 be a smooth Jordan curve in D_0 separating 0 from ∂D_0 and let γ_∞ be a smooth Jordan curve in D_∞ separating ∞ from ∂D_∞. There exist two holomorphic branched covering maps

$$F_0 : \overline{D}_0 \cup A_1 \to \widehat{\mathbb{C}} \setminus \overline{D}(\gamma_\infty) \quad \text{and} \quad F_\infty : \overline{D}_\infty \cup A_n \to D(\gamma_0)$$

satisfying the following conditions:

- $F_0(0) = \infty$ and $F_0 : (\overline{D}_0 \cup A_1) \setminus \{0\} \to \mathbb{C} \setminus \overline{D}(\gamma_\infty)$ is a degree d_2 covering map;
- $F_\infty(\infty) = 0$ and $F_\infty : (\overline{D}_\infty \cup A_n) \setminus \{\infty\} \to D(\gamma_0) \setminus \{0\}$ is a degree d_{n-1} covering map.

We define:

$$G(z) := \begin{cases} f(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus (\overline{D}_0 \cup A_1 \cup \overline{D}_1 \cup \overline{D}_{n-1} \cup A_n \cup \overline{D}_\infty), \\ F_0(z) & \text{if } z \in \overline{D}_0 \cup A_1, \\ F_\infty(z) & \text{if } z \in \overline{D}_\infty \cup A_n, \\ \zeta(z) & \text{if } z \in \overline{D}_1 \cup \overline{D}_{n-1}, \end{cases}$$

where $\zeta : \overline{D}_1 \to \overline{A}(\partial D_\infty, \gamma_\infty)$ and $\zeta : \overline{D}_{n-1} \to \overline{A}(\gamma_0, \partial D_0)$ are annulus-to-annulus quasi-regular interpolations of degrees d_2 and d_{n-1} respectively, such that $G : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular map of degree $d_2 + \cdots + d_{n-1}$.

We define an open set $E := \overline{D}_0 \cup A_1 \cup \overline{D}_\infty \cup A_n$. Then $G(E) \subset E$ and G is holomorphic in $E \cup (\widehat{\mathbb{C}} \setminus G^{-1}(\overline{E}))$. By Lemma 2.1, there exists a quasiconformal mapping $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing 0, 1 and ∞ such that $g := \varphi \circ G \circ \varphi^{-1}$ is a rational map of degree $d_2 + \cdots + d_{n-1}$ having a 2-cycle of super-attracting points $\{0, \infty\}$.

We claim that the Julia set of g is a Cantor set of circles. In fact, according to the surgery construction, the Julia set of g is contained in the union of finite many annuli $W := \varphi(\bigcup_{i=2}^{n-1} A_i)$. Let $(i_0, i_1, \cdots, i_k, \cdots)$ be any infinite sequence satisfying $i_k \in \{2, 3, \cdots, n-1\}$ for all $k \in \mathbb{N}$. We denote

$$J_{i_0 i_1 \cdots i_k \cdots} := \{z \in W : g^{\circ k}(z) \in \varphi(A_{i_k}),\ \forall\, k \in \mathbb{N}\}.$$

Note that the identity $\mathrm{id} : \varphi(A_i) \hookrightarrow W$ is not homotopic to a constant map for any $2 \leqslant i \leqslant n-1$. By [PT00, Lemma 2.4 (Case 2)] and [PT00, Proposition (Case 2)], every $J_{i_0 i_1 \cdots i_k \cdots}$ is a Jordan curve separating 0 from ∞. Intuitively, each $J_{i_0 i_1 \cdots i_k \cdots}$ is the intersection of a sequence of nested annuli by taking preimages of g in W. In particular, according to [McM88, Proposition 7.2],

$$J(g) = \{J_{i_0 i_1 \cdots i_k \cdots} : i_k \in \{2, 3, \cdots, n-1\},\ \forall\, k \in \mathbb{N}\}$$

is a Cantor set of circles and g has the combination $\mathcal{C}' = (\mathrm{III}; d_2, \cdots, d_{n-1})$. Moreover, $\phi(J(g))$ are Julia components of f and $g : J(g) \to J(g)$ is conjugate to $f : \phi(J(g)) \to \phi(J(g))$ by the restriction of $\phi = \varphi^{-1}$. Finally, $\phi(J(g))$ are buried Julia components of f since the forward orbit of $\phi(J(g))$ under f is disjoint with $\partial D_0 \cup D_\infty$. $\square$

Remark. Lemma 5.2 can be seen as an inverse surgery procedure of burying Julia sets in §3 and §4. Moreover, by using a similar surgery as above, the rational map g can be chosen such that if $\mathcal{C} = (\mathrm{III}; d_1, \cdots, d_n) \in \mathscr{C}$, where $n \geqslant 3$ is odd, then $g \in \mathcal{H}_C$ has the combination $\mathcal{C}' = (\mathrm{I}; d_1, \cdots, d_{n-1}) \in \mathscr{C}$; If $\mathcal{C} = (\mathrm{I}; d_1, \cdots, d_n) \in \mathscr{C}$, where $n \geqslant 4$ is even, then $g \in \mathcal{H}_C$ has the combination $\mathcal{C}' = (\mathrm{III}; d_1, \cdots, d_{n-1}) \in \mathscr{C}$.

Proof of Theorem B. Let f_0 be a hyperbolic rational map whose Julia set is a Cantor set of circles with the combination $\mathcal{C}_0 = (\mathrm{I}; 3, 3) \in \mathscr{C}$. Such f_0 can be chosen as $f_0(z) = z^3 + \lambda/z^3$, where $\lambda \neq 0$ is small enough (see [McM88] and [DLU05]). For $n \geqslant 1$, we define a sequence of combinations as following

$$\mathcal{C}_n := (\mathrm{I}; d_n, \cdots, d_1, 3, 3, d_1, \cdots, d_n),$$

where

$$\frac{2}{3} + 2\left(\sum_{i=1}^{n} \frac{1}{d_i}\right) < 1.$$

Let f_n be a hyperbolic rational map whose Julia set is a Cantor set of circles with the combination $\mathcal{C}_n$, where $n \geqslant 1$. The existence of such f_n's are guaranteed by [HP12] (see also [QYY15] for specific examples). For each f_{n+1}, by Lemma 5.2 we obtain a rational map $\widehat{f}_n \in \mathcal{H}_C$ with the same combination $\mathcal{C}_n$ as f_n. According to [QY21, Theorem 2.3], f_n and $\widehat{f}_n$ are quasiconformally conjugate on their corresponding Julia sets. Hence there exists a sequence of quasiconformal mappings $(\phi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}})_{n\geqslant 0}$ such that $\phi_n(J(f_n))$ are buried Julia components of f_{n+1} and $f_n : J(f_n) \to J(f_n)$ is conjugate to $f_{n+1} : \phi_n(J(f_n)) \to \phi_n(J(f_n))$ by the restriction of ϕ_n.

By Lemma 5.1, for $n \geqslant 0$, the conformal dimension of $J(f_n)$ is

$$\dim_C(J(f_n)) = 1 + \alpha_n,$$

where $\alpha_n \in (0, 1)$ is the unique positive root of

$$\frac{2}{3^{\alpha_n}} + 2\left(\sum_{i=1}^{n} \frac{1}{d_i^{\alpha_n}}\right) = 1.$$

In particular $\dim_C(J(f_0)) = 1 + \log 2/\log 3$.

If we choose the sequence $(d_n)_{n\geqslant 1}$ such that it grows to ∞ very fast, then there exists a constant $s_0 < 1$ such that $\alpha_n \leqslant s_0$ for all $n \geqslant 1$. Indeed, one can choose

$$d_n = 9^n \quad \text{and} \quad s_0 = \log(\sqrt{3} + 1)/\log 3.$$

If we choose the sequence $(d_n)_{n\geqslant 1}$ such that it grows to ∞ suitably, then one can obtain that $\lim_{n\to\infty} \alpha_n = 1$. Indeed, one can choose

$$d_n = 3^{n+1} \quad \text{for all } n \geqslant 1.$$

The proof is complete. □

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